Dynamics of optomechanical droplets in a Bose-Einstein condensate

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We investigate numerically a one-dimensional Bose-Einstein condensate illuminated by off-resonant laser light which is retroreflected by a single feedback mirror. Studying the ground states of the system, we find density structures which are self-trapped via the optomechanical action of the diffracted light. We show that these structures are stable and exhibit Newtonian dynamics. We propose that these results allow continuous, nondestructive monitoring of condensate dynamics via the optical intensity and may offer opportunities for optical control and transport of coherent matter via gradients in optical phase alone.

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I. INTRODUCTION

Self-organized patterns and structures that arise due to a combination of optical nonlinearity and diffraction have been predicted and observed in a variety of media [1–10] including specifically atomic vapors [1–6,11–13]. In recent times, there has been significant interest in self-organization phenomena involving cold and ultracold atomic gases, e.g., cold atoms or a Bose-Einstein condensate (BECs) interacting with one or more modes of an optical cavity [14–19], which have resulted in a wide range of new nonlinear and quantum phenomena, e.g., collective atomic recoil lasing [14,20], Dicke superradiance [21], and supersolid formation [22–24]. In these systems, the source of optical nonlinearity is optomechanical, i.e., the center-of-mass motion of the atoms under the mechanical action of light, specifically optical dipole forces. Optomechanical self-structuring of a cold thermal gas has been studied experimentally and theoretically in systems of counterpropagating beams in [25,26] and, as is modeled here, in a single-mirror feedback (SMF) configuration in [27,28], with diffraction of light providing spatial coupling between different parts of the BEC. The concept of optomechanical self-structuring was extended theoretically from the case of a thermal gas to a BEC in [29]. It was shown that a significant difference from the behavior in a classical thermal gas was due to the presence of quantum pressure, i.e., the dispersive nature of the BEC wave function, which acts to damp out density modulations or spatial structure in the BEC. Recent work has shown that in addition to global patterns, the system can display a spatially localized structures termed droplets or quantum droplets both in the SMF configuration [30,31] and in a ring-cavity setup [32]. These droplets are self-bound optomechanical structures consisting of interacting light and matter. They display some similar characteristics to quantum droplets in other systems, e.g., dipolar BECs [33], but are also similar in some respects to other types of spatially localized structures, e.g., spatial solitons [34]. In this paper we study the dynamical behavior of these optomechanical droplets in one dimension in a configuration involving a single feedback mirror, as in [30,31].

II. MODEL

Here we model the optomechanical behavior of a BEC present within a SMF configuration as is shown diagrammatically in Fig. 1. We take an approach similar to that in [29] to model the dilute noninteracting BEC where we use a Schrödinger equation which describes the evolution of the BEC wave function $\Psi(x,t)$ as

$$i\hbar\frac{\partial\Psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} + V(x,t)\Psi(x,t), \qquad (1)$$

where we consider a potential energy V to be given by

$$V(x,t) = \frac{\hbar\delta}{2}[|F|^2 + |B(x,t)|^2] \qquad (2)$$

and where m is the atomic mass; $\delta = \omega - \omega_a$, with ω and ω_a the optical field frequency and atomic transition resonance frequency respectively; $s = |F|^2 + |B(x,t)|^2$ is the saturation parameter due to the forward and backwards fields, which are given by $|F,B|^2 = \frac{I_{F,B}}{I_{\mathrm{sat}}\Delta^2}$, with $I_{F,B}$ the intensity of the forward (F) or backward (B) beam, I_{sat} the saturation intensity on resonance, and $\Delta = \frac{2\delta}{\Gamma}$; and Γ is the decay rate of the atomic transition. It has been assumed that $|\Delta| \gg 1$ and that consequently $s \ll 1$ so that the atoms remain in their ground state. In addition, longitudinal grating effects due to interference between the counterpropagating optical fields on the transverse pattern formation process are neglected.

In order to describe the optical field evolution we assume that the gas is sufficiently thin that diffraction can be neglected, so that the forward field transmitted through the cloud is

$$F_{tr} = \sqrt{p_0}\exp[-i\chi_0 n(x,t)]. \qquad (3)$$

where $p_0 = |F(z=0)|^2$ is the scaled pump intensity incident on the atoms, $\chi_0 = \frac{b_0}{2\Delta}$ is the susceptibility of the cloud, b_0 is the optical thickness of the cloud at resonance, and $n(x,t) =$

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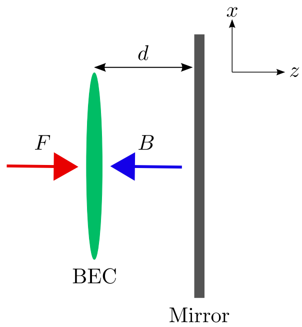

FIG. 1. Schematic diagram of the SMF configuration

$|\Psi(x,t)|^2$ is the local density of the BEC. We consider that the total density is conserved and as such the assumption of a preserved number of atoms is made.

To complete the feedback loop, calculation of the backward field B at the atomic cloud is required. As the field propagates a distance $2d$ from the cloud to the mirror and back, optical diffraction plays a crucial role by converting phase modulations to amplitude modulations and consequently optical dipole forces. The relation between the Fourier components of the forward and backward fields at the cloud is

$$B(q) = \sqrt{R} F_{tr}(q) e^{-iq^2 d/k_0}, \tag{4}$$

where R is the mirror reflectivity, q is the transverse wave number, $k_0 = \frac{2\pi}{\lambda_0}$, and it has been assumed that $q \ll k_0$. It has also been assumed that the propagation time of the light between the BEC and mirror is sufficiently small as to be neglected.

Equations (1), (3), and (4) can be solved self-consistently to describe the mutual interaction of the moving atoms and the optical fields. Numerical integration of these equations is performed to obtain the spatiotemporal dynamics of the BEC within the SMF setup. Transformation to imaginary time $\tau = -it$ is also performed here and the same equations are numerically integrated with an initial Gaussian density distribution to obtain the ground-state solutions.

III. OPTOMECHANICAL PATTERNS AND DROPLETS

A. Optomechanical patterns

The existence of optomechanical patterns in a dilute BEC illuminated by an optical field retroreflected by a single feedback mirror was first predicted in [29] using a one-dimensional model. These patterns form as a result of a self-structuring instability in which a spatially homogeneous optical field and BEC density become unstable, resulting in the spontaneous formation of periodic modulations in both the optical intensity and BEC density. The physical origin of the instability is the Talbot effect, which converts phase modulation in the optical field produced by BEC density fluctuations to intensity modulations and consequently optical dipole forces which increase density modulation. An example of this pattern formation is shown in Fig. 2. The system develops a modulated optical intensity and modulated BEC density with a spatial period of $\Lambda_c = \frac{2\pi}{q_c}$, where

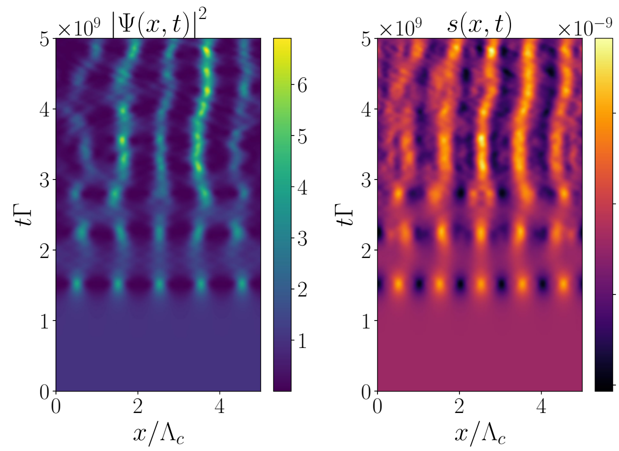

FIG. 2. Example of pattern formation or self-structuring of a BEC in a single-feedback-mirror configuration. The parameters used here are $p_0 = 1.9 \times 10^{-9}$, $\Delta = -100$, $R = 0.99$, $(\omega_r)/\Gamma = 9.88 \times 10^{-9}$, and $b_0 = 20$.

$$q_c = \sqrt{\frac{\pi}{2}\frac{k_0}{d}}. \tag{5}$$

From q_c we define $\omega_r = \frac{\hbar q_c^2}{2m}$, analogous to the recoil frequency associated with momentum changes of $\hbar q_c$.

The reason for this instability is that BEC density modulations (which correspond to refractive index modulations of the BEC) with spatial frequency q_c produce phase modulations in F_{tr} which are in turn converted into intensity modulations of B [see Eq. (4)] [35]. These intensity modulations produce dipole forces, which in turn reinforce density modulations, resulting in positive feedback and instability of the initial homogeneous state. More recently, two-dimensional patterns and droplet formation in a SMF configuration were studied in [30,31], including the effects of direct atom-atom interactions via the BEC scattering length.

B. Optomechanical droplets

1. Stable droplets

In Sec. III A, the initial conditions correspond to a spatially homogeneous optical intensity and BEC density which exhibits a self-structuring instability and evolves into a quasistationary state which consists of a strongly modulated pattern with some temporal variation in the amplitude of the pattern maxima and minima. However, simulations involving imaginary-time propagation demonstrate that the ground states of this system are localized structures of BEC density and corresponding optical intensity, as originally predicted in [30], where a nonlinearity due to a combination of optomechanical forces and atomic collisions was investigated. Here we concentrate on a regime in which effects due to atomic collisions (or atomic scattering length) are negligible and structures are produced due to optomechanical forces alone. The temporal evolution of these stable optomechanical droplets is shown in Figs. 3 and 4 for red detuning ($\Delta < 0$) and blue detuning ($\Delta > 0$), respectively. It can be seen that

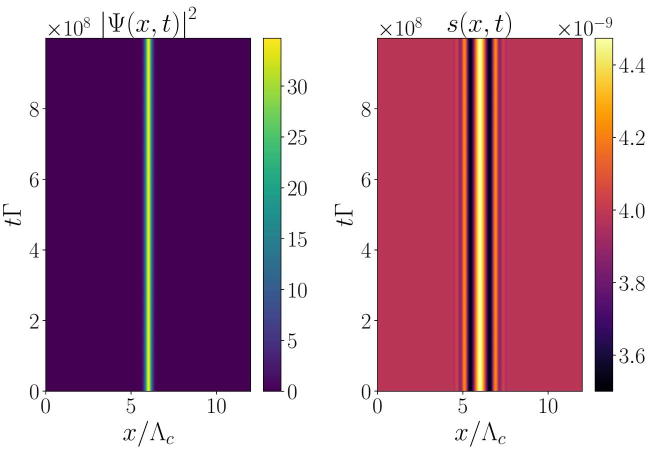

FIG. 3. Evolution of a stable optomechanical droplet for red detuning. The parameters are $p_0 = 2.0 \times 10^{-9}$, $\Delta = -800$, $R = 0.99$, $(\omega_r)/\Gamma = 5.69 \times 10^{-8}$, and $b_0 = 20$

for red detuning, the maxima of the BEC density and optical intensity coincide due to the potential energy of the system being minimized when atoms sit at positions of maximum optical intensity. In contrast, for blue detuning, the maximum of the BEC density coincides with a minimum of optical intensity. The properties of the stable ground-state droplet are determined in full by the optical parameters of the system such as detuning, pump intensity, and mirror distance d (via q_c).

Observation of a fully static droplet in time-dependent simulations using Eqs. (1), (3), and (4) requires the BEC density profile to exactly match the ground-state profile. However, the system continues to support droplet structures with an initial BEC density profile which is perturbed from the ground state. With such initial conditions the system exhibits now dynamical behavior, with density oscillations as the self-imposed trapping potential from the optical intensity continually adjusts to the changing BEC density profile. This behavior is shown in Fig. 5.

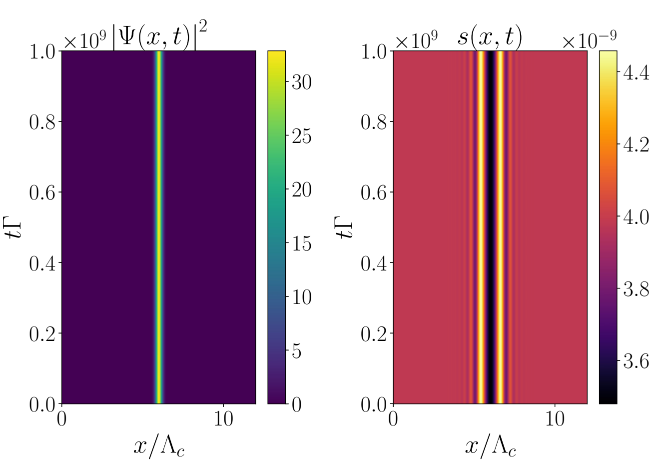

FIG. 4. Evolution of a stable optomechanical droplet for blue detuning. The parameters are $p_0 = 2.0 \times 10^{-9}$, $\Delta = 800$, $R = 0.99$, $(\omega_r)/\Gamma = 5.69 \times 10^{-8}$, and $b_0 = 20$.

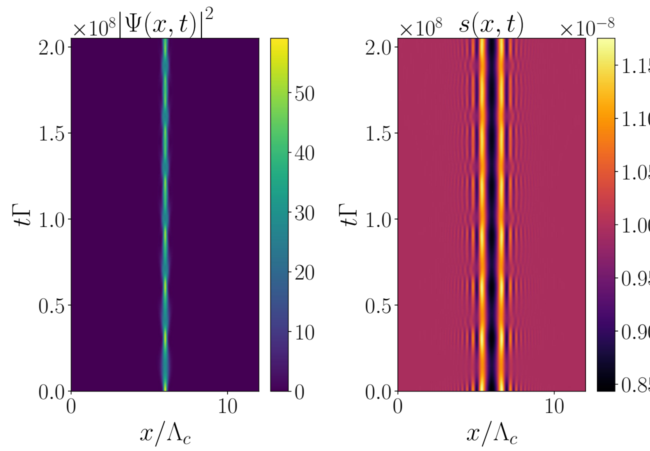

FIG. 5. Evolution of a perturbed blue-detuned droplet, where the initial BEC density has been produced from a Gaussian distribution whose width is adjusted slightly from the ideal value. The parameters are $p_0 = 5.0 \times 10^{-9}$, $\Delta = 800$, $R = 0.99$, $(\omega_r)/\Gamma = 5.69 \times 10^{-8}$, and $b_0 = 20$.

For both the self-structuring patterned state and perturbed droplet profiles we can observe temporal variation consistent with a system which is a superposition of eigenstates rather than in the ground state. In this conservative system the patterned state is a asymptotic state; however, with the inclusion of damping and friction, the system would be expected to relax to the droplet state as it is a ground state of the system, as demonstrated by the imaginary-time simulations.

Quantum pressure in the BEC plays an important role in stabilizing the droplet against compression and is capable of producing a stable droplet even in the absence of other dispersive effects such as finite temperature or repulsive collisions (positive scattering length). It can be shown that the presence of quantum pressure is required to produce a minimized ground-state energy with nonzero droplet width [36,37]. For narrow droplets an additional stabilizing factor is diffraction, which will produce a lower limit to the width of the optical potential associated with the droplet.

2. Single- and multiple-peak droplet structures

Figure 6 shows the dependence of the width and amplitude of the BEC density on pump intensity p_0 when a stable droplet

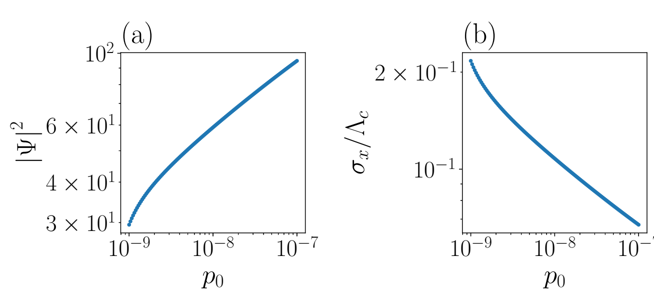

FIG. 6. (a) Droplet width σ_x and (b) peak density $|\Psi|^2$ against pump intensity p_0. Results were calculated through imaginary-time integration with the parameters $R = 0.99$, $(\omega_r)/\Gamma = 1.01 \times 10^{-7}$, $\Delta = 800$, and $b_0 = 20$.

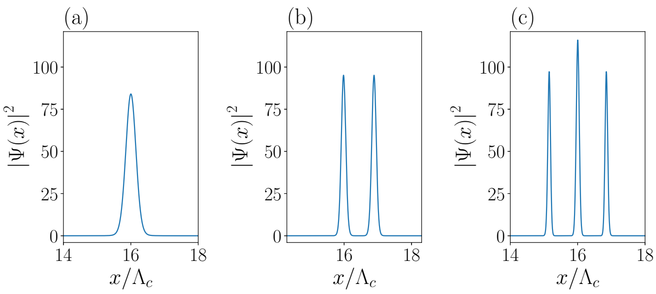

FIG. 7. Example ground-state droplet density profiles of (a) single-, (b) dual-, and (c) triple-peak droplet structures for red detuning, calculated from imaginary-time integration. The parameters are $R = 0.99$, $\langle\omega_r\rangle/\Gamma = 4.05 \times 10^{-7}$, $b_0 = 20$, $\Delta = -800$, and pump intensity p_0 equal to (a) 3.714×10^{-9}, (b) 2.395×10^{-8}, and (c) 9.398×10^{-8}. Shown in (b) is an off-center structural position, consistent with the translational invariance of the system.

forms. It can be seen that as the pump intensity is increased, the droplet narrows and the peak of the BEC density increases. It can also be seen that the width of the droplet has a power-law dependence on p_0, scaling as $\sigma_x \propto {p_0}^{-1/4}$, in agreement with analytical predictions of the droplet width in the limit where $\chi|\Psi|^2 \ll 1$ [36,37].

When the pump is red detuned ($\Delta < 0$), then in addition to the single-peak droplet structures shown previously, structures consisting of multiple density peaks also arise. Examples of these multipeak droplet structures are shown in Figs. 7(b) and 7(c), which show double- and triple-peak droplet structures respectively. Complex, multiple droplet structures were observed in [30,31], but their physical origin is different as their existence was reported only for nonzero BEC scattering length, whereas here we exclusively consider the case for no internal interactions.

These multiple droplet structures have not been observed in numerical simulations for cases involving blue detuning ($\Delta > 0$). The reason for the different behavior of the system when red and blue detuned is due to the combined effect of refraction in the narrow BEC and diffraction between the BEC and mirror, which produces the corresponding optical intensity profile and consequent (dipole) potential energy profile. The BEC acts like a narrow refractive element, which affects the optical phase as described by Eq. (3). Its refractive effect is dependent on χ_0 and consequently on Δ. The resulting diffraction pattern after propagation of the transmitted optical field from the BEC to the mirror and back will also therefore depend on Δ. The most significant difference is in relative amplitudes of the off-axis maxima and minima of the diffraction pattern. As shown in Figs. 3(b) and 4(b), for $\Delta > 0$ the pattern consists of a central minimum, with a series of damped oscillations off-center which are characteristic of Fresnel diffraction. For $\Delta < 0$ the pattern consists of a central maximum, with damped oscillations off-center. As pump intensity p_0 is increased, the width of the BEC decreases as shown in Fig. 6, which causes the amplitude of the off-center diffractive minima and maxima to increase relative to the central one [35].

In the case of red detuning these off-center maxima grow to become the global maxima of the optical intensity profile.

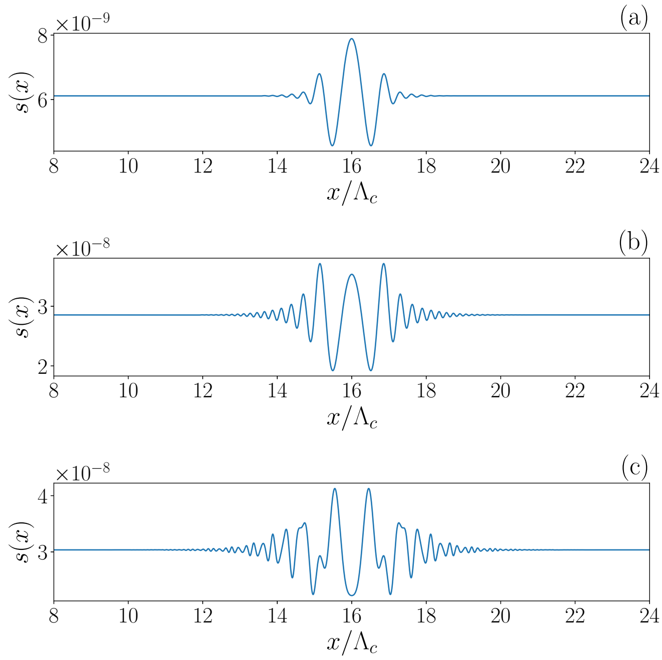

FIG. 8. Ground-state optical field intensity profiles for the red-detuned system with pump amplitudes (a) $p_0 = 3.07 \times 10^{-9}$, (b) $p_0 = 1.43 \times 10^{-8}$, and (c) $p_0 = 1.53 \times 10^{-8}$, calculated from imaginary-time integration. The other parameters are $R = 0.99$, $\langle\omega_r\rangle/\Gamma = 4.05 \times 10^{-7}$, $\Delta = -800$, and $b_0 = 20$.

This results in an energetically favorable configuration when the BEC occupies the off-center locations of peak optical intensity instead of occupying the central local maximum. Figure 8(a) shows the case for a global central peak where a single-peak droplet is the ground-state configuration. Figure 8(b) shows that, for an increased pump amplitude relative to Fig. 8(a), the off-center maxima are now the global maxima. As the pump amplitude is increased further a transition to the two-peak droplet ground state is observed, at which point the optical intensity profile takes the form shown in Fig. 8(c). For blue detuning the off-center minima do not grow to become global minima of the intensity profile for any of the parameters examined here.

3. Dynamic droplets

It has been established that the BEC and optical fields can form a stationary stable droplet. We now consider the dynamics of moving droplets, with first the addition of a uniform velocity. Providing our initial BEC wave function with an additional linear phase gradient will imprint this initial velocity on the droplet:

$$\Psi(x, t = 0) = \sqrt{n_0(x)}e^{imv_0x/\hbar}. \tag{6}$$

Such a phase gradient is given in Eq. (6), where $n_0(x)$ is the initial density profile of the BEC (the ground-state droplet profile), v_0 is the uniform BEC velocity, and m is the mass of each atom in the BEC.

Figures 9 and 10 show the evolution of the BEC and optical fields for red and blue detuning, respectively. These droplets continue to be stable like their static counterparts, which were

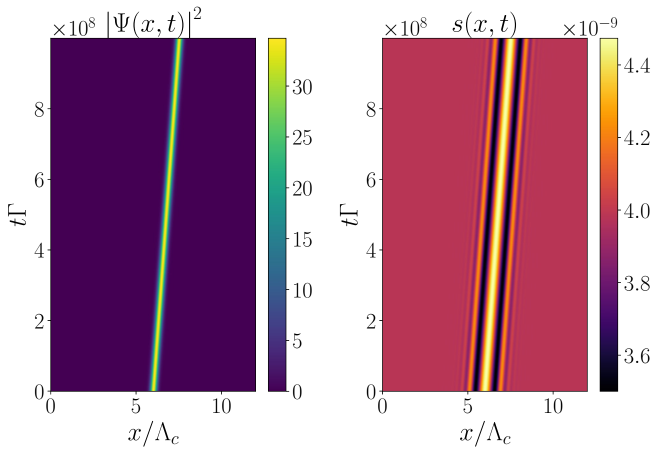

FIG. 9. Uniformly moving droplet for red detuning ($\Delta < 0$). The parameters are identical to those of Fig. 3 with the addition of $v_0 = \hbar q_c/12m$.

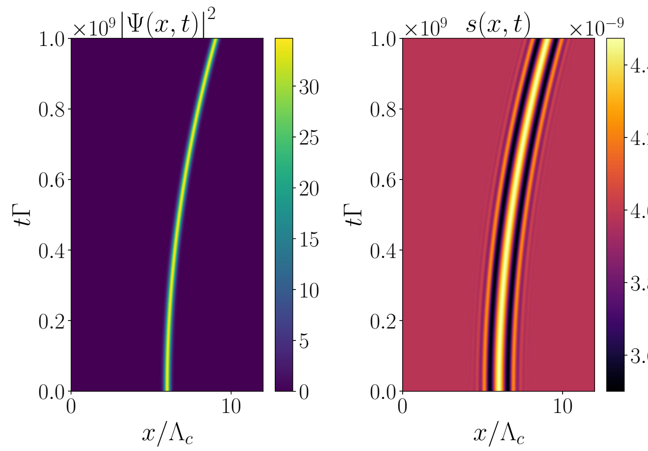

FIG. 11. Uniformly accelerating droplet for red detuning ($\Delta < 0$). The parameters are identical to those of Fig. 3 with the addition of $a = -4.0 \times 10^{-9}(\hbar q_c\Gamma/12m)$.

shown in Figs. 3 and 4. In both cases the optical field distribution also moves with uniform velocity, tracking the uniformly moving BEC. In the case of red detuning (Fig. 9), this results in an optical intensity maximum which always coincides with the BEC, whereas in the case of blue detuning (Fig. 10), an optical intensity minimum always coincides with the BEC.

The dynamics of similar self-trapping BEC structures have been studied for a ring-cavity configuration [32] where a frictionlike force was found to damp out motion of the density structures. This friction arises in the ring-cavity case from the finite response time of the cavity, which allows optical intensity profiles to lag behind changes in BEC density. Although mirror loss is included within our model, the finite time for propagation of light through the system is neglected, resulting in the undamped motion seen here.

Similarly, we can investigate the stability and behavior of these droplets under uniform acceleration. An acceleration can be achieved with the modification of the potential energy given by

$$V(x,t) = \frac{\hbar\delta}{2}[|F|^2 + |B(x,t)|^2] + (ma)x, \tag{7}$$

where a is the constant acceleration. Figures 11 and 12 show the evolution of the BEC and optical fields for the cases of red and blue detuning, respectively. It can be seen that the BEC now accelerates uniformly and, as for the case of uniform motion, the optical field follows this motion with the BEC density coinciding with an optical intensity maximum or minimum for red and blue detuning, respectively.

Figures 9 and 10 for uniform motion and Figs. 11 and 12 for uniform acceleration show that in both cases it is possible to infer the distribution of BEC density continuously via observation of the optical intensity distribution.

It should be noted that although only the motion of single droplet structures has been presented here, the stable multipeak droplet structures display similar behavior under motion, maintaining their structure as they propagate and providing a consistent optical intensity profile dependent on detuning.

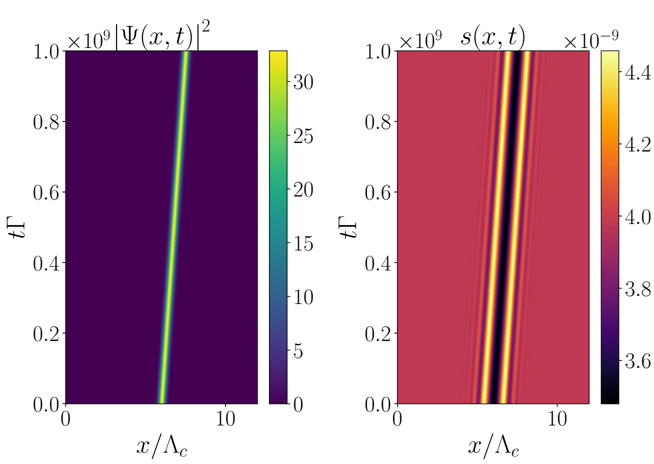

FIG. 10. Uniformly moving droplet for blue detuning ($\Delta > 0$). The parameters are identical to those of Fig. 4 with the addition of $v_0 = \hbar q_c/12m$.

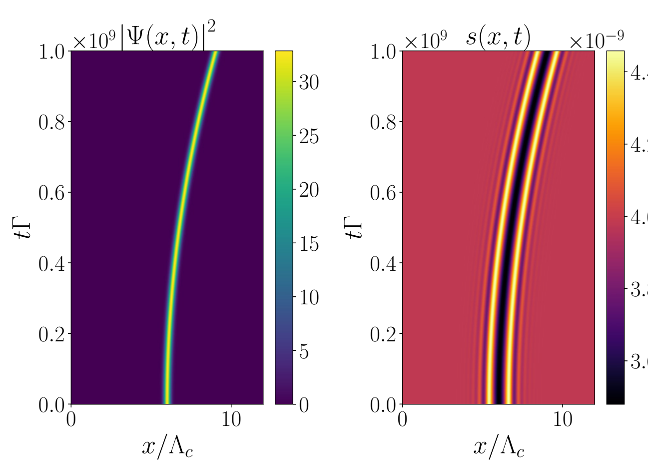

FIG. 12. Uniformly accelerating droplet for blue detuning ($\Delta > 0$). The parameters are identical to those of Fig. 4 with the addition of $a = -4.0 \times 10^{-9}(\hbar q_c\Gamma/12m)$.

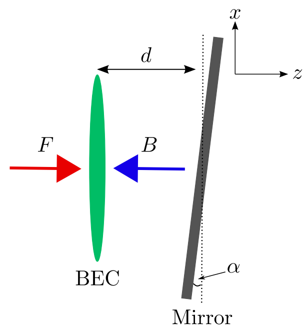

FIG. 13. Schematic diagram of the SMF configuration with a mirror misalignment or tilt, labeled α.

C. Controlling droplet motion using mirror tilt

In the preceding section we demonstrated how imposing a uniform velocity or acceleration on the BEC could produce an optomechanical droplet with a BEC density distribution and optical field distribution which moved with uniform velocity or uniform acceleration, respectively. We now investigate what happens when the mirror in the SMF configuration is not perfectly aligned so that the normal to the mirror and the pump propagation direction are misaligned by a small angle α. This mirror misalignment or tilt is shown schematically in Fig. 13.

In order to simulate the effect of this mirror tilt, we follow the method used in [38], where before calculation of the backward field B using Eq. (4) the forward field is shifted by an amount $\Delta x = 2d\tan(\alpha)$. The effect of a mirror tilt can also be understood as creating a phase gradient in the reflected light. An example of evolution of an initially stationary droplet ($v_0 = a = 0$) with $\Delta x > 0$ is shown in Fig. 14. It can be seen that the effect of the mirror tilt is to produce a constant acceleration on the droplet.

Figure 15 shows the dependence of the droplet acceleration on the mirror-tilt-induced shift Δx. It can be seen that for the smallest mirror tilts, the acceleration produced is

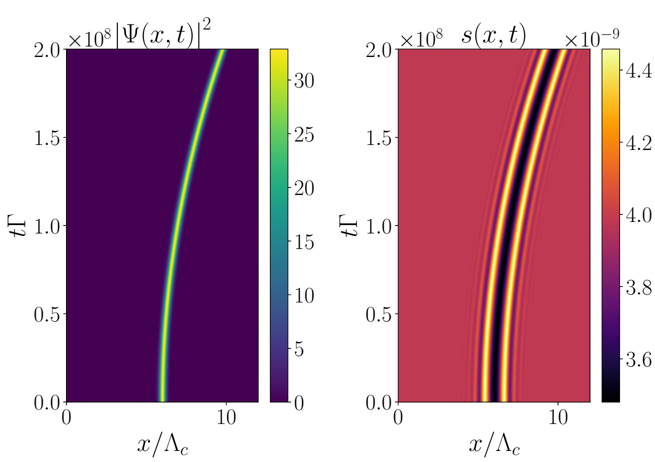

FIG. 14. Evolution of BEC density and optical field intensity when a mirror tilt is present. The parameters are $p_0 = 1.9 \times 10^{-9}$, $\Delta = 800$, $R = 0.99$, $(\omega_r)/\Gamma = 5.69 \times 10^{-8}$, $b_0 = 20$, and $(\Delta_x)/(\Lambda_c) = 1.465 \times 10^{-3}$.

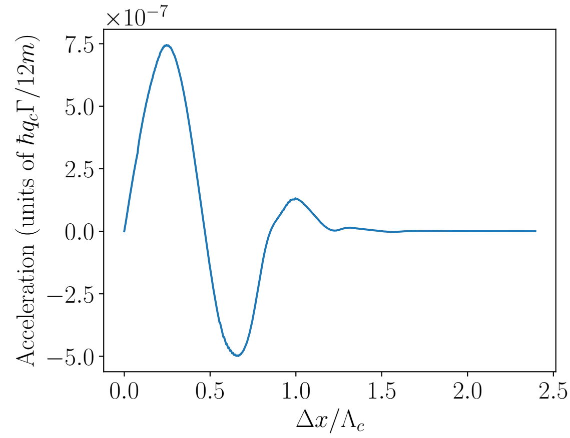

FIG. 15. Dependence of droplet acceleration a_d on mirror-tilt-induced shift $\Delta x/\Lambda_c$. The parameters are $p_0 = 1.9 \times 10^{-9}$, $\Delta = 200$, $R = 0.99$, $(\omega_r)/\Gamma = 1.01 \times 10^{-7}$, and $b_0 = 20$.

approximately proportional to Δx and consequently α. However, as the mirror tilt is increased, there is a region where the acceleration produced changes direction. Similar behavior was observed for dissipative solitons [34]; however, a significant difference between the behavior shown here and that in [34] is that here the droplets exhibit Newtonian dynamics whereas in [34] the solitons exhibited Aristotelian dynamics.

The dynamics of dissipative solitons in phase gradients are known from many such dissipative systems. However, the dynamics are Aristotelian in nature as overdamped motion exhibits a constant velocity in the presence of a constant gradient [34,39,40]. The acceleration consistent with Newtonian motion in the BEC model considered here is a different feature of conservative optomechanical systems, in comparison to dissipative solitons relying on internal degrees of freedom [34,39,40] or optomechanical structures and solitons in the presence of velocity damping [41,42].

Laser solitons in which the medium dynamics is infinitely fast should also exhibit Newtonian dynamics [9,43], but we are not aware of any experimental observation, as typical lasers do not operate in this regime. In contrast, observation in a BEC system, similar to that discussed here, would appear to be very feasible.

In addition to inducing acceleration of the droplet, the application of a finite mirror tilt also decreases the long-term stability of the droplet, with stability of the structures preserved for only very small misalignments.

IV. CONCLUSION

We have investigated the dynamical behavior of optomechanical droplets, self-bound structures which arise due to the interaction between light and a BEC in the presence of a feedback mirror, using a one-dimensional model. We have shown the existence of multipeak droplet profiles from optical interactions alone, when atomic collisions are negligible. We have also shown that by inducing BEC motion with constant velocity and constant acceleration, the optomechanical droplets remain stable and also move with the same velocity and acceleration, respectively, with the BEC density maximum being tracked by the optical field pattern in each case. As the pump is far detuned from resonance, absorption, and therefore heating, of the BEC due to scattering of pump photons is minimized. Consequently, these results may offer possibilities for methods allowing continuous measurement of BEC dynamics. Finally, we demonstrated that by introducing a mirror misalignment or tilt it was possible to induce a constant droplet acceleration. This may offer opportunities for optical control and transport of coherent matter via phase gradients rather than amplitude gradients. Possibilities for future development of the results presented here include investigation of the dynamics of droplets in two dimensions and the inclusion of atomic collisions, i.e., nonzero scattering length in the BEC, as [30] showed that the inclusion of nonzero scattering length leads to other more complex structures, e.g., droplet chains and lattices, in addition to the single-peak and multipeak droplets considered here.

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