Letter

Operator delocalization in quantum networks

Joonho Kim,[1,*] Jeff Murugan,[2,†] Jan Olle,[3,‡] and Dario Rosa[4,§]
[1]Institute for Advanced Study, Princeton, New Jersey 08540, USA
[2]The Laboratory for Quantum Gravity & Strings, Department of Mathematics and Applied Mathematics, University of Cape Town, Cape Town, South Africa
[3]Institut de Física d'Altes Energies (IFAE), The Barcelona Institute of Science and Technology (BIST) Campus UAB, 08193 Bellaterra, Barcelona, Spain
[4]Center for Theoretical Physics of Complex Systems, Institute for Basic Science (IBS), Daejeon 34126, Korea

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We investigate the delocalization of operators in nonchaotic quantum systems whose interactions are encoded in an underlying graph or network. In particular, we study how fast operators of different sizes delocalize as the network connectivity is varied. We argue that these delocalization properties are well captured by Krylov complexity and show, numerically, that efficient delocalization of large operators can only happen within sufficiently connected network topologies. Finally, we demonstrate how this can be used to furnish a deeper understanding of the quantum charging advantage of a class of Sachdev-Ye-Kitaev (SYK)-like quantum batteries.

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I. INTRODUCTION

The conjecture that black holes are the fastest scramblers of information in nature [1] has precipitated a renewed interest into questions of thermalization and ergodicity in quantum systems [2], and ushered in a new era of collaboration between seemingly disparate fields like high energy theory, condensed matter physics, and quantum information. In this regard, one particularly important development in the past five years has been the emergence of the Sachdev-Ye-Kitaev (SYK) model [3],

$$\hat{H}^{(q)}_{\rm SYK} = i^{q/2} \sum_{i_1<\cdots<i_q} J_{i_1\cdots i_q} \hat{\gamma}^{i_1}\cdots\hat{\gamma}^{i_q}, \qquad (1)$$

of disordered Majorana fermions as a canonical framework to study questions from the information-loss paradox in (low-dimensional) quantum gravity to the physics of spin glasses. The SYK model in turn has led to the development of a host of new (or, sometimes, forgotten) tools such as out of time-order correlators (OTOCs), spectral analysis of operators, and computational complexity to attack quantum many-body problems. Indeed, this article arose from our trying to answer the question, *What is it that makes the SYK model so special?* Is it the Majorana fermions? Or its quenched random couplings? Perhaps, it is the all-to-all q-Fermi interactions?

An obvious starting point to answer this question would be to focus on the scrambling properties of the SYK model. Associated with the fact that the SYK$_q$ model (for $q \geqslant 4$) saturates the Maldacena-Shenker-Stanford (MSS) bound [4] on the leading Lyapunov exponent $\lambda_{\rm L} \leqslant 2\pi T$,[1] it was recently argued that scrambling is better understood in terms of the growth of the size of time-evolving operators in the model [5–7]. The idea is that in a scrambling system the probability distribution of the size of the operator, $P_s(t)$, shifts towards larger operators with an initial exponential rate determined by the infinite-temperature chaos exponent.

We start instead from the seemingly very simple observation that scrambling in a many-body system is actually made up of two distinct processes: an initial small operator first grows to a sufficiently large size; at the same time, the grown operator delocalizes over the Hilbert space of large operators. Our main goal in this article will be to study the latter phase only, which we call *operator delocalization*, in as simple (and universal) a setup as possible, to understand how it can be controlled.

To elaborate, in this article we study the SYK$_2$ model. Even though this model is essentially free, the quenched random couplings J_{ij} and Majorana fermions $\hat{\gamma}^i$ endow the system with a rich structure that has garnered much recent attention [7–9]. We go even further and define the model on a graph $G(V, E)$, consisting of a collection of vertices V and edges $E \subseteq V \otimes V$ with the connectivity of the graph encoded into matrix of couplings, J_{ij}, now interpreted as the adjacency matrix of the graph.[2]

The key observation is that, since the SYK$_2$ model is free, we do not expect any operator growth through Hamiltonian evolution [7,11] but this does not mean that the system is trivial. We will show that operator hopping induces nontrivial

*joonhokim@ias.edu
†jeff.murugan@uct.ac.za
‡jolle@ifae.es
§dario_rosa@ibs.re.kr

[1]We will work in units where $\hbar = k_B = 1$.

[2]In the context of scrambling dynamics, the effects of the graph geometry on operator growth have been heavily studied. See, for example, [10].

dynamics of the system which is *heavily* controlled by the underlying graph. We will conjecture that operator delocalization requires two ingredients, i.e., (i) sufficiently nonlocal operators (obtained either by the growing dynamics of initially small operators or directly as initially large operators) and (ii) networks that are able to utilize the nonlocality. At the technical level, we will make use of the notion of *operator complexity*, introduced in [12]. This Krylov, or K-complexity, C_K, describes the delocalization of an operator in a finite dimensional Hilbert space with respect to a specific basis—the Krylov basis—obtained by successive nested commutators.

The following two sections introduce and use the idea of K-complexity to provide supporting evidence for our conjecture. In Sec. IV we show how the conjecture itself, and only the notion of operator delocalization (*without* operator scrambling), can be used to understand the quantum charging advantage exhibited by the SYK quantum batteries introduced in [13]. In our view, this furnishes a concrete example of the utility of studying K-complexity in connection with large operators, going beyond the usual setup in which K-complexity is used to study the dynamics of small operators only.

II. KRYLOV COMPLEXITY FOR FREE MODELS

In this section we review the notion of Krylov complexity [11,12,14]. Our focus, however, will be on the differences that arise between initial operators having small or large size, with the latter not usually considered when dealing with scrambling systems.

To this end, let us start with a given quantum operator, $\hat{\mathcal{O}}$. Our goal will be to efficiently describe its Hamiltonian time evolution,

$$\hat{\mathcal{O}}(t) = e^{i\hat{H}t}\hat{\mathcal{O}}e^{-i\hat{H}t}. \tag{2}$$

We can expand $\hat{\mathcal{O}}(t)$ over the set of the nested commutators of $\hat{\mathcal{O}}$ with $\hat{H}$, called the Krylov space, $\mathscr{H}_{\hat{\mathcal{O}}}$. Out of the Krylov set, we want to find a set of orthonormal operators[3] which can fully reconstruct $\hat{\mathcal{O}}(t)$ at any time t. This subset forms the K-dimensional Krylov basis, $\{\hat{\mathcal{O}}_n\}_{n=0}^{K-1}$, where the subscript n represents the number of commuting operations. In addition, the orthogonalizing coefficients, $\{b_0 \equiv 0,\ b_n\}_{n=1}^{K-1}$, form the set of so-called Lanczos coefficients. In this formalism, the time-evolved operator can be expressed as

$$\hat{\mathcal{O}}(t) = \sum_{n=0}^{K-1} i^n \varphi_n(t)\hat{\mathcal{O}}_n, \tag{3}$$

where $\varphi_n(t)$ satisfy the differential equations

$$\dot{\varphi}_n(t) = b_n\varphi_{n-1}(t) - b_{n+1}\varphi_{n+1}(t). \tag{4}$$

In other words, knowledge of the Lanczos coefficients b_n is enough to determine the operator dynamics. To extract the information encoded in the wave functions $\varphi_n(t)$ in a tractable way, the K-complexity function, $C_K(t)$, can be introduced. It computes the expected number of nested commutators for $\hat{\mathcal{O}}(t)$,

$$C_K(t) = \sum_n n|\varphi_n(t)|^2, \tag{5}$$

with $\sum_n |\varphi_n|^2 = 1$, and plays a pivotal role in our study.

In [12] it has been argued that, by taking a *simple* initial operator, i.e., an operator that can be written as a linear combination of single Majorana fermions, the asymptotic behavior of the coefficients b_n, for $n \lesssim \log D$ can be used to diagnose the chaotic or integrable nature of the Hamiltonian under investigation. In particular, for chaotic models one has $b_n \propto n$,[4] while for integrable models in general one has $b_n \propto n^\alpha$, for some $\alpha < 1$. The extreme case of free models gives $b_n \sim O(1)$.

A crucial consequence of Eqs. (3) and (4) is that, for short enough times, $\hat{\mathcal{O}}(t)$ and $C_K(t)$ are mostly controlled by the b_n's with small n. We then conjecture that at early times there could well be integrable or free models sharing similar physical properties with chaotic models. However, this intuition applies only when considering initial *large* operators, involving products of many $\hat{\gamma}_i$. Such large operators are required because free models, such as SYK_2, do not produce operator growth dynamics; they simply translate operators in the operator space [7,11]. On the other hand, when starting with operators of large size and not requiring any further operator growth, the resulting hopping dynamics can be very close to be chaotic at early times. In particular, we expect that the dynamics of large operators will be highly dependent on the connectivity of the graph which defines the model. In the following section we test this intuition.[5]

III. OPERATOR DELOCALIZATION WITHOUT SCRAMBLING

We now study the time evolution of the K-complexity function, $C_K(t)$, for particular deformations of the SYK_2 model, controlled by the topologies of the graphs over which the models live. Our goal here is to study the extent to which $C_K(t)$ is a good probe to distinguish the properties of the graphs governing the dynamics of the model under investigation.

Consider a set of SYK_2 models defined on graphs, beyond the standard choice of complete graphs that characterize the all-to-all interactions typical of SYK physics. The SYK model is a quantum mechanical model of Majorana fermions in one dimension, consisting of operators $\hat{\gamma}^i$, $i = 1, 2, \ldots, L$, satisfying the Clifford algebra $\{\hat{\gamma}^i,\ \hat{\gamma}^j\} = \delta^{ij}$, with random q-body interactions. In this article, we focus on the case of a quadratic Hamiltonian,

$$\hat{H}^{(2)} = i\sum_{i<j} J_{ij}\hat{\gamma}^i\hat{\gamma}^j. \tag{6}$$

[3]The orthonormality condition is imposed by endowing the space of the operators with the standard Frobenius product $(A|B) = \frac{1}{D}\mathrm{Tr}[A^\dagger B]$, with D being the Hilbert space dimension.

[4]With a logarithmic correction for models defined in 1 dimension which has been studied numerically in very great detail in [15].

[5]It should be stressed that we will focus on early-time dynamics, and only for relatively short times we should expect a qualitative similarity between chaotic systems and non-chaotic systems (but dealing with large operators). For long time evolution, the reader is referred to the discussions in [11,12].

In the usual formulation of the model, the all-to-all coupling constants J_{ij} are randomly extracted from a Gaussian distribution, with vanishing mean and variance $\langle J_{ij}^2 \rangle = J^2/L$, and where the constant J^2 has the dimension of energy. In what follows, we set $J = 1$.

Similar to the sparse SYK models of [16–18], we consider models living on graphs different from the complete graph case described above. To implement this, we replace the matrix of couplings J_{ij} with the adjacency matrix, A_{ij}, of a given graph, $G(L, E)$, with L vertices (one for each Majorana fermion) and E edges. Finally, we multiply each nonvanishing entry of A_{ij} (with $i < j$) with a random number extracted from a Gaussian distribution having vanishing mean and variance $\frac{L-1}{2n_E}$, with n_E denoting the number of edges in $G(L, E)$. This procedure produces a new matrix of couplings, $\tilde{J}_{ij}$, which defines a quadratic Hamiltonian,

$$\hat{H}^{(2)}_{G(L,E)} = i \sum_{i<j} \tilde{J}_{ij} \hat{\gamma}^i \hat{\gamma}^j, \tag{7}$$

in the same fashion as for the complete graph, Eq. (6). To be more concrete, the couplings $\tilde{J}_{ij}$ are explicitly computed by taking the upper diagonal part of the adjacency matrix, A_{ij}, of a given graph and replacing all its unit entries with numbers drawn from a random Gaussian distribution. The lower diagonal part is then obtained by antisymmetry and additional factor of i is added in order to enforce Hermiticity of the Hamiltonian. Among possible networks, *small-world* graphs [19,20] form a distinguished subset. Produced by the so-called Watts-Strogatz algorithm, they parametrically interpolate between a regular lattice and a random Erdös-Renyi graph. The algorithm to generate their adjacency matrix A_{ij}, necessary to build the couplings $\tilde{J}_{ij}$ entering in Eq. (7), is specified in terms of two numbers: an integer k and a probability value $p \in [0, 1]$. For a given value of k, it starts with a regular circulant lattice in which each vertex is connected to its $2k$ nearest neighbors.[6] Edges are subsequently rewired at random with probability p, avoiding self-loops, edge duplication and keeping the graph connected. Examples of small-world networks, and their associated density matrices are shown in Fig. 1. When $p = 0$ and $k = 1$, the resulting SYK_2 model is equivalent to a nearest neighbors tight-binding system, from which no interesting dynamics can arise. On the other hand, by dialling the value of p, networks become highly interconnected and the mean distance between two edges can be very short. We will show that this geometry change has huge impact on the SYK_2 physics.

Given these preliminaries, we have computed the early-time evolution of $C_k(t)$, i.e., for times much shorter than the saturation time, for several choices of k and p and for both small operators (operators of size 1, i.e., the simple operator $\hat{\gamma}^1$) and for large operators (operators having *extensive* size equal to $L/2$, i.e., $\hat{\mathcal{O}}^{(L)} \equiv \prod_{i=1}^{L/2} \hat{\gamma}^i$). For comparison, we have also computed the time evolution of the K-complexity for the fully connected SYK_2 model. In all cases, to remove a possible source of spurious effects, we have normalized the Hamiltonians to have unit bandwidth, i.e., we imposed that the difference between the largest and the smallest eigenvalue is equal to 1. The results are reported in Fig. 2.

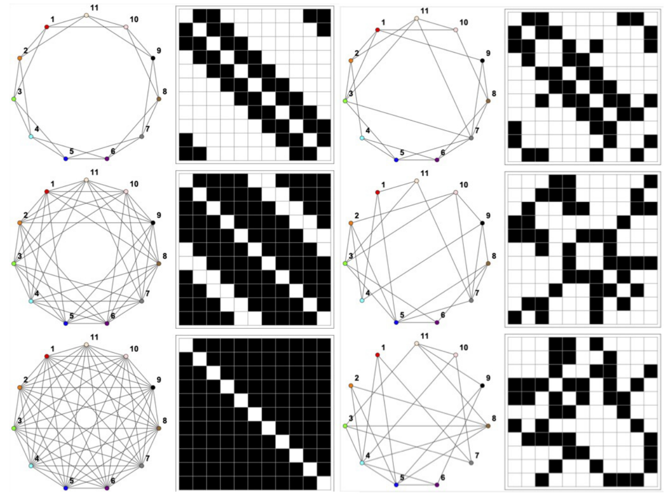

FIG. 1. The Watts-Strogatz algorithm can be visualized best in terms of network diagrams and their associated adjacency matrices, here represented by an $N \times N$ array coded black where a connection exists and white otherwise. In the above, for example, we take $N = 11$. The left set of networks all have $p = 0$ with $k = 2$, 4, and 10, reading from top to bottom. The regularity of these graphs are clear in the adjacency matrices.The set of graphs on the right implement the Watts-Strogatz algorithm on the $k = 2$, $N = 11$ lattice with p increasing from top to bottom. The increasing randomness seen in the graph is reflected in the increasingly crossword-puzzle-like resemblance of the corresponding adjacency matrices.

There are a number of points we wish to draw attention to. First, the underlying graph plays essentially no role for *small* operators. In all cases, $C_K(t)$ displays a slow growth and the full SYK curve is just barely distinguishable from the small-world curves, at both high and low rewiring

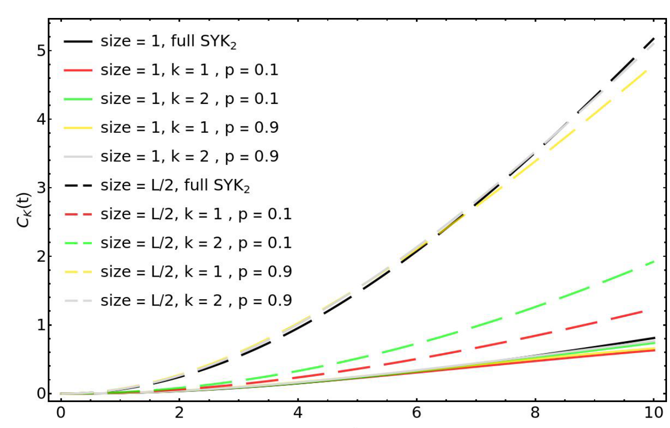

FIG. 2. $C_K(t)$ for systems having $L = 24$. $C_K(t)$ is computed for small (size 1) and large (size $L/2$) operators, for the full SYK_2 model, compared against the Watts-Strogatz Hamiltonians having $k = 1$, 2 and both *low* and *large* rewiring probability ($p = 0.1$ and $p = 0.9$, respectively). The results are averaged over 1000 realizations of disorder and graph.

[6]In the language of graph theory, $2k$ is the fixed degree of each node in the graph.

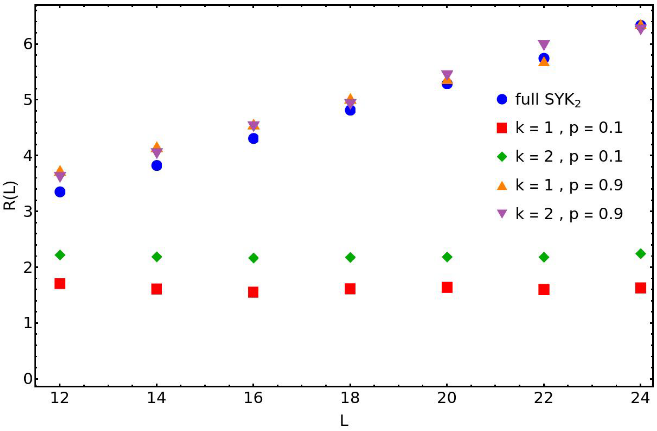

FIG. 3. The quantity $R(L)$, computed at different system sizes and for the full SYK_2 model, compared against the Watts-Strogatz Hamiltonians having $k = 1$, 2 and both *low* and *large* rewiring probability ($p = 0.1$ and $p = 0.9$, respectively). All the results are averaged over 1000 different realizations of disorder and underlying graph.

probabilities. This is just another manifestation of the fact that, irrespective of the underlying quantum network, the SYK_2 model is not a scrambling system. As such, it cannot create any operator size and so does not utilize the connectivity of the underlying graph. The situation changes drastically when *large* operators are involved: here, we see that the evolution of the K-complexity function which, as already mentioned, quantifies the delocalization properties of the system, varies dramatically with a change of the graph topology. In particular, for low rewiring probabilities, the Watts-Strogatz Hamiltonians exhibit much smaller values of $C_K(t)$ compared to the full SYK_2 Hamiltonian. On the other hand, when p is large, the Krylov complexity for quantum small-world graphs is essentially equivalent to the corresponding function for the full SYK_2 model.

Another interesting figure of merit to quantify the ability of a given graph to delocalize large operators is given by the ratio $R(t)$, between $C_K(t)$ for operators of size $L/2$ and operators of size 1. As is evident from its definition, $R(t)$ measures how good a given graph is in utilizing operators of large size, normalized by the delocalizing properties computed for small operators. Interestingly, we note that $R(t)$ is essentially *time independent*. It therefore makes sense to consider instead the quantity $R(L)$, defined as such a constant ratio and computed as a function of the system size, L. Our results are depicted in Fig. 3. The main feature of note is that the difference between the highly connected and poorly connected graph is now *quantitatively* clear; for highly connected graphs, $R(L)$ scales with the system size, a feature shared with the full SYK_2 Hamiltonian. On the other hand, poorly connected graphs do not show any scaling behavior for $R(L)$. This lacking of a scaling happens because, without long range interactions, the early-time physics is dominated by the *local* features of the graph and, in particular, the system size does not affect the dynamics.

Taken together, these results show that, when scrambling dynamics is absent, the topology (and in particular the connectivity) of the graph over which the model is defined becomes the crucial ingredient to understand how large operators delocalize under quantum evolution. We note also that the total number of connections in the model is irrelevant. This is seen from the similarities in delocalization properties of the Watts-Strogatz Hamiltonians and the full SYK_2 model at large p, where the latter has $O(L^2)$ edges against the $O(L)$ edges of the former. This property is the analog, for nonscrambling models, of the results discussed in [16–18] for the sparse SYK_4 models, which exhibit similar behavior to the full model, but with significantly fewer nonvanishing couplings.

IV. AN APPLICATION TO THE QUANTUM CHARGING ADVANTAGE OF SYK-LIKE QUANTUM BATTERIES

The K-complexity framework can be used to understand the quantum charging advantage of SYK-like quantum batteries [13]. We report here the main results, and defer details to the Supplemental Material [21].

An SYK quantum battery (see [22,23] for an overview of the topic) is built by considering a system prepared in the ground state, $|0\rangle$, of a static initial Hamiltonian of the form $\hat{H}_0 = h\sum_{i=1}^{L/2}\hat{\sigma}_i^x$, where h denotes a constant magnetic field, oriented along the x axis (which we will set equal to 1), and $\hat{\sigma}_i^a$, with $a = x, y, z$, are the usual Pauli operators, defined on a spin chain of length $L/2$. At $t = 0$ the system is suddenly coupled, via a standard Jordan-Wigner map, to an SYK Hamiltonian and evolved under the quantum quench. The average charging power of the battery reads

$$P_{\rm av}(t) = \frac{\overline{\langle\psi(t)|\hat{H}_0|\psi(t)\rangle} - \langle 0|\hat{H}_0|0\rangle}{t}, \tag{8}$$

where $|\psi(t)\rangle$ denotes the evolved state at time t, $\langle 0|\hat{H}_0|0\rangle$ is the ground state energy, and $\overline{\langle\psi(t)|\hat{H}_0|\psi(t)\rangle}$ measures the energy stored in the battery (averaged over disorder and possible graph realizations). By quenching with an SYK_4 Hamiltonian (rescaled to unit bandwidth), it was found in [13] that the maximum value of the average power, $P_{\max}$, scales with L, signaling a quantum charging advantage [24–26].

A crucial point, first noticed in [7] and proved in full generality in [26], is that such charging advantage strongly relies on the fact that the $\hat{\sigma}_i^x$, when written in terms of $\hat{\gamma}^i$, have *very large* size. This in turn suggests that charging advantage may also be obtained from SYK_2 models on highly connected graphs.

This intuition can indeed be confirmed for the full SYK_2 model: after considering the ensemble average over the Gaussian couplings, $P_{\rm av}(t)$ takes the form

$$P_{\rm av}(t) \propto \frac{\overline{\varphi_0^{(\hat{H}_0)}(t)} - 1}{t}, \tag{9}$$

where $\overline{\varphi_0^{(\hat{H}_0)}(t)}$ is the (averaged over disorder) first wave function, defined as in Eq. (3), in the Krylov expansion of $\hat{H}_0$. In particular, Eq. (9) expresses the relationship between the average power and the delocalizing properties of the quench Hamiltonian, quantified by the Krylov complexity. For the more general Watts-Strogatz Hamiltonians, Eq. (9), which is valid for a *fixed* graph topology, must be supplemented with an

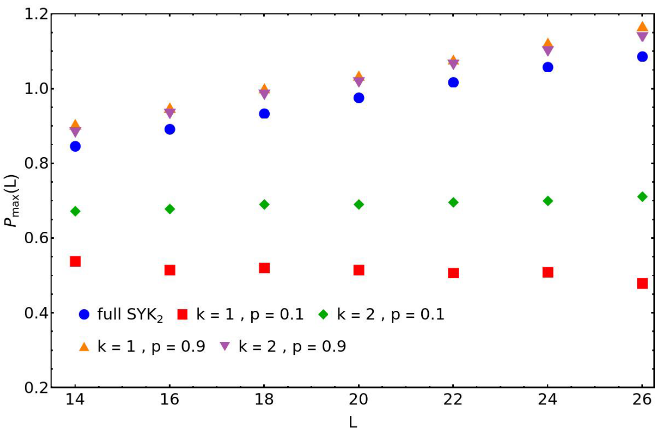

FIG. 4. The maximum charging power, $P_{\max}(L)$ for the same models considered in Fig. 3.

additional average over the graph topology. As an illustration, Fig. 4 displays the result of our numerical computation of $P_{\max}(L)$ for several Watts-Strogatz Hamiltonians, compared against the full SYK_2 charging power. This clearly matches the analogous results obtained in Fig. 3 and demonstrates that K-complexity and the delocalization properties are indeed the relevant quantities to understand the quantum charging advantage of SYK quantum batteries. The scaling, exhibited by highly connected graphs, shows that interaction connectivity is sufficient to obtain such an advantage.

V. CONCLUSIONS

SYK-like models defined on graphs offer a versatile class of quantum systems to explore many-body localization, thermalization, and chaos [16–18,27]. Previous studies that investigated the preservation of chaotic properties by the graph topology have focused primarily on the strongly interacting SYK_4 Hamiltonians. In this paper, to better disambiguate spectral properties from those hinging on the (hyper)graph structure, we have focused instead on the free SYK_2 model defined on various graphs. The spectral properties of the system are then rather trivial, and any nontriviality must be a consequence of the underlying graph structure. Here, we have shown that, as long as operators of *sufficiently large size* are taken into account, the dynamics is far from trivial and in particular the K-complexity function, $C_K(t)$, is highly dependent on the geometry and connectivity of the graph. In turn, this observation has led us to propose the notion of "operator delocalization," describing how large operators delocalize under the operator hopping dynamics [7,11].

As an application of these ideas, we have shown also that the quantum charging advantage of SYK quantum batteries, found in [13], is a direct consequence of operator delocalization and, as such, relies only on the underline graph topology.

There are many intriguing future directions worth pursuing. It would be of significant interest to find other physical quantities which are highly sensitive to operator delocalization only. Another interesting direction would be to understand how operator delocalization depends on the statistics of the evolving operators. Of particular interest, given their relevance in condensed matter systems, would be the study of SYK-like models built from parafermionic operators [28–30].

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[21] See Supplemental Material at http://link.aps.org/supplemental/10.1103/PhysRevA.105.L010201 for more details on the quantum charging advantage of the SYK-like quantum batteries, for a perturbative proof of their charging advantage in the thermodynamic limit and for the description of a particular case of small-world batteries which is analytically solvable.

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