Unusual scenarios in four-wave-mixing instability

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A pump carrier wave in a nonlinear dispersive system may decay by giving birth to blue- and redshifted satellite waves due to modulation or four-wave mixing instability. We analyze situations where the satellites are so different from the carrier wave, that the redshifted satellite either changes its propagation direction ($k < 0$, $\omega > 0$) or even gets a negative frequency ($k, \omega < 0$). Both situations are beyond the envelope approach and require application of the Maxwell equations.

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I. INTRODUCTION

The key property of a dispersive system is the existence of linear small-amplitude waves, e.g., of the form $\mathrm{Re}[e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}]$ with a certain dispersion relation $\omega = \ell(\mathbf{k})$. An increase of power results in a nonlinear wave, which may become unstable [1]. A very common instability scenario is excitation of two growing satellite waves with the reduced and increased frequencies: the Stokes and anti-Stokes (or just red- and blueshifted) spectral lines [2]. The satellite frequencies are equally displaced from the incident one. Starting from the seminal study on modulated water waves [3], the effect was observed in many nonlinear dispersive systems including optical fibers [4], which are in the focus of this work.

A small frequency displacement that is proportional to the incident power is the defining feature of the most common modulation instability (MI). If the new lines are separated from the carrier, no matter how small the incident power, one deals with a four-wave mixing (FWM) instability. In both cases, the parameters of the blueshifted (b) and redshifted (r) satellites are connected to that of the carrier (c) wave by the resonance (phase matching) conditions

$$\omega_b + \omega_r = 2\omega_c, \quad \mathbf{k}_b + \mathbf{k}_r = 2\mathbf{k}_c, \tag{1}$$

where the dispersion relation $\omega = \ell(\mathbf{k})$ must hold for all three waves [5,6].

Conditions (1) are necessary but not sufficient, the sufficient condition for MI was given by Lighthill [7]. With respect to optical fibers, both instability scenarios are described by a generalized nonlinear Schrödinger equation (GNLSE) for the wave envelope [8–15]. Moreover, Lighthill's criterion can be reformulated to cover both MI and FWM regimes [16]. Note that GNLSE actually refers to a class of increasingly complex equations with higher-order dispersion, losses, Raman integral, and self-steepening derivative terms [17–24]. To our knowledge, the most comprehensive "all included" MI analysis was published in Ref. [15]. One can also study MI directly with the full Maxwell equations [25].

This work considers wave instabilities in optical fibers and takes advantage of the fact that fiber dispersion can be engineered [26]. One can manipulate $\ell(\mathbf{k})$ and solutions of the system (1) to excite a wide range of frequencies via the FWM mechanism [27–33]. We aim to answer the question: can the redshifted satellite be so different from the carrier wave that it *either propagates in the opposite direction or gets a negative frequency*, as schematically shown in Fig. 1? In this case, the blueshifted frequency will be greater than $2\omega_c$.

Appearance of the backward wave may resemble the Brillouin scattering with the difference that the FWM instability takes place due to cubic nonlinearities and without any contribution of material waves. Our interest to negative frequencies is motivated by recent papers on classical nonlinear optics (and on water waves [34,35]), where calculation of the excited spectral line leads to a negative frequency. One scenario is scattering of a wave packet at a quickly moving perturbation of the refractive index created by another pulse. One can observe several scattered waves: a standard frequency-shifted backward wave identical to that reflected by a moving mirror [36], forward scattering [37], and an exotic classical Hawking radiation with a negative frequency [38,39]. Another option is the so-called dispersive or Cherenkov radiation emitted by solitons in fibers [40]. A formal calculation of the radiation frequency may lead to a negative value. The positive-frequency partner of the emitted wave was observed in experiment [41,42] and predicted by a novel modification of GNLSE [43,44]. A search for new phenomena involving negative frequencies seems to be interesting and instructive; the stability problem for a nonlinear wave is worth a try.

II. FRAMEWORK

The majority of studies on MI and FWM instability in optical fibers use various versions of GNLSE. The latter is extremely powerful, GNLSE can even be adapted to describe the contribution of negative frequencies [43,44]. Yet, an envelope equation does not fit well to our needs for several reasons.

First, GNLSE describes waves moving in one direction, whereas a possible backward satellite is not covered.

Second, GNLSE approximates medium dispersion by Taylor expansion around a carrier frequency. The expansion is

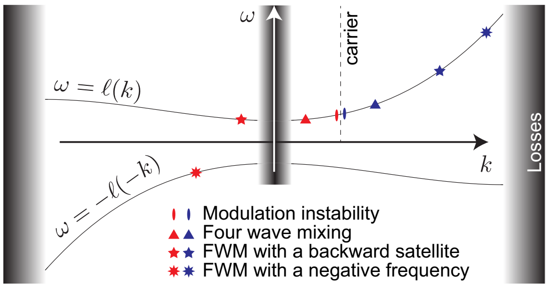

FIG. 1. A schematic dispersion law within a transparency window and possible instability regimes are shown. An anisotropic $\ell(\mathbf{k})$ with $\mathbf{k} = (0, 0, k)$ is used for better visibility. The necessary Eq. (1) selects the feasible instability. A sufficient condition (e.g., that of Lighthill) decides whether the instability develops.

limited by its convergence radius, which is determined by resonances of the dielectric function $\epsilon(\omega, \mathbf{k})$ in the complex ω plane. If the low-frequency resonances are present, the Taylor expansion at ω_c covers neither the red or blue stars in Fig. 1 nor the general relation [45]

$$\epsilon^*(\omega, \mathbf{k}) = \epsilon(-\omega^*, -\mathbf{k}), \tag{2}$$

which we use later.

Third, GNLSE in optical fibers is a space-propagated problem in (t, z) coordinate space. The initial pulse is given for some z and $\forall\, t$, in conflict with the causality principle. The pulse shape is then calculated for a larger z and $\forall\, t$, a possible instability manifests itself by a complex-valued k_z. Dealing with such a delicate question as negative frequencies, it is preferable to have a causal system that evolves in time with either real- or complex-valued ω, i.e., to consider an absolute instability [46]. This point of view is taken in Eq. (2).

For the above reasons, we will follow Ref. [25] and directly employ the Maxwell equations to study the stability problem with the difference being that our system is time-propagated and retains causality, and that both forward and backward waves are covered. Before proceeding, we need to make a few remarks:

i. Given a reasonable incident power, MI satellites are too close to the carrier to have unusual properties. FWM instability is the only case of interest.

ii. Plane waves in what follows will have a real $\mathbf{k}$ and possibly complex ω, which is called "negative" if $\mathrm{Re}(\omega) < 0$. We should modify Eq. (1) for $\omega \in \mathbb{C}$.

iii. $\mathrm{Re}[e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}]$ is invariant with respect to the substitution $\omega \mapsto -\omega^*$, $\mathbf{k} \mapsto -\mathbf{k}$, generated by the complex conjugation, cf. Eq. (2).

iv. Any positive-frequency branch of the dispersion law has a negative-frequency partner

$$\omega = \ell(\mathbf{k}) \text{ comes with } \omega = -\ell^*(-\mathbf{k}). \tag{3}$$

The situation we are interested in is schematically shown in Fig. 1 by five- and eight-pointed stars. Now we can formulate the problem more precisely. Let ω_c, $\mathbf{k}_c$ and ω_b, $\mathbf{k}_b$ belong to the positive-frequency branch in Eq. (3). It is not enough to know whether ω_r, $\mathbf{k}_r$, which come from Eq. (1), can belong to either backward or negative-frequency branch. We shall get a dispersion relation for the satellites and study if they grow up.

III. MODEL EQUATION

A generic electromagnetic wave is described by the wave equation

$$\mu_0 \partial_t^2 \mathbf{D} + \mathrm{rot\,rot}\, \mathbf{E} = 0, \tag{4}$$

where the displacement $\mathbf{D}(t, \mathbf{r})$ and field $\mathbf{E}(t, \mathbf{r})$ are confined by a material relation. We consider an isotropic dispersive dielectric medium with a cubic nonlinearity that is characterized by a single Kerr parameter χ:

$$\frac{1}{\epsilon_0}\mathbf{D} = \mathbf{E} + K \circ \mathbf{E} + \chi(\mathbf{E}\cdot\mathbf{E})\mathbf{E}, \tag{5}$$

where, for simplicity, χ is just a constant and $\mathbf{E}\cdot\mathbf{E}$ is the standard scalar product. The simplest nondispersive nonlinearity is combined with a generic linear dispersion: the term $K \circ \mathbf{E}$ denotes a causal convolution with a suitable kernel $K(t, \mathbf{r})$

$$K \circ \mathbf{E} = \int_0^\infty \int_{\mathbb{R}^3} K(t', \mathbf{r}')\mathbf{E}(t - t', \mathbf{r} - \mathbf{r}')dt'd^3\mathbf{r}'.$$

A wave with $\mathbf{E} \propto e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ yields $K \circ \mathbf{E} = (\epsilon - 1)\mathbf{E}$, where the dielectric function reads [45]

$$\epsilon(\omega, \mathbf{k}) = 1 + \int_0^\infty \int_{\mathbb{R}^3} K(t, \mathbf{r})e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}dt d^3\mathbf{r}. \tag{6}$$

Again, for simplicity and with optical fibers in mind, we consider only one-dimensional (1D) propagation with

$$\mathbf{k} = (0, 0, k), \quad \mathbf{E} = \mathbf{E}(t, z) = (E_x, E_y, 0),$$

and use the notations

$$\epsilon(\omega, k) = \epsilon(\omega, \mathbf{k})|_{\mathbf{k}=(0,0,k)}, \quad \ell(k) = \ell(\mathbf{k})|_{\mathbf{k}=(0,0,k)}.$$

Equations (4) and (5) are then reduced to a single partial differential equation (PDE) for a complex variable Ψ,

$$\partial_t^2(\Psi + K \circ \Psi + \chi|\Psi|^2\Psi) - c^2\partial_z^2\Psi = 0, \tag{7}$$

where $\Psi(t, z) = E_x + iE_y$.

Equation (7) is our starting point. It might look like an envelope equation, but it applies directly to the electric field. Being an exact reduction of (4) and (5), it is not limited by any kind of unidirectional or slowly-varying-envelope approximation. Moreover, Eq. (7) describes causal evolution of both forward and backward waves for an arbitrary medium dispersion. Both positive and negative frequencies are covered such that Eq. (7) is well suited to study two unusual FWM scenarios from Fig. 1.

IV. CARRIER WAVE

For brevity we introduce a kind of generalized dispersion function $\mathfrak{E}(\omega, k)$ and notation for its derivatives:

$$\mathfrak{E} = \epsilon(\omega, k) - \frac{k^2c^2}{\omega^2}, \quad \dot{\mathfrak{E}} = \frac{\partial \mathfrak{E}}{\partial \omega}, \quad \mathfrak{E}' = \frac{\partial \mathfrak{E}}{\partial k}. \tag{8}$$

A small-amplitude $Ae^{i(kz-\omega t)}$ solution to Eq. (7) corresponds to a linear wave with circular polarization and requires

$\mathfrak{E}(\omega, k) = 0$. This yields the main (generally speaking, multivalued) dispersion relation $\omega = \ell(k)$ and its partner (3). The group velocity V and the group-velocity dispersion D are given by the derivatives of an implicit function

$$V = \frac{d\omega}{dk} = -\frac{\mathfrak{E}'}{\dot{\mathfrak{E}}}, \tag{9}$$

$$D = \frac{d^2\omega}{dk^2} = -\frac{\ddot{\mathfrak{E}}V^2 + 2\dot{\mathfrak{E}}'V + \mathfrak{E}''}{\dot{\mathfrak{E}}}, \tag{10}$$

where a bit clumsy form of $D(\omega, k)$ is the price we pay for a general $\epsilon(\omega, k)$.

A possible instability develops upon a weakly nonlinear carrier wave with circular polarization

$$\Psi = A_c e^{i(k_c z - \omega_c t)} \quad \text{with} \quad \sigma = \chi A_c^2 \ll 1, \tag{11}$$

where σ is a dimensionless power parameter. The carrier should belong to a transparency domain, such that $\omega_c,\ k_c \in \mathbb{R}$ and $\mathrm{Im}[\epsilon(\omega_c,\ k_c)] \approx 0$. Equations (7) and (11) yield a nonlinear dispersion relation

$$\mathfrak{E}(\omega_c,\ k_c) + \sigma = 0. \tag{12}$$

The carrier wave has a nonlinear frequency shift ω_{nl}, which is defined such that $\omega_c - \omega_{\text{nl}}$ and k_c satisfy the linear dispersion relation $\mathfrak{E}(\omega_c - \omega_{\text{nl}},\ k_c) = 0$, i.e.,

$$\omega_{\text{nl}} = -\frac{\sigma}{\dot{\mathfrak{E}}_c} + O(\sigma^2). \tag{13}$$

Here and from now on, we use the notations

$$\mathfrak{E}_\xi,\ \dot{\mathfrak{E}}_\xi,\ \mathfrak{E}'_\xi,\ V_\xi,\ D_\xi$$

when the involved quantities are calculated for the carrier wave ($\xi = c$) or its satellites ($\xi = b, r$). We now turn to the carrier-stability problem.

V. DISPERSION RELATION

We consider a small perturbation ψ of the carrier wave (11)

$$\Psi(t, z) = A_c e^{i(k_c z - \omega_c t)} + \psi(t, z),$$

which is subject to a linear PDE yielded by Eq. (7)

$$\partial_t^2(\psi + K \circ \psi + 2\sigma\psi + \sigma e^{2i(k_c z - \omega_c t)}\psi^*) = c^2 \partial_z^2 \psi.$$

We look for a special solution for ψ that combines one blue- and one redshifted satellite

$$\psi = A_b e^{i(k_b z - \omega_b t)} + A_r e^{i(k_r z - \omega_r t)},$$

with $A_b = \text{const}$ and $A_r = \text{const}$. Recall that ω_c and all wave vectors are real. The satellite frequencies ω_b and ω_r may be real or complex. By construction we require, cf. Eq. (1),

$$\omega_b + \omega_r^* = 2\omega_c \text{ and } k_b + k_r = 2k_c, \tag{14}$$

such that the nonhomogeneous term in the PDE for ψ is expressed through the same satellites

$$e^{2i(k_c z - \omega_c t)}\psi^* = A_r^* e^{i(k_b z - \omega_b t)} + A_b^* e^{i(k_r z - \omega_r t)}.$$

The amplitudes A_b and A_r are then nontrivial solutions to a system of two linear homogeneous equations

$$\begin{pmatrix} \mathfrak{E}_b + 2\sigma & \sigma \\ \sigma & \mathfrak{E}_r^* + 2\sigma \end{pmatrix} \begin{pmatrix} A_b \\ A_r^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{15}$$

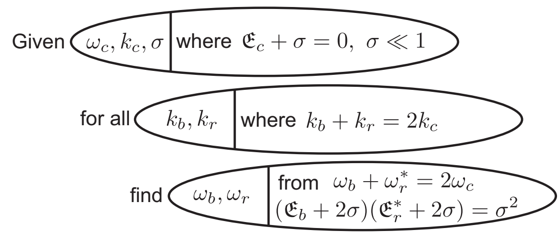

FIG. 2. Summary of the dispersion relation (16). $\mathfrak{E}_\xi$ denotes $\epsilon(\omega_\xi, k_\xi) - k_\xi^2 c^2/\omega_\xi^2$ with $\xi = c, b, r$ for the carrier wave and its blue or red satellites, respectively. $\omega_{b,r} \in \mathbb{C}$ yields instability, σ is a normalized incident power from Eq. (11).

which finally leaves us with the dispersion relation for the satellites

$$(\mathfrak{E}_b + 2\sigma)(\mathfrak{E}_r^* + 2\sigma) = \sigma^2. \tag{16}$$

Equation (16), being *causal and valid for positive and negative frequencies*, contains all we need to know about the instability regimes depicted in Fig. 1.

Let us make a few remarks before continuing. Figure 1 and the standard phase matching conditions (1) apply, of course, to the real parts of the satellite frequencies. The satellites will typically belong to the transparency window, and yet $\omega_{b,r}$ may be complex, which indicates instability. If this is the case, $\mathfrak{E}_b$ and $\mathfrak{E}_r$ are complex as well and the complex conjugation in Eq. (16) is essential. Equations (2) and (14) yield

$$\mathfrak{E}_r^* = \mathfrak{E}(-\omega_r^*, -k_r) = \mathfrak{E}(\omega_b - 2\omega_c, k_b - 2k_c), \tag{17}$$

such that all stable solutions of Eq. (16) can be plotted as a real-valued $\omega_b(k_b)$. The plot contains gaps, where $\omega_b(k_b)$ turns complex because of losses or instability.

It is remarkable that the merged MI/FWM problem gets such a compact formulation as Eq. (16) for an arbitrary $\epsilon(\omega, k)$. On the other hand, the dispersion relation for the satellites that comes from a GNLSE can be solved immediately [8–15]. At most, one is facing a fourth-order equation [25]. In our case, the implicit Eq. (16) requires additional work to be done, as summarized in Fig. 2.

VI. MODULATION-INSTABILITY CASE

The general approach of the previous sections is an overkill for the classical MI, where the carrier wave and both its satellites perfectly fit to a slowly varying envelope approximation and GNLSE. Nevertheless, MI is of course covered by Eq. (16). Note that for $k_{b,r} = k_c$, we have an exact solution $\omega_{b,r} = \omega_c$ of the problem sketched in Fig. 2. In the vicinity of this solution, one can set

$$\omega_b = \omega_c + \Omega, \quad \omega_r = \omega_c - \Omega^*, \quad k_{b,r} = k_c \pm \kappa,$$

such that both phase matching conditions (14) are satisfied, and look for $\Omega(\kappa)$. We expand Eq. (16) with respect to Ω and κ. Two successive iterations yield the classical result

$$(\Omega - \kappa V_c)^2 = \left(2\omega_{\text{nl}} + \frac{D_c\kappa^2}{2}\right)\frac{D_c\kappa^2}{2}, \tag{18}$$

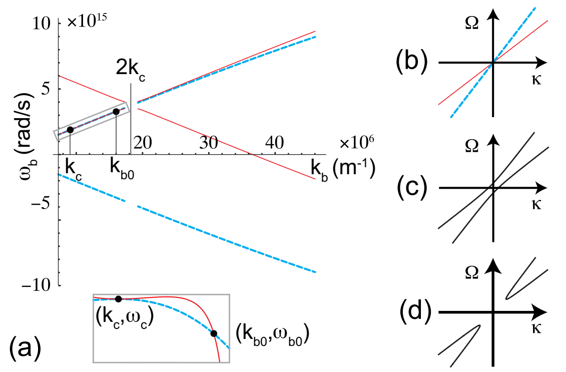

FIG. 3. (a) Solutions of Eq. (20) are plotted as $\omega_b(k_b)$ for $k_b \geqslant k_c$ by thick dashed (the blue-satellite solution) and thin solid (the red-satellite solution) lines. We use a bulk fused silica dispersion [48], take the carrier wave at 1 μm, and set $\sigma = 0$. A gap for $k_b \approx 2k_c$ (i.e., $k_r \approx 0$) appears due to the low-frequency absorption of the red satellite. The FWM instability results from a generic crossing of the blue and red (dashed and solid) curves on the inset. Panels (b)–(d) show stable and unstable reconnections yielded by the right-hand-side of the full Eq. (16).

which contains the group velocity (9) and group velocity dispersion (10) of the carrier wave. The nonlinear frequency shift ω_{nl} was defined in Eq. (13).

For (Lighthill criterion) $\omega_{\text{nl}} D_c < 0$, Eq. (18) describes a MI that evolves in time with the complex modulation frequency Ω and the maximal increment

$$[\text{Im}\,\Omega]_{\max} = |\omega_{\text{nl}}| = \frac{\sigma}{|\dot{\mathfrak{E}}_c|}. \tag{19}$$

Note that a more common MI formulation for fibers is space-propagated and yields complex $\kappa(\Omega)$, see Ref. [47]. Equation (18) describes a corresponding time-propagated absolute instability with the complex $\Omega(\kappa)$.

VII. FOUR-WAVE-MIXING CASE

We are now ready to consider the FWM instability without any reference to GNLSE and dispersion coefficients at carrier frequency. Let us neglect for a moment the right-hand-side of Eq. (16),

$$[\mathfrak{E}(\omega_b, k_b) + 2\sigma][\mathfrak{E}^*(\omega_r, k_r) + 2\sigma] = 0. \tag{20}$$

Solutions of Eq. (20) are split into the "blue" and "red" ones. Within the transparency domain, we get two real-valued implicit functions, which can be plotted on the same (k_b, ω_b) plane using Eq. (17). An example is shown in Fig. 3(a). Assume that these two curves cross each other at some point (k_{b0}, ω_{b0}), see the inset in Fig. 3(a). In the vicinity of this double root one can set

$$\omega_b = \omega_{b0} + \Omega, \qquad \omega_r = \omega_{r0} - \Omega^*, \qquad \Omega = O(\sigma),$$
$$k_r = k_{b0} + \kappa, \qquad k_r = k_{r0} - \kappa, \qquad \kappa = O(\sigma),$$

with the real values of

$$\omega_{r0} = 2\omega_c - \omega_{b0}, \quad k_{r0} = 2k_c - k_{b0}.$$

By expanding over Ω and κ, we get

$$\mathfrak{E}_b = -2\sigma + \dot{\mathfrak{E}}_{b0}\Omega + \mathfrak{E}'_{b0}\kappa + O(\sigma^2),$$
$$\mathfrak{E}_r = -2\sigma - \dot{\mathfrak{E}}_{r0}\Omega^* - \mathfrak{E}'_{r0}\kappa + O(\sigma^2),$$

where the derivatives $\dot{\mathfrak{E}}_{b0,r0}$ and $\mathfrak{E}'_{b0,r0}$ and the corresponding group velocities $V_{b0,r0}$ are real.

We now return to the full Eq. (16), consider the vicinity of the intersection point, and derive

$$(\Omega - V_{b0}\kappa)(\Omega - V_{r0}\kappa) = -\frac{\sigma^2}{\dot{\mathfrak{E}}_{b0}\dot{\mathfrak{E}}_{r0}}. \tag{21}$$

The effect of the right-hand-side in Eq. (21) is that the two lines $\Omega = V_{b0}\kappa$ and $\Omega = V_{r0}\kappa$ [Fig. 3(b)] are now reconnected in one of two possible ways, as shown in Figs. 3(c) and 3(d). A gap, like one in Fig. 3(d), indicates a complex-valued $\Omega(\kappa)$ and results in the FWM instability. The latter occurs if

$$\dot{\mathfrak{E}}_{b0}\dot{\mathfrak{E}}_{r0} > 0, \tag{22}$$

and develops with the maximal increment

$$[\text{Im}\,\Omega]_{\max} = \frac{\sigma}{\sqrt{\dot{\mathfrak{E}}_{b0}\dot{\mathfrak{E}}_{r0}}}. \tag{23}$$

It is remarkable that the MI increment (19) is covered by Eq. (23) for $\omega_{b0} = \omega_{r0} = \omega_c$. Equation (23) is universal. On the other hand, Eq. (22) is very different from the MI criterion because it does not depend on the nonlinear frequency shift.

VIII. EXAMPLES

The main result of the previous section can be summarized as follows:

> An unusual scenario of the FWM instability occurs if an intersection of two curves yielded by Eq. (20) on (k_b, ω_b) plane, like (k_{b0}, ω_{b0}) in the inset in Fig. 3(a), takes place not before but after the attenuation gap at $k_b = 2k_c$.

If so, the redshifted wave-vector $k_{r0} = 2k_c - k_{b0}$ is negative. Two situations are then possible. A positive value of $\omega_{r0} = 2\omega_c - \omega_{b0}$ means a backward satellite, otherwise one deals with a forward negative-frequency wave.

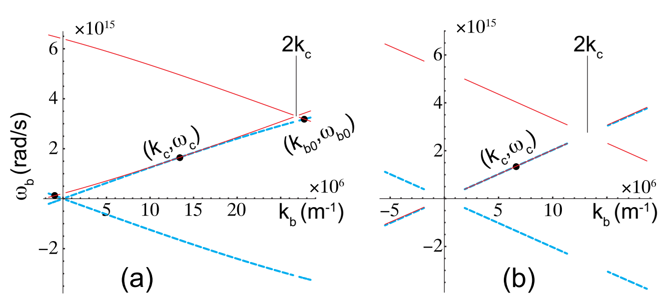

FIG. 4. Solutions of the reduced Eq. (20) are shown for (a) KSR-5 glass with the carrier wave at 1.14 μm and (b) ZBLAN with the carrier wave at 1.4 μm. Change to the full Eq. (16) yields the FWM instability with the backward satellite in panel (a); but it does not yield the negative-frequency wave in panel (b). Notations are as in Fig. 3.

An example of the backward propagating red satellite is shown in Fig. 4(a) for KRS-5 glass [48], which is transparent within 0.6-40 μm, and for a carrier at 1.14 μm. The backward satellite is at 15.6 μm. Similar behavior is expected in several other materials.

FWM instability at a negative frequency is a different story: typical dispersive materials from Ref. [48] proved to be unsuitable. Some systems are very close to the required behavior, e.g., ZBLAN with the carrier wave at 1.4 μm in Fig. 4(b). Here the red (thin solid) and blue (thick dashed) curves are very close to each other. Double roots with negative frequencies should appear for a slightly tuned dispersion law. However, one can demonstrate that even in such a favorable case *the negative-frequency wave is not generated* because the instability condition (22) is not satisfied.

To demonstrate this, we use Eq. (9) to rewrite the inequality (22) as $(\mathfrak{E}'_{b0}/V_{b0})(\mathfrak{E}'_{r0}/V_{r0}) > 0$. As spatial dispersion is small for optical materials, Eq. (8) yields that $\mathfrak{E}' \approx -2kc^2/\omega^2$ and the FWM instability criterion takes the form

$$(k_{b0}/V_{b0})(k_{r0}/V_{r0}) > 0. \tag{24}$$

Examining Fig. 1, we see that inequality (24) is satisfied for the backward satellite but not for the negative-frequency one. The latter satellite does not experience resonant growth.

IX. CONCLUSIONS

Nonlinear waves in dispersive systems are typically decomposed giving birth to new satellite waves, but how far these satellites can go from their origin? Can they go beyond the standard slowly varying envelope approximation and even beyond an extended envelope equation equipped by numerous dispersion coefficients? To address these questions we studied the carrier stability problem using a general material relation (5) and an exact reduction (7) of Maxwell equations. The dielectric function is not expanded at carrier frequency, moreover, the system in question evolves in time in full agreement with the causality principle. This approach resulted in a surprisingly compact dispersion relation for the satellite frequencies (16), which however is implicit and difficult to analyze, as compared with the standard space-propagated modeling of the optical four-wave instabilities. Using geometrical arguments, we revealed the instability criterion (22), and a general expression (23) for the instability increment. In the first place, we have found that the redshifted satellite can reverse its velocity in the laboratory frame and propagate backward to the carrier, as long as the dispersive material is transparent for the infrared radiation. In the second place, we have found that generation of the negative-frequency satellite, while formally possible, does not take place because the wave-mixing instability is switched off for this exotic wave.

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