

**Performance-tradeoff relation for locating two incoherent optical point sources**Jingjing Shao and Xiao-Ming Lu <sup>\*</sup>*School of Sciences, Hangzhou Dianzi University, Hangzhou 310018, China* (Received 11 February 2022; accepted 19 May 2022; published 9 June 2022; corrected 23 December 2022)

The optimal quantum measurements for estimating individual parameters might be incompatible with each other, so they cannot be jointly performed. The tradeoff between the estimation precisions for different parameters can be characterized by information regret: the difference between the Fisher information and its quantum limit. We show that the information-regret-tradeoff relation can give us not only an intuitive picture of the potential for improving the joint scheme for estimating the centroid and the separation but also some clues to the optimal measurements for the sequential scheme. In particular, we show that, for two incoherent point sources with a very small separation, the optimal measurement for the separation must extract little information about the centroid and vice versa.

DOI: [10.1103/PhysRevA.105.062416](https://doi.org/10.1103/PhysRevA.105.062416)**I. INTRODUCTION**

The resolution power of incoherent imaging plays an important role in astronomy observation and biological imaging. The conventional criterion for resolving two incoherent point sources—the Rayleigh criterion [1]—is not a rigorous approach and cannot be used to deal with the randomness from quantum measurements [2]. A modern method of assessing the resolution power utilizes statistical approaches like parameter estimation and hypothesis testing [3–9]. Combining these statistical approaches with quantum mechanics, Tsang *et al.* [7] recently revealed that the resolution power of incoherent imaging in principle can be significantly improved by optimizing the quantum measurement performed on the far field. Concretely, spatial mode demultiplexing (SPADE) has a much better performance than direct imaging for resolving two incoherent point sources [7]. Since then, many efforts have been devoted to pursuing the superresolution of incoherent imaging, e.g., experimental implementations [10–22] and some theoretic generalizations [23–34] (see Ref. [35] for a recent review).

Remarkably, the SPADE measurement, which is designed for estimating the separation between two point sources, requires prior information about the centroid of two point sources and the alignment of the device with the centroid [7]. The prior information about the centroid can be readily obtained, as direct imaging is accurate at estimating the centroid. Grace *et al.* proposed an adaptive two-stage detection scheme that dynamically allocates the resources between the centroid estimation and the separation estimation and showed that it outperforms direct imaging [36]. de Almeida *et al.* investigated the impact of the misalignment of the SPADE device on separation estimation [37]. These works on the sequential estimation of the centroid and the separation of two incoherent point sources demonstrate the feasibility of SPADE.

The centroid and separation of two incoherent point sources can also be jointly estimated by repeating the same quantum measurement on samples. The performance of a quantum measurement for the joint estimation of multiple parameters can be assessed by the Fisher information matrix. However, present-day theoretic tools for studying the quantum limit of the multiparameter estimation, mainly, the quantum Cramér-Rao bounds [38–45], suffer from the incompatibility issue of quantum measurements [46–52]. Chrostowski *et al.* [53] showed that the optimal measurements for individually estimating the centroid and the separation of two incoherent point sources are incompatible. Parniak *et al.* [14] implemented a joint estimation of the centroid and the separation of two incoherent point sources near the quantum limit by using collective measurements on two photons. What kind of tradeoff relation between the estimation precision of the centroid and that of the separation should be obeyed when taking the optimization over quantum measurements remains open.

Recently, Lu and Wang [46] introduced the concept of information regret to characterize the performance of a quantum measurement for estimating individual parameters and established a tradeoff relation between the information regrets for any two parameters of interest. This information-regret-tradeoff relation (IRTR) supplies us with a useful theoretic tool for investigating the multiparameter estimation problem of locating two incoherent optical point sources. In this work, we will show that the IRTR can give us an intuitive picture of the possible improvements of quantum measurements for not only the joint scheme but also the sequential scheme of estimating the centroid and the separation. Moreover, for the case of maximum incompatibility, the IRTR gives an important clue to the optimal quantum measurement for one individual parameter of interest, provided that plenty of prior information about other parameters is available. Particularly, we will show that, for two incoherent point sources with a very small separation, the optimal measurement for the separation must extract little information about the centroid and vice versa.

<sup>\*</sup>lxm@hdu.edu.cn; <http://xmlu.me>

This paper is organized as follows. In Sec. II, we give a brief introduction of the model for locating two incoherent optical point sources, the method of quantum parameter estimation, and the information-regret-tradeoff relation. In Sec. III, we study the IRTR for the quantum estimation problem underlying the resolution of two incoherent optical point sources. In Sec. IV, we use the IRTR to assess several quantum measurements as well as randomly generated ones for estimating the centroid and the separation of two sources. We summarize our work in Sec. V and give the details of some calculations in the Appendix.

## II. PARAMETER ESTIMATION FOR TWO INCOHERENT POINT SOURCES

### A. Model

Focusing on the spatial resolution, we follow Ref. [7] to consider an imaging system with quasimonochromatic scalar paraxial waves, one spatial dimension, and two equally bright incoherent point sources. Assuming that the sources are weak, the one-photon state of the optical field on the image plane provides the dominant information for locating the positions of the sources. The density operator describing the one-photon state with a diffraction-limited imaging system can be written as [7]

$$\rho = \frac{1}{2}(|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|), \quad (2.1)$$

$$|\psi_j\rangle = \int dx \psi_j(x)|x\rangle, \quad j = 1, 2, \quad (2.2)$$

where  $x$  is the image-plane coordinate,  $|x\rangle$  denotes the image-plane position eigenket for a photon, and  $\psi_j(x)$  represents the image-plane wave function of a photon from the  $j$ th source. Furthermore, we assume that the imaging system is spatially invariant such that

$$\psi_j(x) = \psi(x - X_j), \quad (2.3)$$

where  $\psi(x)$  is the normalized point-spread function of the imaging system and  $X_j$  is the unknown position of each source. The resolutions of the two optical point sources can be treated as a problem of estimating the parameters  $X_1$  and  $X_2$ .

### B. Parameter estimation theory

We now briefly review the quantum multiparameter estimation theory. Suppose that the density operator  $\rho$  depends on an unknown vector parameter  $\theta = (\theta_1, \theta_2, \dots, \theta_n)$ . We denote by  $\hat{\theta}_j$  an estimator for  $\theta_j$ , which represents the data processing and maps the observation data to the estimates. The estimation error can be characterized by the error-covariance matrix  $\mathcal{E}$  defined by

$$\mathcal{E}_{jk} := \mathbb{E}_\theta[(\hat{\theta}_j - \theta_j)(\hat{\theta}_k - \theta_k)], \quad (2.4)$$

where the expectation  $\mathbb{E}_\theta[\cdot]$  is taken with respect to the probability distribution of the observation data. A quantum measurement is described by a positive-operator-valued measure (POVM):  $M = \{M_w \mid M_w \geq 0, \sum_w M_w = \mathbb{1}\}$ , with  $w$  denoting the outcome and  $\mathbb{1}$  denoting the identity operator. The probability of obtaining an outcome  $\omega$  is given by  $p(w) =$

$\text{tr}(M_w \rho)$  according to Born's rule in quantum mechanics. For all unbiased estimators, the error-covariance matrix obeys the Cramér-Rao bound (CRB) [54,55]:

$$\mathcal{E} \geq \nu^{-1} F(M)^{-1}, \quad (2.5)$$

where  $\nu$  is the number of experimental repetitions and  $F(M)$  is the Fisher information matrix (FIM) [56] under a quantum measurement  $M$ . The entries of the FIM are defined as

$$F(M)_{jk} := \sum_w \frac{1}{p(w)} \frac{\partial p(w)}{\partial \theta_j} \frac{\partial p(w)}{\partial \theta_k}. \quad (2.6)$$

Note that Eq. (2.5) is a matrix inequality and should be interpreted in the sense of the Loewer order; that is, for two matrices  $A$  and  $B$ , we say that  $A \geq B$  if  $A - B$  is positive semidefinite. The CRB can be asymptotically attained by the maximum-likelihood estimator, whose probability tends to be normal, with the covariance matrix being  $\nu^{-1} F(M)^{-1}$ . Therefore, the FIM characterizes the performance of a quantum measurement for estimating multiple parameters.

Quantum parameter estimation aims at the optimization of estimation precision over quantum measurements. For any quantum measurement, the FIM is bounded as [57,58]

$$F(M) \leq \mathcal{F}, \quad (2.7)$$

where  $\mathcal{F}$  is the quantum Fisher information matrix (QFIM) [38–40,44,59], defined as

$$\mathcal{F}_{jk} := \text{Re tr}(L_j L_k \rho), \quad (2.8)$$

and  $L_j$ , called the symmetric logarithmic derivative (SLD) operator about  $\theta_j$ , is the bounded Hermitian operator satisfying

$$\frac{\partial \rho}{\partial \theta_j} = \frac{1}{2}(L_j \rho + \rho L_j). \quad (2.9)$$

However, the elements of the FIM in general cannot be maximized simultaneously, which is known as the incompatibility problem of quantum multiparameter estimation [46–48,50,60–65].

### C. Regret of Fisher information

The efficiency of a quantum measurement for multiparameter estimation can be characterized by the information-regret matrix introduced in Ref. [66], that is,

$$R(M) = \mathcal{F} - F(M). \quad (2.10)$$

The information-regret matrix is always positive semidefinite according to Eq. (2.7). We henceforth call the  $j$ th diagonal element of  $R(M)$  the *information regret* for the parameter  $\theta_j$ . It will be convenient to use the normalized-square-root (NSR) information regret, which is defined as

$$\Delta_j := \sqrt{\frac{R_{jj}}{\mathcal{F}_{jj}}} = \sqrt{\frac{\mathcal{F}_{jj} - F_{jj}}{\mathcal{F}_{jj}}}, \quad (2.11)$$

to express the tradeoff relation between the information regrets for different parameters. Note that the NSR information regret  $\Delta_j$  must take values in the interval  $[0,1]$ .

The information regrets for different parameters in general cannot simultaneously be diminished to zero by optimizing

over quantum measurements. They obey an IRTR [46]:

$$\Delta_1^2 + \Delta_2^2 + 2\sqrt{1 - \tilde{c}^2}\Delta_1\Delta_2 \geq \tilde{c}^2, \quad (2.12)$$

where  $\tilde{c}$  is a real coefficient defined as

$$\tilde{c} = \frac{\text{tr}[\sqrt{\rho}[L_1, L_2]\sqrt{\rho}]}{2\sqrt{\mathcal{F}_{11}\mathcal{F}_{22}}}, \quad (2.13)$$

with  $|X| := \sqrt{X^\dagger X}$  for an operator  $X$ . This IRTR was derived by first establishing a correspondence relationship between the information regrets and the state-dependent measurement error defined by Ozawa and then invoking the inequalities that Branciard [67] and Ozawa [68] proved for formulating the measurement uncertainty relations. For pure states, the bound Eq. (2.12) can be saturated by a quantum measurement due to a property of Branciard's inequality [67].

The IRTR Eq. (2.12) still holds when  $\tilde{c}$  is replaced by

$$c := \frac{|\text{tr}([L_1, L_2]\rho)|}{2\sqrt{\mathcal{F}_{11}\mathcal{F}_{22}}}. \quad (2.14)$$

The resultant relation is weaker than Eq. (2.12), as we always have  $\tilde{c} \geq c$  for the same parametric density operator. The difference between the two IRTRs can be understood by considering the situations where collective measurements are permitted on  $n$  independent and identically distributed quantum systems. Denote by  $\tilde{c}_n$  the incompatibility coefficient calculated for the density operator  $\rho^{\otimes n}$  of  $n$  copies of quantum states. Reference [69] showed that

$$\lim_{n \rightarrow \infty} \tilde{c}_n = c. \quad (2.15)$$

Using the concept of hierarchical incompatibility measures discussed in Ref. [69], we can say that  $\tilde{c}_1$  (which we will simply denote by  $\tilde{c}$ ) reflects the incompatibility when only independent measurements on each copy are permitted, while  $c$  reflects the incompatibility when all collective measurements on multiple copies are permitted.

The incompatibility coefficients  $c$  and  $\tilde{c}$  can be connected to another incompatibility measure proposed in Ref. [48], namely,

$$\gamma := \|2i\mathcal{F}^{-1}\mathcal{U}\|_\infty, \quad (2.16)$$

where  $\mathcal{U}_{jk} := -(i/4)\text{tr}(\rho[L_j, L_k])$  and  $\|\cdot\|_\infty$  represents the largest eigenvalue of a matrix. Assuming that the QFIM is diagonal, for two parameter estimation problems, it was shown that [48]

$$\gamma = \sqrt{\frac{\det 2\mathcal{U}}{\det \mathcal{F}}}. \quad (2.17)$$

Comparing Eq. (2.17) with Eqs. (2.14) and (2.13), we have

$$\gamma = c \leq \tilde{c}. \quad (2.18)$$

#### D. Estimation error tradeoff

In terms of the estimation errors, the IRTR implies that [46]

$$\gamma_j + \gamma_k - 2\sqrt{1 - \tilde{c}_{jk}^2}\sqrt{(1 - \gamma_j)(1 - \gamma_k)} \leq 2 - \tilde{c}_{jk}^2, \quad (2.19)$$

where  $\gamma_j := 1/(v\mathcal{E}_{jj}\mathcal{F}_{jj})$ , with  $v$  being the number of experiment repetitions. Note that  $\gamma_j$  is proportional to the inverse of

the estimation error  $\mathcal{E}_{jj}$  and is rescaled to the range [0,1] due to the inequality  $\mathcal{E}_{jj} \geq 1/(v\mathcal{F}_{jj})$ . Particularly, we have

$$\frac{1}{v\mathcal{E}_{jj}\mathcal{F}_{jj}} + \frac{1}{v\mathcal{E}_{kk}\mathcal{F}_{kk}} \leq 1 \quad (2.20)$$

when  $\tilde{c}_{jk} = 1$ . This inequality will be used later in our analysis of the joint estimation of the centroid and the separation of two incoherent point optical sources.

### III. IRTR ANALYSIS FOR TWO INCOHERENT POINT SOURCES

Before using the IRTR to analyze the resolution of two incoherent optical point sources, let us give a brief introduction of the QFIM obtained in Ref. [7]. For the problem of locating two equally bright optical point sources, it is convenient to use the centroid and the separation as the parameters of interest, that is,

$$\theta_1 = \frac{X_1 + X_2}{2}, \quad \theta_2 = X_2 - X_1. \quad (3.1)$$

Assume hereafter that the point-spread function  $\psi(x)$  is real. Reference [7] showed that the QFIM for  $\theta_1$  and  $\theta_2$  is given by

$$\mathcal{F} = \begin{pmatrix} 4\kappa - 4\gamma^2 & 0 \\ 0 & \kappa \end{pmatrix}, \quad (3.2)$$

where  $\kappa$  and  $\gamma$  are real numbers, defined as

$$\kappa := \int_{-\infty}^{\infty} dx \left[ \frac{\partial \psi(x)}{\partial x} \right]^2, \quad (3.3)$$

$$\gamma := \int_{-\infty}^{\infty} dx \frac{\partial \psi(x)}{\partial x} \psi(x - \theta_2). \quad (3.4)$$

#### A. Incompatibility coefficient

Now, we use the IRTR to analyze the estimation of the centroid and the separation of two weak incoherent optical point sources. For real point-spread functions, it is easy to see that  $c = 0$  (see the Appendix for the details). Therefore, if collective measurements on multiple samples are permitted, the regrets of Fisher information for both the centroid and the separation can simultaneously vanish. After some algebra, we get the incompatibility coefficient without considering the collective measurements (see the Appendix for the details):

$$\tilde{c}^2 = \frac{\beta^2}{\kappa(\kappa - \gamma^2)}, \quad (3.5)$$

where  $\kappa$  and  $\gamma$  are given by Eqs. (3.3) and (3.4), respectively, and  $\beta$  is given by

$$\begin{aligned} \beta &:= \int_{-\infty}^{\infty} dx \frac{\partial \psi(x - X_1)}{\partial X_1} \frac{\partial \psi(x - X_2)}{\partial X_2} \\ &= \int_{-\infty}^{\infty} dx \frac{\partial \psi(x)}{\partial x} \frac{\partial \psi(x - \theta_2)}{\partial x}. \end{aligned} \quad (3.6)$$

It is easy to see that the incompatibility coefficient  $\tilde{c}$  vanishes when  $\beta = 0$  is satisfied. From the definition of  $\beta$ , we can see that  $\beta = 0$  is equivalent to the fact that the vectors  $\partial|\psi_1\rangle/\partial X_1$  and  $\partial|\psi_2\rangle/\partial X_2$  are orthogonal. It is known that  $\tilde{c} = 0$  is necessary for the saturation of Eq. (2.7) and thus for

the vanishing of the regret matrix [69,70]; however, it remains open whether  $\tilde{c} = 0$  is also sufficient for the saturation of Eq. (2.7).

The most important situation is the estimation of small separations, for which direct imaging has a poor performance and optimal measurement is necessary. Let us consider the limit of  $\theta_2 \rightarrow 0$ ; that is, the separation between the two point sources is infinitesimal. It can be shown from Eq. (3.5) that

$$\lim_{\theta_2 \rightarrow 0} \tilde{c} = 1, \quad (3.7)$$

as  $\lim_{\theta_2 \rightarrow 0} \beta = \kappa$  and  $\lim_{\theta_2 \rightarrow 0} \gamma = 0$  according to their definitions given in Eqs. (3.3), (3.4), and (3.6). Therefore, the incompatibility coefficient approaches its maximum value as the separation decreases to zero. For  $\tilde{c} = 1$ , the IRTR becomes

$$\Delta_1^2 + \Delta_2^2 \geq 1, \quad (3.8)$$

implying that zero information regret about one parameter must lead to the maximal information regret about the other one. We can also use Eq. (2.20) to get the tradeoff relation for the estimation errors. As a result,

$$\frac{1}{4\nu\kappa\mathcal{E}_{11}} + \frac{1}{\nu\kappa\mathcal{E}_{22}} \leq 1 \quad (3.9)$$

must hold in the limiting case of  $\theta_2 \rightarrow 0$ . When the incompatibility coefficient attains its maximum value, the resultant IRTR (3.8) gives us an important clue to optimize quantum measurements for a parameter of interest. In such a case, a quantum measurement that attains  $\Delta_2 = 0$  must satisfy  $\Delta_1 = 1$ , which means that the classical Fisher information about  $\theta_1$  must vanish. Since  $F_{11} = \sum_{\omega} [\partial p(\omega)/\partial \theta_1]^2/p(\omega)$ ,  $F_{11}$  vanishes if and only if

$$\frac{\partial p(\omega)}{\partial \theta_1} = 0 \quad \forall \omega \text{ such that } p(\omega) \neq 0, \quad (3.10)$$

which in turn is equivalent to

$$\text{tr}\left(M_{\omega} \frac{\partial \rho}{\partial \theta_1}\right) = 0 \quad \forall \omega \text{ such that } \text{tr}(M_{\omega}\rho) \neq 0. \quad (3.11)$$

This condition is necessary for attaining the maximum Fisher information about the separation  $\theta_2$  when the separation between two coherent sources is very small. Concretely, in the limiting case of  $\theta_2 \rightarrow 0$ , if we take  $\{|\omega\rangle\}$  as the measurement basis, i.e.,  $M_{\omega} = |\omega\rangle\langle\omega|$ , the above condition is reduced to

$$\text{Re} \langle \omega | \Psi'_{\theta_1} \rangle \langle \Psi_{\theta_1} | \omega \rangle = 0, \quad (3.12)$$

where  $|\Psi_{\theta_1}\rangle = \int \psi(x - \theta_1)|x\rangle dx$  and  $|\Psi'_{\theta_1}\rangle = \int \psi'(x - \theta_1)|x\rangle dx$ , with  $\psi'(x)$  being the first-order derivative of the point-spread function  $\psi(x)$ . In this way, it seems natural to include the fundamental mode  $|\Psi_{\theta_1}\rangle$  and the first derivative mode  $|\Psi'_{\theta_1}\rangle$  in the optimal measurement basis for estimating very small separations. Such a scheme was already used in Ref. [11]. On the other hand, when the separation is very small, the optimal measurement for estimating the centroid must extract little information about the separation. This is consistent with the performance of direct imaging.

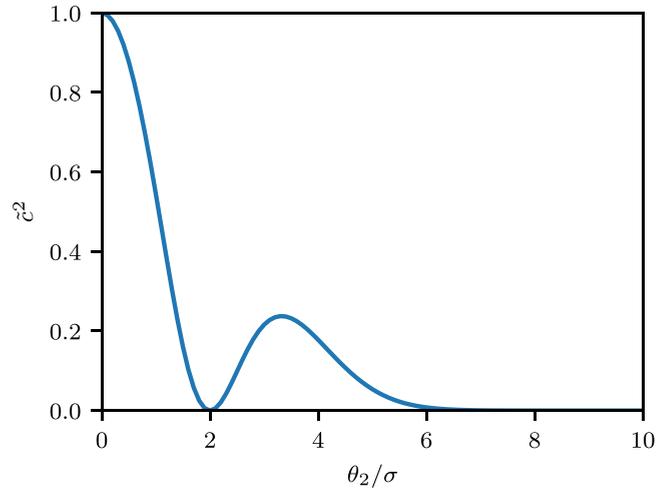


FIG. 1. The incompatibility coefficient versus the separation for a Gaussian point-spread function.

### B. Gaussian point-spread function

We now take the Gaussian point-spread function as a typical example, for which we set

$$\psi(x) = (2\pi\sigma^2)^{-1/4} \exp\left(-\frac{x^2}{4\sigma^2}\right), \quad (3.13)$$

with  $\sigma$  being a characteristic length of the point-spread function. It then follows from Eq. (3.5) that (see the Appendix for the details)

$$\tilde{c}^2 = \left(1 - \frac{\theta_2^2}{4\sigma^2}\right)^2 \left[\exp\left(\frac{\theta_2^2}{4\sigma^2}\right) - \frac{\theta_2^2}{4\sigma^2}\right]^{-1}. \quad (3.14)$$

Figure 1 plots the incompatibility coefficient as a function of  $\theta_2/\sigma$ . It can be seen that  $\tilde{c}$  progressively attains its maximum value of 1 as  $\theta_2 \rightarrow 0$ .

Interestingly, there are two cases where the incompatibility coefficient  $\tilde{c}$  vanishes. In the first case, at large enough separations, e.g.,  $\theta_2 > 6\sigma$  in Fig. 1,  $\tilde{c}$  approximately vanishes. This can be understood by noticing that the one-photon wave functions corresponding to the two point sources are far away from each other, so that the locations of the two point sources can be estimated independently by measurements in different regions in the image plane. Consequently, the centroid and the separation can be jointly estimated with optimal precision. The second case with a zero incompatibility coefficient is at the specific value  $\theta_2 = 2\sigma$ . Whether we can find an optimal measurement that obtains the maximum Fisher information about  $\theta_1$  and  $\theta_2$  simultaneously when the incompatibility coefficient vanishes is still open. Unlike the case of large enough separation, when  $\theta_2 = 2\sigma$ , the derivatives of the state vectors,  $\partial|\psi_1\rangle/\partial X_1$  and  $\partial|\psi_2\rangle/\partial X_2$ , are orthogonal, but  $|\psi_1\rangle$  and  $|\psi_2\rangle$  are not orthogonal. Thus, the measurement and estimation of  $X_1$  and  $X_2$  will interfere with each other. It is nontrivial to find a measurement that is simultaneously optimal for estimating both the centroid and the separation in such a case. The vanishing incompatibility coefficient here makes it promising.

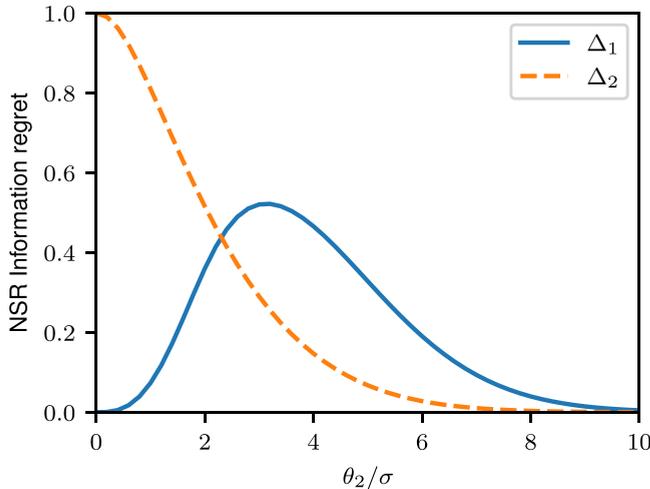


FIG. 2. The NSR information regret versus the separation for direct imaging.

#### IV. MEASUREMENTS FOR JOINT ESTIMATION

Now, we study the performance of some specific quantum measurements for estimating the centroid and the separation of two incoherent point sources with the help of the IRTR.

##### A. Direct imaging

Let us first consider direct imaging, which measures the intensity distribution of the optical field on the image plane. The probability density function for observing a photon at the position  $x = \omega$  is given by

$$p(\omega) = \frac{1}{2}[|\psi(\omega - X_1)|^2 + |\psi(\omega - X_2)|^2], \quad (4.1)$$

where  $X_1$  and  $X_2$  are related to the centroid  $\theta_1$  and the separation  $\theta_2$  as  $X_1 = \theta_1 - \theta_2/2$  and  $X_2 = \theta_1 + \theta_2/2$ . For Gaussian point-spread functions, we numerically compute the FIM about the parameters  $\theta_1$  and  $\theta_2$  via Eq. (2.6) and then compute their NSR information regrets via Eq. (2.11). Figure 2 plots the NSR information regrets of direct imaging for estimating the centroid and the separation of two incoherent optical point sources. It can be seen that the information regret for estimating the separation of the two sources is large in the sub-Rayleigh region, which was shown in a previous work [4] and was dubbed Rayleigh's curse [7,31]. Note that for each parameter, its information regret can be reduced to zero. So Fig. 2 in fact reflects the possible improvements implied by the quantum CRB. However, reducing the information regret for one parameter may be accompanied by the inevitable increase of the information regret for the other parameter, which we need the IRTR to deal with.

Using the IRTR, we can obtain an intuitive picture of the possible improvements of the quantum measurement for the joint estimation of the centroid and the separation of two incoherent point sources. It can be seen from Fig. 3 that, for small separations (e.g., see the first three panels in the top row of Fig. 3), although direct imaging has a poor performance for estimating the separation, it already attains or gets close to the IRTR. Therefore, in such cases, any substantial decrease in the information regret for estimating the separation must be obtained at the cost of increasing the information regret for the centroid, in comparison with direct imaging. When the separation of two point sources is close or equal to  $2\sigma$  (see the fourth panel in the top row of Fig. 3), the incompatibility coefficient  $\tilde{c}$  is very small, indicating that it is possible to get a better joint estimation of the centroid and the separation than with direct imaging. When the separation is large enough (see

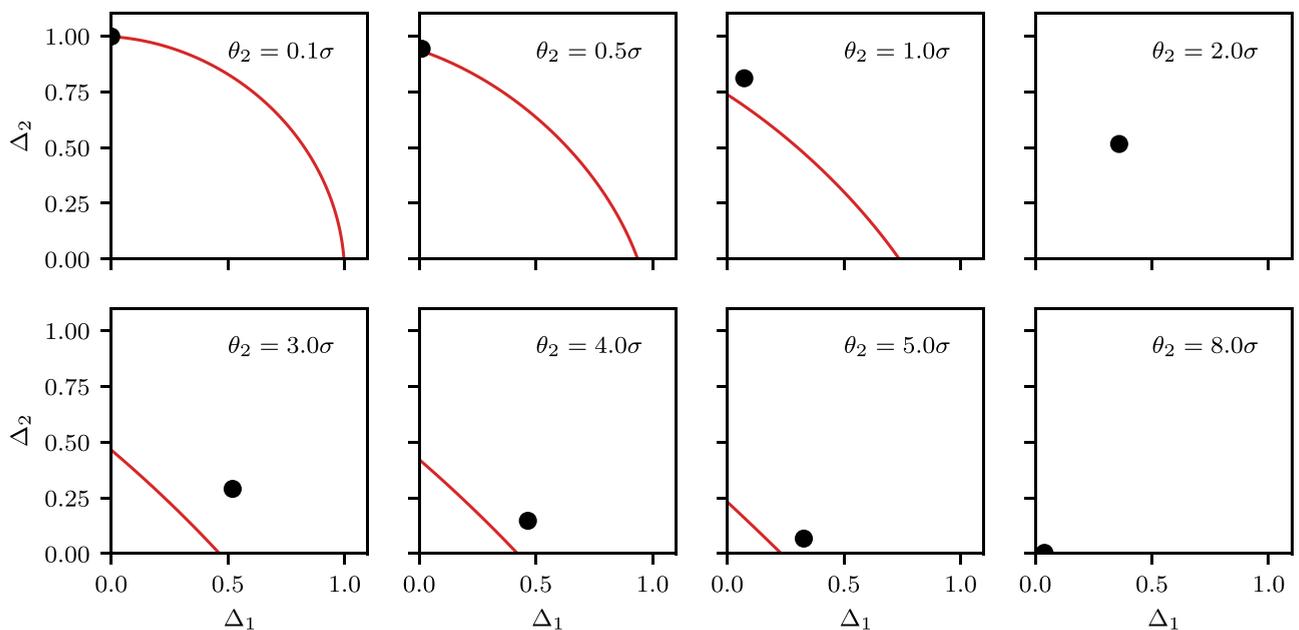


FIG. 3. The IRTR and the information regrets of direct imaging for different separations. The black circles represent the combination  $(\Delta_1, \Delta_2)$  for direct imaging. The regions below the IRTR (the red solid curve) are unattainable by quantum measurements. For  $\theta_2 = 2\sigma$  (the fourth panel in the top row),  $\tilde{c}$  equals zero, and thus, there is no IRTR. For  $\theta_2 = 8\sigma$  (the fourth panel in the bottom row),  $\tilde{c}$  is so small that the IRTR is invisible.

the fourth panel in the bottom row in Fig. 3), the information regrets for both the centroid and the separation are small.

### B. SPADE

We now consider the SPADE measurement introduced in Ref. [7], which takes the Hermite-Gaussian modes as the basis to decompose and measure the image-plane optical field. We set the origin of the coordinate system on the image plane to the origin of the Hermite-Gaussian modes. The wave function of the  $q$ th Hermite-Gaussian mode with the same characteristic length  $\sigma$  as the Gaussian point-spread function is given by

$$\phi_q(x) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \frac{1}{\sqrt{2^q q!}} H_q\left(\frac{x}{\sqrt{2}\sigma}\right) \exp\left(-\frac{x^2}{4\sigma^2}\right), \quad (4.2)$$

where  $H_q$  is the Hermite polynomial. For the Gaussian point-spread function, the probability of obtaining the outcome  $q$  corresponding to the  $q$ th Hermite-Gaussian mode state  $|\phi_q\rangle$  is given by

$$\begin{aligned} p(q) &= \frac{1}{2} |\langle\phi_q|\psi_1\rangle|^2 + \frac{1}{2} |\langle\phi_q|\psi_2\rangle|^2 \\ &= \frac{1}{2q!} [\alpha_1^{2q} \exp(-\alpha_1^2) + \alpha_2^{2q} \exp(-\alpha_2^2)], \end{aligned} \quad (4.3)$$

with  $\alpha_1 = X_1/2\sigma = (\theta_1 - \theta_2/2)/2\sigma$  and  $\alpha_2 = X_2/2\sigma = (\theta_1 + \theta_2/2)/2\sigma$ . In Ref. [7], assuming that the SPADE device is aligned with the centroid of two incoherent point sources (i.e.,  $\theta_1 = 0$ ) such that  $\alpha_1 = -\alpha_2$  in Eq. (4.3), it was shown that the SPADE is optimal for estimating the separation of the two point sources in the sense that the information regret for  $\theta_2$  vanishes. This optimal strategy requires the prior information or preliminary estimation of the centroid of two incoherent point sources.

In fact, the SPADE also extracts Fisher information about the centroid from the one-photon state when there is a misalignment between the origin of the SPADE device and the centroid of the two incoherent point sources. In Fig. 4, we numerically compute and plot the information regrets of the SPADE for estimating  $\theta_1$  and  $\theta_2$  as well as the IRTR for a small separation  $\theta_2 = 0.1\sigma$  and different values of the centroid  $\theta_1$ . We can see that, for this small separation, the information regret for  $\theta_2$  increases but the information regret for  $\theta_1$  decreases as the misalignment increases. This means that, for two closely placed incoherent point sources, the SPADE shows good performance for estimating the separation when it is aligned with the centroid of the two sources and good performance for estimating the centroid when the misalignment becomes large. The latter can be understood by considering the limit case in which the origin of the Hermite-Gaussian modes is distant from the centroid of the two close point sources, i.e.,  $\theta_1 \gg \theta_2$ . In such a case, the probability of the SPADE measurement outcome is approximately given by

$$p(q) \approx \frac{1}{q!} \left(\frac{\theta_1}{2\sigma}\right)^{2q} \exp\left(-\frac{\theta_1^2}{4\sigma^2}\right), \quad (4.4)$$

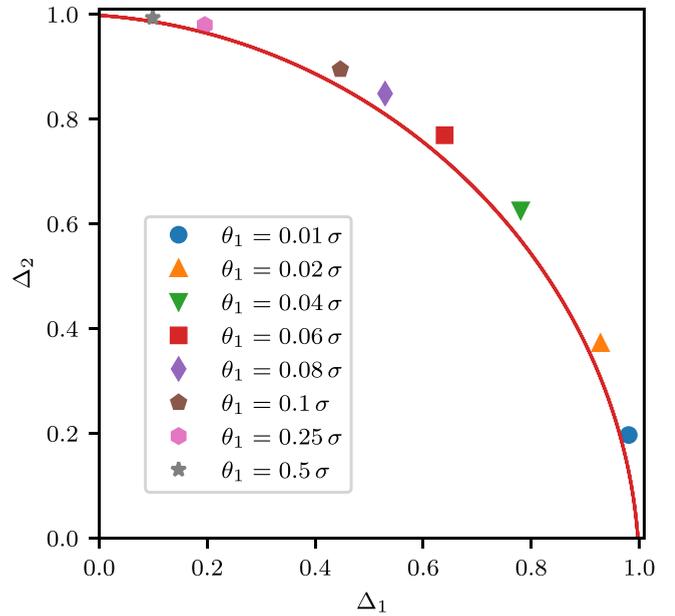


FIG. 4. The IRTR (red solid curve) and the information regrets of the SPADE measurement for estimating the centroid and separation of two incoherent point sources. Here, the information regrets are numerically computed at  $\theta_2 = 0.1\sigma$ .

meaning that the effect of the SPADE for two close incoherent point sources is like that for one point source with two times the intensity. The Fisher information for the above probability distribution for the parameter  $\theta_1$  is  $\sigma^{-2}$ , which equals its quantum limit. Therefore, the SPADE can also be used to estimate the centroid for two closely placed incoherent point sources.

### C. Random POVMs on the minimal subspace

Note that the Hilbert space associated with the one-photon state given by Eq. (2.1) is infinite-dimensional. Both the (continuum) direct imaging and the (complete) SPADE have an infinite number of possible outcomes. Nevertheless, the FIM and the CRB are relevant to only the minimal subspace  $\mathcal{S}_\theta$  supporting the density operators and their derivatives. This subspace  $\mathcal{S}_\theta$  is spanned by the state vectors  $|\psi_j\rangle$  and their derivatives  $\partial|\psi_j\rangle/\partial X_j$  for  $j = 1, 2$ . According to Naimark's dilation theorem [71], the optimization over all quantum measurements on the underlying Hilbert space can be equivalently done by optimizing over the POVMs on the four-dimensional subspace  $\mathcal{S}_\theta$ .

The subscript of  $\mathcal{S}_\theta$  indicates that this subspace depends on the value of the vector parameter  $\theta = (\theta_1, \theta_2)$ , which is to be estimated. The POVMs on  $\mathcal{S}_\theta$  also in general depend on  $\theta$ . This fact does not make the study of the POVMs on  $\mathcal{S}_\theta$  useless for the following reasons. First, recall that the (quantum) Fisher information is defined at a fiducial parameter point and measures the distinguishability of the parameters for the quantum states in the first-order infinitesimal neighborhood of the fiducial state [57,72]. The distinguishability characterized by the FIM under a  $\theta$ -dependent POVM could be attained with prior information about the fiducial parameter point, which may be obtained by a ‘‘preestimation’’ stage. Second, it is

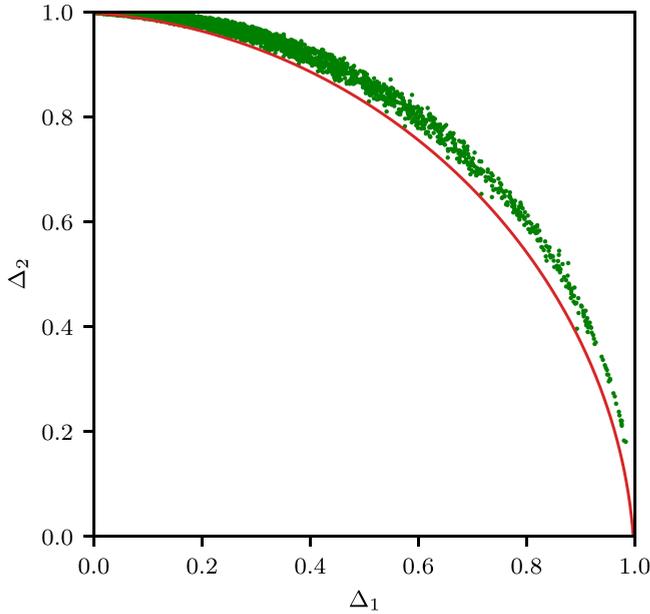


FIG. 5. The IRTR (red solid curve) and the information regrets of random orthogonal projection measurements on the relevant subspace. We generated 10 000 random real-valued orthonormal bases with the Haar measure. Here,  $\theta_2 = 0.1\sigma$ .

possible to implement a  $\theta$ -dependent POVM on  $\mathcal{S}_\theta$  through a  $\theta$ -independent POVM on the whole Hilbert space, although a systematic method for such a task is still lacking.

We now investigate the performance of randomly generated POVMs on  $\mathcal{S}_\theta$  in extracting the Fisher information for both the centroid and the separation of two incoherent point sources. For the optimization of the Fisher information, it is sufficient to consider rank-1 POVMs because all POVMs can be obtained by coarse graining of rank-1 POVMs and coarse graining cannot increase the Fisher information. For simplicity, we consider only the orthogonal projection measurement on  $\mathcal{S}_\theta$  in this work and leave the optimization over general rank-1 POVMs to future work. Furthermore, since the point-spread function is assumed to be real valued in this work, we consider only the spatial mode whose wave functions are real.

The random orthogonal projection measurement on  $\mathcal{S}_\theta$  is produced as follows. First, we apply the Gram-Schmidt process on the set  $\{\psi_1(x), \psi_2(x), \frac{\partial\psi_1(x)}{\partial X_1}, \frac{\partial\psi_2(x)}{\partial X_2}\}$  to get an orthonormal basis  $\{\phi_j(x) \mid j = 1, 2, 3, 4\}$  for  $\mathcal{S}_\theta$ . Note that  $\phi_j(x)$  are all real valued, as the wave functions  $\psi_1(x)$  and  $\psi_2(x)$  are real valued. Then, we generate 10 000 random four-dimensional orthogonal matrices [73] with the Haar measure [74].

Figure 5 plots the NSR information regrets as well as the IRTR for the 10 000 random orthogonal projection measurements mentioned above, where the separation of two incoherent point sources is chosen to be  $\theta_2 = 0.1\sigma$ . From the distributions of the information regrets  $\Delta_1$  and  $\Delta_2$ , we can see that the orthogonal projection measurement with a real-valued function in the subspace  $\mathcal{S}_\theta$  nearly attains the IRTR.

## V. CONCLUSION

In this work, we have used the IRTR to investigate the estimation of the centroid and the separation of two incoherent weak optical point sources. We have obtained an analytic expression of the incompatibility coefficients, with which we can paint an intuitive picture of the possible improvement of a measurement for estimating the centroid and the separation. We demonstrated that when the incompatibility coefficient is close or equal to its maximum value, corresponding to very small separations, the optimal measurement for the separation must extract little information about the centroid and vice versa.

## ACKNOWLEDGMENTS

This work is supported by the National Natural Science Foundation of China (Grants No. 61871162, No. 11805048, and No. 11935012).

## APPENDIX: CALCULATION OF THE INCOMPATIBILITY MEASURE

Here, we give detailed derivations of Eqs. (3.5) and (3.14) for the incompatibility coefficient  $\tilde{c}$ . According to the definition of  $\tilde{c}$ , we need the SLD operators  $L_1$  and  $L_2$  for  $\theta_1$  and  $\theta_2$ . Following Ref. [7], the SLD operators can be represented as four-dimensional matrices with an orthonormal basis for the subspace that support the density operators and their derivatives with respect to  $\theta_1$  and  $\theta_2$ . A specific orthonormal basis is given by [7]

$$|e_1\rangle = \frac{1}{\sqrt{2(1-\delta)}}(|\psi_1\rangle - |\psi_2\rangle), \quad (\text{A1})$$

$$|e_2\rangle = \frac{1}{\sqrt{2(1+\delta)}}(|\psi_1\rangle + |\psi_2\rangle), \quad (\text{A2})$$

$$|e_3\rangle = \frac{1}{\eta_3} \left[ \frac{1}{\sqrt{2}} \left( \frac{\partial|\psi_1\rangle}{\partial X_1} + \frac{\partial|\psi_2\rangle}{\partial X_2} \right) - \frac{\gamma}{\sqrt{1-\delta}} |e_1\rangle \right], \quad (\text{A3})$$

$$|e_4\rangle = \frac{1}{\eta_4} \left[ \frac{1}{\sqrt{2}} \left( \frac{\partial|\psi_1\rangle}{\partial X_1} - \frac{\partial|\psi_2\rangle}{\partial X_2} \right) + \frac{\gamma}{\sqrt{1+\delta}} |e_2\rangle \right], \quad (\text{A4})$$

where  $\kappa$ ,  $\gamma$ , and  $\beta$  are given by Eqs. (3.3), (3.4), and (3.6), respectively; other coefficients are defined as

$$\delta := \int dx \psi(x - X_1) \psi(x - X_2), \quad (\text{A5})$$

$$\eta_3 := \sqrt{\kappa + \beta - \frac{\gamma^2}{1-\delta}}, \quad (\text{A6})$$

$$\eta_4 := \sqrt{\kappa - \beta - \frac{\gamma^2}{1+\delta}}. \quad (\text{A7})$$

With this basis, the density matrix for the image-plane one-photon state is given by

$$\rho = \begin{pmatrix} \frac{1-\delta}{2} & 0 & 0 & 0 \\ 0 & \frac{1+\delta}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (\text{A8})$$

The SLD operators with respect to the centroid  $\theta_1$  and separation  $\theta_2$  are represented by

$$L_1 = \begin{pmatrix} 0 & \frac{2\gamma\delta}{\sqrt{1-\delta^2}} & 0 & \frac{2\eta_4}{\sqrt{1-\delta}} \\ \frac{2\gamma\delta}{\sqrt{1-\delta^2}} & 0 & \frac{2\eta_3}{\sqrt{1+\delta}} & 0 \\ 0 & \frac{2\eta_3}{\sqrt{1+\delta}} & 0 & 0 \\ \frac{2\eta_4}{\sqrt{1-\delta}} & 0 & 0 & 0 \end{pmatrix}, \quad (\text{A9})$$

$$L_2 = \begin{pmatrix} \frac{-\gamma}{1-\delta} & 0 & \frac{-\eta_3}{\sqrt{1-\delta}} & 0 \\ 0 & \frac{\gamma}{1+\delta} & 0 & \frac{-\eta_4}{\sqrt{1+\delta}} \\ \frac{-\eta_3}{\sqrt{1-\delta}} & 0 & 0 & 0 \\ 0 & \frac{-\eta_4}{\sqrt{1+\delta}} & 0 & 0 \end{pmatrix}. \quad (\text{A10})$$

Because the matrices for  $\rho$ ,  $L_1$ , and  $L_2$  are all real valued, it is easy to see that

$$|\text{tr}([L_1, L_2]\rho)| = |\text{Im tr}(L_1 L_2 \rho)| = 0, \quad (\text{A11})$$

implying that the quantity  $c$  defined in Eq. (2.14) vanishes. By calculating the eigenvalues of  $\sqrt{\rho}[L_1, L_2]\sqrt{\rho}$ , it can be shown that

$$\text{tr}|\sqrt{\rho}[L_1, L_2]\sqrt{\rho}| = 4|\beta|, \quad (\text{A12})$$

which implies Eq. (3.5). For the Gaussian point-spread function given by Eq. (3.13), it can be shown that

$$\kappa = \frac{1}{4\sigma^2}, \quad \gamma = -\frac{\theta_2}{4\sigma^2} \exp\left(-\frac{\theta_2^2}{8\sigma^2}\right), \quad (\text{A13})$$

and

$$\beta = -\frac{\theta_2^2 - 4\sigma^2}{16\sigma^4} \exp\left(-\frac{\theta_2^2}{8\sigma^2}\right). \quad (\text{A14})$$

Substituting the above expressions for  $\kappa$ ,  $\gamma$ , and  $\beta$  into Eq. (3.5), we get Eq. (3.14).

- 
- [1] L. Rayleigh, XXXI. Investigations in optics, with special reference to the spectroscope, *Philos. Mag. Ser. 5* **8**, 261 (1879).
- [2] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics* (Addison-Wesley, Reading, MA, 1963), Vol. 1, Sec. 30-4.
- [3] J. L. Harris, Resolving power and decision theory, *J. Opt. Soc. Am.* **54**, 606 (1964).
- [4] S. Ram, E. S. Ward, and R. J. Ober, Beyond Rayleigh's criterion: A resolution measure with application to single-molecule microscopy, *Proc. Natl. Acad. Sci. USA* **103**, 4457 (2006).
- [5] J. Chao, E. S. Ward, and R. J. Ober, Fisher information theory for parameter estimation in single molecule microscopy: Tutorial, *J. Opt. Soc. Am. A* **33**, B36 (2016).
- [6] A. J. den Dekker and A. van den Bos, Resolution: A survey, *J. Opt. Soc. Am. A* **14**, 547 (1997).
- [7] M. Tsang, R. Nair, and X.-M. Lu, Quantum Theory of Super-resolution for Two Incoherent Optical Point Sources, *Phys. Rev. X* **6**, 031033 (2016).
- [8] J. S. Sidhu and P. Kok, Quantum metrology of spatial deformation using arrays of classical and quantum light emitters, *Phys. Rev. A* **95**, 063829 (2017).
- [9] M. Tsang, Conservative classical and quantum resolution limits for incoherent imaging, *J. Mod. Opt.* **65**, 1385 (2018).
- [10] W.-K. Tham, H. Ferretti, and A. M. Steinberg, Beating Rayleigh's Curse by Imaging Using Phase Information, *Phys. Rev. Lett.* **118**, 070801 (2017).
- [11] M. Paúr, B. Stoklasa, Z. Hradil, L. L. Sánchez-Soto, and J. Rehacek, Achieving the ultimate optical resolution, *Optica* **3**, 1144 (2016).
- [12] F. Yang, A. Taschilina, E. S. Moiseev, C. Simon, and A. I. Lvovsky, Far-field linear optical superresolution via heterodyne detection in a higher-order local oscillator mode, *Optica* **3**, 1148 (2016).
- [13] Z. S. Tang, K. Durak, and A. Ling, Fault-tolerant and finite-error localization for point emitters within the diffraction limit, *Opt. Express* **24**, 22004 (2016).
- [14] M. Parniak, S. Borówka, K. Boroszko, W. Wasilewski, K. Banaszek, and R. Demkowicz-Dobrzański, Beating the Rayleigh Limit Using Two-Photon Interference, *Phys. Rev. Lett.* **121**, 250503 (2018).
- [15] M. Paúr, B. Stoklasa, J. Grover, A. Krzic, L. L. Sánchez-Soto, Z. Hradil, and J. Reháček, Tempering Rayleigh's curse with PSF shaping, *Optica* **5**, 1177 (2018).
- [16] Y. Zhou, J. Yang, J. D. Hassett, S. M. H. Rafsanjani, M. Mirhosseini, A. N. Vamivakas, A. N. Jordan, Z. Shi, and R. W. Boyd, Quantum-limited estimation of the axial separation of two incoherent point sources, *Optica* **6**, 534 (2019).
- [17] M. Paúr, B. Stoklasa, D. Koutný, J. Reháček, Z. Hradil, J. Grover, A. Krzic, and L. L. Sánchez-Soto, Reading out Fisher information from the zeros of the point spread function, *Opt. Lett.* **44**, 3114 (2019).
- [18] P. Boucher, C. Fabre, G. Labroille, and N. Treps, Spatial optical mode demultiplexing as a practical tool for optimal transverse distance estimation, *Optica* **7**, 1621 (2020).
- [19] S. A. Wadood, K. Liang, Y. Zhou, J. Yang, M. A. Alonso, X.-F. Qian, T. Malhotra, S. M. H. Rafsanjani, A. N. Jordan, R. W. Boyd, and A. N. Vamivakas, Experimental demonstration of superresolution of partially coherent light sources using parity sorting, *Opt. Express* **29**, 22034 (2021).
- [20] W. Larson and B. E. A. Saleh, Two-lens anisotropic image-inversion system for interferometric information processing, *Opt. Express* **29**, 15403 (2021).
- [21] K. Santra, V. Nguyen, E. A. Smith, J. W. Petrich, and X. Song, Localization of nonblinking point sources using higher-order-mode detection and optical heterodyning: Developing a strategy for extending the scope of molecular, super-resolution imaging, *J. Phys. Chem. B* **125**, 3092 (2021).
- [22] A. A. Pushkina, G. Maltese, J. I. Costa-Filho, P. Patel, and A. I. Lvovsky, Superresolution Linear Optical Imaging in the Far Field, *Phys. Rev. Lett.* **127**, 253602 (2021).
- [23] R. Nair and M. Tsang, Interferometric superlocalization of two incoherent optical point sources, *Opt. Express* **24**, 3684 (2016).
- [24] R. Nair and M. Tsang, Far-Field Superresolution of Thermal Electromagnetic Sources at the Quantum Limit, *Phys. Rev. Lett.* **117**, 190801 (2016).

- [25] M. Tsang, Subdiffraction incoherent optical imaging via spatial-mode demultiplexing, *New J. Phys.* **19**, 023054 (2017).
- [26] C. Lupo and S. Pirandola, Ultimate Precision Bound of Quantum and Subwavelength Imaging, *Phys. Rev. Lett.* **117**, 190802 (2016).
- [27] J. Rehacek, M. Paúr, B. Stoklasa, Z. Hradil, and L. L. Sánchez-Soto, Optimal measurements for resolution beyond the Rayleigh limit, *Opt. Lett.* **42**, 231 (2017).
- [28] X.-M. Lu, H. Krovi, R. Nair, S. Guha, and J. H. Shapiro, Quantum-optimal detection of one-versus-two incoherent optical sources with arbitrary separation, *npj Quantum Inf.* **4**, 64 (2018).
- [29] M. Tsang, Quantum limit to subdiffraction incoherent optical imaging, *Phys. Rev. A* **99**, 012305 (2019).
- [30] S. Zhou and L. Jiang, Modern description of Rayleigh's criterion, *Phys. Rev. A* **99**, 013808 (2019).
- [31] L. Peng and X.-M. Lu, Generalization of Rayleigh's criterion on parameter estimation with incoherent sources, *Phys. Rev. A* **103**, 042601 (2021).
- [32] B. Wang, L. Xu, J. Li, and L. Zhang, Quantum-limited localization and resolution in three dimensions, *Photonics Res.* **9**, 1522 (2021).
- [33] Y. L. Len, C. Datta, M. Parniak, and K. Banaszek, Resolution limits of spatial mode demultiplexing with noisy detection, *Int. J. Quantum Inf.* **18**, 1941015 (2020).
- [34] C. Datta, M. Jarzyna, Y. L. Len, K. Łukanowski, J. Kołodyński, and K. Banaszek, Sub-Rayleigh resolution of two incoherent sources by array homodyning, *Phys. Rev. A* **102**, 063526 (2020).
- [35] M. Tsang, Resolving starlight: A quantum perspective, *Contemp. Phys.* **60**, 279 (2019).
- [36] M. R. Grace, Z. Dutton, A. Ashok, and S. Guha, Approaching quantum-limited imaging resolution without prior knowledge of the object location, *J. Opt. Soc. Am. A* **37**, 1288 (2020).
- [37] J. O. de Almeida, J. Kołodyński, C. Hirche, M. Lewenstein, and M. Skotiniotis, Discrimination and estimation of incoherent sources under misalignment, *Phys. Rev. A* **103**, 022406 (2021).
- [38] C. Helstrom, Minimum mean-squared error of estimates in quantum statistics, *Phys. Lett. A* **25**, 101 (1967).
- [39] C. Helstrom, The minimum variance of estimates in quantum signal detection, *IEEE Trans. Inf. Theory* **14**, 234 (1968).
- [40] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).
- [41] H. Yuen and M. Lax, Multiple-parameter quantum estimation and measurement of nonselfadjoint observables, *IEEE Trans. Inf. Theory* **19**, 740 (1973).
- [42] A. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory*, 2nd ed. (Edizioni della Normale, Pisa, 2011).
- [43] X.-M. Lu, Z. Ma, and C. Zhang, Generalized-mean Cramér-Rao bounds for multiparameter quantum metrology, *Phys. Rev. A* **101**, 022303 (2020).
- [44] J. Liu, H. Yuan, X.-M. Lu, and X. Wang, Quantum Fisher information matrix and multiparameter estimation, *J. Phys. A* **53**, 023001 (2020).
- [45] E. Polino, M. Valeri, N. Spagnolo, and F. Sciarrino, Photonic quantum metrology, *AVS Quantum Sci.* **2**, 024703 (2020).
- [46] X.-M. Lu and X. Wang, Incorporating Heisenberg's Uncertainty Principle into Quantum Multiparameter Estimation, *Phys. Rev. Lett.* **126**, 120503 (2021).
- [47] S. Ragy, M. Jarzyna, and R. Demkowicz-Dobrzański, Compatibility in multiparameter quantum metrology, *Phys. Rev. A* **94**, 052108 (2016).
- [48] A. Carollo, B. Spagnolo, A. A. Dubkov, and D. Valenti, On quantumness in multi-parameter quantum estimation, *J. Stat. Mech.* (2019) 094010.
- [49] J. Suzuki, Explicit formula for the Holevo bound for two-parameter qubit-state estimation problem, *J. Math. Phys.* **57**, 042201 (2016).
- [50] I. Kull, P. A. Guérin, and F. Verstraete, Uncertainty and trade-offs in quantum multiparameter estimation, *J. Phys. A* **53**, 244001 (2020).
- [51] J. S. Sidhu and P. Kok, Geometric perspective on quantum parameter estimation, *AVS Quantum Sci.* **2**, 014701 (2020).
- [52] A. Candeloro, M. G. A. Paris, and M. G. Genoni, On the properties of the asymptotic incompatibility measure in multiparameter quantum estimation, *J. Phys. A* **54**, 485301 (2021).
- [53] A. Chrostowski, R. Demkowicz-Dobrzański, M. Jarzyna, and K. Banaszek, On super-resolution imaging as a multiparameter estimation problem, *Int. J. Quantum Inf.* **15**, 1740005 (2017).
- [54] H. Cramér, *Mathematical Methods of Statistics* (Princeton University Press, Princeton, NJ, 1946).
- [55] C. R. Rao, Information and the accuracy attainable in the estimation of statistical parameters, *Bull. Calcutta Math. Soc.* **37**, 81 (1945).
- [56] R. A. Fisher, On the mathematical foundations of theoretical statistics, *Philos. Trans. R. Soc. London, Ser. A* **222**, 309 (1922).
- [57] S. L. Braunstein and C. M. Caves, Statistical Distance and the Geometry of Quantum States, *Phys. Rev. Lett.* **72**, 3439 (1994).
- [58] F. Hiai and D. Petz, *Introduction to Matrix Analysis and Applications*, Universitext (Springer, Cham, 2014).
- [59] J. Liu, X.-X. Jing, W. Zhong, and X.-G. Wang, Quantum Fisher information for density matrices with arbitrary ranks, *Commun. Theor. Phys.* **61**, 45 (2014).
- [60] P. J. D. Crowley, A. Datta, M. Barbieri, and I. A. Walmsley, Tradeoff in simultaneous quantum-limited phase and loss estimation in interferometry, *Phys. Rev. A* **89**, 023845 (2014).
- [61] M. D. Vidrighin, G. Donati, M. G. Genoni, X.-M. Jin, W. S. Kolthammer, M. Kim, A. Datta, M. Barbieri, and I. A. Walmsley, Joint estimation of phase and phase diffusion for quantum metrology, *Nat. Commun.* **5**, 3532 (2014).
- [62] M. Szczykulska, T. Baumgratz, and A. Datta, Multi-parameter quantum metrology, *Adv. Phys.: X* **1**, 621 (2016).
- [63] R. Demkowicz-Dobrzański, W. Górecki, and M. Guţă, Multiparameter estimation beyond quantum Fisher information, *J. Phys. A* **53**, 363001 (2020).
- [64] J. S. Sidhu, Y. Ouyang, E. T. Campbell, and P. Kok, Tight Bounds on the Simultaneous Estimation of Incompatible Parameters, *Phys. Rev. X* **11**, 011028 (2021).
- [65] J. Suzuki, Information geometrical characterization of quantum statistical models in quantum estimation theory, *Entropy* **21**, 703 (2019).
- [66] Y.-N. Lu, Y.-R. Zhang, G.-Q. Liu, F. Nori, H. Fan, and X.-Y. Pan, Observing Information Backflow from Controllable Non-Markovian Multichannels in Diamond, *Phys. Rev. Lett.* **124**, 210502 (2020).

- [67] C. Branciard, Error-tradeoff and error-disturbance relations for incompatible quantum measurements, *Proc. Natl. Acad. Sci. USA* **110**, 6742 (2013).
- [68] M. Ozawa, Error-disturbance relations in mixed states, [arXiv:1404.3388](https://arxiv.org/abs/1404.3388).
- [69] H. Chen, Y. Chen, and H. Yuan, Hierarchical incompatibility measures in multi-parameter quantum estimation, [arXiv:2109.05807](https://arxiv.org/abs/2109.05807).
- [70] J. Yang, S. Pang, Y. Zhou, and A. N. Jordan, Optimal measurements for quantum multiparameter estimation with general states, *Phys. Rev. A* **100**, 032104 (2019).
- [71] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer Academic, Dordrecht, 1995), Sec. 9-6.
- [72] S. L. Braunstein, C. M. Caves, and G. J. Milburn, Generalized uncertainty relations: Theory, examples, and Lorentz invariance, *Ann. Phys. (NY)* **247**, 135 (1996).
- [73] We utilize the function `SCIPY.STATS.ORTHO_GROUP` from the open-source PYTHON library SCIPY to generate the random orthogonal matrices.
- [74] F. Mezzadri, How to generate random matrices from the classical compact groups, *Not. Am. Math. Soc.* **54**, 592 (2007).
- Correction:* Partial derivatives in Eqs. (2.6), (2.9), (3.3), (3.4), (3.6), (3.10), and (3.11) and in the fourth paragraph of Sec. IV C were misrepresented due to a conversion error and have been set right.