Quantum cost of dense coding and teleportation

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Quantum cost is a key ingredient in evaluating the quality of quantum protocols from a practical viewpoint. We show that the quantum cost of a d-dimensional dense coding protocol is equal to $d+3$ when transmitting the classical message (0,0) and $d+4$ when transmitting another classical message. It appears as linear growth with the dimension and thus makes sense for implementation. In contrast, the quantum cost of high-dimensional teleportation protocols is equal to 13, which is the maximum value of the cost for the two-dimensional case. As an application, we establish the relation between the quantum cost and fidelity of dense coding protocols in terms of four typical noise scenarios.

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I. INTRODUCTION

In recent decades, quantum communication has been a prominent application of quantum mechanics. Dense coding was first proposed by Bennett *et al.* [1]. It is a fascinating method to transmit two bits of classical information using quantum resource like entanglement. One year later, quantum teleportation [2] was proposed to realize reliable transmission of an unknown state. Recently, several applications of quantum communication such as secure quantum key distribution [3,4] have been successfully deployed. Teleportation of photonic qubits over long distances of up to 1400 km through an uplink channel has been reported [5]. A demonstration of teleportation from photons to the vibrations of nanomechanical resonators has been proposed [6,7]. Superdense teleportation has been implemented by photon pairs to communicate a specific class of single-photon ququarts with an average fidelity of 87.0% [8]. The probabilistic implementation of a nonlocal operation using a nonmaximally entangled state has been developed [9]. In addition to the practical application, quantum teleportation provides a new perspective to redesign the classical communication system model [10]. Dense coding and teleportation are generalized with quantum states in high-dimensional Hilbert space [11,12], as the qudit states with higher robustness to noise improve the channel capacity and the information security. The relation between quantum error-correcting codes in heterogeneous systems and quantum information masking is indicated [13]. A scheme for teleportation of arbitrarily high-dimensional photonic quantum states has been proposed, and the averaged fidelity is calculated to be 75% in current experiments [14]. Since the high-dimensional unitary operations are more difficult to implement in physical experiments, it is necessary to measure the implementation cost of quantum protocols by calculating the quantum cost. The quantum cost of an arbitrary gate was first introduced by Barenco *et al.* [15]. Generally, the quantum cost of a circuit is the sum of the cost of each gate used in designing the circuit. The higher the quantum cost, the more complex the execution of the circuit. Quantum cost is a common figure of merit to evaluate and compare different circuits. It is the key to evaluating the quality of protocols both theoretically and experimentally. Since the quantum cost was first proposed, efforts have been made to calculate the cost of unitary gates and quantum circuits. A procedure has been presented to optimize distributed quantum circuits in terms of teleportation cost for a predetermined partitioning [16]. An efficient method has been proposed to reduce the number of teleportation requirements based on the commuting of quantum gates [17]. In a recent work, the quantum cost of teleporting a single qubit message among six different entangled channels was calculated and compared [18]. However, the situation may become more complex when we consider the quantum cost of higher-dimensional teleportation protocols.

In this paper we analyze the quantum cost of d-dimensional dense coding and teleportation protocols. We obtain that the quantum cost of the d-dimensional dense coding protocol is equal to $d+3$ when transmitting the classical message (0,0) and $d+4$ when transmitting another classical message. As for the teleportation protocol, we generalize the two-dimensional Pauli X gate to d-dimensional gates, which are implemented on the controlled qudits to recover the information. By adding these appropriate gates on the circuits, we obtain that the quantum cost of all high-dimensional teleportation protocols is equal to 13, which is the maximum value of the quantum cost for the two-dimensional case. The quantum protocol will finally need hardware to realize. The implementation cost of a quantum circuit increases with its quantum cost. Our results show that the physical implementation cost of the high-dimensional dense coding protocol grows linearly with the dimension. As an application of the quantum cost, we show that the fidelities of dense coding decrease with the increase of its

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quantum cost. Since the quantum cost of d-dimensional teleportation remains 13 for any $d \geqslant 3$, our results show that the demand for a practical device to implement high-dimensional teleportation remains the same, without regard to the implementation cost for each gate influenced by the dimension.

Dense coding and teleportation protocols have been extended via multipartite entangled states, such as Greenberger-Horne-Zeilinger states [19,20], W states [21], cluster states [22], and genuine multiparticle entangled states [23]. An explicit scheme has been designed for the teleportation of an n-qubit quantum state. Its experimental realization is performed using a five-qubit superconductivity-based IBM quantum computer with high fidelity [24]. A scheme of $1 \rightarrow 2$ optimal universal asymmetric quantum telecloning for pure multiqubit states is proposed [25]. Since the multipartite entangled states can be regarded as bipartite states, our study of bipartite high-dimensional dense coding and teleportation influences the multipartite case.

The rest of this paper is organized as follows. In Sec. II we introduce some basic concepts and list the basic gates used in this paper. Based on those, we decompose the nonbasic gates into basic gates and calculate the quantum cost of each gate. In Secs. III and IV we analyze the quantum cost of high-dimensional dense coding and teleportation protocols, respectively. We show the application of quantum cost in Sec. V. We show the quantum cost in experiment in Sec. VI. Finally, we summarize in Sec. VII.

II. PRELIMINARIES

In this section we review some basic concepts, decompose the nonbasic gates, and calculate their quantum cost. In Sec. II A we introduce the concept of quantum cost and its computation. In Sec. II B we show the basic gates used in this paper. In Sec. II C we show the decomposition of nonbasic gates, from which we obtain the quantum cost of each gate used in the dense coding and teleportation protocols.

A. Quantum cost

The quantum cost of a circuit is obtained by adding up the cost of each gate in the circuit. An arbitrary gate can be decomposed into several basic gates and the cost of basic gates is considered to be a unit cost, regardless of their internal structure. That is to say, we consider that the cost of a basic gate is 1. If a gate can be decomposed into n basic gates, then the quantum cost of the gate is equal to n. When we refer to the quantum cost of a protocol, we mean the quantum cost of the corresponding circuit. Mohammadi and Eshghi [26] have proposed two prescriptions for the calculation of quantum cost.

(i) Implement a circuit using only the quantum primitive gates and count them.

(ii) Synthesize a circuit using the gates whose quantum cost is specified. Add up the quantum cost of each gate in the circuit to obtain the total quantum cost of the circuit.

In this paper we follow prescription (ii). We consider a gate primitive if it maps decomposable states to decomposable states. This means that the primitive gate cannot generate entanglement. Obviously, some gates used in the dense coding and teleportation protocols should have the ability to generate entanglement and hence they are not primitive, for example, the CNOT gate. In addition, some gates used in the two protocols can be prepared by the gates whose quantum cost is specified.

B. Basic gate

Barenco *et al.* considered all single-qubit gates and the CNOT gate as the basic gates in the two-dimensional case and showed that we can realize the controlled operations by at most six basic gates [15]. They showed that the CNOT gate along with single-qubit gates may be assembled to do any quantum computation. The basic qubit gate can be extended to the basic qudit gate. Brylinski and Brylinski [27] proposed that the collection of all one-qudit gates together with a two-qudit imprimitive gate is universal, i.e., every n-qudit gate can be approximated with arbitrary accuracy by this collection of gates. Hence, all the single-qudit gates and the two-qudit imprimitive gate are the basic gates in the d-dimensional case.

In this paper we consider the single-qudit unitary gates, i.e., H_d, $H_d^\dagger$, $U_{mn,d}$, and $P_{k,d}$, and the two-qudit imprimitive CNOT gate as the basic gates. That is to say, the two-qudit gates should be decomposed with the help of the CNOT gate. The expressions of the two-dimensional basic gates are shown in Sec. II B 1 and those of the d-dimensional basic gates are given in Sec. II B 2.

1. Two-dimensional basic gates

We list some two-dimensional basic gates used in this paper: the two-dimensional Hadamard gate

$$H_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \tag{1}$$

the CNOT gate.

$$U_{\text{CNOT},2} = |0,0\rangle\langle 0,0| + |0,1\rangle\langle 0,1| + |1,1\rangle\langle 1,0| + |1,0\rangle\langle 1,1|, \tag{2}$$

and the Pauli X, Y, and Z matrices

$$\sigma_X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{3}$$

The quantum cost of these two-dimensional basic gates is equal to 1.

2. The *d*-dimensional basic gates $d > 2$

We show the d-dimensional basic gates which can be considered as the generalization of two-dimensional basic gates.

We set $\omega = e^{2\pi i/d}$. The d-dimensional Hadamard gate is

$$H_d = \frac{1}{\sqrt{d}} \sum_{x,y=0}^{d-1} \omega^{xy} |x\rangle\langle y|. \tag{4}$$

When $d = 2$, we have $H_d = H_2$. The quantum cost of the H_d gate is equal to 1.

The $H_d^\dagger$ gate is used to recover the message in a dense coding protocol. The expression of this gate is

$$H_d^\dagger = \frac{1}{\sqrt{d}} \sum_{x,y=0}^{d-1} \omega^{x(d-y)} |x\rangle\langle y|. \tag{5}$$

It is a single-qudit basic gate and its quantum cost is equal to 1.

The $U_{mn,d}$ gates are used to implement the operation corresponding to the classical message to be transmitted in a dense coding protocol. This kind of gate is expressed as

$$U_{mn,d} = \sum_{u=0}^{d-1} \omega^{mu} |u\rangle\langle n \oplus u|, \tag{6}$$

where $m, n = 0, 1, \ldots, d-1$ and $\oplus$ denotes sum modulo d.

The following $P_{k,d}$ gates play an important role in the quantum teleportation protocol:

$$P_{k,d} = \sum_{s=0}^{d-1} |s\rangle\langle k \oplus (d-s)|. \tag{7}$$

The quantum cost of $P_{k,d}$ gates is equal to 1. Note that when $d = 2$, we have $P_{0,2} = I$ and $P_{1,2} = \sigma_X$. So they are the generalization of two-dimensional basic gate.

The CNOT gate performs the transformation $|a, b\rangle \rightarrow |a, a \oplus b\rangle$. The expression of this gate is

$$U_{\mathrm{CNOT},d} = \sum_{x,y=0}^{d-1} |x, y \oplus x\rangle\langle x, y|. \tag{8}$$

When $d = 2$, we have $U_{\mathrm{CNOT},d} = U_{\mathrm{CNOT},2}$. The quantum cost of the CNOT gate is equal to 1.

We show its function on the controlled qudit, which will be used later. If the controlled qudit is $|k\rangle$, then the CNOT gate performs the operation $X_{k,d}$ on the controlled qudit

$$X_{k,d}|j\rangle = |j \oplus k\rangle, \tag{9}$$

where

$$X_{k,d} = \sum_{s=0}^{d-1} |s \oplus k\rangle\langle s|. \tag{10}$$

C. Some gates prepared by the basic gates

We prepare the controlled-Z and CNOT† gates by the basic gates and show their quantum cost. They are used in the dense coding and teleportation protocols later.

The controlled-Z gate is a two-qudit gate. It can be prepared with the help of a CNOT gate and two Hadamard gates, i.e.,

$$U_{\mathrm{CZ},d} = (I \otimes H_d) U_{\mathrm{CNOT},d} (I \otimes H_d). \tag{11}$$

The gate can be decomposed into three basic gates. The quantum cost of it is equal to 3.

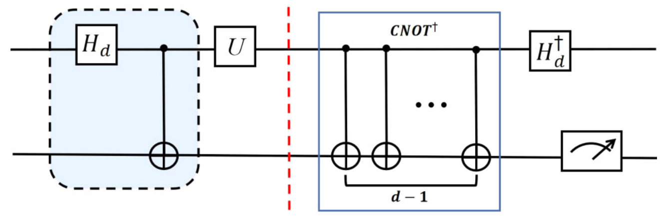

FIG. 1. Protocol for the d-dimensional dense coding. The two left gates are used to prepare Bell states $|\phi_1\rangle$ with the qudit $|0, 0\rangle$. The gates to the right of the dashed line are used by Bob to recover the message.

If the controlled qudit is $|k\rangle$, then the controlled-Z gate performs the operation $Z_{k,d}$ on the controlled qudit, where

$$Z_{k,d} = H_d X_{k,d} H_d \quad (k = 0, 1, \ldots, d-1). \tag{12}$$

We can verify that

$$Z_{k,d} = \sum_{j=0}^{d-1} \omega^{kj} |j\rangle\langle d - j|.$$

When $d = 2$, we have $Z_{0,2} = I_2$ and $Z_{1,2} = \sigma_Z$. The two-qudit controlled-Z gate is the generalization of the two-qubit controlled-Z gate.

The CNOT† gate is used to recover the message in dense coding. It is used to implement the operation $|a, b\rangle \rightarrow |a, b \oplus (d-1)a\rangle$. For all the two-qudit gates, the only basic gate is the CNOT gate. Hence, the CNOT† gate should be decomposed with the help of the CNOT gate. We can obtain that it can be prepared by $d - 1$ CNOT gates, i.e.,

$$U_{\mathrm{CNOT},d}^\dagger = \sum_{x,y=0}^{d-1} |x, y \oplus (d-1)x\rangle\langle x, y| = (U_{\mathrm{CNOT},d})^{d-1}. \tag{13}$$

The quantum cost of the CNOT† gate is equal to $d - 1$.

III. QUANTUM COST OF DENSE CODING

In this section we show the quantum cost of the high-dimensional dense coding protocol. Suppose Alice and Bob share the d-dimensional Bell channel $|\phi_1\rangle = \frac{1}{\sqrt{d}} \sum_{k=0}^{d-1} |k, k\rangle$. The first qudit belongs to Alice and the second one belongs to Bob. Alice wants to send two dits of a classical message to Bob. The protocol is shown in Fig. 1.

The two-dit classical message may be one of the elements in the set $\{(0, 0), (0, 1), \ldots, (d-1, d-1)\}$. If Alice wants to send the classical message (m, n) to Bob, the $U_{mn,d}$ gate implements appropriate operations on her qudit. The operation of the gate on $|\phi_1\rangle$ is

$$\begin{aligned}(U_{mn,d} \otimes I)|\phi_1\rangle &= \left(\sum_{u=0}^{d-1} \omega^{mu} |u\rangle\langle n \oplus u|\right) \otimes I)|\phi_1\rangle \\ &= |\phi_{md+n+1}\rangle,\end{aligned} \tag{14}$$

where $m, n = 0, 1, \ldots, d-1$. The classical message and corresponding operation are shown in Table I in detail.

TABLE I. Classical message and corresponding operation implemented by the $U_{mn,d}$ gate on the channel $|\phi_1\rangle$. The last column contains the quantum cost of the d-dimensional dense coding protocol, $d \geqslant 3$.

Classical message	$(U_{mn,d} \otimes I)|\phi_1\rangle$	Quantum cost
(0,0)	$|\phi_1\rangle = \frac{1}{\sqrt{d}}(|0,0\rangle + |1,1\rangle + \cdots + |d-1,d-1\rangle)$	$d+3$
(0,1)	$|\phi_2\rangle = \frac{1}{\sqrt{d}}(|0,1\rangle + |1,2\rangle + \cdots + |d-1,0\rangle)$	$d+4$
$\vdots$	$\vdots$	$\vdots$
$(0, d-1)$	$|\phi_d\rangle = \frac{1}{\sqrt{d}}(|0,d-1\rangle + |1,0\rangle + \cdots + |d-1,d-2\rangle)$	$d+4$
$\vdots$	$\vdots$	$\vdots$
$(d-1, 0)$	$|\phi_{d^2-d+1}\rangle = \frac{1}{\sqrt{d}}(|0,0\rangle + \omega^{d-1}|1,1\rangle + \cdots + \omega|d-1,d-1\rangle)$	$d+4$
$(d-1, 1)$	$|\phi_{d^2-d+2}\rangle = \frac{1}{\sqrt{d}}(|0,1\rangle + \omega^{d-1}|1,2\rangle + \cdots + \omega|d-1,0\rangle)$	$d+4$
$\vdots$	$\vdots$	$\vdots$
$(d-1, d-1)$	$|\phi_{d^2}\rangle = \frac{1}{\sqrt{d}}(|0,d-1\rangle + \omega^{d-1}|1,0\rangle + \cdots + \omega|d-1,d-2\rangle)$	$d+4$

Then Alice transmits her qudit to Bob. Bob tries to recover the message by two gates $H_d^\dagger$ and CNOT†. We can verify that

$$(H_d^\dagger \otimes I)U_{\text{CNOT},d}^\dagger|\phi_{md+n+1}\rangle = |m,n\rangle. \tag{15}$$

Finally, Bob performs the measurement on his qudits and obtains the classical message. The quantum cost of the final measurement is equal to 1.

Now we show the quantum cost of the protocol in Fig. 1. The key is to analyze the quantum cost of the $U_{mn,d}$ gate. When $d=2$, we have $H_2^\dagger = H_2$ and $\text{CNOT}_2^\dagger = \text{CNOT}_2$. Alice sends one of the classical messages from (0,0),(0,1),(1,0),(1,1) to Bob. When Alice sends the classical message (m,n) to Bob, the quantum cost of the dense coding protocol ($D_{mn,2}$) is

$$\begin{aligned} D_{mn,2} &= D(H_2) + D(U_{mn,2}) + D(\text{CNOT}_2) + D(M) \\ &= 2\times 1 + DC(U_{mn,2}) + 2\times 1 + 1 \\ &= D(U_{mn,2}) + 5, \end{aligned} \tag{16}$$

where $D(X)$ is the total quantum cost of all the X gates used in this dense coding protocol and $D(M)=1$ is the cost of the final measurement. Based on the corresponding gate $U_{mn,2}$ shown in Table II, we have

$$D(U_{00,2}) = 0, D(U_{01,2}) = 1, D(U_{10,2}) = 1, D(U_{11,2}) = 1. \tag{17}$$

We now obtain the quantum cost of the two-dimensional protocol (shown in Table II). We consider the case $d>2$ for any d. In Fig. 1 we see that the H_d, CNOT, $U_{mn,d}$, CNOT†, and $H_d^\dagger$ gates are used once in the protocol. In Sec. II B 2 we have shown that $D(\text{CNOT}_d^\dagger) = d-1$. When Alice wants to send classical message (m,n) to Bob, the quantum cost of the d-dimensional protocol is

$$\begin{aligned} D_{mn,d} &= D(H_d) + D(\text{CNOT}_d) + D(U_{mn,d}) + D(\text{CNOT}_d^\dagger) \\ &\quad + D(H_d^\dagger) + D(M) \\ &= 1 + 1 + D(U_{mn,d}) + (d-1) + 1 + 1 \\ &= D(U_{mn,d}) + d + 3, \end{aligned} \tag{18}$$

where $m,n = 0,1,\ldots,d-1$. Note that $U_{00,d} = I$ and $U_{mn,d} \neq I$ for $m \neq 0$ or $n \neq 0$. We have

$$D(U_{mn,d}) = \begin{cases} 0 & \text{if } m,n=0 \\ 1 & \text{otherwise.} \end{cases} \tag{19}$$

Hence, for any $d \geqslant 2$, we have

$$D_{mn,d} = \begin{cases} d+3 & \text{if } m,n=0, \\ d+4 & \text{otherwise.} \end{cases} \tag{20}$$

The quantum cost of dense coding appears to grow linearly with the dimension d. Thus the physical implementation cost of dense coding increases with the dimension. Further, the fidelity of this protocol is related to its cost, which will be analyzed in Sec. V.

The H_d, CNOT, $U_{mn,d}$, and $H_d^\dagger$ gates are the basic gates used in this protocol. The nonbasic gate CNOT† is prepared by $d-1$ CNOT gates. Hence, four kinds of basic gates are used in the circuit for the dense coding protocol.

IV. QUANTUM COST OF TELEPORTATION

In this section we show the quantum cost of d-dimensional teleportation protocol. The single-qudit quantum message is written as $|M_d\rangle = \sum_{j=0}^{d-1}\alpha_j|j\rangle$, where $\alpha_j \in \mathbb{C}$ and $\sum_{j=0}^{d-1}|\alpha_j|^2 = 1$. Alice and Bob share the maximally

TABLE II. Classical message and corresponding operation implemented by the $U_{mn,2}$ gate on the channel $|\varphi_1\rangle$. The last two columns contain the quantum cost and the number of sorts of gates used in the quantum circuit for the two-dimensional case.

Classical message	Operation $U_{mn,2}$	$(U_{mn,2} \otimes I)|\varphi_1\rangle$	Quantum cost	Sorts of basic gates
(0,0)	$U_{00,2} = I$	$|\varphi_1\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle + |1,1\rangle)$	5	2
(0,1)	$U_{01,2} = \sigma_X$	$|\varphi_2\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle + |1,0\rangle)$	6	3
(1,0)	$U_{10,2} = \sigma_Z$	$|\varphi_3\rangle = \frac{1}{\sqrt{2}}(|0,0\rangle - |1,1\rangle)$	6	3
(1,1)	$U_{11,2} = i\sigma_Y$	$|\varphi_4\rangle = \frac{1}{\sqrt{2}}(|0,1\rangle - |1,0\rangle)$	6	3

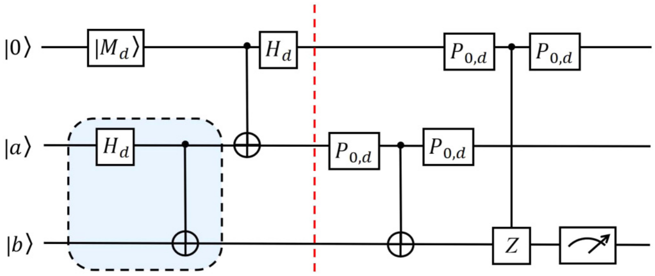

FIG. 2. Teleportation protocol via the d-dimensional Bell state $|\phi_1\rangle = \frac{1}{\sqrt{d}}\sum_{k=1}^{d-1}|k,k\rangle$. The unitary gates to recover the message are shown to the right of the dashed line. Three kinds of basic gates are used in the circuit: H_d, CNOT, and $P_{0,d}$.

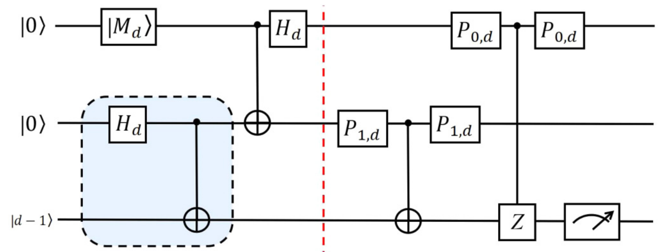

FIG. 3. Teleportation protocol via the d-dimensional Bell state $|\phi_d\rangle$. The unitary gates to recover the message are shown to the right of the dashed line. Four kinds of basic gates are used in the circuit: H_d, CNOT, $P_{0,d}$, and $P_{1,d}$.

entangled state $|\phi_1\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|k,k\rangle$ as the channel. So they are in the state

$$|\xi_d\rangle = |M_d\rangle \otimes |\phi_1\rangle = \left[\sum_{j=0}^{d-1}\alpha_j|j\rangle_1\right]\otimes\left[\frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|k,k\rangle_{2,3}\right]. \tag{21}$$

Qudits 1 and 2 of the combined state belong to Alice and qudit 3 belongs to Bob. Alice and Bob implement the operations on their qudits. The final state is shown as follows:

$$\begin{aligned}&(H_d\otimes I\otimes I)(U_{\mathrm{CNOT},d}\otimes I)|\xi_d\rangle\\ &\quad=\sum_{m,n=0}^{d-1}\frac{|m,n\rangle_{1,2}}{d}\left[\sum_{j=0}^{d-1}\alpha_j\omega^{mj}|n\oplus(d-j)\rangle_3\right].\end{aligned} \tag{22}$$

Based on the state given in (22), we obtain that when Alice's measurement is $|m,n\rangle$, Bob should apply the local unitary operation $U_{mn}^{(0,0)} = Z_{d-m}X_{d-n}$ to his qudit, where

$$U_{m,n}^{(0,0)} = \sum_{s=0}^{d-1}\omega^{ms}|d-s\rangle\langle s\oplus n|. \tag{23}$$

Finally, Bob performs the measurement and completes the teleportation process. The quantum cost of the final measurement is equal to 1.

We analyze the quantum cost of the teleportation protocol via the channel $|\phi_1\rangle$. In Fig. 2 we show that two H_d gates, three CNOT gates, four $P_{0,d}$ gates, and one controlled-Z gate are used in this protocol. In Sec. II B we have shown that the quantum cost of the H_d, CNOT, and $P_{0,d}$ gates is equal to 1 and the cost of the controlled-Z gate is equal to 3. We add up the quantum cost of each gate and the final measurement and obtain the quantum cost of the d-dimensional protocol

$$\begin{aligned}T_d &= T(H_d)+T(\mathrm{CNOT}_d)+T(P_{0,d})+T(\mathrm{CNOT}_d)+T(M)\\ &= 2\times1+3\times1+4\times1+1\times3+1=13,\end{aligned} \tag{24}$$

where $T(X)$ is the total quantum cost of the X gate used in this protocol and $T(M)=1$ is the cost of the final measurement. Three sorts of basic gates (H_d, $P_{0,d}$, and CNOT) are used in the teleportation protocol.

Next we analyze the quantum cost of the teleportation protocol via other d-dimensional Bell channels. In the quantum circuit shown in Fig. 2 we have $|a\rangle, |b\rangle \in \{|0\rangle, |1\rangle, \dots, |d-1\rangle\}$. Choosing all combinations of $|a\rangle$ and $|b\rangle$, we obtain d^2 kinds of Bell channels,

$$|\phi_{ad+b+1}\rangle = \frac{1}{\sqrt{d}}\sum_{x=0}^{d-1}\omega^{xa}|x, b\oplus x\rangle. \tag{25}$$

Via the channel $|\phi_{ad+b+1}\rangle$, when Alice's measurement result is $|m,n\rangle$, the corresponding operations that Bob should apply are

$$U_{m,n}^{(a,b)} = Z_{a\oplus(d-m)}X_{(d-b)\oplus(d-n)}, \tag{26}$$

which are shown in Table III in detail. Hence, the gates implemented on the first and second qudits are $P_{a,d}$ and $P_{d\oplus(d-b),d}$. They transform the controlled qudit of controlled-Z and CNOT gates into appropriate values, respectively. The type of gates $P_{k,d}$ is the only difference between the teleportation protocols via different channels. The numbers of gates $P_{k,d}$ in quantum circuits via different Bell channels are all equal to 4. For example, comparing Figs. 2 and 3, we see that the difference between the teleportation protocols via $|\phi_1\rangle$ and $|\phi_d\rangle$ is the type of gates the $P_{k,d}$ gates implement on the controlled qudit of the CNOT gate. The quantum cost of the teleportation protocol via all the Bell channels $|\phi_u\rangle$ ($u = 1, 2, \dots, d^2$) is equal to 13. This shows that the quantum cost of teleportation remains 13, which is the maximum value of the teleportation cost for the two-dimensional case. The demand for a practical device to implement teleportation in high-dimensional space remains the same, without regard to the increase of implementation cost for each gate. One or two kinds of $P_{k,d}$, H_d, and CNOT gates are used in the circuit. Hence, three or four kinds of basic gates are used in the teleportation protocol via all the Bell channels.

When $d = 2$, the quantum cost is different from the case $d > 2$, as we have $P_{0,2} = I_2$, which is used four times in Fig. 2. Hence, the quantum cost of the teleportation protocol via the two-dimensional channel $|\varphi_1\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ is

$$T_2 = T_d - T(P_{0,d}) = 13 - 4 = 9.$$

That is the reason why the quantum cost of teleportation protocols via the two-dimensional channels varies from 9 to 13. The quantum cost of teleportation protocols via four two-dimensional channels and Bob's recovery operations are given in Table IV. It is in agreement with previous result for the case of dimension 2 in [18]. The quantum cost of dense

TABLE III. Controlled operations that Bob should apply to recover the message $|M_d\rangle$ via the d-dimensional Bell channels. When the controlled qubit is $|k\rangle$, the CNOT and controlled-Z gates perform the operations $X_{k,d}$ and $Z_{k,d}$ on Bob's qudit. Here X_k and Z_k represent $X_{k,d}$ and $Z_{k,d}$ given in (10) and (13) ($k = 0, 1, \ldots, d-1$).

Alice's measurement	$\|\phi_1\rangle$	$\|\phi_2\rangle$	$\cdots$	$\|\phi_d\rangle$	$\cdots$	$\|\phi_{d^2-d+1}\rangle$	$\|\phi_{d^2-d+2}\rangle$	$\cdots$	$\|\phi_{d^2}\rangle$
$\|0,0\rangle$	Z_0X_0	Z_0X_{d-1}	$\cdots$	Z_0X_1	$\cdots$	$Z_{d-1}X_0$	$Z_{d-1}X_{d-1}$	$\cdots$	$Z_{d-1}X_1$
$\|0,1\rangle$	Z_0X_{d-1}	Z_0X_{d-2}	$\cdots$	Z_0X_0	$\cdots$	$Z_{d-1}X_{d-1}$	$Z_{d-1}X_{d-2}$	$\cdots$	$Z_{d-1}X_0$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\|0,d-1\rangle$	Z_0X_1	Z_0X_0	$\cdots$	Z_0X_2	$\cdots$	$Z_{d-1}X_1$	$Z_{d-1}X_0$	$\cdots$	$Z_{d-1}X_2$
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\|d-1,0\rangle$	Z_1X_0	Z_1X_{d-1}	$\cdots$	Z_1X_1	$\cdots$	Z_0X_0	Z_0X_{d-1}	$\cdots$	Z_0X_1
$\|d-1,1\rangle$	Z_1X_{d-1}	Z_1X_{d-2}	$\cdots$	Z_1X_0	$\cdots$	Z_0X_{d-1}	Z_0X_{d-2}	$\cdots$	Z_0X_0
$\vdots$	$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\vdots$		$\vdots$
$\|d-1,d-1\rangle$	Z_1X_1	Z_1X_0	$\cdots$	Z_1X_2	$\cdots$	Z_0X_1	Z_0X_0	$\cdots$	Z_0X_2

coding shows a linear growth with the dimension, while that of teleportation is a constant. The reason is the difference between the recovery of the message in the two protocols. For the dense coding protocol, a CNOT† gate is required to recover the message. The CNOT† gate is prepared by $d-1$ CNOT gates, which is the only two-qudit basic gate and thus its cost is equal to $d-1$. Hence, the dense coding cost shows the linear growth. For the teleportation cost, one or two kinds of $P_{k,d}$ gates are employed to transform the controlled qudits of CNOT and controlled-Z gates to appropriate values, so the recovery operations are implemented on Bob's qudit correctly. The six gates along with one measurement shown in Fig. 2 to the right of the dashed line is sufficient to recover the message, and thus the teleportation cost is a constant.

V. APPLICATIONS

In this section we introduce an application of the quantum cost. As we all know, some unavoidable interaction of the communication channel with the environment leads to the loss of accuracy of the protocol. In order to assess the reliability of a protocol, the fidelity of the protocol is proposed [28]. It gives the closeness between the ideal state Alice wants to send and the final state under a noisy channel. By considering the channel under four classes of noise, we find the relation between the quantum cost and its fidelity. In Sec. IV we obtained that the quantum cost of teleportation remains 13, regardless of the dimension. The fidelity of the teleportation protocol has nothing to do with its quantum cost, as the fidelity is related to the dimension d and error probability p. On the other hand, we will show that the fidelity of the dense coding protocol under four classes of noise decreases with the increase of quantum cost.

In order to calculate the fidelity of the dense coding protocol, we briefly introduce four classes of noise for d-dimensional case . Suppose d^2 Weyl operators U_{mn} are defined as

$$U_{mn} = \sum_{j=0}^{d-1} \omega_d^{jm} |j \oplus n\rangle\langle j|. \tag{27}$$

In analogy to the two-dimensional noise, four classes of noise and its corresponding Kraus operators are shown as follows, where p is the probability that the error occurs: (i) dit-flip noise, where $E_{00} = \sqrt{1-p}U_{00}$ and $E_{01} = \sqrt{\frac{p}{d-1}}U_{01}, \ldots, E_{0,d-1} = \sqrt{\frac{p}{d-1}}U_{0,d-1}$; (ii) d-phase-flip noise, where $E_{00} = \sqrt{1-p}U_{00}$ and $E_{10} = \sqrt{\frac{p}{d-1}}U_{10}, \ldots, E_{d-1,0} = \sqrt{\frac{p}{d-1}}U_{d-1,0}$; (iii) dit-phase-flip noise, where $E_{00} = \sqrt{1-p}U_{00}$ and $E_{mn} = \frac{\sqrt{p}}{d-1}U_{mn}$, with $1 \leqslant m, n \leqslant d-1$; and (iv) depolarizing noise, where $E_{00} = \sqrt{1 - \frac{d^2-1}{d^2}p}U_{00}$ and $E_{mn} = \frac{\sqrt{p}}{d}U_{mn}$, with $0 \leqslant m, n \leqslant d-1$ for $(m,n) \neq (0,0)$. These four classes of noise contain the information about the effects of the system-environment interaction. Given an arbitrary system initially prepared in a state $\rho = \sum_{\vec{k}\vec{l}} \rho_{\vec{k}\vec{l}} |\vec{k}\rangle\langle\vec{l}|$ for the number of subsystems N, $\vec{k} = (k_1, \ldots, k_N)$, and $0 \leqslant k_j \leqslant d-1$, the action of a set of Kraus operators $E_{\vec{k}\vec{l}} = E_{k_1 l_1} \otimes \cdots \otimes E_{k_N l_N}$ transforms ρ into ρ'. The evolution can be modeled by the trace-preserving map $\rho \to \rho' = \sum_{\vec{k}\vec{l}} E_{\vec{k}\vec{l}} \rho E_{\vec{k}\vec{l}}^\dagger$, where the $E_{\vec{k}\vec{l}}$'s satisfy the completeness relation $\sum_{\vec{k}\vec{l}} E_{\vec{k}\vec{l}} E_{\vec{k}\vec{l}}^\dagger = I$.

TABLE IV. Here we set $d = 2$ to obtain the operations that Bob should apply for the two-dimensional case. The last column contains the quantum cost of the two-dimensional teleportation protocol via different channels. Here X_k and Z_k represent $X_{k,2}$ and $Z_{k,2}$ ($k = 0, 1$).

			Alice's measurement				
$\|a\rangle$	$\|b\rangle$	Two-dimensional Bell states	$\|0,0\rangle$	$\|0,1\rangle$	$\|1,0\rangle$	$\|1,1\rangle$	Quantum cost
$\|0\rangle$	$\|0\rangle$	$\|\varphi_1\rangle = \frac{1}{\sqrt{2}}(\|0,0\rangle + \|1,1\rangle)$	$Z_0X_0 = I_2$	$Z_0X_1 = \sigma_X$	$Z_1X_0 = \sigma_Z$	$Z_1X_1 = \sigma_Z\sigma_X$	9
$\|0\rangle$	$\|1\rangle$	$\|\varphi_2\rangle = \frac{1}{\sqrt{2}}(\|0,1\rangle + \|1,0\rangle)$	$Z_0X_1 = \sigma_X$	$Z_0X_0 = I_2$	$Z_1X_1 = \sigma_Z\sigma_X$	$Z_1X_0 = \sigma_Z$	11
$\|1\rangle$	$\|0\rangle$	$\|\varphi_3\rangle = \frac{1}{\sqrt{2}}(\|0,0\rangle - \|1,1\rangle)$	$Z_1X_0 = \sigma_Z$	$Z_1X_1 = \sigma_Z\sigma_X$	$Z_0X_0 = I_2$	$Z_0X_1 = \sigma_X$	11
$\|1\rangle$	$\|1\rangle$	$\|\varphi_4\rangle = \frac{1}{\sqrt{2}}(\|0,1\rangle - \|1,0\rangle)$	$Z_1X_1 = \sigma_Z\sigma_X$	$Z_1X_0 = \sigma_Z$	$Z_0X_1 = \sigma_X$	$Z_0X_0 = I_2$	13

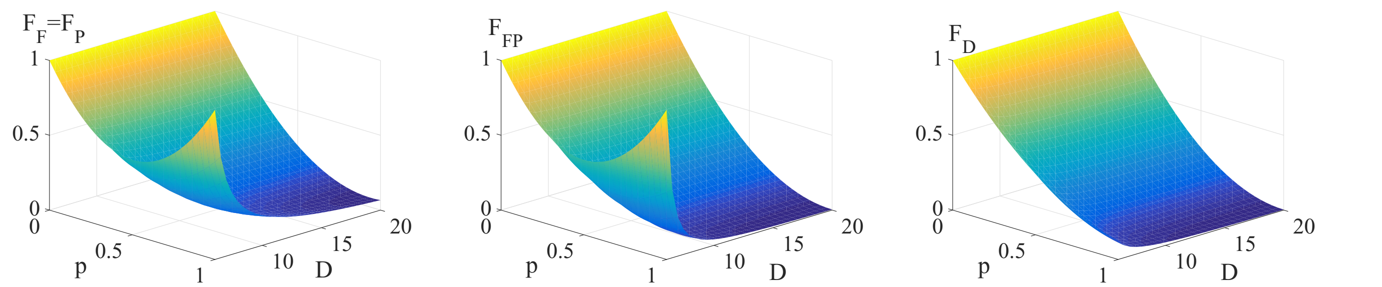

FIG. 4. Fidelities of dense coding under a scenario in which the channel is affected by dit-flip $\mathcal{F}_F$, d-phase-flip $\mathcal{F}_P$, dit-phase-flip $\mathcal{F}_{FP}$, and depolarizing noise $\mathcal{F}_D$, respectively. The axes represent the fidelity, error probability p, and the quantum cost D, for $0 \leqslant p \leqslant 1$ and $6 \leqslant D \leqslant 20$.

We calculate the fidelity of dense coding for the cases in which each qudit of the channel is affected by the noise. The calculation is shown in Appendix A in detail. In Sec. III we obtained the fidelity when Alice wants to send (m, n) to Bob, the quantum cost of the d-dimensional dense coding in (20). For simplicity, we consider that Alice sends a classic message other than (0,0). The quantum cost is $D = D_{mn,d} = d + 4$. We establish the relation between the fidelity of the dense coding protocol under dit-flip noise $\mathcal{F}_F$, d-phase-flip noise $\mathcal{F}_P$, dit-phase-flip noise $\mathcal{F}_{FP}$, and depolarizing noise $\mathcal{F}_D$,

$$\mathcal{F}_F = \mathcal{F}_P = (1-p)^2 + \frac{p^2}{d-1} = (1-p)^2 + \frac{p^2}{D-5}, \tag{28}$$

$$\mathcal{F}_{FP} = (1-p)^2 + \frac{p^2}{(d-1)^2} = (1-p)^2 + \frac{p^2}{(D-5)^2}, \tag{29}$$

$$\begin{aligned}\mathcal{F}_D &= \left(1 - \frac{d^2-1}{d^2}p\right)^2 + \frac{(d-1)^2 p^2}{d^4} \\ &= \left(1 - \frac{(D-4)^2-1}{(D-4)^2}p\right)^2 + \frac{(D-5)^2 p^2}{(D-4)^4}. \end{aligned} \tag{30}$$

We compare the fidelities under four kinds of noise so as to obtain their influence on dense coding. The details are shown in Appendix B. Generally, the higher the fidelity is, the less the noise affects the protocol. Comparing the expressions (28)–(30), we find that with the same error probability p, we have $\mathcal{F}_F = \mathcal{F}_P > \mathcal{F}_{FP}$ for any $D > 6$. Next we compare $\mathcal{F}_D$ with $\mathcal{F}_F$ ($=\mathcal{F}_P$) and $\mathcal{F}_{FP}$. The influence of depolarizing noise tends to be in the middle of dit-flip (phase-flip) noise and dit-phase-flip noise with the increase of quantum cost D for arbitrary $0 < p < 1$. On the other hand, the influence of depolarizing noise tends to be larger with the increase of p, and the growth rate of it is greater than that of other three kinds of noise. The reason is that the expressions of four kinds of noise are different, which is determined by the properties of noise. The Weyl operators U_{mn} in the expressions represent dit-flip and phase-flip operations and the coefficients of them represent the probability with which the operator acts on the channel. The fidelities of dense coding for the noise scenario are plotted in Fig. 4. The figures show that the increase of quantum cost of the dense coding protocol will result in the loss of its fidelity for the noise scenario. With the increase of D, the reduction rate of fidelity with respect to p becomes larger. Hence, the quantum cost of a protocol is one of the indicators of its fidelity. The more gates or complicated gates we employ in a protocol, the higher the quantum cost will be and thus the lower the fidelity will be. This inspires us that decreasing the quantum cost would be one of the useful strategies to improve the fidelity of the high-dimensional dense coding protocol.

VI. QUANTUM COST IN EXPERIMENT

In this section we consider the experimental setup that realizes the protocols so as to show the relation between the exact implementation cost and the quantum cost practically. Based on the correspondence between the gates and the optical devices that realize them, we show that the quantum cost can be obtained by the character of the specific experimental system. The higher the quantum cost is, the higher the implementation cost will be in the practical experiment. Hence, the quantum cost is a meaningful figure of merit to evaluate the cost of a protocol both theoretically and practically. Next we take the experimental realization of dense coding as an example to explain in more detail.

We show the experiment setup of dense coding and its corresponding gates in the protocol. Dense coding was first realized in the optical system using pairs of photons entangled in polarization [29]. Three parts are included in their experimental setup: the source that generates the entangled Bell state, Bob's encoding station, and Alice's Bell-state analyzer. For the source part, a UV pump and a nonlinear β barium borate crystal is employed to generate the entangled state, which corresponds to the Hadamard and CNOT gates used to prepare the entangled state in Fig. 1. Bob's encoding station includes a half waveplate (HWP) retardation and a quarter waveplate (QWP). This part corresponds to the U gate in Fig. 1. Alice's Bell-state analyzer consists of one beam splitter, two two-channel polarizers, and four detectors. This part corresponds to the gates to the right of the dashed line in Fig. 1. The correspondence between the gates and experiment setup shows that the realization of gates requires corresponding optical experimental devices and operations. The higher the quantum cost is, the higher the implementation cost will be in the practical experiment.

We have proposed a framework to calculate the quantum cost of a protocol in Secs. III and IV. For a specific experimental system, the calculation may change according to its character. In reality, limited sorts of operations are considered to be implemented by the optical devices and such operations correspond to the basic gates for this experiment. Thus, the universal set including all the basic gates is determined by the specific experiment setup and the quantum cost changes with it. For example, only two elements are employed for the encoding station in this experiment: the QWP and HWP, which correspond to the σ_Z and σ_X gates, respectively. So the σ_Z and σ_X gates are basic gates for encoding. If Bob wants to send classical message (0,1), (1,1), (0,0), or (1,0), his setting is nothing (I), 90° QWP (σ_Z), 45° HWP (σ_X), and 90° QWP and 45° HWP ($\sigma_Z\sigma_X$), respectively. Hence, the quantum costs for the encoding part are in turn 0, 1, 1, and 2. On the other hand, the quantum cost of each gate can be determined by its realization requirement in experiment, such as the number of devices and operations requirement. If it requires more optical devices to realize a two-qubit gate than a single-qubit gate in a specific experiment, then the quantum cost of the two-qubit gate will be larger than that of the single-qubit gate. Now we calculate the quantum cost in this experiment by considering a number of optical devices: The quantum cost of the source part is 2 (that of the encoding part analyzed above) and that of the analyzer part is 7. Thus, if Bob wants to send a classical message (0,1), (1,1), (0,0), or (1,0), the quantum cost is in turn equal to 9, 10, 10, and 11.

VII. CONCLUSION

We have analyzed the quantum cost of high-dimensional dense coding and teleportation protocols. Our results of the teleportation protocol have generalized the results recently shown for the two-dimensional case [18]. We have obtained that the quantum cost of the d-dimensional dense coding protocol is equal to $d+3$ when transmitting the classical message (0,0) and $d+4$ when transmitting another classical message, showing a linear increase with the dimension. Four kinds of basic gates are used in the dense coding protocol. The quantum cost of high-dimensional teleportation remains 13, which is the maximum value of the quantum cost of two-dimensional case. Three or four kinds of basic gates are used in the teleportation protocol. As an application of our main result, we have been able to establish a relation between the fidelity of the dense coding protocol and its quantum cost. The higher the quantum cost is, the lower the fidelity of the protocol will be for the four kinds of noise scenario.

Many problems arising from this paper can be further explored. The quantum cost of other high-dimensional protocols may be obtained and the relation between the fidelity and quantum cost can be established, for example, the two-step quantum direct communication protocol [30] and the protocol for quantum secure direct communication with superdense coding [31]. This work offers a strategy to improve the fidelity of protocols. In addition, we have studied the quantum cost of protocols by bipartite entangled states, which can be extended to multipartite states. Whether there is a relation between the quantum cost and entanglement cost in protocols is left as an open question.

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APPENDIX A: CALCULATION OF DENSE CODING FIDELITY

We present the calculation of fidelity of dense coding for four kinds of noise scenario: dit-flip, d-phase-flip, dit-phase-flip, and depolarizing noise.

Suppose Alice wants to send a classical message (m, n) to Bob. They share the channel $\rho = |\phi_1\rangle\langle\phi_1|$, where $|\phi_1\rangle = \frac{1}{\sqrt{d}}\sum_{k=0}^{d-1}|k,k\rangle$. The $U_{mn,d}$ gate shown in (6) implements the operation that Alice performs on her qudit. In the noise-free environment, i.e., when the channel ρ is not affected by any kind of noise, the final state of Bob's two qudits is

$$\rho_{mn} = (H_d^\dagger \otimes I)U_{\text{CNOT}}^\dagger(U_{mn} \otimes I)\rho(U_{mn}^\dagger \otimes I)U_{\text{CNOT}}(H_d \otimes I). \tag{A1}$$

We obtain the operations in the dense coding protocol

$$\begin{aligned} U_{\text{DC}} &= (H_d^\dagger \otimes I)U_{\text{CNOT}}^\dagger(U_{mn} \otimes I) \\ &= \frac{1}{\sqrt{d}}\sum_{s,u,x=0}^{d-1}\omega^{(m-x)s}|x, u\oplus(d-s)\rangle\langle n\oplus s, u|. \end{aligned} \tag{A2}$$

Next we show the calculation of dense coding fidelities under four kinds of noise.

1. Fidelity under dit-flip and phase-flip noise

The action of dit-flip noise transforms the channel ρ into ρ_F':

$$\begin{aligned} \rho_F' &= \sum_{j,q=0}^{d-1}(E_{0j}\otimes E_{0q})\rho(E_{0j}\otimes E_{0q})^\dagger \\ &= \sum_{j,q=0}^{d-1}(E_{0j}\otimes E_{0q})|\phi_1\rangle\langle\phi_1|(E_{0j}\otimes E_{0q})^\dagger. \end{aligned} \tag{A3}$$

Hence, Bob's two qudits are transformed into the states

$$\begin{aligned} \rho_{mn,F}' &= U_{\text{DC}}\rho_F' U_{\text{DC}}^\dagger \\ &= \sum_{j,q=0}^{d-1}U_{\text{DC}}(E_{0j}\otimes E_{0q})|\phi_1\rangle\langle\phi_1|(E_{0j}\otimes E_{0q})^\dagger U_{\text{DC}}^\dagger. \end{aligned}$$

For any $j, q = 1, 2, \dots, d-1$ we have

$$|\xi_{00,F}\rangle = U_{\text{DC}}(E_{00}\otimes E_{00})|\phi_1\rangle = \frac{1-p}{d}\sum_{s,x=0}^{d-1}\omega^{(m-x)s}|x, n\rangle, \tag{A4}$$

$$\begin{aligned} |\xi_{0q,F}\rangle &= U_{\text{DC}}(E_{00}\otimes E_{0q})|\phi_1\rangle \\ &= \frac{\sqrt{p(1-p)}}{d\sqrt{d-1}}\sum_{s,x=0}^{d-1}\omega^{(m-x)s}|x, n\oplus q\rangle, \end{aligned} \tag{A5}$$

$$\begin{aligned}|\xi_{j0,F}\rangle &= U_{\rm DC}(E_{0j}\otimes E_{00})|\phi_1\rangle\\ &= \frac{\sqrt{p(1-p)}}{d\sqrt{d-1}}\sum_{s,x=0}^{d-1}\omega^{(m-x)s}|x,n\oplus(d-j)\rangle,\end{aligned}\tag{A6}$$

$$\begin{aligned}|\xi_{jq,F}\rangle &= U_{\rm DC}(E_{0j}\otimes E_{0q})|\phi_1\rangle\\ &= \frac{p}{d(d-1)}\sum_{s,x=0}^{d-1}\omega^{(m-x)s}|x,n\oplus q\oplus(d-j)\rangle,\end{aligned}\tag{A7}$$

where we do not normalize $|\xi_{jq,F}\rangle$ for convenience. Hence,

$$\rho'_{mn,F}=\sum_{j,q=0}^{d-1}|\xi_{jq,F}\rangle\langle\xi_{jq,F}|.\tag{A8}$$

By considering an arbitrary classical message (m,n) that Alice wants to send, the fidelity of dense coding under dit-flip noise is

$$\mathcal{F}_F=\mathrm{Tr}\{|m,n\rangle\langle m,n|\rho'_{mn,F}\}=(1-p)^2+\frac{p^2}{d-1}.\tag{A9}$$

Next we calculate the dense coding fidelity under phase-flip noise. Based on the definition of phase-flip noise shown in Sec. V, we have

$$|\xi_{00,P}\rangle = U_{\rm DC}(E_{00}\otimes E_{00})|\phi_1\rangle=\frac{1-p}{d}\sum_{s,x=0}^{d-1}\omega^{(m-x)s}|x,n\rangle,\tag{A10}$$

$$\begin{aligned}|\xi_{0q,P}\rangle &= U_{\rm DC}(E_{00}\otimes E_{0q})|\phi_1\rangle\\ &= \frac{\sqrt{p(1-p)}}{d\sqrt{d-1}}\sum_{s,x=0}^{d-1}\omega^{(m-x)s+q(n+s)}|x,n\rangle,\end{aligned}\tag{A11}$$

$$\begin{aligned}|\xi_{j0,P}\rangle &= U_{\rm DC}(E_{0j}\otimes E_{00})|\phi_1\rangle\\ &= \frac{\sqrt{p(1-p)}}{d\sqrt{d-1}}\sum_{s,x=0}^{d-1}\omega^{(m-x)s+j(n+s)}|x,n\rangle,\end{aligned}\tag{A12}$$

$$\begin{aligned}|\xi_{jq,P}\rangle &= U_{\rm DC}(E_{0j}\otimes E_{0q})|\phi_1\rangle\\ &= \frac{p}{d(d-1)}\sum_{s,x=0}^{d-1}\omega^{(m-x)s+(j+q)(n+s)}|x,n\rangle,\end{aligned}\tag{A13}$$

where $j,q=1,2,\dots,d-1$. Note that

$$\rho'_{mn,P}=\sum_{j,q=0}^{d-1}|\xi_{jq,P}\rangle\langle\xi_{jq,P}|,\tag{A14}$$

$$\mathcal{F}_P=\mathrm{Tr}\{|m,n\rangle\langle m,n|\rho'_{mn,P}\}.\tag{A15}$$

By analogous calculation, we obtain that $\mathcal{F}_P=\mathcal{F}_F=(1-p)^2+\frac{p^2}{d-1}$.

2. Fidelity under dit-phase-flip and depolarizing noise

The action of dit-phase-flip noise transforms the channel ρ into ρ'_{FP}:

$$\begin{aligned}\rho'_{FP} &= \sum_{t,j,v,q=0}^{d-1}(E_{tj}\otimes E_{vq})\rho(E_{tj}\otimes E_{vq})^\dagger\\ &= \sum_{t,j,v,q=0}^{d-1}(E_{tj}\otimes E_{vq})|\phi_1\rangle\langle\phi_1|(E_{tj}\otimes E_{vq})^\dagger.\end{aligned}\tag{A16}$$

Hence, Bob's two qudits are transformed into the new state

$$\begin{aligned}\rho'_{mn,FP} &= U_{\rm DC}\rho'_{FP}U_{\rm DC}^\dagger\\ &= \sum_{t,j,v,q=0}^{d-1}U_{\rm DC}(E_{tj}\otimes E_{vq})|\phi_1\rangle\langle\phi_1|(E_{tj}\otimes E_{vq})^\dagger U_{\rm DC}^\dagger.\end{aligned}\tag{A17}$$

For any $t,j,v,q=1,2,\dots,d-1$ we have

$$|\xi_{0000,FP}\rangle=U_{\rm DC}(E_{00}\otimes E_{00})|\phi_1\rangle=\frac{1-p}{d}\sum_{s,x=0}^{d-1}\omega^{(m-x)s}|x,n\rangle,\tag{A18}$$

$$\begin{aligned}|\xi_{00vq,FP}\rangle &= U_{\rm DC}(E_{00}\otimes E_{vq})|\phi_1\rangle\\ &= \frac{\sqrt{p(1-p)}}{d\sqrt{d-1}}\sum_{s,x=0}^{d-1}\omega^{(m-x)s+v(n+s)}|x,n\oplus q\rangle,\end{aligned}\tag{A19}$$

$$\begin{aligned}|\xi_{tj00,FP}\rangle &= U_{\rm DC}(E_{0j}\otimes E_{00})|\phi_1\rangle\\ &= \frac{\sqrt{p(1-p)}}{d\sqrt{d-1}}\sum_{s,x=0}^{d-1}\omega^{(m-x)s+t(n+s-j)}\\ &\quad\times|x,n\oplus(d-j)\rangle,\end{aligned}\tag{A20}$$

$$\begin{aligned}|\xi_{tjvq,FP}\rangle &= U_{\rm DC}(E_{tj}\otimes E_{vq})|\phi_1\rangle\\ &= \frac{p}{d(d-1)^2}\sum_{s,x=0}^{d-1}\omega^{(m-x)s+(t+v)(n+s-j)}\\ &\quad\times|x,n\oplus q\oplus(d-j)\rangle,\end{aligned}\tag{A21}$$

where we do not normalize $|\xi_{tjvq,F}\rangle$ for convenience. Hence,

$$\rho'_{mn,FP}=\sum_{t,j,v,q=0}^{d-1}|\xi_{tjvq,FP}\rangle\langle\xi_{tjvq,FP}|.\tag{A22}$$

By considering an arbitrary classical message (m,n) that Alice wants to send, the fidelity of dense coding under dit-phase-flip noise is

$$\mathcal{F}_{FP}=\mathrm{Tr}\{|m,n\rangle\langle m,n|\rho'_{mn,FP}\}=(1-p)^2+\frac{p^2}{(d-1)^2}.\tag{A23}$$

Next we show the calculation of fidelity corresponding to the depolarizing noise. Based on the definition of the noise shown in Sec. V, we have

$$\begin{aligned}|\xi_{0000,D}\rangle &= U_{\mathrm{DC}}(E_{00}\otimes E_{00})|\phi_1\rangle \\ &= \frac{1}{d}\left(1-\frac{d^2-1}{d^2}p\right)\sum_{s,x=0}^{d-1}\omega^{(m-x)s}|x,n\rangle,\end{aligned} \tag{A24}$$

$$\begin{aligned}|\xi_{00vq,D}\rangle &= U_{\mathrm{DC}}(E_{00}\otimes E_{vq})|\phi_1\rangle = \frac{\sqrt{p}}{d^2}\sqrt{1-\frac{d^2-1}{d^2}p}\sum_{s,x=0}^{d-1} \\ &\times \omega^{(m-x)s+v(n+s)}|x,n\oplus q\rangle,\end{aligned} \tag{A25}$$

$$\begin{aligned}|\xi_{tj00,D}\rangle &= U_{\mathrm{DC}}(E_{0j}\otimes E_{00})|\phi_1\rangle \\ &= \frac{\sqrt{p}}{d^2}\sqrt{1-\frac{d^2-1}{d^2}p}\sum_{s,x=0}^{d-1}\omega^{(m-x)s+t(n+s-j)} \\ &\times |x,n\oplus(d-j)\rangle,\end{aligned} \tag{A26}$$

$$\begin{aligned}|\xi_{tjvq,D}\rangle &= U_{\mathrm{DC}}(E_{tj}\otimes E_{vq})|\phi_1\rangle \\ &= \frac{p}{d^3}\sum_{s,x=0}^{d-1}\omega^{(m-x)s+(t+v)(n+s-j)}|x,n\oplus q\oplus(d-j)\rangle,\end{aligned} \tag{A27}$$

where $t, j, v, q = 1, 2, \dots, d-1$. Note that

$$\rho'_{mn,D} = \sum_{t,j,v,q=0}^{d-1}|\xi_{tjvq,D}\rangle\langle\xi_{tjvq,D}|, \tag{A28}$$

$$\mathcal{F}_D = \mathrm{Tr}\{|m,n\rangle\langle m,n|\rho'_{mn,D}\}. \tag{A29}$$

After analogous calculation, we obtain that

$$\mathcal{F}_D = \left(1-\frac{d^2-1}{d^2}p\right)^2 + \frac{(d-1)^2p^2}{d^4}. \tag{A30}$$

APPENDIX B: COMPARISON OF DENSE CODING FIDELITIES UNDER DIFFERENT NOISE

We compare the fidelity of the dense coding protocol under dit-flip noise $\mathcal{F}_F$, d-phase-flip noise $\mathcal{F}_P$, dit-phase-flip noise $\mathcal{F}_{FP}$, and depolarizing noise $\mathcal{F}_D$ so as to show the influence of these four kinds of noise on dense coding.

Comparing the expressions (28)–(30), we find that with the same error probability p, we have $\mathcal{F}_F = \mathcal{F}_P > \mathcal{F}_{FP}$ for any $D > 6$. We compare $\mathcal{F}_D$ with $\mathcal{F}_F$ ($=\mathcal{F}_P$) and $\mathcal{F}_{FP}$, by calculating $\mathcal{F}_F - \mathcal{F}_D$ and $\mathcal{F}_{FP} - \mathcal{F}_D$. We set

$$a_i = \frac{1}{(D-5)^i} + \frac{D^2-6D+6}{(D-4)^4}(i=1,2), b = \frac{-2}{(D-4)^2}. \tag{B1}$$

The value of $-\frac{b}{a_i}$ with respect to the quantum cost D is shown in Fig. 5. After some calculations, we obtain that

$$\mathcal{F}_F - \mathcal{F}_D = a_1p^2 + bp, \tag{B2}$$

$$\mathcal{F}_{FP} - \mathcal{F}_D = a_2p^2 + bp. \tag{B3}$$

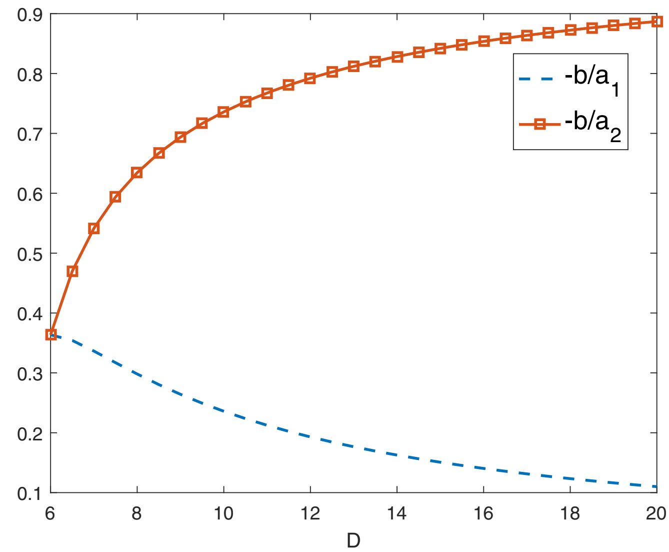

FIG. 5. Value of $-\frac{b}{a_i}$ $(i = 1, 2)$ with respect to the quantum cost D. The blue dashed line and red line with squares represent the values of $-\frac{b}{a_1}$ and $-\frac{b}{a_2}$, respectively.

Hence,

$$\begin{aligned}&\mathcal{F}_{FP} < \mathcal{F}_F = \mathcal{F}_P < \mathcal{F}_D \quad \text{for } 0 < p < -\frac{b}{a_1}, \\ &\mathcal{F}_{FP} < \mathcal{F}_D < \mathcal{F}_F = \mathcal{F}_P \quad \text{for } -\frac{b}{a_1} < p < -\frac{b}{a_2}, \\ &\mathcal{F}_D < \mathcal{F}_{FP} < \mathcal{F}_F = \mathcal{F}_P \quad \text{for } -\frac{b}{a_2} < p < 1.\end{aligned} \tag{B4}$$

The influence of noise can be evaluated with the fidelity. Generally, the higher the fidelity is, the less influence the noise has. First, we compare $\mathcal{F}_D$ with the three other kinds of fidelities with the constant p and variable quantum cost D. Fidelity $\mathcal{F}_D$ is between $\mathcal{F}_{FP}$ and $\mathcal{F}_P$ when the error probability p is in the interval $(-\frac{b}{a_1}, -\frac{b}{a_2})$ in Eq. (B4). This shows that $-\frac{b}{a_1} < -\frac{b}{a_2}$. With the increase of D, $-\frac{b}{a_1}$ decreases and $-\frac{b}{a_2}$ increases in Fig. 5. We obtain that when D is large enough, $-\frac{b}{a_1}$ approaches 0 and $-\frac{a_2}{b}$ approaches 1 by calculation. Hence, the increase of quantum cost can widen the interval. That is to say, when D is large enough, $\mathcal{F}_{FP} < \mathcal{F}_D < \mathcal{F}_F = \mathcal{F}_P$ holds with arbitrary $0 < p < 1$ in Eq. (B4). The influence of depolarizing noise tends to be in the middle of dit-flip (phase-flip) noise and dit-phase-flip noise with the increase of quantum cost for arbitrary $0 < p < 1$.

On the other hand, we compare $\mathcal{F}_D$ with the three other kinds of fidelities with the constant D and variable p. From Eq. (B4) we obtain that $\mathcal{F}_D$ is the largest of all kinds of fidelities for $0 < p < -\frac{b}{a_1}$ and it becomes the smallest one for $-\frac{b}{a_2} < p < 1$. Hence, the influence of depolarizing noise is less than other noise for $0 < p < -\frac{b}{a_1}$, it is in the middle of other noise for $-\frac{b}{a_1} < p < -\frac{b}{a_2}$, and it is larger than the three other kinds of noise for $p > \frac{b}{a_2}$. The influence of depolarizing noise tends to be larger with the increase of p and its growth rate is greater than that of the three other kinds of noise.

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