Fourth-order moment of the light field in the atmosphere for moderate and strong turbulence

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The collisionless Boltzmann equation is used to describe the intensity correlations in partially saturated and fully saturated regimes in terms of photon distribution function in the phase space. Explicit expression for fourth moment of the light fields is obtained for the case of moderate and strong turbulence. Such expression consists of two terms accounting for two regions in the phase space that independently contribute to the correlation function. It is shown that the present solution agrees with previous results for the fully saturated regime. Additionally it embodies the effect of partially saturated radiation where the correlations of photon trajectories are important and the magnitude of the scintillation index is well above the unity. The fourth moment is used to study the fluctuations of transmittance, which consider the effect of finite detector aperture.

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I. INTRODUCTION

Propagation of light in atmospheric channel is an essential part of many new-age applied areas such as quantum key distribution [1,2], satellite-ground communication [3,4], quantum teleportation [5–7], propagation of entangled, and squeezed states [8–11]. Altogether these areas contribute to the development of the next-generation quantum and classical communication systems including quantum internet [12], quantum protocols [13,14], etc. However, spatiotemporal properties of light in the atmosphere are modified drastically in a way of propagation limiting current applications.

Fluctuations of refractive index in Earth's atmosphere introduce random distortions to the phase of waves. Since the range of sizes of optical inhomogeneities is very wide, from millimeters (inner scale of turbulence, l_0) to hundreds of meters (outer scale of turbulence, L_0), laser beam is exposed to the bunch of various negative effects: beam spreading, beam wandering, fragmentation, beam jitter, intensity fluctuations, etc. [15–19]. All of them affect statistical and spatiotemporal properties of the light radiation causing additional to absorption and scattering losses in atmosphere and impairing the performance of free-space communication systems.

Although intensity fluctuations play a critical role in the transmission of optical signal, their description remains one of the most challenging problems in free-space optics. Due to multiple scattering on atmospheric inhomogeneities primary optical waves are gradually randomized so initially coherent laser radiation acquires some properties of Gaussian statistics [20]. Such gradual change of statistical properties complicates theoretical analysis of correlation properties (fourth-order moment) of propagating radiation. One of such obstacle is so-called saturation of intensity fluctuations [21]. In the course of propagation scintillation index, the inverse of signal-to-noise ratio, after steep growing and hitting the maximum value, starts to decline and asymptotically approach unity. Further we refer to the asymptotic regime as fully saturated and to the region between maximum and asymptotic values for intensity fluctuations as partially saturated regime. Such levels of saturation are naturally derived from degree of randomization of primary optical waves.

Some general approaches involving equations of evolution for fourth-order moments were proposed in early studies [15,16]. However, the applicability of these equations is limited due to their complexity. Besides that, correlations of intensity were studied in terms of scintillation index and covariance function [18,22–24]. Nevertheless, there is still no rigorous theory of fourth moment for optical fields in atmosphere for moderate and strong turbulence (partially saturated and fully saturated regimes).

Here we use the method of photon distribution function (PDF) in the phase space [25] to describe laser radiation in atmosphere. PDF is defined as photon density in coordinate-momentum space (phase space) and can be regarded as quantum generalization of classic radiant intensity in radiative transfer methods [16,26]. Intensity of the light and correlation properties of radiation are derived from PDF moments. The method of PDF was successfully applied to the problem of light propagation in atmospheric channels [27–30]. Particularly it proved to be effective for study of intensity fluctuations (scintillation) in the range of moderate turbulence regime [31,32] and fourth-order moment problem in asymptotic case of large propagation distances, $z \to \infty$ [33]. In the current paper, we use the Boltzmann kinetic equation for PDF for the description of partially saturated and saturated regimes where only substantial change of photon momenta due to atmospheric turbulence takes place [25]. In this case the effect of turbulence is enclosed in random force originating from the gradient of refractive index.

The paper is devoted to derivation of the expression for fourth moment applicable for the case of moderate-to-strong

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and strong turbulence regimes and analysis of role of degrees of randomization of the primary optical waves. In contrast to the fully saturated model [33], the present model describes much wider range of atmospheric channels. It includes description of channels where intensity fluctuations are the biggest, i.e., the scintillation index is significantly above unity. Such objectives are viable for much applied research including intensity correlation [34,35], enhanced focusing [36,37], different imaging problems [38–42], reconstruction of the probability distribution of transmittance [43], and its applications to communication protocols [44]. In this paper we account for the effect of multiple collisions with turbulent inhomogeneities, which leads to the change of radiation statistics to the Gaussian one. In this case only particular volume in the phase space contributes to the correlation function.

The remainder of this paper is organized as follows. In Sec. II, we provide review of photon distribution function approach applied to laser beam propagation in atmosphere. In Sec. III, explicit expression for fourth moment of the light fields and its analysis are presented. Section IV is devoted to estimation of transmittance fluctuations and its dependence on the size of detector aperture. In Appendix, we give detailed derivation of the expressions for fourth-moment terms.

II. PRELIMINARIES

Photon distribution function

The photon distribution function resembling the idea of distribution functions in physics of solids [45,46] is given by [25,47]

$$\hat{f}(\mathbf{r},\mathbf{q},t)=\frac{1}{V}\sum_{\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}b^{\dagger}_{\mathbf{q}+\mathbf{k}/2}b_{\mathbf{q}-\mathbf{k}/2}, \tag{1}$$

where $b^{\dagger}_{\mathbf{q}}$ and $b_{\mathbf{q}}$ are the quantum amplitudes of bosonic photon field with the wave vector $\mathbf{q}$; $V\equiv L_xL_yL_z\equiv SL_z$ is the normalizing volume. All operators are given in the Heisenberg representation. The laser beam propagates in the z direction. It is assumed that $\mathbf{k}_\perp,\mathbf{q}_\perp\ll q_0$ where q_0 is the wave vector corresponding to the central frequency ω_0 of radiation $\omega_0=cq_0$, c is the speed of light in a vacuum. Such assumption justifies the paraxial approximation. The initial polarization of light left out of consideration in this case as for a wide range of propagation distances it remains almost constant (see Ref. [48]).

The Hamiltonian of photons in a medium with a fluctuating refractive index could be derived from representation of energy in inhomogeneous media [49]

$$H=\sum_{\mathbf{k}}\hbar\omega_{\mathbf{k}}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}}-\sum_{\mathbf{k},\mathbf{k}'}\hbar\omega_{\mathbf{k}}n_{\mathbf{k}'}b^{\dagger}_{\mathbf{k}}b_{\mathbf{k}+\mathbf{k}'}, \tag{2}$$

where $\hbar\omega_{\mathbf{k}}\equiv\hbar ck$ is the photon energy, and $n_{\mathbf{k}}$ is the Fourier transform of the refractive index fluctuations $\delta n(\mathbf{r})$. The Fourier transform is defined by

$$n_{\mathbf{k}}=\frac{1}{V}\int dVe^{i\mathbf{k}\mathbf{r}}\delta n(\mathbf{r}). \tag{3}$$

Usually, δn is assumed to be a Gaussian random variable with known covariance $\langle\delta n(\mathbf{r})\delta n(\mathbf{r}')\rangle$. The covariance is defined by its Fourier transform, $\psi(\mathbf{g})$, with respect to the difference $\mathbf{r}-\mathbf{r}'$. In a statistically homogeneous atmosphere it can be written as

$$\langle\delta n(\mathbf{r}-\mathbf{r}')\delta n(0)\rangle=\int d\mathbf{g}e^{-i\mathbf{g}(\mathbf{r}-\mathbf{r}')}\psi(\mathbf{g}). \tag{4}$$

The evolution equation for PDF is derived from Heisenberg's equation. Although in more general case its evolution is gathered by Boltzmann-Langevin equation [31], which takes into account the whole range of possible changes in photon momentum due to collisions with atmosphere inhomogeneities, for reasonably long distances (see Ref. [32]) description with collisionless Boltzmann equation

$$\partial_t\hat{f}(\mathbf{r},\mathbf{q},t)+\mathbf{c}_{\mathbf{q}}\cdot\partial_{\mathbf{r}}\hat{f}(\mathbf{r},\mathbf{q},t)+\mathbf{F}(\mathbf{r})\cdot\partial_{\mathbf{q}}\hat{f}(\mathbf{r},\mathbf{q},t)=0 \tag{5}$$

is justified, where $\mathbf{c}_{\mathbf{q}}=\partial_{\mathbf{q}}\omega_{\mathbf{q}}$. In this case the effect of atmospheric turbulence is enclosed in random smooth force $\mathbf{F}(\mathbf{r})=\omega_0\partial_{\mathbf{r}}n(\mathbf{r})$.

The general solution of (5) is obtained by characteristics method

$$\hat{f}(\mathbf{r},\mathbf{q},t)=\phi\left\{\mathbf{r}-\int_0^t dt'\frac{\partial\mathbf{r}(t')}{\partial t'};\mathbf{q}-\int_0^t dt'\frac{\partial\mathbf{q}(t')}{\partial t'}\right\}, \tag{6}$$

where evolution of PDF is described in terms of the classical trajectories of photons

$$\frac{\partial\mathbf{r}(t')}{\partial t'}=\mathbf{c}[\mathbf{q}(t')], \tag{7}$$

$$\frac{\partial\mathbf{q}(t')}{\partial t'}=\mathbf{F}[\mathbf{r}(t')], \tag{8}$$

the function $\phi(\mathbf{r},\mathbf{q})$ is the initial value of $\hat{f}(\mathbf{r},\mathbf{q},t)$ in the aperture plane of the source, i.e.,

$$\phi(\mathbf{r},\mathbf{q})=\frac{1}{V}\sum_{\mathbf{k}}e^{-i\mathbf{k}\mathbf{r}}\left(b^{+}_{\mathbf{q}+\frac{\mathbf{k}}{2}}b_{\mathbf{q}-\frac{\mathbf{k}}{2}}\right)\Big|_{t=0}\equiv\sum_{\mathbf{k}}e^{-i\mathbf{k}\mathbf{r}}\phi(\mathbf{k},\mathbf{q}). \tag{9}$$

Under paraxial approximation atmosphere mostly affects divergence of the beam, $\mathbf{q}_\perp$, and has negligible influence on the longitudinal components (z axis). Then, Eq. (6) can be written as

$$\hat{f}(\mathbf{r},\mathbf{q},t)=\phi\left\{\mathbf{r}-\mathbf{c}_{\mathbf{q}}t+\frac{c}{q_0}\int_0^t dt't'\mathbf{F}_\perp[\mathbf{r}(t')];\right.$$
$$\left.\mathbf{q}^{-}\int_0^t dt'\mathbf{F}_\perp[\mathbf{r}(t')]\right\}. \tag{10}$$

Assuming the initial configuration for laser radiation is known, the first and second moments of $\hat{f}$, which describe beam intensity and its correlations, could be obtained by means of the iterative procedure for powers of $\mathbf{F}_\perp(\mathbf{r})$ [25,32].

III. INTENSITY CORRELATIONS

The intensity (density of photons in the spatial domain) is derived from operator $\hat{f}(\mathbf{r},\mathbf{q},t)$ by summation over all values of $\mathbf{q}$

$$\hat{I}(\mathbf{r},t)=\sum_{\mathbf{q}}\hat{f}(\mathbf{r},\mathbf{q},t)=\frac{1}{V}\sum_{\mathbf{q},\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{r}}b^{\dagger}_{\mathbf{q}+\mathbf{k}/2}b_{\mathbf{q}-\mathbf{k}/2}. \tag{11}$$

The second-order moment for the field operators $\Gamma_2(\mathbf{r}) \equiv \langle \hat{I}(\mathbf{r}, t)\rangle$, respectively. Consequently, the intensity correlations, or fourth-order moment, is defined by

$$\Gamma_4(\mathbf{r}, \mathbf{r}') \equiv \langle \hat{I}(\mathbf{r}, t)\hat{I}(\mathbf{r}', t)\rangle = \frac{1}{V^2}\sum_{\substack{\mathbf{q},\mathbf{k},\\ \mathbf{q}',\mathbf{k}'}} e^{-i(\mathbf{k}\cdot\mathbf{r}+\mathbf{k}'\cdot\mathbf{r}')}\big\langle b^\dagger_{\mathbf{q}+\frac{\mathbf{k}}{2}} b_{\mathbf{q}-\frac{\mathbf{k}}{2}} b^\dagger_{\mathbf{q}'+\frac{\mathbf{k}'}{2}} b_{\mathbf{q}'-\frac{\mathbf{k}'}{2}}\big\rangle. \tag{12}$$

From here on the averaging $\langle \dots \rangle$ includes the quantum-mechanical averaging of operators $\hat{I}$ and averaging over different configurations of atmospheric turbulence. Both averaging can be performed independently.

It was shown in the recent research [33] that due to saturation effect for fluctuations in asymptotic case of large distances, $z \to \infty$, fourth moment may be expressed via second moments

$$\begin{aligned}\big\langle b^\dagger_{\mathbf{q}+\frac{\mathbf{k}}{2}} b_{\mathbf{q}-\frac{\mathbf{k}}{2}} b^\dagger_{\mathbf{q}'+\frac{\mathbf{k}'}{2}} b_{\mathbf{q}'-\frac{\mathbf{k}'}{2}}\big\rangle &\approx \big\langle b^\dagger_{\mathbf{q}+\frac{\mathbf{k}}{2}} b_{\mathbf{q}-\frac{\mathbf{k}}{2}}\big\rangle\big\langle b^\dagger_{\mathbf{q}'+\frac{\mathbf{k}'}{2}} b_{\mathbf{q}'-\frac{\mathbf{k}'}{2}}\big\rangle + \big\langle b^\dagger_{\mathbf{q}+\frac{\mathbf{k}}{2}} b_{\mathbf{q}'-\frac{\mathbf{k}'}{2}}\big\rangle\big\langle b^\dagger_{\mathbf{q}'+\frac{\mathbf{k}'}{2}} b_{\mathbf{q}-\frac{\mathbf{k}}{2}}\big\rangle \\ &= n_{\mathbf{q}} n_{\mathbf{q}'}\delta_{\mathbf{k},0}\delta_{\mathbf{k}',0} + n_{\mathbf{q}+\frac{\mathbf{k}}{2}} n_{\mathbf{q}-\frac{\mathbf{k}}{2}}\delta_{\mathbf{q},\mathbf{q}'}\delta_{\mathbf{k},-\mathbf{k}'},\end{aligned} \tag{13}$$

where $n_{\mathbf{q}} \equiv \langle b^\dagger_{\mathbf{q}} b_{\mathbf{q}}\rangle$. Also it is assumed that initial laser radiation is in a multiphoton coherent state, so the shot-noise term is omitted. Expression (13) is legitimate if the amplitudes $b^\dagger$ and b obey Gaussian statistics. In other words for $t \to \infty$ it is assumed that each primary coherent electromagnetic wave experiences multiple scatterings by randomly distributed turbulent eddies [20,23] and the radiation becomes fully saturated.

At large but finite z, in partially saturated regime, field amplitudes are also effected by random scattering on atmospheric inhomogeneities but still preserve some properties of initial statistics. In this case one should consider nondiagonal terms in four-wave correlations and two regions of wave vectors space, derived from two terms of Eq. (13), where pair correlations of the field operators should be taken into account [25]: (i) $k, k' \leqslant R_b^{-1}$, (ii) $|\mathbf{q}-\mathbf{q}'+(\mathbf{k}+\mathbf{k}')/2|$, $|\mathbf{q}-\mathbf{q}'+(\mathbf{k}+\mathbf{k}')/2| \leqslant R_b^{-1}$, where $R_b^2 \equiv \langle \mathbf{r}^2\rangle_T$ (see Ref. [33]) is the turbulent part of beam radius, $R_b^2 = 4z^3c\alpha/(3\omega_0^2)$, $\alpha = 0.5\pi\omega_0^2 c^{-1}\int d\mathbf{g} g^2\psi(g)$, and r_0 is the initial radius of the beam. The region (ii) considers correlation between different pairs of waves $b^\dagger_{\mathbf{q}+\mathbf{k}/2}$, $b_{\mathbf{q}'-\mathbf{k}'/2}$ and $b^\dagger_{\mathbf{q}'+\mathbf{k}'/2}$, $b_{\mathbf{q}-\mathbf{k}/2}$. Such region could be distinguished from (i) only if turbulent contribution to divergence of the beam, $\mathbf{q}_\perp$, is sufficiently large. In other words, turbulent term should be dominant to initial radius of the beam and the diffraction term, $R_b^2 > r_0^2, 4z^2q_0^{-2}r_0^{-2}$, to distinguish certain levels of saturation of the fluctuations, i.e., partially saturated and fully saturated regimes. Regions (i) and (ii) approach each other when the turbulence effect becomes weaker, and in the limit of small turbulence they join into a single region [23,28].

Exploiting the approach from Refs. [25,32], for these two regions corresponding terms of fourth moment are obtained for the case of Gaussian beams (see Appendix for details), $\Gamma_4(\mathbf{r}, \mathbf{r}') = \Gamma_4^{(i)}(\mathbf{r}, \mathbf{r}') + \Gamma_4^{(ii)}(\mathbf{r}, \mathbf{r}')$:

$$\begin{aligned}\Gamma_4^{(i)}(\mathbf{r}, \mathbf{r}') = 2\pi C\int_0^\infty d\tilde{q}\tilde{q}\Big[F_1^{\tilde{q},\rho_\parallel}F_2^{\tilde{q},\rho_\parallel}\big(F_3^{\tilde{q},\rho_\parallel}H_1^{\tilde{q},\rho_\parallel} - {G_1^{\tilde{q},\rho_\parallel}}^2\big)\big(F_4^{\tilde{q},\rho_\parallel}H_2^{\tilde{q},\rho_\parallel} - {G_2^{\tilde{q},\rho_\parallel}}^2\big)\Big]^{-\frac{1}{2}} \\ \times \exp\left\{-\frac{\Big(\tilde{q} - \rho_\parallel\frac{G_1^{\tilde{q},\rho_\parallel}}{2F_3^{\tilde{q},\rho_\parallel}}\Big)^2}{\Big(H_1^{\tilde{q},\rho_\parallel} - {G_1^{\tilde{q},\rho_\parallel}}^2/F_3^{\tilde{q},\rho_\parallel}\Big)}\right\}\exp\left\{-\frac{\rho_\perp^2 H_2^{\tilde{q},\rho_\parallel}}{\Big(H_2^{\tilde{q},\rho_\parallel}F_4^{\tilde{q},\rho_\parallel} - {G_2^{\tilde{q},\rho_\parallel}}^2\Big)}\right\}\exp\left\{-\left(\frac{\rho_\parallel'^2}{F_1^{\tilde{q},\rho_\parallel}} + \frac{\rho_\perp'^2}{F_2^{\tilde{q},\rho_\parallel}} + \frac{\rho_\parallel^2}{4F_3^{\tilde{q},\rho_\parallel}}\right)\right\},\end{aligned} \tag{14}$$

$$\begin{aligned}\Gamma_4^{(ii)}(\mathbf{r}, \mathbf{r}') = 2\pi C\int_0^\infty d\tilde{q}\tilde{q}\Big[F_1^{\tilde{q}}F_2^{\tilde{q}}\big(F_3^{\tilde{q}}H_1^{\tilde{q}} - {G_1^{\tilde{q}}}^2\big)\big(F_4^{\tilde{q}}H_2^{\tilde{q}} - {G_2^{\tilde{q}}}^2\big)\Big]^{-\frac{1}{2}} \\ \times \exp\left\{-\frac{\tilde{q}^2}{H_1^{\tilde{q}} - {G_1^{\tilde{q}}}^2/F_3^{\tilde{q}}}\right\}\exp\left\{-\left(\frac{\rho_\parallel'^2}{F_1^{\tilde{q}}} + \frac{\rho_\perp'^2}{F_2^{\tilde{q}}}\right)\right\}\exp\left\{-i2\tilde{q}\rho_\parallel\frac{q_0}{z}\right\},\end{aligned} \tag{15}$$

where $\boldsymbol{\rho} = \mathbf{r} - \mathbf{r}'$, $\boldsymbol{\rho}' = (\mathbf{r} + \mathbf{r}')/2$ are two-dimensional (2D) vectors transverse to propagation direction z; F, G, H are functions of $\rho_\parallel$ and $\tilde{q}$ for (i) and functions of $\tilde{q}$ for (ii) contributions to Γ_4; C is constant derived from total flux. It is worth to emphasize that while intensity correlations are evaluated for two points $\mathbf{r}$ and $\mathbf{r}'$, the expression for fourth moment is expressed via the difference $\mathbf{r} - \mathbf{r}'$ and the center-of-mass position $(\mathbf{r} + \mathbf{r}')/2$. Although the dependence on $\boldsymbol{\rho}$, which accounts for the correlations of different trajectories, is quite intricate, the dependence on $\boldsymbol{\rho}'$ has a simple Gaussian form. The latter could be favorable for the calculation of integral quantities (transmittance fluctuations, beam wandering, etc.) that consider spatial distribution of the radiation in detector aperture plane.

Fourth moment $\Gamma_4(\mathbf{r}, \mathbf{r}')$ characterize spatial correlation properties of the laser beam in (x, y) plane in atmosphere. Functions F, G, H (see Appendix) incorporate the effect of correlation for different photon trajectories with $\{\mathbf{r}, \mathbf{q}\}$ and $\{\mathbf{r}', \mathbf{q}'\}$ on intensity correlations. As it was shown in Refs. [25,32] such correlations of trajectories are responsible for intensity fluctuations in the range of moderate and strong turbulence.

For the case of asymptotically large distances, $z \to \infty$, cross-correlation term vanishes [see (A7), (A8), (A13)–(A20)] due to randomization of the particle displacements from the straight lines, so functions F, G, and H do not depend on $\tilde{q}$ and ρ. Therefore, the values of functions are expressed via R_b: $F \approx R_b^2/2$, $G \approx 3R_b^2/4$, $H \approx 3R_b^2/2$. (Free-space terms are omitted since turbulence is assumed to give dominant contribution.) Also since cross correlation is vanished the integration over directions for $\tilde{\mathbf{q}}$ is preserved

$$\Gamma_4^{(i)}(\mathbf{r},\mathbf{r}') = C\iint d\tilde{\mathbf{q}}[F(FH-G^2)]^{-1}\exp\left\{-\frac{\left(\tilde{\mathbf{q}}-(\mathbf{r}-\mathbf{r}')\frac{G}{2F}\right)^2}{(H-G^2/F)}\right\}\exp\left\{-\left(\frac{\mathbf{r}^2}{2F}+\frac{\mathbf{r}'^2}{2F}\right)\right\}, \tag{16}$$

$$\Gamma_4^{(ii)}(\mathbf{r},\mathbf{r}') = C\iint d\tilde{\mathbf{q}}[F(FH-G^2)]^{-1}\exp\left\{-\frac{\tilde{\mathbf{q}}^2}{(H-G^2/F)}\right\}\exp\left\{-\left(\frac{(\mathbf{r}+\mathbf{r}')^2}{4F}\right)\right\}\exp\left\{-i2\tilde{\mathbf{q}}(\mathbf{r}-\mathbf{r}')\frac{q_0}{z}\right\}, \tag{17}$$

In this case it is easy to perform integration analytically. The contribution $\Gamma_4^{(i)}(\mathbf{r},\mathbf{r}')$ can be expressed via average intensity, $\langle\hat{I}(\mathbf{r})\rangle \propto \exp\{-(\frac{\mathbf{r}^2}{R_b^2})\}/R_b^2$ (see Ref. [33]) and $\Gamma_4^{(ii)}(\mathbf{r},\mathbf{r}')$ has a simple Gaussian form, so

$$\Gamma_4(\mathbf{r},\mathbf{r}') = \langle\hat{I}(\mathbf{r},t)\rangle\langle\hat{I}(\mathbf{r}',t)\rangle + C\pi\left(\frac{1}{F}\right)^2 \times \exp\left[-\frac{(\mathbf{r}+\mathbf{r}')^2}{4F} - \frac{(\mathbf{r}-\mathbf{r}')^2 q_0^2(H-G^2/F)}{z^2}\right], \tag{18}$$

which is exactly the result of Ref. [33], taking into account that $q_0^2(H-G^2/F)/z^2 \approx \langle\mathbf{q}^2\rangle_T/8$, where change of photon momentum caused by atmospheric turbulence $\langle\mathbf{q}^2\rangle_T = 4\alpha t$, and corresponding relation between constants.

Applicability of approximation

The applicability of expressions (14) and (15) is inherent from main approximations of the approach, i.e., collisionless Boltzmann equation and the concept of photon trajectories. As it was pointed out in Ref. [25] all components of the momentum of photons should be much bigger than characteristic wave vectors of turbulence. Effect of turbulence is negligible for q_z because of its large values, so one should consider only the transverse momentum. Since the upper limit of the spectrum is defined by the inner scale of the turbulence, l_0, the relation $\langle\mathbf{q}^2\rangle_T l_0^2$ should be large enough. On the other hand, throughout the paper we account for the effect of correlation of photon trajectories, so such concept should be justified. To consider photons as particles, whose density in the $(\mathbf{r},\mathbf{q})$ domain is defined by the distribution function $\hat{f}(\mathbf{r},\mathbf{q},t)$, the uncertainty of the momentum $\mathbf{q}$ should be small [32]. The value of the uncertainty can be estimated from the definition of the distribution function (1) as $\mathbf{k}/2$. Initial values of $\mathbf{q}$ and $\mathbf{k}/2$ are equal and proportional to inverse of initial radius r_0 [25]. For large distances and Tatarskii spectrum [19], $\psi(\mathbf{g}) = 0.033C_n^2\exp\{-(gl_0/2\pi)^2\}g^{-11/3}$, where C_n^2 is known as the index-of-refraction structure constant, such uncertainty is estimated by the relation

$$\langle\mathbf{q}^2\rangle_T R_b^2 \approx 15q_0^2 l_0^{-2/3}C_n^4 z^4, \tag{19}$$

which also should be large compared to unity.

IV. FLUCTUATIONS OF TRANSMITTED RADIATION

For many practical cases, e.g., development classical and quantum communication, the fluctuation of transmittance in Earth's atmosphere is a key parameter that defines the properties of atmospheric channel [50–52]. The magnitude of fluctuations is estimated via variance

$$\sigma_\eta^2 = \frac{\langle\hat{\eta}^2\rangle - \langle\hat{\eta}\rangle^2}{\langle\hat{\eta}\rangle^2}, \tag{20}$$

where transmittance of the optical channel is defined as

$$\hat{\eta} = (4C\pi^3)^{-\frac{1}{2}}\int_{\mathcal{A}} d\mathbf{r}\hat{I}(\mathbf{r},t) \tag{21}$$

and accounts for the finite size of detector aperture. In the following, we adopt the normalizing condition $\langle\hat{\eta}\rangle = 1$ for area $\mathcal{A}$ much larger than the beam cross section. The variance (20) could be also considered as aperture-averaged scintillation index.

Two moments for $\hat{\eta}$ are defined as [53,54]

$$\langle\hat{\eta}\rangle = (4C\pi^3)^{-\frac{1}{2}}\int_{\mathcal{A}} d\mathbf{r}\Gamma_2(\mathbf{r}), \tag{22}$$

$$\langle\hat{\eta}^2\rangle = (4C\pi^3)^{-1}\int_{\mathcal{A}} d\mathbf{r}\int_{\mathcal{A}} d\mathbf{r}'\Gamma_4(\mathbf{r},\mathbf{r}'). \tag{23}$$

To obtain the fluctuations of transmitted radiation σ_η^2 we calculate numerically (fivefold integration) $\langle\hat{\eta}^2\rangle = \langle\hat{\eta}^2\rangle^{(i)} + \langle\hat{\eta}^2\rangle^{(ii)}$ for circular aperture with radius R using the expressions (14) and (15).

First of all, it is informative to compare approximations where partially saturated (PS) and fully saturated (FS) regimes were assumed (Fig. 1) for atmospheric channels considered in Ref. [33]. Such comparison allows us to both estimate the accuracy of asymptotic approximation, $z \to \infty$, and to emphasize the differentiation of correlation properties for two regimes. For partially saturated radiation scintillations are slightly larger because of the additional contribution of nondiagonal terms in fourth moment, which incorporate larger region of the phase space and account for residual effect of the initial statistics of radiation. Remarkably, the values of aperture-averaged scintillations differ from unity for pointlike aperture. That is a natural outcome for partially saturated regime where radiation still preserves some properties of initial statistics and does not fully acquire Gaussian statistics. In contrast to approximation of fully saturated radiation there is clear dependence of the values of scintillations, σ_η^2, for small

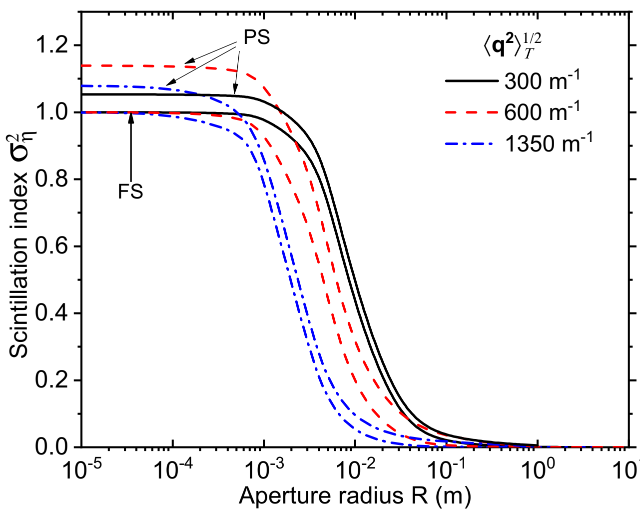

FIG. 1. Effect of nondiagonal terms in Γ_4 on aperture-averaged scintillations. In each pair of lines bottom curve depicts the results from Ref. [33], where it is assumed that intensity fluctuations are fully saturated; top curve represents the results for (14) and (15). Dash-dotted lines: $z = 20\,\text{km}$, $C_n^2 = 2.5 \times 10^{-14}\,\text{m}^{-2/3}$, Rytov variance $\sigma_R^2 = 62$; dashed lines: $z = 17\,\text{km}$, $C_n^2 = 5.8 \times 10^{-15}\,\text{m}^{-2/3}$, $\sigma_R^2 = 60$; solid lines: $z = 100\,\text{km}$, $C_n^2 = 2.5 \times 10^{-16}\,\text{m}^{-2/3}$, $\sigma_R^2 = 66$. Common parameters of the beam and channel for all curves: $r_0 = 0.01\,\text{m}$, $l_0/2\pi = 10^{-3}\,\text{m}$, $q_0 = 10^7\,\text{m}^{-1}$.

aperture sizes on Rytov parameter ($\sigma_R^2 = 1.23 C_n^2 z^{11/6} q_0^{7/6}$). Consistent with other studies, there is such size of aperture where the detector could not be considered as pointlike and steep reduction of the fluctuations is observed. These values are strongly dependent on the transverse momentum of photons $\langle \mathbf{q}^2\rangle_T$. In addition, unlike the fully saturated approximation the beam spreading R_b^2 also plays significant role to the behavior of aperture-averaged scintillations via both (14) and (15). Figure 2 shows that the present result more adequately accounts for the values of $\Gamma_4^{(i)}$ and corresponding correlation length. Since Ref. [33] considers fully saturated regime, $\Gamma_4^{(i)}$ does not contribute to the fluctuations there. In contrast term (14) has a sizable effect till the sizes of aperture are less than beam radius.

Also, there are two basic properties for scintillations averaging that are preserved in the current approximation: for pointlike detectors, $\mathbf{r} = \mathbf{r}' = 0$, (14) and (15) contribute equally, which repeats the result of previous works [25,33]; for aperture sizes reasonably larger than beam radius the fluctuations of transmittance tend to zero.

For atmospheric channels under partially saturated regime (Fig. 3) asymptotic result (18) significantly differs from one that considers nondiagonal terms in the expression for Γ_4. First of all, $\Gamma_4^{(i)}$ term contributes substantially to the values of transmittance fluctuations. Particularly it is responsible for long tails of the curves at large detector apertures. This effect is reminiscent of the leveling effect mentioned in Ref. [55], where there are two characteristic scales, $\rho_0 \propto \langle \mathbf{q}^2\rangle_T^{1/2}$ and $\rho_0 z/q_0 \propto R_b$, which define correlation properties of the light radiation. Particularly one may see from Fig. 3 that correlation length in term $\Gamma_4^{(ii)}$ being proportional to $\langle \mathbf{q}^2\rangle_T^{1/2}$ is much

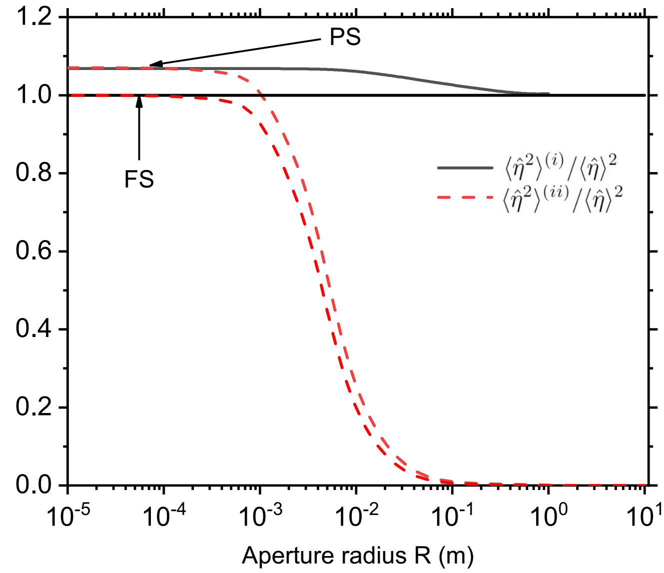

FIG. 2. Contributions of two regions (i) (solid lines) and (ii) (dashed lines) for partially saturated (top pair of curves) and fully saturated regimes (bottom pair of curves) for $\langle \mathbf{q}^2\rangle_T^{1/2} = 600\,\text{m}^{-1}$ from Fig. 1.

smaller than correlation length of $\Gamma_4^{(i)}$, which comes from the values of beam radius.

Figures 4 and 5 depict aperture effect on scintillations in relation to values of Rytov parameter for fixed C_n^2 and z, respectively. One might see that both magnitude of fluctuations and steepness of its decrease with larger sizes of the detector aperture strongly depends on parameters of the atmospheric channel. Generally, for bigger values of Rytov parameter the effect of saturation of the fluctuations is responsible for decrease of σ_η^2. That is, magnitude of intensity fluctuations decrease for atmospheric channels with stronger turbulence regime.

V. SUMMARY AND CONCLUSIONS

The fourth moment of light fields in atmosphere was derived under collisionless Boltzmann equation approximation where smooth random force represents the effect of turbulence on the laser beam. Such approximation is justified for the case of moderate and strong turbulence where fluctuations are saturated for particular degree. Randomization of the electromagnetic waves allows us to distinguish two separate regions in the phase space that contribute to noise level in optical system and its correlation properties. For the case of partially saturated regime intensity correlations are not factorized to product of two-wave correlations and additional nondiagonal terms should be considered. For Gaussian beams the effect of photon trajectories correlations is enclosed in a set of functions that define distribution of photon density in the phase space. Values of such functions strongly depend on properties of the atmospheric channel.

The fourth moment is applied for the problem of transmittance fluctuations. It is shown that current calculations describe more wide range of parameters of the atmospheric channels. There are two characteristic scales whose relation

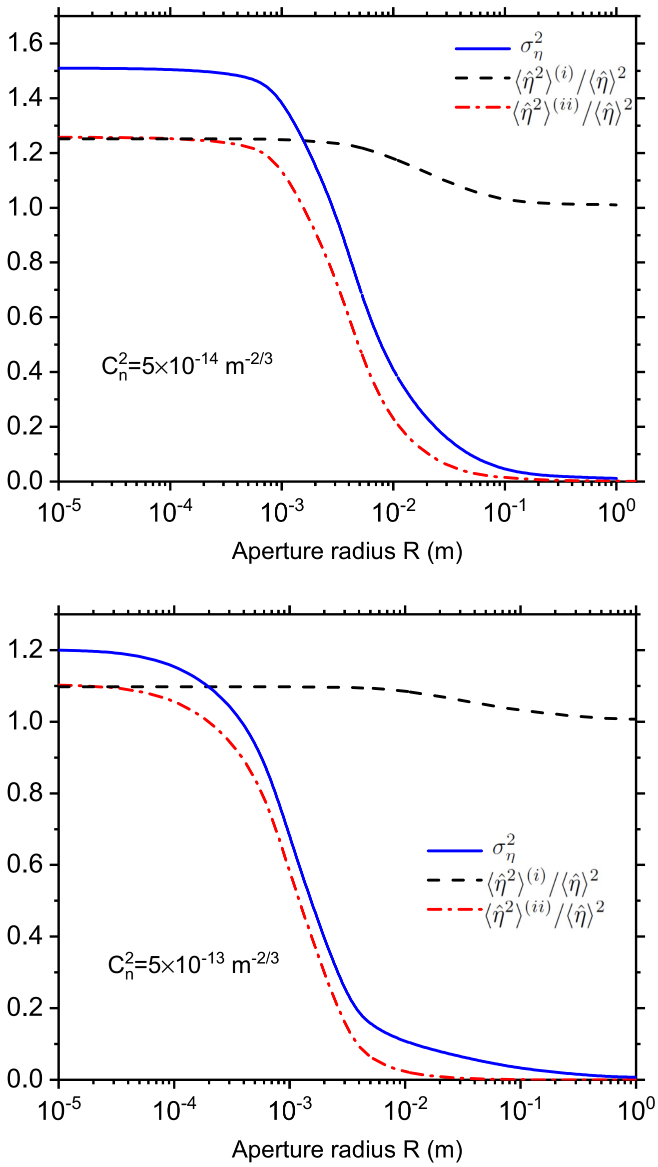

FIG. 3. Aperture-averaged scintillation index vs. radius of detector aperture. Parameters for both graphs: $r_0 = 0.01$ m, $l_0/2\pi = 10^{-3}$ m, $q_0 = 1.29 \times 10^7$ m^{-1}, $z = 3$ km.

defines correlation length for detected radiation. Found results could be useful for practical purposes.

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APPENDIX: CALCULATION OF Γ_4

In the main text we consider Gaussian beams and Tatarskii spectrum for the fluctuations of index of refraction. Therefore,

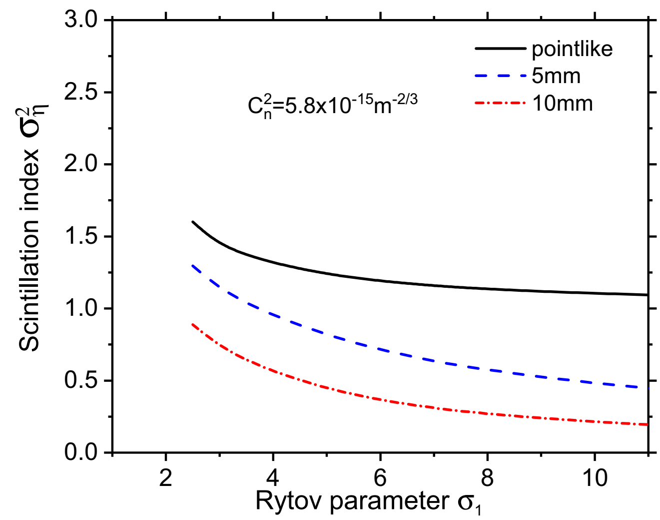

FIG. 4. Aperture-averaged scintillation index vs. Rytov parameter. Parameters of the channel: $r_0 = 0.01$ m, $l_0/2\pi = 10^{-3}$ m, $q_0 = 10^7$ m^{-1}.

for Gaussian beam the initial configuration of radiation defines [33]

$$\langle \phi(\mathbf{k}, \mathbf{q}) \rangle_{qm} = \frac{2\pi r_0^2}{V L_x L_y} \langle b^\dagger b \rangle_{qm} e^{-(q_\perp^2 + k_\perp^2/4) r_0^2/2}, \qquad \text{(A1)}$$

where the symbol $\langle \ldots \rangle_{qm}$ indicates a quantum-mechanical averaging of operators in the angle brackets. Coefficient C' can be obtained if the total photon flux is known. Tatarskii spetrum is defined as

$$\psi(\mathbf{g}) = 0.033 C_n^2 \exp\left\{-(g l_0')^2\right\} g^{-11/3}, \qquad \text{(A2)}$$

where $l_0' = l_0/2\pi$.

In the similar manner to the works [25,32] we can write down the general expression for fourth moment in terms of

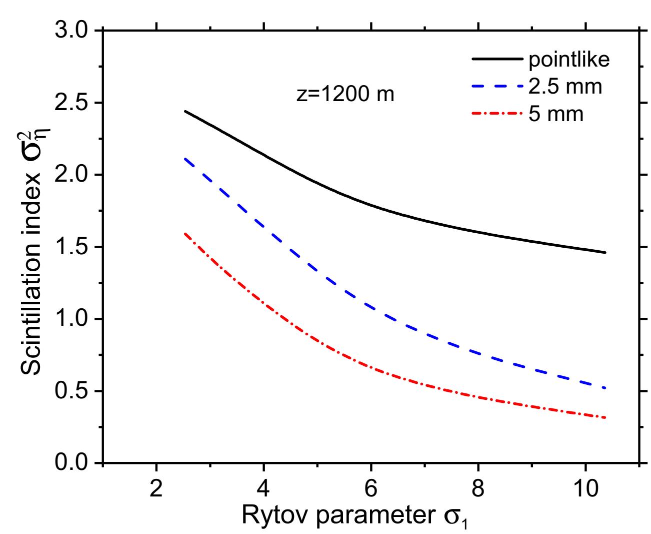

FIG. 5. Aperture-averaged scintillation index vs. Rytov parameter, $q_0 = 1.29 \times 10^7$ m^{-1}, other parameters are the same as in Fig. 3.

random fluctuating force $\mathbf{F}_\perp$

$$\langle I(\mathbf{r},t)I(\mathbf{r}',t)\rangle = \left(\frac{2\pi r_0^2}{VL_xL_y}\right)^2 \langle b^\dagger b^\dagger bb\rangle_{qm} \sum_{\mathbf{q},\mathbf{k},\mathbf{q}',\mathbf{k}'} \Big\langle e^{-i\mathbf{k}[\mathbf{r}-\mathbf{c_q}t+\frac{c}{q_0}\int_0^t dt't'\mathbf{F}_\perp(\mathbf{r}(\mathbf{q},t'))]-i\mathbf{k}'[\mathbf{r}'-\mathbf{c}_{\mathbf{q}'}t+\frac{c}{q_0}\int_0^t dt't'\mathbf{F}_\perp(\mathbf{r}(\mathbf{q}',t'))]} \times \exp\left(-\left(Q^2+Q'^2+\frac{k^2+k'^2}{4}\right)\frac{r_0^2}{2}\right)\Big\rangle_{\mathrm{atm}}, \tag{A3}$$

where $\mathbf{Q}=\mathbf{q}-\int_0^t dt'\mathbf{F}_\perp(\mathbf{r}(\mathbf{q},t))$ and $\mathbf{Q}'=\mathbf{q}'-\int_0^t dt'\mathbf{F}_\perp[\mathbf{r}'(\mathbf{q}',t)]$ are the solutions of evolution equations for momenta, $\langle\ldots\rangle_{\mathrm{atm}}$ indicates averaging over different configurations of atmosphere (refractive index reliefs). It is reasonable to express all turbulent parts in the exponent in linear form for $\mathbf{F}_\perp$. So the factor $e^{-(Q_\perp^2+Q_\perp'^2)\frac{r_0^2}{2}}$ in (A3) is expressed in the integral form as

$$e^{-(Q_\perp^2+{Q'_\perp}^2)\frac{r_0^2}{2}} = \int \frac{d\mathbf{p}\,d\mathbf{p}'}{(2\pi r_0^2)^2} e^{i\mathbf{p}\cdot\mathbf{Q}_\perp+i\mathbf{p}'\cdot\mathbf{Q}'_\perp-(p^2+p'^2)/2r_0^2}. \tag{A4}$$

In this case (A3) can be expressed as

$$\langle I(\mathbf{r},t)I(\mathbf{r}',t)\rangle = \frac{\langle b^\dagger b^\dagger bb\rangle_{qm}}{(VL_xL_y)^2}\int d\mathbf{p}d\mathbf{p}' \sum_{\mathbf{q},\mathbf{k},\mathbf{q}',\mathbf{k}'} e^{-i\mathbf{k}[\mathbf{r}-\mathbf{c_q}t]-i\mathbf{k}'[\mathbf{r}'-\mathbf{c}_{\mathbf{q}'}t]} e^{-(k^2+k'^2)r_0^2/8} e^{i\mathbf{p}\cdot\mathbf{q}_\perp+i\mathbf{p}'\cdot\mathbf{q}'_\perp-(p^2+p'^2)/2r_0^2}\langle M\rangle_{\mathrm{atm}}, \tag{A5}$$

where the factor

$$M = \exp\left(-i\int_0^t dt'\left\{\left(\mathbf{p}+\mathbf{k}t'\frac{c}{q_0}\right)\mathbf{F}[\mathbf{r}(\mathbf{q},t')] + \left(\mathbf{p}'+\mathbf{k}'t'\frac{c}{q_0}\right)\mathbf{F}[\mathbf{r}(\mathbf{q}',t')]\right\}\right) \tag{A6}$$

includes all fluctuating parts in fourth moment. (For the sake of brevity from here on we omit $\perp$ notation for perpendicular to propagation direction.) Using similar to Refs. [25,32] approach, the average of M is expressed as

$$\langle M\rangle_{\mathrm{atm}} = \exp\left\{-\pi^2 q_0^2\int_0^z dx\int d\mathbf{g}\psi(\mathbf{g})\Big[((\mathbf{p}+\mathbf{k}x/q_0)\cdot\mathbf{g})^2 + ((\mathbf{p}'+\mathbf{k}'x/q_0)\cdot\mathbf{g})^2 + 2(\mathbf{p}+\mathbf{k}x/q_0)\cdot\mathbf{g}(\mathbf{p}'+\mathbf{k}'x/q_0)\cdot\mathbf{g}\exp\left\{i\mathbf{g}\cdot\boldsymbol{\Delta r} - \frac{R_b^2}{480{l'_0}^2}\Delta r^2(1-x/z)^3 g^2\left[1+\frac{R_b^2(1-x/z)^3}{672{l'_0}^2}\right]\right\}\Big]\right\}, \tag{A7}$$

where $g=|\mathbf{g}|$ and we also used Markov approximation index-of-refraction spectrum is δ correlated in the direction of propagation, which was rigorously justified in Ref. [32]. After integration over direction of $\mathbf{g}$

$$\langle M\rangle_{\mathrm{atm}} = \exp\Big\{-0.033C_n^2\pi^2q_0^2\int_0^z dx\int_0^\infty dg g^{-2/3}e^{-g^2l_0^2}\Big[(\mathbf{p}+\mathbf{k}x/q_0)^2+(\mathbf{p}'+\mathbf{k}'x/q_0)^2 + (2(p_\parallel+k_\parallel x/q_0)(p'_\parallel+k'_\parallel x/q_0)(J_0(g\Delta r)-J_2(g\Delta r)) + 2(p_\perp+k_\perp x/q_0)(p'_\perp+k'_\perp x/q_0)(J_0(g\Delta r)+J_2(g\Delta r))) \times \exp\left\{-\frac{R_b^2}{480{l'_0}^2}\Delta r^2(1-x/z)^3g^2\left[1+\frac{R_b^2(1-x/z)^3}{672{l'_0}^2}\right]\right\}\Big]\Big\}, \tag{A8}$$

where J_0 and J_2 are zeroth- and second-order Bessel functions, vector $\boldsymbol{\Delta r}=(\mathbf{r}-\mathbf{r}')-(\mathbf{q}-\mathbf{q}')(z-x)/q_0$, $\Delta r=|\boldsymbol{\Delta r}|$. The indices $\{_\parallel\}$ and $\{_\perp\}$ indicate the parallel and perpendicular to $\boldsymbol{\Delta r}$ components of the corresponding 2D vectors. The first two terms in square brackets represent correlation of waves with same pairs $\{\mathbf{r},\mathbf{q}\}$ and $\{\mathbf{r}',\mathbf{q}'\}$. The terms that include Bessel functions consider cross correlation of photon trajectories ($\{\mathbf{r},\mathbf{q}\}$ and $\{\mathbf{r}',\mathbf{q}'\}$ are different), which were extensively reviewed in Ref. [32]. In (A7) and (A8) we omit the dependence on angle between $\boldsymbol{\Delta r}$ and $\mathbf{g}$ (see Eq. (40) in Ref. [32]) in the last exponent as its contribution is small. After substitution of (A8) to (A5) and change of variables, most of integrations could be performed analytically. As a result expression for fourth moment can be written as

$$\Gamma_4^{(i)}(\mathbf{r},\mathbf{r}') = 2\pi C\int_0^\infty d\tilde{q}\tilde{q}\big[F_1^{\tilde{q},\rho_\parallel}F_2^{\tilde{q},\rho_\parallel}\big(F_3^{\tilde{q},\rho_\parallel}H_1^{\tilde{q},\rho_\parallel} - {G_1^{\tilde{q},\rho_\parallel}}^2\big)\big(F_4^{\tilde{q},\rho_\parallel}H_2^{\tilde{q},\rho_\parallel} - {G_2^{\tilde{q},\rho_\parallel}}^2\big)\big]^{-\frac{1}{2}}$$
$$\times \exp\left\{-\frac{\Big(\tilde{q}-\rho_\parallel\frac{G_1^{\tilde{q},\rho_\parallel}}{2F_3^{\tilde{q},\rho_\parallel}}\Big)^2}{\big(H_1^{\tilde{q},\rho_\parallel} - {G_1^{\tilde{q},\rho_\parallel}}^2/F_3^{\tilde{q},\rho_\parallel}\big)}\right\}\exp\left\{-\frac{\rho_\perp^2 H_2^{\tilde{q},\rho_\parallel}}{\big(H_2^{\tilde{q},\rho_\parallel}F_4^{\tilde{q},\rho_\parallel} - {G_2^{\tilde{q},\rho_\parallel}}^2\big)}\right\}\exp\left\{-\left(\frac{\rho_\parallel'^2}{F_1^{\tilde{q},\rho_\parallel}} + \frac{\rho_\perp'^2}{F_2^{\tilde{q},\rho_\parallel}} + \frac{\rho_\parallel^2}{4F_3^{\tilde{q},\rho_\parallel}}\right)\right\}, \tag{A9}$$

functions F, G, and H are defined as

$$F_{1,2,3,4} = \frac{r_0^2}{4} + \frac{z^2}{q_0^2 r_0^2} + \varphi_{1,2,3,4} \tag{A10}$$

$$G_{1,2} = \frac{z^2}{q_0^2 r_0^2} + \gamma_{1,2} \tag{A11}$$

$$H_{1,2} = \frac{z^2}{q_0^2 r_0^2} + \chi_{1,2} \tag{A12}$$

$$\varphi_1 = \alpha\int_0^1 d\tau\tau^2\left[1\boxplus\left(1+\beta\Delta\tilde{r}^2\tau^3\right)^{-\frac{1}{6}}\left[{}_1F_1\left(\frac{1}{6},1;\frac{-\Delta\tilde{r}^2}{4{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\boxplus\frac{\Delta\tilde{r}^2}{48{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}{}_1F_1\left(\frac{7}{6},3;\frac{-\Delta\tilde{r}^2}{{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\right]\right] \tag{A13}$$

$$\varphi_2 = \alpha\int_0^1 d\tau\tau^2\left[1\boxplus(1+\beta\Delta\tilde{r}^2\tau^3)^{-\frac{1}{6}}\left[{}_1F_1\left(\frac{1}{6},1;\frac{-\Delta\tilde{r}^2}{4{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\boxminus\frac{\Delta\tilde{r}^2}{48{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}{}_1F_1\left(\frac{7}{6},3;\frac{-\Delta\tilde{r}^2}{{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\right]\right] \tag{A14}$$

$$\varphi_3 = \alpha\int_0^1 d\tau\tau^2\left[1\boxminus(1+\beta\Delta\tilde{r}^2\tau^3)^{-\frac{1}{6}}\left[{}_1F_1\left(\frac{1}{6},1;\frac{-\Delta\tilde{r}^2}{4{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\boxminus\frac{\Delta\tilde{r}^2}{48{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}{}_1F_1\left(\frac{7}{6},3;\frac{-\Delta\tilde{r}^2}{{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\right]\right] \tag{A15}$$

$$\varphi_4 = \alpha\int_0^1 d\tau\tau^2\left[1\boxminus(1+\beta\Delta\tilde{r}^2\tau^3)^{-\frac{1}{6}}\left[{}_1F_1\left(\frac{1}{6},1;\frac{-\Delta\tilde{r}^2}{4{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\boxplus\frac{\Delta\tilde{r}^2}{48{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}{}_1F_1\left(\frac{7}{6},3;\frac{-\Delta\tilde{r}^2}{{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\right]\right] \tag{A16}$$

$$\gamma_1 = \alpha\int_0^1 d\tau\tau\left[1-(1+\beta\Delta\tilde{r}^2\tau^3)^{-\frac{1}{6}}\left[{}_1F_1\left(\frac{1}{6},1;\frac{-\Delta\tilde{r}^2}{4{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\boxminus\frac{\Delta\tilde{r}^2}{48{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}{}_1F_1\left(\frac{7}{6},3;\frac{-\Delta\tilde{r}^2}{{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\right]\right] \tag{A17}$$

$$\gamma_2 = \alpha\int_0^1 d\tau\tau\left[1-(1+\beta\Delta\tilde{r}^2\tau^3)^{-\frac{1}{6}}\left[{}_1F_1\left(\frac{1}{6},1;\frac{-\Delta\tilde{r}^2}{4{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\boxplus\frac{\Delta\tilde{r}^2}{48{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}{}_1F_1\left(\frac{7}{6},3;\frac{-\Delta\tilde{r}^2}{{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\right]\right] \tag{A18}$$

$$\chi_1 = \alpha\int_0^1 d\tau\left[1-(1+\beta\Delta\tilde{r}^2\tau^3)^{-\frac{1}{6}}\left[{}_1F_1\left(\frac{1}{6},1;\frac{-\Delta\tilde{r}^2}{4{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\boxminus\frac{\Delta\tilde{r}^2}{48{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}{}_1F_1\left(\frac{7}{6},3;\frac{-\Delta\tilde{r}^2}{{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\right]\right] \tag{A19}$$

$$\chi_2 = \alpha\int_0^1 d\tau\left[1-(1+\beta\Delta\tilde{r}^2\tau^3)^{-\frac{1}{6}}\left[{}_1F_1\left(\frac{1}{6},1;\frac{-\Delta\tilde{r}^2}{4{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\boxplus\frac{\Delta\tilde{r}^2}{48{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}{}_1F_1\left(\frac{7}{6},3;\frac{-\Delta\tilde{r}^2}{{l_0'}^2(1+\beta\Delta\tilde{r}^2\tau^2)}\right)\right]\right], \tag{A20}$$

and parameters are defined as $\alpha = \frac{3}{2}R_b^2$, $\beta = \frac{R_b^2}{480{l_0'}^4}[1 + \frac{R_b^2\tau^3}{672{l_0'}^2}]$, $\Delta\tilde{r} = \rho_\parallel - 2\tilde{q}\tau$, and ${}_1F_1(a,b;z) = \sum_{n=0}^\infty \frac{a^{(n)}z^n}{b^{(n)}n!}$ is a confluent hypergeometric function (Kummer's function), $a^{(n)}$, $b^{(n)}$ are the Pochhammer symbols. Addition and subtraction operations are highlighted with squares, $\boxplus$ and $\boxminus$, to emphasize the difference between corresponding functions. Here notations $\boldsymbol{\rho} = \mathbf{r} - \mathbf{r}'$, $\boldsymbol{\rho}' = (\mathbf{r} + \mathbf{r}')/2$ are used since $\mathbf{r}$ and $\mathbf{r}'$ enters expression for Γ_4 only in such combinations (see more details in main text). Functions (A13)–(A20) depend on $\rho_\parallel$ and $\tilde{q}$.

For the region of the phase space defined with conditions (ii) $|\mathbf{q}-\mathbf{q}'+(\mathbf{k}+\mathbf{k}')/2|$, $|\mathbf{q}-\mathbf{q}'+(\mathbf{k}+\mathbf{k}')/2| \leqslant R_b^{-1}$ one should perform change of indices similar to one in Refs. [27,33]

$$\mathbf{k} \to \mathbf{q}-\mathbf{q}'+\frac{\mathbf{k}+\mathbf{k}'}{2}, \quad \mathbf{k}' \to \mathbf{q}'-\mathbf{q}+\frac{\mathbf{k}+\mathbf{k}'}{2}$$

$$\mathbf{q} \to \frac{1}{2}\left(\mathbf{q}+\mathbf{q}'+\frac{\mathbf{k}-\mathbf{k}'}{2}\right), \quad \mathbf{q}' \to \frac{1}{2}\left(\mathbf{q}+\mathbf{q}'-\frac{\mathbf{k}-\mathbf{k}'}{2}\right).$$

Then $\Gamma_4^{(ii)}$ is expressed as

$$\Gamma_4^{(ii)}(\mathbf{r},\mathbf{r}') = \sum_{\mathbf{q},\mathbf{q}'} e^{i(\mathbf{q}'-\mathbf{q})\cdot(\mathbf{r}-\mathbf{r}')}\left\langle f\left(\frac{\mathbf{r}+\mathbf{r}'}{2},\mathbf{q}\right) f\left(\frac{\mathbf{r}+\mathbf{r}'}{2},\mathbf{q}'\right)\right\rangle, \tag{A21}$$

which is reminiscent of the asymptotic expression for fluctuations of intensity [33]. However, for smaller distances, accounting for partial saturation of fluctuations, it is not factorized to the product of first moments for PDF. Calculation of (A8) is similar to one for term (A5). However since PDFs in (A8) depend on the same coordinate $(\mathbf{r}+\mathbf{r}')/2$ both autocorrelations and cross-correlation terms in (A8) do not include space coordinates. After integrations

$$\Gamma_4^{(ii)}(\mathbf{r},\mathbf{r}') = 2\pi C \int_0^\infty d\tilde{q}\tilde{q}\left[F_1^{\tilde{q}}F_2^{\tilde{q}}\left(F_3^{\tilde{q}}H_1^{\tilde{q}}-G_1^{\tilde{q}\,2}\right)\left(F_4^{\tilde{q}}H_2^{\tilde{q}}-G_2^{\tilde{q}\,2}\right)\right]^{-\frac{1}{2}} \exp\left\{-\frac{\tilde{q}^2}{H_1^{\tilde{q}}-G_1^{\tilde{q}\,2}/F_3^{\tilde{q}}}\right\}$$
$$\times \exp\left\{-\left(\frac{\rho_\parallel'^2}{F_1^{\tilde{q}}}+\frac{\rho_\perp'^2}{F_2^{\tilde{q}}}\right)\right\} \exp\left\{-i2\tilde{q}\rho_\parallel\frac{q_0}{z}\right\}, \tag{A22}$$

where functions F, G, and H does not depend on $\mathbf{r}$, $\mathbf{r}'$ and one should put $2\tilde{q}\tau$ instead of $\Delta\tilde{r}$ in corresponding expressions.

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