Relation between non-Markovianity and Landauer's principle

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(Received 23 December 2021; revised 19 April 2022; accepted 24 May 2022; published 16 June 2022)

The dynamical role of system-environment correlations will lead to the violation of Landauer's principle and may result in a non-Markovian dynamics of the system. Here, we employ entropic quantities called the telescopic relative entropy, to detect non-Markovianity [N. Megier, Phys. Rev. Lett. **127**, 030401 (2021)], and study the relation between information backflow of non-Markovian dynamics and the validity of Landauer's principle. We consider a class of energy-conserving system-environment interactions in qubit systems, and we prove that the violation of Landauer's principle corresponds to information backflow of a thermalizing process without coherence and vice versa. Then we study the effect of quantum coherence of the system on the relationship between non-Markovianity and Landauer's principle, and we find that coherence will break the corresponding relation in the information backflow and violation of Landauer's principle. Based on this, we explain the physical mechanism of non-Markovianity in this irreversible process of heat dissipation and information erasure. We further numerically confirm the above theoretical results by means of a collision model.

DOI: 10.1103/PhysRevA.105.062429

I. INTRODUCTION

Non-Markovian open-system dynamics has recently received considerable attention [1–3], including the formulation of figures of merit for its characteristics [4–9] and the first step towards its experimental assessment [10–13]. While a full understanding of the origin of non-Markovianity [14,15] and the formulation of a universal characterization of its implications are the subjects of current investigations [16–21], the relevance of non-Markovianity for the assessment of the properties of nonequilibrium quantum systems has been recently recognized [22–25]. In particular, the role of non-Markovian effects in logically irreversible processes has recently attracted much attention [26,27] in light of the relation between Landauer's principle and information processing at both the classical and quantum level [28–30]. Landauer's principle [28] states that the dissipated heat of a system in the information erasure process is lower bounded by a change in the information-theoretic entropy of the system. In the quantum domain, although this statement is validated in the Markovian process, it can be violated in the non-Markovian one [26,31,32]. Thus interesting questions are raised: What is the relationship between non-Markovianity and Landauer's principle? Specifically, must the violation of Landauer's principle be accompanied by information backflow?

A wide variety of different definitions of quantum non-Markovianity have been proposed in recent years, and the most widespread ones are based on the divisibility property of the dynamical map [6,33–36], the monotonicity of the trace distance between two distinct reduced states [4,7,36], the change in the volume of accessible reduced states [9], and the process tensor formalism [19,37,38]. Furthermore, entropic quantities have also been used to detect non-Markovianity [16,39,40]. In the approach based on trace distance, its increase in time indicates a backflow of information to the open system, resulting in an enhanced reduced-state distinguishability and representing the distinctive trait of memory effects in the dynamics. The revivals of distinguishability are related to the establishment of system-environment correlations and changes in the environmental state depending on the initial system state [14,15,41–43], and the proof of the connection of the distinguishability revivals with correlations and environmental state changes as formulated via the trace distance essentially relies on the triangle inequality [14], so that one may think that it only holds when distance quantifiers are used. To the contrary, recently, Megier *et al.* have shown that such a connection can be maintained also when considering entropic quantifiers [44]. Specifically, by focusing on the entropic quantifier of non-Markovianity, named telescopic relative entropy (TRE), they derived an upper bound to the variation of the reduced state distinguishability determined by the system-environment correlations and the environmental states.

In this paper, we consider an irreversible non-Markovian thermalization process of a qubit system to shed further light on the interplay between environment memory effects and logical irreversibility in nonequilibrium processes. We connect Landauer's principle with non-Markovianity detected by entropic quantities, and the corresponding relation between the violation of Landauer's principle and the occurrence of information backflow in the thermalization process with-

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out coherence is proved. Based on this, the corresponding physical interpretation of non-Markovianity is obtained. We find that the quantum coherence will break the relevance of non-Markovianity to Landauer's principle; furthermore, we investigate how coherence breaks this connection. Finally, we show the above theoretical results numerically.

This paper is organized as follows. In Sec. II, the non-Markovianity and Landauer's principle are introduced. In Sec. III we study the relation between Landauer's principle and the information backflow of non-Markovian dynamics generally, and some numerical confirmation for a collision model is given in Sec. IV. The paper ends with Sec. V, where we draw our conclusions.

II. NON-MARKOVIANITY AND LANDAUER'S PRINCIPLE

A. Non-Markovianity

Trace-distance-based measure of non-Markovianity. The trace distance between two quantum states is one of the most important measures of distinguishability of quantum states [4], which is given by

$$D\big(\rho_S^1, \rho_S^2\big) := \tfrac{1}{2}\mathrm{Tr}\big|\rho_S^1 - \rho_S^2\big|, \quad (1)$$

where $|A| = \sqrt{A^\dagger A}$ for any operator A. For the time evolution of a quantum state described by completely positive, trace-preserving (CPTP) maps $\{\Phi_{0\to t}\}$ which are time homogeneous and Markovian (they keep no record of the initial time, $\Phi_{0\to t} = \Phi_t$) [45], the trace distance decreases monotonically with time, i.e., $D(\Phi_t[\rho_S^1], \Phi_t[\rho_S^2]) \leqslant D(\rho_S^1, \rho_S^2)$, for any pair of states ρ_S^1 and ρ_S^2. In contrast, if the evolution of the trace distance becomes positive in some time intervals, the time evolution is non-Markovian,

$$D\big(\rho_S^1(t), \rho_S^2(t)\big) - D\big(\rho_S^1(s), \rho_S^2(s)\big) > 0, \quad (2)$$

for some $t > s$ and a pair $\rho_S^1(0)$ and $\rho_S^2(0)$. The revivals in the trace distance correspond to revivals in distinguishability, which can be interpreted as information backflow. Note that the total amount of information at time t is a constant and can be identified as the distinguishability of the states of both system and environment $I_{\mathrm{tot}}(t) = D(\rho_{SE}^1(t), \rho_{SE}^2(t))$. This quantity can be naturally written as the sum of two contributions referring to the information that can be obtained by performing local measurements only, i.e., $I_S(t) = D(\rho_S^1(t), \rho_S^2(t))$ and $I_E(t) = D(\rho_{SE}^1(t), \rho_{SE}^2(t)) - D(\rho_S^1(t), \rho_S^2(t))$, which are information accessed by the system and environment, respectively. Although $\frac{d}{dt}I_{\mathrm{tot}}(t) = \frac{d}{dt}(I_S(t) + I_E(t)) = 0$, the revivals in the system information can take place, i.e., $I_S(t) > I_S(s)$ for $t > s$, and this can be interpreted as information backflow. This interpretation is substantiated by the following inequality [43,44]:

$$\begin{aligned} D\big(\rho_S^1(t), \rho_S^2(t)\big) - D\big(\rho_S^1(s), \rho_S^2(s)\big) &\leqslant D\big(\rho_E^1(s), \rho_E^2(s)\big) \\ &+ D\big(\rho_{SE}^1(s), \rho_S^1(s)\otimes\rho_E^1(s)\big) \\ &+ D\big(\rho_{SE}^2(s), \rho_S^2(s)\otimes\rho_E^2(s)\big), \end{aligned} \quad (3)$$

where $t \geqslant s$. It appears that the information backflow associated with the revival of the trace distance can be traced back to the establishment of correlations between system and environment as well as the changes in the state of the environment.

TRE-based measure of non-Markovianity. Relative entropy is a fundamental quantity in statistical mechanics and information theory, and it also plays a distinguished role in quantum thermodynamics. The expression of the quantum relative entropy first introduced by Umegaki [46] reads

$$\mathcal{D}(\rho^1||\rho^2) := \mathrm{Tr}(\rho^1\log\rho^1 - \rho^1\log\rho^2), \quad (4)$$

where ρ^1 and ρ^2 are two density matrices. As is known, although the quantum relative entropy is the most relevant quantum f divergence distinguishing quantum states [47], it is not bounded and can diverge in finite dimensions. To overcome this difficulty, the TRE has been proposed as regularized versions [47–50]. The TRE is defined as

$$S_\mu(\rho^1, \rho^2) = \log(1/\mu)^{-1}\mathcal{D}(\rho^1||\mu\rho^1 + (1-\mu)\rho^2) \quad (5)$$

and is actually independent of the logarithm basis used in the definition. The telescopic parameter $\mu \in (0, 1)$ gives the amount of mixing between ρ^1 and ρ^2. The main property of the TRE, which distinguishes it from the standard quantum relative entropy, is its boundedness, i.e., $0 \leqslant S_\mu(\rho^1, \rho^2) \leqslant 1$. Moreover, TRE inherits from the quantum relative entropy the joint convexity and the contractivity under (C)PTP maps [50], $S_\mu(\Phi_t[\rho^1], \Phi_t[\rho^2]) \leqslant S_\mu(\rho^1, \rho^2)$. The change in TRE can be expressed as [44]

$$\begin{aligned} &S_\mu\big(\rho_S^1(t), \rho_S^2(t)\big) - S_\mu\big(\rho_S^1(s), \rho_S^2(s)\big) \\ &\quad\leqslant \kappa_\mu\big\{S_\mu^{1/4}\big(\rho_E^1(s), \rho_E^2(s)\big) + S_\mu^{1/4}\big(\rho_{SE}^1(s), \rho_S^1(s)\otimes\rho_E^1(s)\big) \\ &\qquad + S_\mu^{1/4}\big(\rho_{SE}^2(s), \rho_S^2(s)\otimes\rho_E^2(s)\big)\big\}, \end{aligned} \quad (6)$$

with $\kappa_\mu = [2\mu^2\log^3(1/\mu)]^{-1/4}$ and $t \geqslant s$. Note that the boundedness of TRE allows us to introduce a well-defined non-Markovianity measure as for the trace distance [1], and this bound Eq. (6) permits a full-fledged interpretation of TRE as a quantifier of information backflow. Thus the dynamic process is identified as non-Markovian, if and only if

$$S_\mu\big(\rho_S^1(t), \rho_S^2(t)\big) - S_\mu\big(\rho_S^1(s), \rho_S^2(s)\big) > 0, \quad (7)$$

for some $t > s$ and a pair $\rho_S^1(0)$ and $\rho_S^2(0)$.

B. Landauer's principle

Landauer's principle relates entropy decrease and heat dissipation during logically irreversible processes [27,51], and the four assumptions needed for Landauer's principle are as follows: (a) the process involves a system S and its environment E; (b) the environment E is initially in a thermal state, $\rho_E = e^{-\beta\hat{H}_E}/\mathrm{Tr}[e^{-\beta\hat{H}_E}] =: \eta^{(\beta)}$, where $\hat{H}_E$ is the Hamiltonian of the environment and $\beta \in (0, +\infty)$ is the inverse temperature; (c) the system S and the environment E are initially uncorrelated, $\rho_{SE}(0) = \rho_S(0)\otimes\rho_E(0)$; and (d) the process proceeds through a joint unitary evolution, $\rho'_{SE} = U\rho_{SE}(0)U^\dagger$.

For a process as just described, the entropy decrease of the system is expressed as

$$\Delta S = S(\rho_S(0)) - S(\rho'_S), \quad (8)$$

where $\rho'_S = \mathrm{Tr}_E[\rho'_{SE}]$ and $S(\rho) = -\mathrm{Tr}[\rho\log\rho]$ is the von Neumann entropy of a density matrix ρ. The heat transferred to the environment

$$\Delta Q_E = \mathrm{Tr}[\hat{H}_E(\rho'_E - \rho_E)], \tag{9}$$

which corresponds to the increase in internal energy of the thermal environment, where $\rho'_E = \mathrm{Tr}_S[\rho'_{SE}]$. The equality form of Landauer's principle can be expressed as [27]

$$\beta \Delta Q_E = \Delta S + I(\rho'_{SE}) + \mathcal{D}(\rho'_E || \rho_E), \tag{10}$$

where $I(\rho'_{SE}) := S(\rho'_S) + S(\rho'_E) - S(\rho'_{SE})$ is the mutual information which characterizes the correlations between the system and environment. Due to the non-negativity of the mutual information and the relative entropy, Landauer's principle can be written as

$$\beta \Delta Q_E - \Delta S \geqslant 0. \tag{11}$$

This embodies an open-system formulation of Landauer's principle for the heat dissipation and entropy decrease.

III. NON-MARKOVIANITY ACCOMPANIED BY LANDAUER'S PRINCIPLE

The following discussion is based on assumptions (a)–(d) mentioned above. We consider a qubit (system) coupled to a thermal bath, and the evolution of the system can be described by a unitary operator $\hat{U} = e^{-i\hat{H}^{\mathrm{int}}t}$, where $\hat{H}^{\mathrm{int}}$ is the interaction Hamiltonian, and we set $\hbar = 1$ throughout this paper. Specifically, the process can be expressed as

$$\rho_S(0) \rightarrow \rho_S(t) = \mathrm{Tr}_E[\rho_{SE}(t)], \tag{12}$$

with $\rho_{SE}(t) = \hat{U}(\rho_S(0) \otimes \rho_E(0))\hat{U}^{\dagger}$. We then say that E induces thermalization on S if, irrespective of the specific choice of $\rho_S(0)$, S will be driven by $\Phi_{0\rightarrow t}$ into the equilibrium configuration state $\eta_S^{(\beta)} := e^{-\beta \hat{H}_S}/\mathrm{Tr}[e^{-\beta \hat{H}_S}]$ in the limit $t \rightarrow \infty$; here, $\hat{H}_S$ is the Hamiltonian of S and $\beta = 1/T$. We consider energy-conserving interactions between system and environment, i.e., $[\hat{H}_{\mathrm{int}}, \hat{H}_S + \hat{H}_E] = 0$; then the heat absorbed by S can be legitimately identified with the increases in the local energy of S [52–54],

$$\Delta Q_S = \mathrm{Tr}[\hat{H}_S(\rho_S(t) - \rho_S(0))]. \tag{13}$$

The conservation of the total energy in the composite system "$S + E$" leads to $\Delta Q_S = -\Delta Q_E$. The derivation of the Landauer inequality in Eq. (11) can be generalized to bound the differential heat increment $dQ_E(t) := \mathrm{Tr}[\hat{H}_E(\rho_E(t + dt) - \rho_E(t))]$ in terms of the corresponding differential entropy increase $dS(t) := S(\rho_S(t + dt)) - S(\rho_S(t))$ [55],

$$\beta \dot{Q}_E - \dot{S} \geqslant 0, \tag{14}$$

where $\dot{Q}_E := \frac{d}{dt}Q_E(t)$ and $\dot{S} := \frac{d}{dt}S(t)$.

A. Relation between Landauer's principle and information backflow with diagonal states of the system

We consider a pair of initial states of a system without coherence, $\rho_S^1(0)$ and $\rho_S^2(0)$ (their off-diagonal elements are zero). We assume that the system undergoes the same process Eq. (12) for the pair of initial states; then a pair of linearly independent states without coherence during the time evolution is obtained [i.e., $\rho_S^1(t)$ and $\rho_S^2(t)$ are linearly independent], until the system thermalizes completely (namely, the system is in a thermal equilibrium state $\eta_S^{(\beta)}$). According to Ref. [55] we obtain

$$\dot{S} - \beta \dot{Q}_E = \frac{d}{dt}\mathcal{D}\big(\rho_S^1(t) || \eta_S^{(\beta)}\big), \tag{15}$$

and a proof is provided in the Appendix. It is obvious that the left-hand side and right-hand side of Eq. (15) are always zero in the limit $t \rightarrow \infty$; thus we discuss the transient period before the system is completely thermalized in the following. Due to the linear independence of $\rho_S^1(t)$ and $\rho_S^2(t)$ mentioned above, $\eta_S^{(\beta)}$ in Eq. (15) can be written as

$$\eta_S^{(\beta)} = \mu \rho_S^1(t) + \lambda \rho_S^2(t); \tag{16}$$

here $\mu, \lambda \in (0, 1)$, and note that they are functions of t, i.e., $\mu = \mu(t)$, $\lambda = \lambda(t)$. As the density matrix is trace 1, namely, $\mathrm{Tr}\eta_S^{(\beta)} = \mu\, \mathrm{Tr}[\rho_S^1(t)] + \lambda\, \mathrm{Tr}[\rho_S^2(t)] = \mu + \lambda$, we thus obtain $\lambda = 1 - \mu$. Then Eq. (15) can be rewritten as

$$\dot{S} - \beta \dot{Q}_E = \frac{d}{dt}\mathcal{D}\big(\rho_S^1(t) || \mu \rho_S^1(t) + (1-\mu)\rho_S^2(t)\big), \tag{17}$$

and from Eq. (5), the right-hand side of Eq. (17) can be written as

$$\begin{aligned} \frac{d}{dt}\bigg[\log \frac{1}{\mu} S_\mu\big(\rho_S^1(t), \rho_S^2(t)\big)\bigg] &= \frac{\partial}{\partial \mu}\bigg[\log \frac{1}{\mu} S_\mu\big(\rho_S^1(t), \rho_S^2(t)\big)\bigg]\frac{d\mu}{dt} \\ &\quad + \log \frac{1}{\mu}\frac{\partial}{\partial t}\big[S_\mu\big(\rho_S^1(t), \rho_S^2(t)\big)\big]; \end{aligned} \tag{18}$$

as mentioned above, it is independent of the logarithm basis here. It can be seen that the first term on the right-hand side of Eq. (18) can be written as $\frac{\partial}{\partial \mu}\mathcal{D}(\rho_S^1(t)||\eta_S^{(\beta)})\frac{d\mu}{dt}$, which is found to be zero. Therefore we can obtain

$$\dot{S} - \beta \dot{Q}_E = \log \frac{1}{\mu}\frac{\partial}{\partial t}\big[S_\mu\big(\rho_S^1(t), \rho_S^2(t)\big)\big], \tag{19}$$

where $\log \frac{1}{\mu} > 0$ $(0 < \mu < 1)$ and $\mu = \mu(t)$ is a function of t as mentioned above in Eq. (16). This shows that Eq. (19) links non-Markovianity which is detected by TRE to Landauer's principle. Specifically, a non-Markovian evolution $[\sigma = \frac{\partial}{\partial t}S_\mu(\rho_S^1(t), \rho_S^2(t)) > 0]$ corresponds to the violation of Landauer's principle ($\dot{S} - \beta \dot{Q}_E > 0$), and the converse is also true that if Landauer's principle is violated ($\dot{S} - \beta \dot{Q}_E > 0$), the information backflow must happen ($\sigma > 0$).

B. The mechanism of non-Markovianity

Taking the derivative of the equality version of Landauer's principle [Eq. (10)], we obtain

$$\beta \dot{Q}_E - \dot{S} = \frac{d}{dt}I(\rho_{SE}(t)) + \frac{d}{dt}\mathcal{D}(\rho_E(t)||\rho_E(0)). \tag{20}$$

Putting Eq. (19) into Eq. (20), we obtain

$$\begin{aligned} \log \frac{1}{\mu}\frac{\partial}{\partial t}\big[S_\mu\big(\rho_S^1(t), \rho_S^2(t)\big)\big] \\ = -\bigg[\frac{d}{dt}I(\rho_{SE}(t)) + \frac{d}{dt}\mathcal{D}(\rho_E(t)||\rho_E(0))\bigg]. \end{aligned} \tag{21}$$

From Eq. (21), it is apparent that $\frac{\partial}{\partial t}[S_\mu(\rho_S^1(t)||\rho_S^2(t))] > 0$ if and only if $\frac{d}{dt}I(\rho_{SE})(t) + \frac{d}{dt}\mathcal{D}(\rho_E(t)||\rho_E(0)) < 0$. The left-hand side of Eq. (21) quantifies the amount of information gained or lost by the pair of reduced states of the system, and any information backflow towards the system in a time interval will thus result in a positive value of it. As the composite system (system + thermal bath) is closed, this information comes from the establishment of correlations between system and environment and/or from the changes in the state of the environment [right-hand side of Eq. (21)]. This result is similar to that of Eqs. (3) and (6) in spirit, namely, the actual physical mechanism behind the occurrence of memory effects in quantum dynamics is the establishment of correlations or changes in the environmental states.

C. Effects of quantum coherence

In this section, we consider the effect of quantum coherence on the relation between the information backflow of non-Markovian dynamics and Landauer's principle. We consider a pair of initial states of a system with coherence, $\rho_{\rm coh}^1(0)$ and $\rho_{\rm coh}^2(0)$, and we also assume that the system undergoes the same process Eq. (12) for this pair of initial states. Then we obtain the corresponding pair of states with coherence during the time evolution, $\rho_{\rm coh}^1(t)$ and $\rho_{\rm coh}^2(t)$, which can be expressed as

$$\rho_{\rm coh}^{1(2)}(t) = \sigma_{\rm dia}^{1(2)}(t) + \chi_{\rm coh}^{1(2)}(t), \tag{22}$$

where $\sigma_{\rm dia}^{1(2)}(t)$ is a diagonal state which has the same energy as $\rho_{\rm coh}^{1(2)}(t)$, i.e., $\mathrm{Tr}[\hat{H}_S\sigma_{\rm dia}^{1(2)}(t)] = \mathrm{Tr}[\hat{H}_S\rho_{\rm coh}^{1(2)}(t)]$, and $\chi_{\rm coh}^{1(2)}(t) = \rho_{\rm coh}^{1(2)}(t) - \sigma_{\rm dia}^{1(2)}(t)$. By means of the linearly independence of $\sigma_{\rm dia}^1(t)$ and $\sigma_{\rm dia}^2(t)$, $\eta_S^{(\beta)}$ in Eq. (15) satisfies

$$\eta_S^{(\beta)} = \mu\sigma_{\rm dia}^1(t) + (1-\mu)\sigma_{\rm dia}^2(t), \tag{23}$$

where $\mu = \mu(t) \in (0, 1)$; then from Eq. (22) we obtain

$$\begin{aligned}\eta_S^{(\beta)} + \mu\chi_{\rm coh}^1(t) + (1-\mu)\chi_{\rm coh}^2(t)\\ = \mu\rho_{\rm coh}^1(t) + (1-\mu)\rho_{\rm coh}^2(t).\end{aligned} \tag{24}$$

Here, we define a quantity κ as

$$\kappa = \frac{d}{dt}\mathcal{D}\big(\rho_{\rm coh}^1(t)||\eta_S^{(\beta)} + \mu\chi_{\rm coh}^1(t) + (1-\mu)\chi_{\rm coh}^2(t)\big), \tag{25}$$

and from Eqs. (22) and (5), Eq. (25) can be rewritten as

$$\kappa = \frac{d}{dt}\left[\log\frac{1}{\mu}S_\mu\big(\rho_{\rm coh}^1(t), \rho_{\rm coh}^2(t)\big)\right] = \log\frac{1}{\mu}\frac{\partial}{\partial t}\big[S_\mu\big(\rho_{\rm coh}^1(t), \rho_{\rm coh}^2(t)\big)\big] + \varepsilon, \tag{26}$$

where $\varepsilon = \frac{\partial}{\partial\mu}[\mathcal{D}(\rho_{\rm coh}^1(t)||\eta_S^{(\beta)} + \mu\chi_{\rm coh}^1(t) + (1-\mu)\chi_{\rm coh}^2(t))]\frac{d\mu}{dt}$. Now Eq. (15) can be expressed as

$$\dot{S} - \beta\dot{Q}_E = \log\frac{1}{\mu}\frac{\partial}{\partial t}\big[S_\mu\big(\rho_{\rm coh}^1(t), \rho_{\rm coh}^2(t)\big)\big] + \xi_{\rm coh}, \tag{27}$$

where $\xi_{\rm coh} = \frac{d}{dt}\mathrm{Tr}\left[\rho_{\rm coh}^1(t)\log\frac{\eta_S^{(\beta)} + \mu\chi_{\rm coh}^1(t) + (1-\mu)\chi_{\rm coh}^2(t)}{\eta_S^{(\beta)}}\right] + \varepsilon$. It can be seen that the first term on the right-hand side of Eq. (27) is the contribution of TRE and the second term $\xi_{\rm coh}$ is the effects of quantum coherence. This is different from the cases of linearly independent thermal states [Eq. (19)]. In other words, the relation of Landauer's principle and the information backflow of non-Markovian dynamics Eq. (19) is no longer valid due to quantum coherence. If there is no quantum coherence during the time evolution, i.e., $\chi_{\rm coh}^1(t) = \chi_{\rm coh}^2(t) = 0$, $\xi_{\rm coh} = 0$ and Eq. (27) is reduced to Eq. (19).

IV. NUMERICAL VERIFICATION

We now showcase our findings by numerical calculations. Let us consider a qubit (system S) that couples to a hierarchical environment E, which contains an auxiliary qubit A_Q and a large collection of N identical noninteracting ancillas (qubits) $\{\mathcal{R}_1, \mathcal{R}_2, \ldots, \mathcal{R}_N\}$ (see Fig. 1); E is initially in the product state $\rho_E = \eta_{A_Q}^{(\beta)} \otimes_{j=1}^N \eta_j^{(\beta)}$, i.e., A_Q and $\mathcal{R}_N$ are initially in thermal diagonal states. The Hamiltonians of the system and a generic environment particle including the auxiliary qubit and ancillas are $\hat{H}_{S(E)} = \omega_0\hat{\sigma}_z/2$, with ω_0 and $\hat{\sigma}_z$ being the transition frequency and the Pauli matrices, respectively. The evolution of system S and its interaction with the environment are as follows: S and A_Q interact first, and subsequently, A_Q collides with the individual ancilla $\mathcal{R}_n$; then this process is repeated. The general scheme is illustrated in Fig. 1. The assumption of a large collection of ancillas implies that A_Q never interacts twice with the same ancilla, i.e., at each collision the state of the ancilla is refreshed. This process is implemented through the unitary operator

$$\hat{U}_{SE} = \hat{\mathcal{U}}_{A_Q,\mathcal{R}_j}\hat{\mathcal{U}}_{S,A_Q}, \tag{28}$$

where $\hat{\mathcal{U}}_{S,A_Q} = e^{-i\hat{H}_{S,A_Q}^{\rm int}t}$, $\hat{\mathcal{U}}_{A_Q,\mathcal{R}_j} = e^{-i\hat{H}_{A_Q,\mathcal{R}_j}^{\rm int}t}$, and $\hat{H}_{S,A_Q}^{\rm int}$ and $\hat{H}_{A_Q,\mathcal{R}_j}^{\rm int}$ are the interactions between S and A_Q ("S-A_Q") and the interactions between A_Q and $\mathcal{R}_j$ ("A_Q-$\mathcal{R}_j$"), respectively. We consider a coherent interaction between the bipartite systems, $\hat{H}_{S,A_Q(A_Q,\mathcal{R}_j)}^{\rm int} = g_{1(2)}(\hat{\sigma}_x\hat{\sigma}_x + \hat{\sigma}_y\hat{\sigma}_y + \hat{\sigma}_z\hat{\sigma}_z)$, with $g_{1(2)}$ being a coupling constant. In this case, we can write the unitary time-evolution operator $\hat{\mathcal{U}}_{S,A_Q}$ in Eq. (28) as [56]

$$\hat{\mathcal{U}}_{S,A_Q}(\gamma) = (\cos\gamma)\hat{\mathbb{I}} + i(\sin\gamma)\hat{S}_{sw}, \tag{29}$$

where $\gamma = 2g_1t$ is a dimensionless interaction strength, $\hat{\mathbb{I}}$ is the identity operator, and $\hat{S}_{sw}$ is the two-particle swap operator whose action is $|\psi_1\rangle\otimes|\psi_2\rangle \to |\psi_2\rangle\otimes|\psi_1\rangle$ for all $|\psi_1\rangle, |\psi_2\rangle \in \mathbb{C}^2$. Note that in the ordered basis

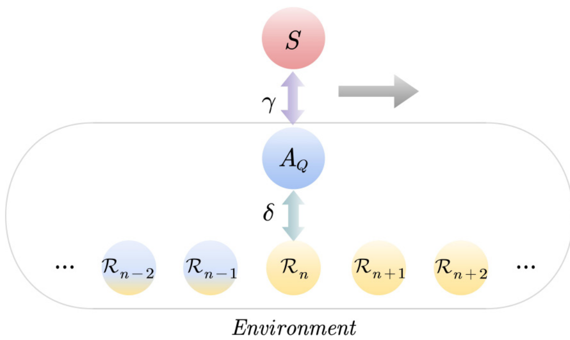

FIG. 1. Sketch of the protocol of system S plus a hierarchical environment: After A_Q interacts with $\mathcal{R}_n$, it collides with S and is then directed to $\mathcal{R}_{n+1}$.

$\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$, $\hat{S}_{sw}$ reads [57]

$$\hat{S}_{sw} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{30}$$

Similarly, $\hat{\mathcal{U}}_{A_Q,\mathcal{R}_j}(\delta) = (\cos\delta)\hat{\mathbb{I}} + i(\sin\delta)\hat{S}_{sw}$, with $\delta \neq \gamma$ in general. As mentioned above, the dynamics of system S consists of sequential system-environment interaction interspersed with interenvironmental collisions. Thus the system is brought from step n to step $n+1$ through the process

$$\rho_n^{S,A_Q} \otimes \eta_{n+1}^{(\beta)} \rightarrow \rho_{n+1}^{SE} = \hat{U}_{SE}\big(\rho_n^{S,A_Q} \otimes \eta_{n+1}^{(\beta)}\big)\hat{U}_{SE}^{\dagger}, \tag{31}$$

where ρ_n^{S,A_Q} is the state of S-A_Q after the nth interaction. Hence after the $(n+1)$th interaction, we can obtain the reduced states, $\rho_{n+1}^{S,A_Q} = \mathrm{Tr}_{\mathcal{R}}[\rho_{n+1}^{SE}]$ (the state of S-A_Q), $\rho_{n+1}^{S} = \mathrm{Tr}_{A_Q}[\rho_{n+1}^{S,A_Q}]$ (the state of S), and $\rho_{n+1}^{A_Q} = \mathrm{Tr}_S[\rho_{n+1}^{S,A_Q}]$ (the state of A_Q), where $\mathrm{Tr}_x[\cdots]$ means the trace of x's degrees of freedom.

We consider a pair of orthogonal initial states of system S

$$|\psi_+\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}|1\rangle,$$
$$|\psi_-\rangle = \sin\frac{\theta}{2}|0\rangle - \cos\frac{\theta}{2}|1\rangle, \tag{32}$$

where $\theta \in [0, \frac{\pi}{2}]$.

A. Diagonal states of the system

Firstly, we consider the case of a pair of linearly independent states without coherence during the time evolution (Sec. III A), and we choose $\theta = 0$ in Eq. (32); namely, the pair of initial states of the system is $|0\rangle$ and $|1\rangle$. From numerical calculations we find that when the initial state is given, μ does not change with time and is a constant depending on the given initial state. In Fig. 2, we plot $\dot{S} - \beta\dot{Q}_E$, $\frac{d}{dt}D(\rho_S^1(t), \rho_S^2(t))$, and $\frac{d}{dt}S_\mu(\rho_S^1(t), \rho_S^2(t))$ against the number of collisions n for a pair of initial states $|0\rangle$ and $|1\rangle$ ($\mu = 0.378$), and it presents a perfect correspondence between them; that is, the ranges of positive and negative of these three quantities are completely consistent with each other (see inset for a magnified view). Specifically, a non-Markovian evolution $[\frac{d}{dt}D(\rho_S^1(t), \rho_S^2(t)) > 0, \frac{d}{dt}S_\mu(\rho_S^1(t), \rho_S^2(t)) > 0]$ corresponds to the violation of Landauer's principle ($\dot{S} - \beta\dot{Q}_E > 0$), and the converse is also true that the violation of Landauer's principle ($\dot{S} - \beta\dot{Q}_E > 0$) corresponds to information backflow.

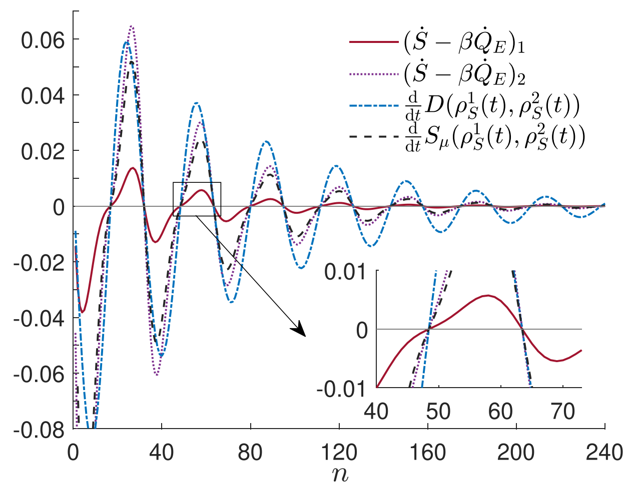

FIG. 2. We consider a pair of initial states of system $|0\rangle$ and $|1\rangle$ with $\mu = 0.378$ in Eq. (19). $(\dot{S} - \beta\dot{Q}_E)_1$ for initial state $|0\rangle$ (brown solid line) and $(\dot{S} - \beta\dot{Q}_E)_2$ for $|1\rangle$ (purple dotted line), $\frac{d}{dt}D(\rho_S^1(t), \rho_S^2(t))$ (blue dash-dotted line), and $\frac{d}{dt}S_\mu(\rho_S^1(t), \rho_S^2(t))$ (black dashed line). $\gamma = 0.03\pi$, $\delta = 0.045\pi$, $\beta = 1$, and $\omega_0 = 1$. The inset is a magnified display.

B. System states with coherence

We now consider the effect of coherence on the relation between Landauer's principle and information backflow. In Fig. 3, we plot $\dot{S} - \beta\dot{Q}_E$, $\frac{d}{dt}D(\rho_S^1(t), \rho_S^2(t))$, and $\frac{d}{dt}S_\mu(\rho_S^1(t), \rho_S^2(t))$ against the number of collisions n for initial states of a system with coherence, i.e., Eq. (32) with $\theta = 2\pi/5$ ($\mu = 0.896$). In contrast to the case of initial states of a system

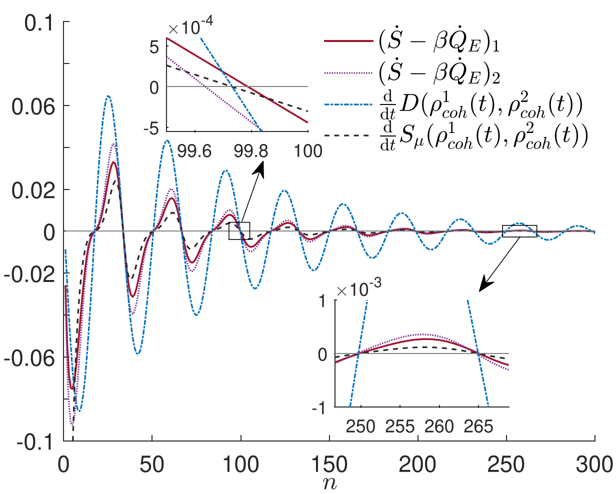

FIG. 3. The pair of initial states of the system in Eq. (32) with $\theta = 2\pi/5$ ($\mu = 0.896$). $(\dot{S} - \beta\dot{Q}_E)_1$ for initial state $|\psi_+\rangle$ (brown solid line) and $(\dot{S} - \beta\dot{Q}_E)_2$ for $|\psi_-\rangle$ (purple dotted line), $\frac{d}{dt}D(\rho_S^1(t), \rho_S^2(t))$ (blue dash-dotted line), and $\frac{d}{dt}S_\mu(\rho_S^1(t), \rho_S^2(t))$ (black dashed line). $\gamma = 0.03\pi$, $\delta = 0.045\pi$, $\beta = 1$, and $\omega_0 = 1$. The insets are magnified views of quantities as indicated in the plot.

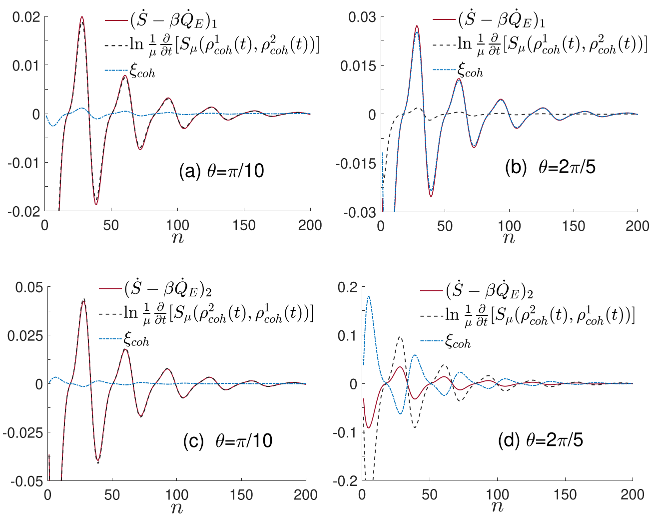

FIG. 4. The pair of initial states of the system in Eq. (32) with (a) $\theta = \pi/10$ ($\mu = 0.629$), (b) $\theta = 2\pi/5$ ($\mu = 0.896$), (c) $\theta = \pi/10$ ($\mu = 0.371$), and (d) $\theta = 2\pi/5$ ($\mu = 0.104$). $\ln \frac{1}{\mu}\frac{\partial}{\partial t}[S_\mu(\rho^1_{\rm coh}(t), \rho^2_{\rm coh}(t))]$ [the first term on the right-hand side of Eq. (27) with logarithm base e] can reflect non-Markovianity measured by TRE, and $\xi_{\rm coh}$ in Eq. (27) is the contribution of quantum coherence; the other parameters are the same as in Fig. 3.

without coherence discussed above, these three quantities no longer have a strict correspondence between them. Specifically, the violation of Landauer's principle ($\dot{S} - \beta\dot{Q}_E > 0$) does not always correspond to backflow of information [$\frac{d}{dt}D(\rho^1_S(t), \rho^2_S(t)) > 0$, $\frac{d}{dt}S_\mu(\rho^1_S(t), \rho^2_S(t)) > 0$], which can be seen from the upper inset of Fig. 3, and this is the effect of quantum coherence [$\xi_{\rm coh}$ in Eq. (27)]. In the following, we study the contribution of TRE [the first term on the right-hand side of Eq. (27)] and quantum coherence ($\xi_{\rm coh}$) to $\dot{S} - \beta\dot{Q}_E$ in Eq. (27). In Fig. 4, we show each quantity in Eq. (27) for initial states $|\psi_+\rangle$ and $|\psi_-\rangle$ in Eq. (32). It can be seen that the term $\ln \frac{1}{\mu}\frac{\partial}{\partial t}[S_\mu(\rho^1_{\rm coh}(t), \rho^2_{\rm coh}(t))]$ plays a major role in Landauer's principle ($\dot{S} - \beta\dot{Q}_E$) if the effect of quantum coherence is weak (small $\xi_{\rm coh}$), and $\dot{S} - \beta\dot{Q}_E$ and $\ln \frac{1}{\mu}\frac{\partial}{\partial t}[S_\mu(\rho^1_{\rm coh}(t), \rho^2_{\rm coh}(t))]$ are almost coincident with each other when $\xi_{\rm coh}$ is small enough. In contrast to this, $\xi_{\rm coh}$ breaks the corresponding relation Eq. (19) when $\xi_{\rm coh}$ is big. However, the decoherence effect caused by the environment would make the system be thermalized; in a long-time limit, $\rho^1_{\rm coh}(t) \to \rho^1_S(t)$ in Eq. (19), $\xi_{\rm coh}$ in Eq. (27) tends to zero, and Eq. (27) is reduced to Eq. (19). In other words, the correspondence between the information backflow of non-Markovian dynamics and the violation of Landauer's principle is reestablished in the long-time limit, and this is shown in the lower inset of Fig. 3.

For the pair of initial states $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ and $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$ [Eq. (32) with $\theta = \pi/2$], from numerical calculations we find that the time-varying density matrices $\rho^1_{\rm coh}(t)$ and $\rho^2_{\rm coh}(t)$ in Eq. (22) satisfy the relation $\sigma^{1(2)}_{\rm dia}(t) = (\rho^1_{\rm coh}(t) + \rho^2_{\rm coh}(t))/2$, and Eq. (23) does not hold now. With the help of the "Pythagoras theorem" of relative entropy [58], Eq. (15) can be rewritten as $\dot{S} - \beta\dot{Q}_E = \frac{d}{dt}\mathcal{D}(\rho^1_{\rm coh}(t)||\sigma^1_{\rm dia}(t)) + \frac{d}{dt}\mathcal{D}(\sigma^1_{\rm dia}(t)||\eta^{(\beta)}_S)$, and from Eq. (5),

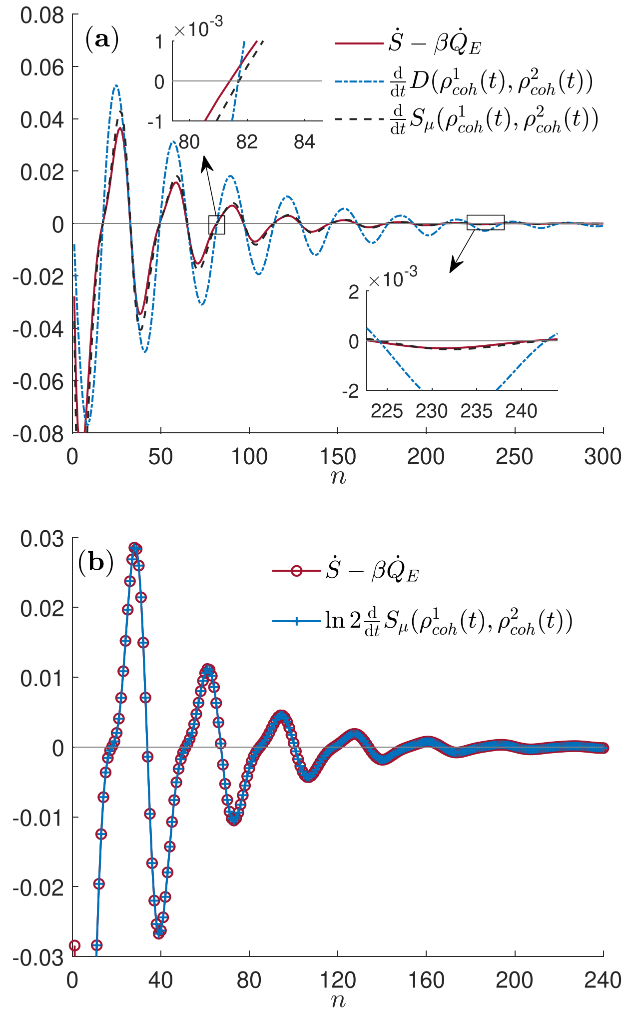

FIG. 5. The pair of initial states of the system is $|+\rangle$ and $|-\rangle$ with $\mu = 0.5$, $\gamma = 0.03\pi$, $\delta = 0.045\pi$, and $\omega_0 = 1$. (a) $\dot{S} - \beta\dot{Q}_E$ for initial state $|+\rangle$ ($|-\rangle$) (brown solid line), $\frac{d}{dt}D(\rho^1_S(t), \rho^2_S(t))$ (blue dash-dotted line), and $\frac{d}{dt}S_\mu(\rho^1_S(t), \rho^2_S(t))$ (black dashed line), and $\beta = 1$. The insets provide a magnified display of quantities as indicated in the plot. (b) Eq. (33) in the high-temperature limit $\beta = 1/100$.

we obtain

$$\dot{S} - \beta\dot{Q}_E = \log 2\frac{d}{dt}S_\mu\big(\rho^1_{\rm coh}(t), \rho^2_{\rm coh}(t)\big) + \frac{d}{dt}\mathcal{D}\big(\sigma^1_{\rm dia}(t)||\eta^{(\beta)}_S\big), \tag{33}$$

where $\mu = 0.5$. It can be seen that the first term on the right-hand side of Eq. (33) can be expressed as the time derivative of TRE and the second term is the contribution of the quantum coherence of the system. In Fig. 5(a) we plot $\dot{S} - \beta\dot{Q}_E$, $\frac{d}{dt}D(\rho^1_S(t), \rho^2_S(t))$, and $\frac{d}{dt}S_\mu(\rho^1_S(t), \rho^2_S(t))$ against the number of collisions n for two initial states $|+\rangle$ and $|-\rangle$. As expected, the violation of Landauer's principle ($\dot{S} - \beta\dot{Q}_E > 0$) does not always correspond to the information backflow of non-Markovian evolution [$\frac{d}{dt}D(\rho^1_S(t), \rho^2_S(t)) > 0$, $\frac{d}{dt}S_\mu(\rho^1_S(t), \rho^2_S(t)) > 0$], and this can be seen from the upper inset of Fig. 5(a). The one-to-one relation of the violation of Landauer's principle and information backflow is reestablished in the process of decoherence [the lower inset of Fig. 5(a)]. When we consider a high-temperature limit, the second term on the right-hand side of Eq. (33) tends to zero, and Eq. (33) becomes

$$\dot{S} - \beta\dot{Q}_E = \log 2 \frac{d}{dt} S_\mu\big(\rho^1_{\rm coh}(t)||\rho^2_{\rm coh}(t)\big), \quad (34)$$

with $\mu = 0.5$; the numerical results of Eq. (34) with logarithm base e are shown in Fig. 5(b).

V. CONCLUSION

In this paper, we have investigated the relation between the information backflow of non-Markovian dynamics and Landauer's principle. Specifically, we have considered a qubit coupled to its thermal environment, and we have proved that there is a one-to-one correspondence between the violation of Landauer's principle and the backflow of information for thermal initial states of system. However, this correspondence does not hold for initial states of a system with coherence, and the relevance of Landauer's principle and information backflow is reestablished during the dynamical process of the decoherence. We have verified our findings through numerical calculations.

Note that in this paper we have used qubit systems to investigate the relationship between the information backflow of non-Markovian dynamics and Landauer's principle. The reason to consider this simple model is that exact solutions can be obtained for a general class of linearly independent thermal diagonal states during the time evolution. This class of models may reveal the relation between the information backflow and Landauer's principle that is not satisfied by general non-Markovian systems. We expect that the properties revealed in this paper can help one to gain some insight into the relation between non-Markovianity and Landauer's principle and can inspire more work in investigating their connections.

ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (Grants No. 12147174, No. 11775019, and No. 61835013) and the National Key R&D Program of China (Grants No. 2021YFA1400900, No. 2021YFA0718300, and No. 2021YFA1400243).

APPENDIX

According to the expression of quantum relative entropy given in Eq. (4), at any time t,

$$\begin{aligned}\mathcal{D}\big(\rho^1_S(t)||\eta^{(\beta)}_S\big) - \mathcal{D}\big(\rho^1_S(0)\big|\big|\eta^{(\beta)}_S\big) &= \mathrm{Tr}\big[\rho^1_S(t)\log\rho^1_S(t) \\ &\quad - \rho^1_S(t)\log\eta^{(\beta)}_S - \rho^1_S(0) \\ &\quad \times \log\rho^1_S(0) + \rho^1_S(0)\log\eta^{(\beta)}_S\big],\end{aligned} \quad (A1)$$

where $\mathrm{Tr}[\rho^1_S(t)\log\rho^1_S(t) - \rho^1_S(0)\log\rho^1_S(0)] = -S(\rho^1_S(t)) + S(\rho^1_S(0)) = \Delta S$, and

$$\begin{aligned}&\mathrm{Tr}\big[\rho^1_S(0)\log\eta^{(\beta)}_S - \rho^1_S(t)\log\eta^{(\beta)}_S\big] \\ &\quad = \mathrm{Tr}\big[\big(\rho^1_S(0) - \rho^1_S(t)\big)\big(-\beta\hat{H}_S - \hat{\mathbb{I}}\log\mathrm{Tr}(e^{-\beta\hat{H}_S})\big)\big] \\ &\quad = \mathrm{Tr}\big[-\beta\big(\rho^1_S(0) - \rho^1_S(t)\big)\hat{H}_S\big] \\ &\qquad + \mathrm{Tr}\big[\big(\rho^1_S(0) - \rho^1_S(t)\big)\big(-\hat{\mathbb{I}}\log\mathrm{Tr}(e^{-\beta\hat{H}_S})\big)\big] \\ &\quad = -\beta\Delta Q_E.\end{aligned} \quad (A2)$$

Thus Eq. (A1) can be written as

$$\Delta S - \beta\Delta Q_E = \mathcal{D}\big(\rho^1_S(t)||\eta^{(\beta)}_S\big) - \mathcal{D}\big(\rho^1_S(0)||\eta^{(\beta)}_S\big). \quad (A3)$$

For time $t + dt$, we have

$$\begin{aligned}\Delta S(t+dt) - \beta\Delta Q_E(t+dt) = &\,\mathcal{D}\big(\rho^1_S(t+dt)||\eta^{(\beta)}_S\big) \\ &- \mathcal{D}\big(\rho^1_S(0)||\eta^{(\beta)}_S\big);\end{aligned} \quad (A4)$$

from Eqs. (A3) and (A4) in the limit $dt \to 0$, we obtain

$$\begin{aligned}\dot{S} - \beta\dot{Q}_E &= \lim_{dt\to 0}\big[\mathcal{D}\big(\rho^1_S(t+dt)||\eta^{(\beta)}_S\big) - D\big(\rho^1_S(t)||\eta^{(\beta)}_S\big)\big]/dt \\ &= \frac{d}{dt}\mathcal{D}\big(\rho^1_S(t)||\eta^{(\beta)}_S\big).\end{aligned} \quad (A5)$$

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