Counting statistics of energy transport across squeezed thermal reservoirs

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A general formalism for computing the full counting statistics of energy exchanged between "N" squeezed thermal photon reservoirs weakly coupled to a cavity with "M" photon modes is presented. The formalism is based on the two-point measurement scheme and is applied to two simple special cases: the relaxation dynamics of a single mode cavity in contact with a single squeezed thermal photon reservoir and the steady-state energy transport between two squeezed thermal photon reservoirs coupled to a single cavity mode. Using analytical results, it is found that the short time energy exchange statistics is significantly affected by noncommutativity of the initial energy measurements with the reservoirs' squeezed states, and may lead to negative probabilities if not accounted for properly. Furthermore, it is found that for the single reservoir setup, generically there is no transient or steady-state Jarzynski-Wójcik energy exchange fluctuation theorem. In contrast, for the two reservoir cases, although there is no generic transient energy exchange fluctuation theorem, a steady-state energy exchange fluctuation theorem with a nonuniversal affinity is found to be valid. Statistics of energy currents are further discussed.

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I. INTRODUCTION

Fluctuations of observables in physical systems are ubiquitous. These fluctuations, seemingly arbitrary, carry a great amount of information related to the underlying physical processes. For example, fluctuations of observables in systems at equilibrium are known to be related to their responses to weak perturbations through the fluctuation-dissipation theorem. These relations are valid only for systems close to equilibrium [1–4]. The past three decades of research have shown that the fluctuations in physical systems, even far from equilibrium, satisfy universal relations, dubbed fluctuation theorems [5–8]. These fluctuation theorems have been demonstrated for various nonequilibrium systems, such as heat and charge transport in nanometer-sized junctions such as nanoelectronic quantum dot junctions, cavity photonic systems, and nanosized hybrid electro-optical, and electromechanical systems.

The fluctuation theorems are microscopic expressions of the second law of thermodynamics [5–8]. Although generic fluctuation theorems for the stochastic entropy production can be obtained for systems prepared in generic initial states [5,7,9], relating the stochastic entropy production to the fluctuating physical observables of the system is generally not obvious. For the traditionally considered transport setups, the stochastic entropy production can be related to the fluctuating particle and energy fluxes [5,10]. For these setups, the entropy production fluctuation theorems, referred to as the transient and steady-state energy and particle exchange fluctuation theorems, are traditionally derived based on the assumptions that the system's initial state is a canonical (local) equilibrium state and the dynamics is microreversible [5]. To our knowledge, not much work has been done in exploring the existence of such energy and particle exchange fluctuation theorems for systems specially prepared in noncanonical initial states. One such special class of noncanonical states of recent interest has been squeezed thermal states of photons. Squeezed thermal states of bosonic reservoirs have been used to enhance the efficiencies of heat engines [11]. It was shown that quantum heat engines with squeezed reservoirs can have efficiencies that surpass Carnot efficiency [12–14], a universal efficiency bound for engines working with thermal reservoirs, and allow work extraction even from a single reservoir [15]. Later works have established generalized Carnot-type bounds on the efficiencies of engines with squeezed reservoirs [16–19]. More recently, a non-Abelian generalization of the standard linear-response transport theory is proposed for studying the transport of energy and squeezing fluxes through squeezed reservoirs [20]. Some of these predictions have been realized in a recent experiment [21].

Although fluctuation theorems for the entropy production are derived for systems prepared in noncanonical initial states [22,23], it is not clear whether the transient or

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steady-state energy and particle exchange fluctuation theorems [5–9], which are useful in establishing various universal identities for out-of-equilibrium systems [24–28], are valid for systems prepared in such initial states. Motivated by these questions, in this work we study the statistics of energy transport and explore the question of the existence of transient and steady-state energy exchange fluctuation theorem in a simple model system consisting of "M" photon modes of a cavity coupled to "N" squeezed thermal photon reservoirs. It is to be noted that the phase-decohered state of a single bosonic mode prepared in a squeezed thermal state can be described by an effective temperature [16]. Such an effective temperature also appears in the definition of the thermodynamic affinity for the energy transport through a qubit system coupled to a squeezed thermal reservoir, for which an energy exchange fluctuation theorem may be valid [18]. It is not clear if the existence of an energy exchange fluctuation theorem is a generic feature or a result of a special qubit system, for which the populations in the qubit eigenbasis are decoupled from the coherences. As we discuss in this work, the qubit system indeed is a possible exception. It is also to be noted that even for canonical reservoirs and with microreversible dynamics, new fluctuation theorems, different from the traditional ones, may emerge; for example, particle exchange fluctuation theorems for superconducting systems coupled to normal metals [29].

This work is organized as follows. After introducing the model system in Sec. II, a description of the two-point measurement protocol for energy exchange statistics and a formal key result for the moment-generating function derived using the weak-coupling master equation approach are presented in Sec. III. These are then followed by the application of the results obtained in Sec. III to two simple model systems in Sec. IV. Finally, the conclusions are presented. Details of the computations are relegated to the Appendix.

II. MODEL SYSTEM

The model system considered in this work consists of a cavity having M photon modes, weakly coupled to N photon reservoirs. The Hamiltonian describing the system is

$$H = \underbrace{\sum_{i,j=1}^{M} b_{Si}^{\dagger} h_{S_{ij}} b_{Sj}}_{H_S} + \sum_{\alpha=1}^{N} \underbrace{\sum_{k\in\alpha} \epsilon_{\alpha k} b_{\alpha k}^{\dagger} b_{\alpha k}}_{H_\alpha} + i \sum_{\alpha=1}^{N} \underbrace{\sum_{k\in\alpha}\sum_{i=1}^{M} g_{Si\alpha k}[b_{\alpha k}^{\dagger} b_{Si} - b_{Si}^{\dagger} b_{\alpha k}]}_{H_{S\alpha}}. \quad (1)$$

Here $b_{Si}^{\dagger}$ (b_{Si}) and $b_{\alpha k}^{\dagger}$ ($b_{\alpha k}$) are bosonic creation (annihilation) operators for creating (annihilating) a photon in the "ith" cavity mode and in the "kth" mode in the αth photonic reservoir, respectively, and $h_{S_{ij}} = \epsilon_{Si}\delta_{ij}$. A schematic of the model considered is displayed in Fig. 1.

Initially, at time $t = 0$, it is assumed that the cavity photon modes and the photon reservoirs are not coupled and are prepared in individual squeezed thermal states, i.e., the full density matrix of the whole system at initial time is assumed

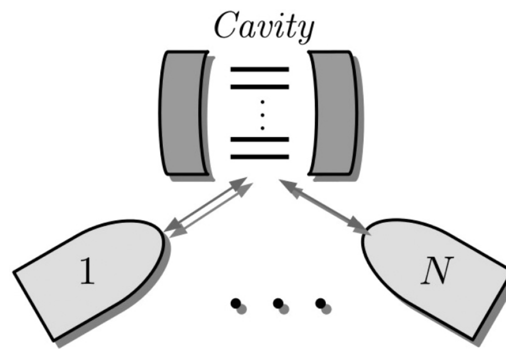

FIG. 1. Schematic of the model considered. The model consists of an "M" mode cavity prepared in a squeezed thermal state coupled to "N" reservoirs prepared in squeezed thermal states.

to be of the product (uncorrelated) form,

$$\rho(0) = \rho_S(0) \otimes_{\alpha=1}^{N} \rho_\alpha(0), \quad (2)$$

where

$$\rho_\alpha(0) = S_\alpha^{\dagger} \frac{e^{-\beta_\alpha H_\alpha}}{\mathbf{Tr}[e^{-\beta_\alpha H_\alpha}]} S_\alpha, \quad (3)$$

for $\alpha = S, 1, \dots, N$ and $S_\alpha = e^{-(1/2)\sum_{r\in\alpha} Z_\alpha [e^{i\phi_\alpha} b_{\alpha r}^{\dagger 2} - e^{-i\phi_\alpha} b_{\alpha r}^{2}]}$, being the squeezing operator [30–33]. For the sake of simplicity, the squeezing amplitude, $Z_\alpha \geqslant 0$, and the phase, $\phi_\alpha \in [-\pi, +\pi)$, of each subsystem (i.e., system and reservoirs) are assumed to be mode independent.

In order to study fluctuations of energy transfer from the system into squeezed thermal reservoirs, in the next section, we construct full distribution of energy transfer using the two-point measurement scheme [5,6,34–36].

III. MOMENT-GENERATING FUNCTION

The cavity and the reservoirs prepared in uncorrelated squeezed thermal states are coupled at time $t = 0$ (by turning on $H_{S\alpha}$) leading to the flow of energy between the system and the reservoirs. The joint probability distribution for the amount of energy flowing, $\mathbf{\Delta e} = (\Delta e_1 \ \cdots \ \Delta e_N)^T$, into each of the reservoirs in time t, can be written as

$$P[\mathbf{\Delta e}, t] = \frac{1}{(2\pi)^N} \int_{\boldsymbol{\chi}\in\mathbb{R}^N} d^N\boldsymbol{\chi}\, \mathcal{Z}[\boldsymbol{\chi}, t] e^{i\boldsymbol{\chi}^T \mathbf{\Delta e}}, \quad (4)$$

where $\mathcal{Z}[\boldsymbol{\chi}, t]$ is the moment-generating function, which within the two-point measurement scheme [5,6,34–36] is obtained as

$$\mathcal{Z}[\boldsymbol{\chi}, t] = \frac{1}{(2\pi)^N} \int_{\boldsymbol{\lambda}\in\mathbb{R}^N} d^N\boldsymbol{\lambda}\, \tilde{\mathcal{Z}}[\boldsymbol{\chi}, \boldsymbol{\lambda}, t], \quad (5)$$

with

$$\tilde{\mathcal{Z}}[\boldsymbol{\chi}, \boldsymbol{\lambda}, t] = \mathbf{Tr}_{S+B}[e^{-(i/\hbar)H[\boldsymbol{\lambda}+\boldsymbol{\chi}/2]t} \rho(0) e^{(i/\hbar)H[\boldsymbol{\lambda}-\boldsymbol{\chi}/2]t}], \quad (6)$$

where $\boldsymbol{\chi} = (\chi_1 \ \cdots \ \chi_N)^T$ keeps track of the energy flow, $\mathbf{\Delta e}$, from the system into the reservoirs, and $\boldsymbol{\lambda} = (\lambda_1 \ \cdots \ \lambda_N)^T$ carries the information of the initial projective measurement of the reservoirs' energies. The integral over $\boldsymbol{\lambda}$ in Eq. (5) is necessary because the initial density matrices of the reservoirs do not commute with the initial projective energy measurements on the reservoirs. This integral essentially projects out the initial coherences between isolated reservoirs' energy eigenstates which are destroyed by the initial projective measurements on the reservoirs [37]. It is crucial to note that the above procedure of implementing initial projections should be treated with caution, as naively using $\tilde{\mathcal{Z}}[\mathbf{0},\boldsymbol{\lambda},t]=1$ from Eq. (6) in Eq. (5) leads to divergence. However, it can be made meaningful by a physical limiting procedure discussed at the end of this section.

The counting-field-dependent Hamiltonian in Eq. (6) is defined as

$$H[\boldsymbol{\chi}]=\sum_{i,j=1}^{M}b_{Si}^{\dagger}h_{Sij}b_{Sj}+\sum_{\alpha=1}^{N}\sum_{k\in\alpha}\epsilon_{\alpha k}b_{\alpha k}^{\dagger}b_{\alpha k}+i\sum_{\alpha=1}^{N}\sum_{k\in\alpha}\sum_{i=1}^{M}g_{Si\alpha k}[e^{-i\epsilon_{\alpha k}\chi_{\alpha}}b_{\alpha k}^{\dagger}b_{Si}-e^{i\epsilon_{\alpha k}\chi_{\alpha}}b_{Si}^{\dagger}b_{\alpha k}]. \tag{7}$$

$\tilde{\mathcal{Z}}[\boldsymbol{\chi},\boldsymbol{\lambda},t]$ defined in Eq. (6) can be recast as

$$\tilde{\mathcal{Z}}[\boldsymbol{\chi},\boldsymbol{\lambda},t]=\mathbf{Tr}_S\big[\rho_{SI}^{\Lambda}(t)\big], \tag{8}$$

with the counting-field-dependent system's reduced density matrix $[\rho_{SI}^{\Lambda}(t)]$, in the interaction picture, defined as

$$\rho_{SI}^{\Lambda}(t)=e^{(i/\hbar)H_S t}\mathbf{Tr}_B[e^{-(i/\hbar)H[\boldsymbol{\lambda}+\boldsymbol{\chi}/2]t}\rho(0)e^{(i/\hbar)H[\boldsymbol{\lambda}-\boldsymbol{\chi}/2]t}]e^{-(i/\hbar)H_S t}.$$

Here the subscripts S and I of $\rho_{SI}^{\Lambda}(t)$ represent that $\rho_{SI}^{\Lambda}(t)$ is the system's reduced density matrix in the interaction picture, and the superscript Λ represents its dependence on the counting fields ($\boldsymbol{\chi}$ and $\boldsymbol{\lambda}$). We note that if the counting fields ($\boldsymbol{\chi}$ and $\boldsymbol{\lambda}$) are set to zero, $\rho_{SI}^{\Lambda}(t)$ is just the system's reduced density matrix, whose trace is unity.

By invoking Born-Markov-secular approximations (and also neglecting the Lamb shifts), a counting-field-dependent generalized Lindblad quantum master equation can be derived for $\rho_{SI}^{\Lambda}(t)$ [30,38–43]. This is given as

$$\begin{aligned}\frac{\partial}{\partial t}\rho_{SI}^{\Lambda}(t)=&-\sum_{\alpha=1}^{N}\mathbf{B}_S^{T}\left\{e^{i\mathbf{h}_S[\lambda_\alpha+\chi_\alpha/2]}\boldsymbol{\Gamma}_\alpha\boldsymbol{\sigma}_y\left[\mathbf{D}_\alpha+\frac{i}{2}\boldsymbol{\sigma}_y\right]\boldsymbol{\sigma}_y e^{i\mathbf{h}_S[\lambda_\alpha-\chi_\alpha/2]}\right\}\rho_{SI}^{\Lambda}(t)\mathbf{B}_S\\&+\frac{1}{2}\sum_{\alpha=1}^{N}\mathbf{B}_S^{T}\left\{e^{i\mathbf{h}_S[\lambda_\alpha+\chi_\alpha/2]}\boldsymbol{\Gamma}_\alpha\boldsymbol{\sigma}_y\left[\mathbf{D}_\alpha-\frac{i}{2}\boldsymbol{\sigma}_y\right]\boldsymbol{\sigma}_y e^{i\mathbf{h}_S[\lambda_\alpha+\chi_\alpha/2]}\right\}\mathbf{B}_S\rho_{SI}^{\Lambda}(t)\\&+\frac{1}{2}\sum_{\alpha=1}^{N}\rho_{SI}^{\Lambda}(t)\mathbf{B}_S^{T}\left\{e^{i\mathbf{h}_S[\lambda_\alpha-\chi_\alpha/2]}\boldsymbol{\Gamma}_\alpha\boldsymbol{\sigma}_y\left[\mathbf{D}_\alpha-\frac{i}{2}\boldsymbol{\sigma}_y\right]\boldsymbol{\sigma}_y e^{i\mathbf{h}_S[\lambda_\alpha-\chi_\alpha/2]}\right\}\mathbf{B}_S,\end{aligned} \tag{9}$$

where $\mathbf{B}_S=(b_{S1}^{\dagger}\ \cdots\ b_{SM}^{\dagger}\ \ b_{S1}\ \cdots\ b_{SM})^T$ is a vector of creation and annhilation operators of the system, $\mathbf{h}_S=\sigma_z\otimes h_S$, $\boldsymbol{\sigma}_{x,y,z}=\sigma_{x,y,z}\otimes I_{M\times M}$, with $\sigma_{x,y,z}$ being Pauli matrices and $I_{M\times M}$ being the $M\times M$ identity matrix, $\boldsymbol{\Gamma}_\alpha=I\otimes\Gamma_\alpha$ (for brevity $I_{2\times2}$ is denoted by I) with the system-reservoir coupling matrix elements defined as

$$\Gamma_{\alpha ij}=\begin{cases}\frac{2\pi}{\hbar}\sum_{k\in\alpha}g_{Si\alpha k}g_{Sj\alpha k}\delta(\epsilon_{\alpha k}-\epsilon_{Si}) & \text{if } \epsilon_{Si}=\epsilon_{Sj}\\ 0 & \text{if } \epsilon_{Si}\neq\epsilon_{Sj},\end{cases}$$

and $\mathbf{D}_\alpha$, which carries the information of the reservoirs' states, is defined as $\mathbf{D}_\alpha=-i\boldsymbol{\sigma}_y e^{-i\mathbf{S}_\alpha\boldsymbol{\sigma}_y}[n_\alpha(\mathbf{h}_S)+\frac{1}{2}\mathbf{I}]e^{i\mathbf{S}_\alpha\boldsymbol{\sigma}_y}$ ($\mathbf{I}=I_{2M\times2M}$) with $\mathbf{S}_\alpha=Z_\alpha\boldsymbol{\sigma}_z e^{i\boldsymbol{\sigma}_z\phi_\alpha}$, $n_\alpha(x)=(e^{\beta_\alpha x}-1)^{-1}$. The solution of Eq. (9), supplemented with the initial condition $\rho_{SI}^{\Lambda}(t)|_{t=0}=\rho_S(0)$, when used in Eq. (8), gives $\tilde{\mathcal{Z}}[\boldsymbol{\chi},\boldsymbol{\lambda},t]$.

We solve for $\rho_{SI}^{\Lambda}(t)$ using the Wigner phase-space representation, which is used to obtained an analytical expression for $\tilde{\mathcal{Z}}[\boldsymbol{\chi},\boldsymbol{\lambda},t]$. The final result of the detailed computations presented in the Appendix is

$$\tilde{\mathcal{Z}}[\boldsymbol{\chi},\boldsymbol{\lambda},t]=\frac{e^{(1/2)\mathbf{Tr}[\Gamma]t}}{\sqrt{\mathbf{Det}[\mathbb{U}_{11}(t)+\mathbb{U}_{12}(t)\mathbf{D}_S]}}. \tag{10}$$

Here $\Gamma=\sum_{\alpha=1}^{N}\Gamma_\alpha$, $\mathbb{U}_{xy}(t)$ are $2M\times2M$ matrices defined as the 2×2 blocks of $\mathbb{U}(t)$, defined as

$$\mathbb{U}(t)=\begin{pmatrix}\mathbb{U}_{11}(t) & \mathbb{U}_{12}(t)\\ \mathbb{U}_{21}(t) & \mathbb{U}_{22}(t)\end{pmatrix}=e^{-\mathbb{H}\boldsymbol{\Sigma}t}, \tag{11}$$

with the standard symplectic matrix defined as $\boldsymbol{\Sigma}=i\sigma_y\otimes\mathbf{I}_{2M\times2M}$ and the (2×2 block partitioned) complex symmetric matrix $\mathbb{H}$ defined as

$$\mathbb{H}=\frac{1}{2}\sum_{\alpha=1}^{N}\mathbb{V}[\chi_\alpha,\lambda_\alpha]^T[\sigma_x\otimes\boldsymbol{\Gamma}_\alpha+(I-\sigma_z)\otimes(\boldsymbol{\Gamma}_\alpha\mathbf{D}_\alpha)]\mathbb{V}[\chi_\alpha,\lambda_\alpha], \tag{12}$$

with

$$\mathbb{V}[\chi,\lambda]=e^{i\sigma_z\otimes\mathbf{h}_S\lambda}\left\{I\otimes\cos\left[\frac{1}{2}\mathbf{h}_S\chi\right]-\frac{1}{4}(5\sigma_x-3i\sigma_y)\otimes\left(\boldsymbol{\sigma}_y\sin\left[\frac{1}{2}\mathbf{h}_S\chi\right]\right)\right\}. \tag{13}$$

Finally, $\mathbf{D}_S$, the covariance matrix of the Wigner function corresponding to the system's initial squeezed thermal state [44], is defined as $\mathbf{D}_S=-i\boldsymbol{\sigma}_y e^{-i\mathbf{S}_S\boldsymbol{\sigma}_y}[n_S(\mathbf{h}_S)+\frac{1}{2}\mathbf{I}]e^{i\mathbf{S}_S\boldsymbol{\sigma}_y}$, with $\mathbf{S}_S=Z_S\boldsymbol{\sigma}_z e^{i\boldsymbol{\sigma}_z\phi_S}$ and $n_S(x)=(e^{\beta_S x}-1)^{-1}$.

The above expression for $\tilde{\mathcal{Z}}[\boldsymbol{\chi},\boldsymbol{\lambda},t]$, Eq. (10), can be considered as the dissipative generalization of the Levitov-Lesovik-Klich formula [45,46].

We note that in Ref. ([47]), a method, similar in spirit as here, was developed for computing long-time statistics of fluxes through quantum harmonic networks. It is to be emphasized that the presented approach allows one to compute statistics at all times. Furthermore, the approach is based on the microscopic master equation and the two-point measurement scheme applied to the measurements of the reservoirs' energies, whereas Ref. [47] is based on the counting of quantum jumps [48] of a system described by a generic phenomenological master equation. For the master equations of the type considered in this work, Eq. (14) [a special case of Eq. (9)] with the counting fields set to zero, if quantum jump unraveling is performed in the generalized squeezed basis, as is done in Ref. [23], each quantum jump can be interpreted as leading to a quanta of entropy flowing into or out of the reservoirs. However, it is not clear how to unravel such master equations, where the populations and coherences in the system's energy eigenstate basis are coupled into quantum trajectories for studying statistics of energy exchange with the reservoirs.

It is important to note that for $\boldsymbol{\chi}=\mathbf{0}$, $\mathbb{U}_{12}(t)$ [defined in Eq. (11)] reduces to a $2M\times 2M$ null matrix and $\mathbb{U}_{11}(t)=e^{(1/2)[I\otimes\Gamma]t}$, and thus Eq. (10) gives $\tilde{\mathcal{Z}}[\mathbf{0},\boldsymbol{\lambda},t]=1$. This indicates that $\mathcal{Z}[\mathbf{0},t]$ [defined in Eq. (5)] is a divergent quantity. This divergence of $\mathcal{Z}[\mathbf{0},t]$ is not an artifact of the Markov approximation used here. As already pointed out, it can be seen from the initial definition of $\mathcal{Z}[\boldsymbol{\chi},t]$ [Eq. (5)] by using $\tilde{\mathcal{Z}}[\mathbf{0},\boldsymbol{\lambda},t]=1$. Furthermore, it turns out that for the simple models discussed in the next section, $\mathcal{Z}[\boldsymbol{\chi},t]$, for $\boldsymbol{\chi}\neq\mathbf{0}$, itself diverges as a result of the Markov approximation used here [49]. For it to represent a meaningful moment-generating function, we have to renormalize it, so that the resultant probability function is normalized and meaningful. This renormalization can be achieved by dividing the value of $\mathcal{Z}[\boldsymbol{\chi},t]$ by $\mathcal{Z}[0,t]$. Since both these quantities diverge, this renormalization is performed after regularizing $\mathcal{Z}[\boldsymbol{\chi},t]$ by introducing a cutoff on the $\boldsymbol{\lambda}$ integral and taking the cutoff to infinity after division. This introduced cutoff can be thought of as arising physically, by working with reservoirs with mode frequencies that are equally spaced with a small spacing ($\bar{\epsilon}$) (for which initial projection can be implemented by $\boldsymbol{\lambda}$ integrals with an ultraviolet cutoff $|\boldsymbol{\lambda}_k|\leqslant\frac{\pi}{\bar{\epsilon}}$), which is sent to zero eventually. This renormalization is done case by case in the following.

In the next section, we apply the general results obtained in this section to two special cases, both with the single cavity mode coupled either to a single reservoir or to two reservoirs.

IV. APPLICATION TO SIMPLE MODELS

We now specialize to the case of a cavity with a single photon mode, i.e., we apply the results presented in the previous section to the case $M=1$.

For this case h_S, Γ_α, and $I_{M\times M}$ become scalars and $\mathbf{h}_S$, $\mathbf{S}_S$, $\mathbf{D}_S$, $\mathbf{S}_\alpha$, and $\mathbf{D}_\alpha$ become 2×2 matrices, $\mathbb{U}(t)$ and $\boldsymbol{\Sigma}$ reduce to 4×4 matrices and, hence, $\mathbb{U}_{xy}(t)$ are 2×2 matrices.

Moreover, the counting-field-dependent Lindblad master equation can be written as

$$\begin{aligned}\frac{\partial}{\partial t}\rho_{SI}^{\Lambda}(t)=\sum_{\alpha=1,2}\frac{\Gamma_\alpha}{2}\Big\{&N_\alpha\big[2e^{i\chi_\alpha\epsilon}b^\dagger\rho_{SI}^{\Lambda}(t)b-\{bb^\dagger,\rho_{SI}^{\Lambda}(t)\}\big]+(1+N_\alpha)\big[2e^{-i\chi_\alpha\epsilon}b\rho_{SI}^{\Lambda}(t)b^\dagger-\{b^\dagger b,\rho_{SI}^{\Lambda}(t)\}\big]\\&-\Delta_\alpha e^{-i(2\lambda_\alpha\epsilon-\phi_\alpha)}\big[2b\rho_{SI}^{\Lambda}(t)b-e^{-i\chi_\alpha\epsilon}bb\rho_{SI}^{\Lambda}(t)-e^{i\chi_\alpha\epsilon}\rho_{SI}^{\Lambda}(t)bb\big]\\&-\Delta_\alpha e^{i(2\lambda_\alpha\epsilon-\phi_\alpha)}\big[2b^\dagger\rho_{SI}^{\Lambda}(t)b^\dagger-e^{i\chi_\alpha\epsilon}b^\dagger b^\dagger\rho_{SI}^{\Lambda}(t)-e^{-i\chi_\alpha\epsilon}\rho_{SI}^{\Lambda}(t)b^\dagger b^\dagger\big]\Big\},\end{aligned}\tag{14}$$

with $N_\alpha=\cosh(2Z_\alpha)(n_\alpha+\frac{1}{2})-\frac{1}{2}$ and $\Delta_\alpha=\sinh(2Z_\alpha)(n_\alpha+\frac{1}{2})$ with $n_\alpha=(e^{\beta_\alpha\epsilon}-1)^{-1}$ ($\alpha=1,\ 2,$ and S). Further, $b\equiv b_{S1}$ and $\epsilon_1\equiv\epsilon$.

Below we consider two simple cases: the first one consisting of only one photon reservoir, while the second case is with two photon reservoirs.

A. Single mode coupled to a single reservoir

For a single photon mode cavity coupled to a single squeezed thermal photon reservoir, i.e., the case $N=1$ and $M=1$, the explicit expression for the auxiliary-generating function, $\tilde{\mathcal{Z}}[\chi_1,\lambda_1,t]$, for the energy flow from the cavity into the reservoir can be obtained as [a more formal expression is given in Eqs. (A21) and (A22) of the Appendix]

$$\begin{aligned}\tilde{\mathcal{Z}}[\chi_1,\lambda_1,t]=e^{\Gamma_1 t/2}\Bigg\{&\prod_{x=\pm}\left(\cosh\left[\frac{\Gamma_1 t}{2}\right]+\sinh\left[\frac{\Gamma_1 t}{2}\right]\Lambda_x^S[\chi_1]\right)+4[1-e^{-\Gamma_1 t}]\Delta_1\Delta_S[(e^{i\epsilon\chi_1}-1)\\&+(e^{-i\epsilon\chi_1}-1)]\sin^2\left[\epsilon\lambda_1+\frac{\phi_1-\phi_S}{2}\right]\Bigg\}^{-1/2},\end{aligned}\tag{15}$$

with

$$\Lambda_\pm^S[\chi_1]=1-2\{[N_1\pm\Delta_1][(1+N_S)\pm\Delta_S](e^{i\epsilon\chi_1}-1)+[(1+N_1)\pm\Delta_1][N_S\pm\Delta_S](e^{-i\epsilon\chi_1}-1)\}.\tag{16}$$

The moment-generating function for energy released from the system into the reservoir in time t is then given by integrating over λ_1 [defined in Eq. (5)] as

$$\mathcal{Z}[\chi_1,t]=\frac{1}{2\pi}\int_{-\infty}^{+\infty}d\lambda_1\tilde{\mathcal{Z}}[\chi_1,\lambda_1,t]. \tag{17}$$

Since $\tilde{\mathcal{Z}}[\chi_1,\lambda_1,t]$ in Eq. (15) is a periodic function of λ_1 with period $\frac{2\pi}{\epsilon}$, i.e., $\tilde{\mathcal{Z}}[\chi_1,\lambda_1+\frac{2\pi}{\epsilon},t]=\tilde{\mathcal{Z}}[\chi_1,\lambda_1,t]$, $\mathcal{Z}[\chi_1,t]$ becomes divergent. To make sense of it as a moment-generating function, we have to renormalize it. As discussed at the end of Sec. III, this is done by introducing a cutoff, $|\lambda_1|\leqslant\frac{\pi}{\bar{\epsilon}}$, and renormalizing $\mathcal{Z}[\chi_1,t]$ by $\mathcal{Z}[0,t]$ and taking the limit $\bar{\epsilon}\to 0$ as

$$\begin{aligned}\mathcal{Z}[\chi_1,t]&=\lim_{\bar{\epsilon}\to 0}\frac{\frac{1}{2\pi}\int_{-\pi/\bar{\epsilon}}^{+\pi/\bar{\epsilon}}d\lambda_1\tilde{\mathcal{Z}}[\chi_1,\lambda_1,t]}{\frac{1}{2\pi}\int_{-\pi/\bar{\epsilon}}^{+\pi/\bar{\epsilon}}d\lambda_1\tilde{\mathcal{Z}}[0,\lambda_1,t]}\\&=\frac{\epsilon}{2\pi}\int_{-\pi/\epsilon}^{+\pi/\epsilon}d\lambda_1\tilde{\mathcal{Z}}[\chi_1,\lambda_1,t].\end{aligned} \tag{18}$$

To arrive at the second equality, we have used the periodic property of $\tilde{\mathcal{Z}}[\chi_1,\lambda_1,t]$ and $\tilde{\mathcal{Z}}[0,\lambda_1,t]=1$. The λ_1 integral in the second equality can be analytically performed for $\tilde{\mathcal{Z}}[\chi_1,\lambda_1,t]$ given in Eq. (15). This gives $\mathcal{Z}[\chi_1,t]$ in terms of complete elliptic function of the first kind with the argument which is a complicated function of χ_1. Since this expression is not amenable to further analysis, we do not provide it here. However, we note that, for the case when the initial states of system and reservoir are thermal, i.e., $Z_1=Z_S=0$, this expression for $\mathcal{Z}[\chi_1,t]$ agrees with the expressions previously reported in the literature [42,50,51] and the probability distribution function for the energy flow from the system into the reservoir satisfies the Jarzynski-Wójcik energy exchange fluctuation theorem [52].

Using $\mathcal{Z}[\chi_1,t]$, the cumulants of the energy flow from the system into the reservoir can be obtained. The average energy flow in time t is given as

$$\langle\Delta e_1\rangle=(1-e^{-\Gamma_1 t})[N_S-N_1]. \tag{19}$$

When the squeezing of the system and the reservoir are absent ($Z_1=Z_S=0$), i.e., the system's initial state and the reservoir's state are thermal states, then $\langle\Delta e_1\rangle=(1-e^{-\Gamma_1 t})[n_S-n_1]$. Comparing Eq. (19) with this allows us to define an effective (inverse) temperature in the presence of squeezing as

$$\tilde{\beta}_\alpha=\frac{1}{\epsilon}\ln\left(N_\alpha^{-1}+1\right). \tag{20}$$

We note that the above effective temperature coincides with the effective temperature for a phase-averaged state of a single bosonic mode prepared in a squeezed thermal state [16]. As $N_\alpha\geqslant n_\alpha$ and $\ln(x)$ is a monotonically increasing function, $\tilde{\beta}_\alpha^{-1}\geqslant\beta_\alpha^{-1}$. Hence, it is tempting to attribute the effect of squeezing to the enhancement of the effective temperature of the reservoir. Using Eq. (20), the energy flow in the presence of squeezing can be expressed as $\langle\Delta e_1\rangle=(1-e^{-\Gamma_1 t})[\tilde{n}_S-\tilde{n}_1]$ with $\tilde{n}_\alpha=(e^{\tilde{\beta}_\alpha\epsilon}-1)^{-1}$.

If the system's and reservoir's states could be described by thermal states (after initial energy projective measurement) with effective temperatures, then the energy flow from the system into the reservoir would satisfy the Jarzynski-Wójcik transient energy exchange fluctuation theorem in terms of the same effective temperatures. However, it turns out from the following discussion that the fluctuation theorem for the energy flow is absent. Although the average energy flow can be described in terms of effective temperatures, the same is not true for the fluctuations. For instance, the second cumulant of the energy flow in time t is given by

$$\begin{aligned}\langle\Delta e_1^2\rangle-\langle\Delta e_1\rangle^2&=(1-e^{-\Gamma_1 t})^2\left[(N_S-N_1)^2+\Delta_S^2+\Delta_1^2\right]\\&\quad+(1-e^{-\Gamma_1 t})[N_S(1+N_1)+N_1(1+N_S)],\end{aligned} \tag{21}$$

and cannot be expressed in terms of the effective temperature in the form $\langle\Delta e_1^2\rangle-\langle\Delta e_1\rangle^2=(1-e^{-\Gamma_1 t})^2[(\tilde{n}_S-\tilde{n}_1)^2]+(1-e^{-\Gamma_1 t})[\tilde{n}_S(1+\tilde{n}_1)+\tilde{n}_1(1+\tilde{n}_S)]$, as obtained for the thermal case. This should be contrasted with a qubit coupled to a squeezed thermal reservoir, where it is possible to define an effective temperature such that the fluctuations of energy flow are of the same form as that of the thermal case and the energy exchange fluctuation theorem holds with an effective temperature [18].

Note that, in the long-time limit ($\Gamma_1 t\to\infty$), as the system reaches the same state as that of the reservoir, the energy ceases to flow from the system into the reservoir and hence the energy flow and its fluctuations saturate to finite values. As a consequence, $\tilde{\mathcal{Z}}[\chi_1,\lambda_1,t]$, given in Eq. (15), becomes independent of time, indicating that the statistics of the energy flowing from the system into the reservoir becomes independent of time. This is a generic feature of a finite system coupled to a single reservoir.

Owing to the periodicity, $\mathcal{Z}[\chi_1+\frac{2\pi}{\epsilon},t]=\mathcal{Z}[\chi_1,t]$ [Eq. (15) along with Eq. (18)], the probability function for the energy flow from system into the reservoir acquires a Dirac comb structure, i.e., $P[\Delta e_1,t]=\sum_{n\in\mathbb{Z}}p[n,t]\delta[\Delta e_1-n\epsilon]$, with $p[n,t]=\frac{1}{2\pi}\int_{-\pi}^{+\pi}d\chi_1\mathcal{Z}[\frac{\chi_1}{\epsilon},t]e^{i\chi_1 n}$. $p[n,t]$ is the probability of n quanta of energy transferred from the system to the reservoir.

The λ_1 dependence in $\tilde{\mathcal{Z}}[\chi_1,\lambda_1,t]$, which is integrated out to obtain the moment-generating function in Eq. (18), contains information of the initial energy projective measurement on the reservoir. This λ_1 integral has two important roles. Firstly, this makes $\mathcal{Z}[\chi_1,t]$ independent of the initial reservoir's and system's squeezing phases, ϕ_1 and ϕ_S, respectively. Hence the energy flow statistics is independent of these phases. This can be seen by performing a change of variables, $\lambda_1\to\lambda_1-(\frac{\phi_1-\phi_S}{2\epsilon})$, in the λ_1 integral appearing in Eq. (18) along with the expression for $\tilde{\mathcal{Z}}[\chi_1,\lambda_1,t]$ given in Eq. (15). Secondly, the λ_1 integral is crucial for probability function $p[n,t]$ to be meaningful. If we set $\lambda_1=0$ to obtain $\mathcal{Z}[\chi_1,t]=\tilde{\mathcal{Z}}[\chi_1,0,t]$, which is equivalent to the assumption that the initial energy projection commutes with the initial state of the reservoir, which is not the case here, we observe that the resulting moment-generating function $\mathcal{Z}[\chi_1,t]$ may lead to negative probabilities, $p[n,t]$, for certain events (n values). This is evident from the plots shown in the upper panel (and the inset) of Fig. 2, where negative probabilities are clearly evident

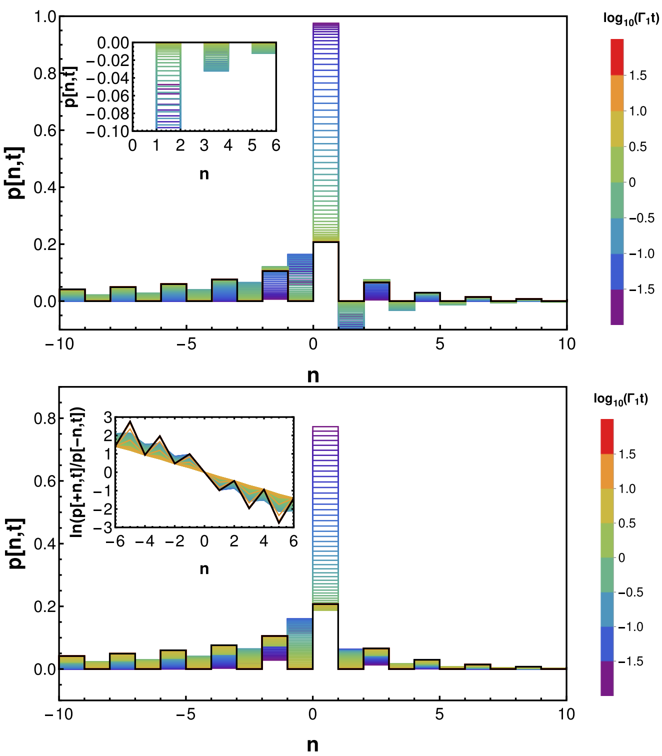

FIG. 2. Probability distribution function for the number of quanta of energy released from the system into the reservoir in time t for a range of $\Gamma_1 t$. The plot in the upper panel is obtained using $\tilde{\mathcal{Z}}[\chi_1, 0, t]$ as the moment-generating function (i.e., ignoring the noncommutativity of the initial projection and the initial reservoir's state) with the inset showing the region where $p[n,t]$ becomes negative. The plot in the lower panel is obtained using $\mathcal{Z}[\chi_1, t]$ (i.e., properly accounting for the initial projection) with the plots in the inset displaying $\ln([p[n,t]/p[-n,t])$ vs n. Black curves in both plots and their insets correspond to $\Gamma_1 t \to \infty$. Parameters used are $\beta_1\epsilon = 10.0$, $\beta_S\epsilon = 20.0$, $Z_1 = 2.0$, $Z_S = 1.0$, and $\phi_1 - \phi_S = \pi$.

for short timescales. The weight of negative probabilities decreases as time increases. In the long-time limit ($\Gamma_1 t \to \infty$), it can be shown that $\mathcal{Z}[\chi_1, t] = \tilde{\mathcal{Z}}[\chi_1, \lambda_1, t] = \tilde{\mathcal{Z}}[\chi_1, 0, t]$, making the long-time statistics of energy flow independent of the initial energy projection, as it should be since the system reaches a well-defined state, the same as that of the reservoir. More precisely, the initial noncommutativity of the reservoir's density matrix with energy projective measurements does not affect the long-time statistics. The figure in the lower panel uses the proper moment-generating function, $\mathcal{Z}[\chi_1, t]$, obtained by accounting for the initial energy projection of the reservoir and gives the correct positive semidefinite distribution function $p[n,t]$ for all times. The negative probabilities observed previously in the statistics of charge flow between superconductors [53–55] were attributed to the interference of transition amplitudes corresponding to different realizations (quantum trajectories) leading to the same energy and or particle change of the reservoir, but starting in different initial states [56–62]. These negative probabilities were also recently discussed in the works attempting to go beyond the standard two-point measurement scheme used here [63–65]. Finally, it is important to note that for the case when either the system's initial state or the reservoir's state is not squeezed, i.e., $Z_S = 0$ or $Z_1 = 0$, $\tilde{\mathcal{Z}}[\chi_1, \lambda_1, t]$ given in Eq. (15) becomes independent of λ_1. Hence for this case, as expected, the statistics of energy flow is not affected by the noncommutative nature of the reservoir's density matrix with the initial projective measurement of the reservoir's energy.

The inset in the lower panel of Fig. 2 shows a nonlinear relationship between $\ln(p[+n,t]/p[-n,t])$ and n, indicating that the stochastic energy flow generically does not satisfy the Jarzynski-Wójcik energy exchange fluctuation theorem [52] both at finite times as well as in the $\Gamma_1 t \to \infty$ limit. However, for a special choice of parameters,

$$Z_1 = \frac{\ln[1+2n_1]}{2} \quad \text{and} \quad Z_S = \frac{\ln[1+2n_S]}{2}, \tag{22}$$

for which $\Delta_1 = N_1$ and $\Delta_S = N_S$, the long-time ($\Gamma_1 t \to \infty$) moment-generating function [obtained using Eq. (15) in Eq. (18)],

$$\mathcal{Z}[\chi_1, \infty] = \{1 - 4 \times [N_1(1+2N_S)(e^{i\epsilon\chi_1} - 1) + (1+2N_1)N_S(e^{-i\epsilon\chi_1} - 1)]\}^{-1/2}, \tag{23}$$

exhibits the Jarzynski-Wójcik energy exchange fluctuation theorem, $\frac{p[n,\infty]}{p[-n,\infty]} = e^{\alpha_{1S}\epsilon n}$, as a result of the Gallavotti-Cohen symmetry, $\mathcal{Z}[-\chi_1 - i\alpha_{1S}, \infty] = \mathcal{Z}[\chi_1, \infty]$ [5], with the affinity $\alpha_{1S} = \frac{1}{\epsilon}\ln\frac{N_S(1+2N_1)}{(1+2N_S)N_1}$.

B. Single mode coupled to two reservoirs

Here, we consider a system with a single photon mode coupled to two squeezed thermal photon reservoirs, i.e., we discuss the $M = 1$ and $N = 2$ case. Unlike the $N = 1$ case, this allows one to study the fluctuations of energy flux in the nonequilibrium steady state.

Explicit expression for the auxiliary-generating function $[\tilde{\mathcal{Z}}[\boldsymbol{\chi}, \boldsymbol{\lambda}, t]$ with $\boldsymbol{\chi} = (\chi_1 \quad \chi_2)^T$ and $\boldsymbol{\lambda} = (\lambda_1 \quad \lambda_2)^T$] for the energy flow into two reservoirs at arbitrary times (transients and steady state) are given in Eqs. (A23)–(A27) of the Appendix. Similar to the last section, $\tilde{\mathcal{Z}}[\boldsymbol{\chi}, \boldsymbol{\lambda}, t]$ is a periodic function of both λ_1 and λ_2 with period $\frac{2\pi}{\epsilon}$. Hence the joint moment-generating function $\mathcal{Z}[\boldsymbol{\chi}, t] = \int_{\boldsymbol{\lambda}\in\mathbb{R}^2} \frac{d^2\boldsymbol{\lambda}}{(2\pi)^2} \tilde{\mathcal{Z}}[\boldsymbol{\chi}, \boldsymbol{\lambda}, t]$ diverges. We renormalize $\mathcal{Z}[\boldsymbol{\chi}, t]$ in a similar way as in the previous section (see Appendix for details).

The moment-generating function $\mathcal{Z}[\boldsymbol{\chi}, t]$, as a result of initial projective measurement ($\boldsymbol{\lambda}$ integrals), becomes independent of squeezing phases of the initial states of the system (ϕ_S) and both the reservoirs (ϕ_1 and ϕ_2) (see Appendix for details).

For further analysis, it is convenient to consider the joint statistics of $\Delta e_s = (\Delta e_1 + \Delta e_2)$ and $\Delta e_r = \frac{1}{2}(\Delta e_1 - \Delta e_2)$, which, in the weak system-reservoir coupling limit considered in this work, can be interpreted as the net energy flow out of the system (Δe_s) and the net energy flow (Δe_r) between the two reservoirs, respectively. The joint moment-generating function for these stochastic quantities can be obtained as $\bar{\mathcal{Z}}[\chi_r, \chi_s, t] = \mathcal{Z}[\boldsymbol{\chi}, t]|_{\chi_{1/2}\to\chi_s\pm\chi_r/2}$, where χ_r and χ_s are parameters conjugate to Δe_r and Δe_s, respectively.

The marginal moment-generating function corresponding to Δe_s, $\mathcal{Z}_s[\chi_s, t] = \bar{\mathcal{Z}}[0, \chi_s, t]$, becomes independent of time in $t \to 0$ limit, indicating that the fluctuations of energy flow out of the system saturate with time as the system reaches steady state (see Appendix for details).

From here onwards, we confine ourselves to the steady state and only discuss the statistics of the energy flow from the reservoir "2" into the reservoir "1" (Δe_r), i.e., we only analyze the marginal distribution function $P[\Delta e_r, t]$ in the $t \to \infty$ limit.

In the steady-state limit, the $\boldsymbol{\lambda}$ integrals implementing initial energy projective measurements on reservoirs can be performed using the saddle-point approximation. This is sketched in the Appendix. This gives the steady-state scaled cumulant-generating function for the energy flow from the reservoir "2" into the reservoir "1" as

$$\mathcal{F}[\chi_r] = \lim_{t\to\infty} \frac{\ln \mathcal{Z}_r[\chi_r, t]}{t} = \frac{\Gamma_1 + \Gamma_2}{2} \sum_{x=\pm} \left[\frac{1 - \Lambda_x^{12}[\chi_r]}{2}\right], \tag{24}$$

with

$$\Lambda_{\pm}^{12}[\chi_r] = \sqrt{1 - \mathbb{T}\{[N_1 \pm \Delta_1][(1+N_2) \pm \Delta_2](e^{i\epsilon\chi_r} - 1) + [(1+N_1) \pm \Delta_1][N_2 \pm \Delta_2](e^{-i\epsilon\chi_r} - 1)\}}, \tag{25}$$

where $\mathbb{T} = \frac{4\Gamma_1\Gamma_2}{(\Gamma_1+\Gamma_2)^2}$.

The statistics of energy flux flowing between the two reservoirs can be computed using the above scaled-cumulant-generating function. The steady-state average flux is obtained as $\lim_{t\to\infty} \frac{\langle \Delta e_r \rangle}{t} = \frac{\Gamma_1\Gamma_2}{\Gamma_1+\Gamma_2}[N_2 - N_1]$. For $N_1 = N_2 = N$, the energy flux between the reservoirs vanishes; however, it turns out that the probability function is not symmetric (i.e., skewed) around the origin ($n = 0$), as the third cumulant, $\lim_{t\to\infty} \frac{\langle \Delta e_r^3 \rangle_c}{t} = 6\frac{\Gamma_1^2\Gamma_2^2}{(\Gamma_1+\Gamma_2)^3}[1+2N][\Delta_2^2 - \Delta_1^2]$, is nonzero. Hence according to the two-point measurement scheme analysis, two squeezed thermal reservoirs can be considered at mutual equilibrium (fluctuating energy flow is time-reversal symmetric, i.e., energy flow between the two systems in both the directions is equally likely) if their temperatures and squeezing amplitudes are the same, although their phases may be different.

The marginal distribution function for the energy flow between reservoirs is then given as $P[\Delta e_r, t] = \int_{-\infty}^{+\infty} \frac{d\chi_r}{2\pi} e^{\mathcal{F}[\chi_r]t + i\chi_r \Delta e_r} = \sum_{n\in\mathbb{Z}} p[n,t]\delta[\Delta e_r - n\epsilon]$, with $p[n,t] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} d\chi_r e^{\mathcal{F}[\chi_r/\epsilon]t + i\chi_r n}$. The second equality in the above equation is a result of the periodicity of $\mathcal{F}[\chi_r + \frac{2\pi}{\epsilon}] = \mathcal{F}[\chi_r]$.

In the long-time limit, we can define a large-deviation rate function $J[\frac{n}{t}] = -\lim_{t\to\infty} \frac{\ln p[n,t]}{t}$, such that $p[n,t] \overset{t\to\infty}{\approx} e^{-J[\frac{n}{t}]t}$ [5,66].

The marginal probability function and the corresponding rate function for the energy flow between reservoirs in the long-time limit are plotted in the upper and the lower panels of Fig. 3, respectively. The insets of these plots show, respectively, $\ln \frac{p[+n,t]}{p[-n,t]}$ vs n and $J[-\frac{n}{t}] - J[\frac{n}{t}]$ vs $\frac{n}{t}$, which are both linear functions indicating the presence of the Gallavotti-Cohen symmetry, $\mathcal{F}[-\chi_r - i\alpha_{12}] = \mathcal{F}[\chi_r]$, and hence the steady-state fluctuation theorem $\frac{p[n,\infty]}{p[-n,\infty]} = e^{\alpha_{12}\epsilon n}$ for the marginal probability ($p[n,t]$). Although we were not able to identify an analytical form for the thermodynamic affinity (α_{12}) for a general parameter set, due to the complexity of the steady-state cumulant-generating function, Eq. (24) along with Eq. (25), our extensive numerical computations indicate the existence of a well-defined thermodynamic affinity and a steady-state energy exchange fluctuation theorem. However, we note that, for a special set of parameters, $Z_1 = \frac{\ln[1+2n_1]}{2}$ and $Z_2 = \frac{\ln[1+2n_2]}{2}$, such that $\Delta_1 = N_1$ and $\Delta_2 = N_2$ (hence $\Lambda_-[\chi_r] = 1$), an analytical expression for the thermodynamic affinity, $\alpha_{12} = \frac{1}{\epsilon}\ln\frac{N_2(1+2N_1)}{(1+2N_2)N_1}$, can be identified. Our numerical calculations for a general parameter set indicate that the affinity (α_{12}) is generically not a universal function of the reservoir parameters (temperatures and squeezing ampli-

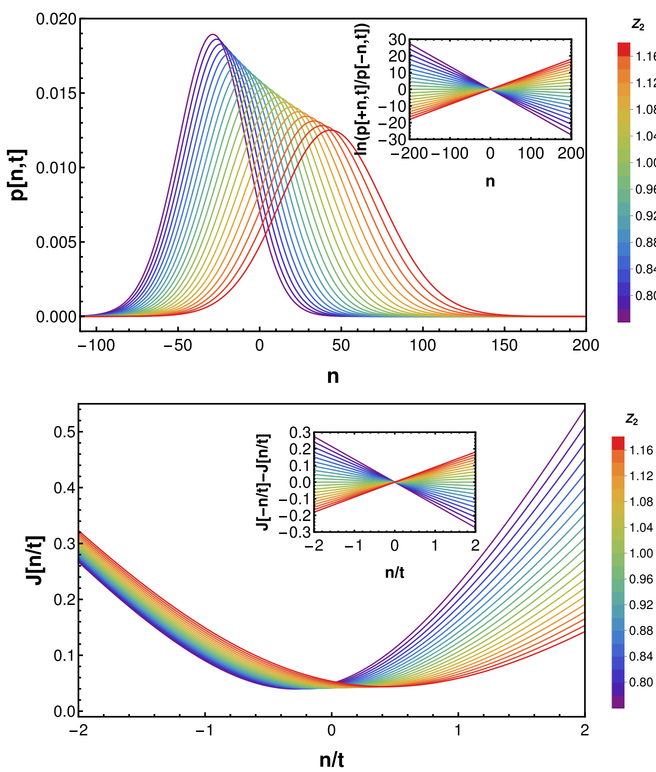

FIG. 3. Upper panel: Marginal probability distribution function ($p[n,t]$) and (lower panel) the corresponding large deviation rate function ($J[n/t]$) for the number of quanta of energy exchanged between the reservoirs for a range of Z_2 values in steady state. Parameters used are $(\frac{\Gamma_1+\Gamma_2}{2})t = 100.0$, $\mathbb{T} = 1$, $\beta_1\epsilon = \beta_2\epsilon = 100.0$, and $Z_1 = 1.0$. Inset (upper panel) shows the linearity of $\ln(p[+n,t]/p[-n,t])$ vs n and (lower panel) the linearity of $J[-\frac{n}{t}] - J[\frac{n}{t}]$ vs $\frac{n}{t}$ ($t \equiv [(\Gamma_1+\Gamma_2)/2]t$).

tudes). Although it is independent of the system-reservoir couplings, it depends on the cavity mode frequency, which is also evident from the above analytically identified expression for the affinity for the special set of parameters.

Thus unlike in the single reservoir case, where the long-time energy exchange (Jarzynski-Wójcik) fluctuation theorem, for the energy flow between the system and the reservoir, was recovered only for a special set of parameters, in the two-reservoir case, our numerical results indicate that the steady-state energy exchange fluctuation theorem (with a nonuniversal affinity), for the steady-state energy flow between the two reservoirs, is satisfied for all numerically explored parameter values, although explicit analytical expression for thermodynamic affinity could only be identified for a special set of parameters.

V. CONCLUSION

A formalism for the analytical computation of the full counting statistics of energy exchanged between a cavity weakly coupled to an arbitrary number of squeezed thermal photon reservoirs within the two-point measurement scheme is developed. The crucial result of the formalism is Eq. (10) for the moment-generating function for the energy exchange, which can be considered as the dissipative generalization of the Levitov-Lesovik-Klich formula. This formula is applied to two model systems: a single mode cavity in contact with a single squeezed thermal reservoir and two squeezed thermal reservoirs. It is found that the careful treatment of the initial projective measurement is necessary for getting physically meaningful probabilities for energy transport statistics at short times, although irrelevant for long-time scales. Generically, the full distribution function for the energy transfers is found not to satisfy transient energy exchange fluctuation theorems. For the single-reservoir case, a special parameter regime given by Eq. (22) is identified for which a steady-state Jarzynski-Wójcik energy exchange fluctuation theorem, for the energy flow from the system into the reservoir, is satisfied. Contrary to this, for the two-reservoir case, numerical results indicate that the steady-state energy exchange fluctuation theorem, for the steady-state energy flow between the two reservoirs, with a nonuniversal affinity is valid always. Furthermore, the analysis of the cumulants indicates that it is generically not possible to describe squeezed thermal reservoirs with an effective temperature, and two-squeezed thermal reservoirs cannot be considered as at equilibrium (fluctuating energy flow is time-reversal symmetric, i.e., energy flow between the two systems in both the directions is equally likely) even if there is no average energy flux between them and can be considered at equilibrium only if their temperatures and squeezing amplitudes are the same.

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APPENDIX

1. Derivation of Eq. (10) using Wigner phase-space representation

Instead of solving Eq. (9) for $\rho_{SI}^{\Lambda}(t)$, we find it convenient to solve for the counting-field-dependent system's Wigner function, $\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon},t]$ $[\boldsymbol{\Upsilon}=(\gamma_1^* \;\; \cdots \;\; \gamma_M^* \;\; \gamma_1 \;\; \cdots \;\; \gamma_M)^T]$, in the interaction picture [30,39,43,67–71]. This is defined as the Fourier transform of the Weyl (symmetric ordered moment) generating function [30,39,43,67–70] for the system,

$$\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon},t]=\frac{1}{\pi^{2M}}\int\mathcal{D}[\mathbf{W}]\underbrace{\mathbf{Tr}_S\big[e^{i\mathbf{W}^{\dagger}\mathbf{B}_S}\rho_{SI}^{\Lambda}(t)\big]}_{\text{Weyl generating function}}e^{-i\mathbf{W}^{\dagger}\boldsymbol{\Upsilon}},\tag{A1}$$

where $\mathbf{W}=(w_1^* \;\; \cdots \;\; w_M^* \;\; w_1 \;\; \cdots \;\; w_M)^T$ and $\int\mathcal{D}[\boldsymbol{W}]=\int_{-\infty}^{+\infty}d[\mathrm{Re}(w_1)\int_{-\infty}^{+\infty}d[\mathrm{Im}(w_1)]\cdots\int_{-\infty}^{+\infty}d[\mathrm{Re}(w_M)\int_{-\infty}^{+\infty}d[\mathrm{Im}(w_M)]$.

$\tilde{\mathcal{Z}}[\boldsymbol{\chi},\boldsymbol{\lambda},t]$ defined in Eq. (8) is then expressed in terms of $\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon},t]$ as

$$\tilde{\mathcal{Z}}[\boldsymbol{\chi},\boldsymbol{\lambda},t]=\int\mathcal{D}[\boldsymbol{\Upsilon}]\,\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon},t],\tag{A2}$$

where the shorthand notation $\int\mathcal{D}[\boldsymbol{\Upsilon}]=\int_{-\infty}^{+\infty}d[\mathrm{Re}(\gamma_1)\int_{-\infty}^{+\infty}d[\mathrm{Im}(\gamma_1)]\cdots\int_{-\infty}^{+\infty}d[\mathrm{Re}(\gamma_M)\int_{-\infty}^{+\infty}d[\mathrm{Im}(\gamma_M)]$ is introduced.

Using the counting-field-dependent Lindblad quantum master equation given in Eq. (9), an evolution equation for the counting-field-dependent system's Wigner function, $\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon},t]$, is obtained as [30,39,43,71]

$$\frac{\partial}{\partial t}\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon},t]=\frac{1}{2}\bigg[\begin{pmatrix}\boldsymbol{\Upsilon}\\\boldsymbol{\nabla}_{\boldsymbol{\Upsilon}}\end{pmatrix}^T\mathbb{H}\begin{pmatrix}\boldsymbol{\Upsilon}\\\boldsymbol{\nabla}_{\boldsymbol{\Upsilon}}\end{pmatrix}+\mathbf{Tr}[\Gamma]\bigg]\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon},t],\tag{A3}$$

where $\boldsymbol{\nabla}_{\boldsymbol{\Upsilon}}=(\frac{\partial}{\partial\gamma_1^*} \;\; \cdots \;\; \frac{\partial}{\partial\gamma_M^*} \;\; \frac{\partial}{\partial\gamma_1} \;\; \cdots \;\; \frac{\partial}{\partial\gamma_M})^T$; Γ and $\mathbb{H}$ are defined below Eq. (10).

The parabolic partial differential equation, Eq. (A3), can be analytically solved. A brief description of two methods that can be used to solve this class of equations is given in the following section. It is to be noted that similar types of partial differential equations also appeared in the studies of heat current fluctuations through classical harmonic chains [72,73] and work statistics of driven classical harmonic oscillators subjected to thermal noise [74,75]. Also, a related partial differential equation is encountered in the study of the work statistics of a degenerate parametric amplification process [76].

The solution of Eq. (A3), obtained in the following section, is given in terms of a Green's function as

$$\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon}, t] = \int \mathcal{D}[\boldsymbol{\Upsilon}']\mathbb{G}[\boldsymbol{\Upsilon}, t|\boldsymbol{\Upsilon}', 0]\mathbb{P}_S[\boldsymbol{\Upsilon}', 0], \tag{A4}$$

with the Green's function given by

$$\mathbb{G}[\boldsymbol{\Upsilon}, t|\boldsymbol{\Upsilon}', 0] = \frac{1}{\pi^M}\frac{e^{(1/2)\mathbf{Tr}[\Gamma]t}}{\sqrt{\mathbf{Det}[\mathbb{U}_{21}(t)\boldsymbol{\sigma}_x]}}e^{-(1/2)\{\boldsymbol{\Upsilon}^T[\mathbb{U}_{12}(t)\mathbb{U}_{22}(t)^{-1}]\boldsymbol{\Upsilon}+[\boldsymbol{\Upsilon}-\mathbb{U}_{22}(t)\boldsymbol{\Upsilon}']^T[\mathbb{U}_{21}(t)\mathbb{U}_{22}(t)^T]^{-1}[\boldsymbol{\Upsilon}-\mathbb{U}_{22}(t)\boldsymbol{\Upsilon}']\}}. \tag{A5}$$

$\mathbb{U}_{xy}(t)$ and $\boldsymbol{\Sigma}$ are defined in and below Eq. (11), respectively.

The Wigner function corresponding to the system's initial state, the squeezed thermal state [44], is given as

$$\mathbb{P}_S[\boldsymbol{\Upsilon}, 0] = \frac{1}{\pi^M}\frac{1}{\sqrt{\mathbf{Det}[\mathbf{D}_S\boldsymbol{\sigma}_x]}}e^{-(1/2)\boldsymbol{\Upsilon}^T\mathbf{D}_S{}^{-1}\boldsymbol{\Upsilon}}, \tag{A6}$$

with $\mathbf{D}_S$ defined below Eq. (13). Using this in Eq. (A4) and performing $\boldsymbol{\Upsilon}'$ Gaussian integral along with the use of identities derived from the symplectic property of $\mathbb{U}(t)$, i.e., $\mathbb{U}(t)^T\boldsymbol{\Sigma}\mathbb{U}(t) = \boldsymbol{\Sigma}$, an explicit form of the time-dependent Wigner function is obtained. This is given as

$$\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon}, t] = \frac{1}{\pi^M}\frac{e^{(1/2)\mathbf{Tr}[\Gamma]t}e^{-(1/2)\boldsymbol{\Upsilon}^T\{[\mathbb{U}_{11}(t)+\mathbb{U}_{12}(t)\mathbf{D}_S][\mathbb{U}_{21}(t)+\mathbb{U}_{22}(t)\mathbf{D}_S]^{-1}\}\boldsymbol{\Upsilon}}}{\sqrt{\mathbf{Det}[[\mathbb{U}_{21}(t)+\mathbb{U}_{22}(t)\mathbf{D}_S]\boldsymbol{\sigma}_x]}}. \tag{A7}$$

Using this in Eq. (A2), and performing the Gaussian $\boldsymbol{\Upsilon}$ integral gives Eq. (10).

2. Counting-field-independent Wigner function of the system

The Wigner function of the system, in the interaction picture, for the case $\boldsymbol{\chi} = \boldsymbol{\lambda} = \mathbf{0}$, i.e., in the absence of two-point measurements, is given as

$$\mathbb{P}_{SI}^{0}[\boldsymbol{\Upsilon}, t] = \frac{1}{\pi^M}\frac{1}{\sqrt{\mathbf{Det}[\mathbf{D}_S(t)\boldsymbol{\sigma}_x]}}e^{-(1/2)\boldsymbol{\Upsilon}^T\mathbf{D}_S(t)\boldsymbol{\Upsilon}}, \tag{A8}$$

with

$$\mathbf{D}_S(t) = e^{-(1/2)[\sum_{\alpha=1}^N\boldsymbol{\Gamma}_\alpha]t}\mathbf{D}_S e^{-(1/2)[\sum_{\alpha=1}^N\boldsymbol{\Gamma}_\alpha]t} + \int_0^t ds\, e^{-(1/2)[\sum_{\alpha=1}^N\boldsymbol{\Gamma}_\alpha]s}\left[\sum_{\alpha=1}^N\boldsymbol{\Gamma}_\alpha\mathbf{D}_\alpha\right]e^{-(1/2)[\sum_{\alpha=1}^N\boldsymbol{\Gamma}_\alpha]s}. \tag{A9}$$

3. Methods for solving PDEs in Eq. (A3)

In this section, we provide a brief sketch of two methods to solve the parabolic partial differential equation of the form in Eq. (A3).

a. Method I

In this section we sketch a way to solve parabolic partial differential equations of the form

$$\frac{\partial}{\partial t}\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon}, t] = \frac{1}{2}\left[\begin{pmatrix}\boldsymbol{\Upsilon}\\ \boldsymbol{\nabla}_{\boldsymbol{\Upsilon}}\end{pmatrix}^T\mathbb{H}\begin{pmatrix}\boldsymbol{\Upsilon}\\ \boldsymbol{\nabla}_{\boldsymbol{\Upsilon}}\end{pmatrix} + \mathbf{Tr}[\Gamma]\right]\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon}, t], \tag{A10}$$

with the initial condition $\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon}, t]|_{t=0} = \mathbb{P}_S[\boldsymbol{\Upsilon}, 0]$. Here $\boldsymbol{\Upsilon} = (\gamma_1^* \;\cdots\; \gamma_M^* \;\; \gamma_1 \;\cdots\; \gamma_M)^T$, $\boldsymbol{\nabla}_{\boldsymbol{\Upsilon}} = (\frac{\partial}{\partial\gamma_1^*} \;\cdots\; \frac{\partial}{\partial\gamma_M^*} \;\; \frac{\partial}{\partial\gamma_1} \;\cdots\; \frac{\partial}{\partial\gamma_M})^T$, and $\mathbb{H} = \begin{pmatrix}\mathbb{H}_{11} & \mathbb{H}_{12}\\ \mathbb{H}_{21} & \mathbb{H}_{22}\end{pmatrix}$ is a 2×2 block partitioned $4M\times 4M$ complex symmetric matrix independent of $\boldsymbol{\Upsilon}$ and t. If $\mathbb{H}_{11} = \mathbf{O}_{2M\times 2M}$, the above equation is of the standard Ornstein-Uhlenbeck form, whose solution can be found in the Fourier domain by using the method of characteristics [43,77–80]. For $\mathbb{H}_{11} \neq \mathbf{O}_{2M\times 2M}$, the quadratic term in the above equation can be eliminated using the transformation [81]

$$\mathbb{P}_{SI}^{\Lambda}[\boldsymbol{\Upsilon}, t] = e^{(1/2)[\boldsymbol{\Upsilon}^T\mathbb{R}(t)\boldsymbol{\Upsilon}+\mathbf{Tr}[\Gamma]t]}\bar{\mathbb{P}}_{SI}^{\Lambda}[\boldsymbol{\Upsilon}, t], \tag{A11}$$

where without loss of generality, we can assume $\mathbb{R}^T(t) = \mathbb{R}(t)$ and the requirement that $\mathbb{R}(t)$ satisfies the following Riccati matrix differential equation [82–88],

$$\frac{d}{dt}\mathbb{R}(t) = \mathbb{R}(t)\mathbb{H}_{22}\mathbb{R}(t) + \mathbb{H}_{12}\mathbb{R}(t) + \mathbb{R}(t)\mathbb{H}_{21} + \mathbb{H}_{11}, \tag{A12}$$

with the initial condition $\mathbb{R}(t)|_{t=0} = \mathbf{O}_{2M\times 2M}$. The solution of this Riccati matrix differential equation is given as

$$\mathbb{R}(t) = -\mathbb{U}_{12}(t)\mathbb{U}_{22}(t)^{-1}, \tag{A13}$$

where $\mathbb{U}_{xy}(t)$ are $2M \times 2M$ matrices defined as the 2×2 blocks of $\mathbb{U}(t)$, defined as

$$\mathbb{U}(t) = \begin{pmatrix} \mathbb{U}_{11}(t) & \mathbb{U}_{12}(t) \\ \mathbb{U}_{21}(t) & \mathbb{U}_{22}(t) \end{pmatrix} = e^{-\mathbb{H}\,\boldsymbol{\Sigma} t}, \tag{A14}$$

with the standard symplectic matrix defined as $\boldsymbol{\Sigma} = i\sigma_y \otimes \mathbf{I}_{2M\times 2M}$.

Using the symplectic property $\mathbb{U}^T(t)\boldsymbol{\Sigma}\mathbb{U}(t) = \boldsymbol{\Sigma}$ and the equation $\frac{d}{dt}\mathbb{U}(t) = -\mathbb{H}\,\boldsymbol{\Sigma}\,\mathbb{U}(t)$ [with $\mathbb{U}(t)|_{t=0} = \mathbf{I}_{4M\times 4M}$], $\bar{\mathbb{P}}^{\Lambda}_{SI}[\boldsymbol{\Upsilon}, t]$ can be shown to satisfy the following parabolic partial differential equation of Ornstein-Uhlenbeck type,

$$\frac{\partial}{\partial t}\bar{\mathbb{P}}^{\Lambda}_{SI}[\boldsymbol{\Upsilon}, t] = \frac{1}{2}\left[\begin{pmatrix} \boldsymbol{\Upsilon} \\ \boldsymbol{\nabla}_{\boldsymbol{\Upsilon}} \end{pmatrix}^T \begin{pmatrix} \mathbf{O} & \mathbb{H}_{12} + \mathbb{R}(t)\mathbb{H}_{22} \\ \mathbb{H}_{21} + \mathbb{H}_{22}\mathbb{R}(t) & \mathbb{H}_{22} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Upsilon} \\ \boldsymbol{\nabla}_{\boldsymbol{\Upsilon}} \end{pmatrix}\right]\bar{\mathbb{P}}^{\Lambda}_{SI}[\boldsymbol{\Upsilon}, t], \tag{A15}$$

where $\mathbf{O}$ is the $(2M \times 2M)$-dimensional null matrix. This equation can be solved in Fourier space using the method of characteristics [43,77–80] [simplifications while applying this procedure can be achieved by using the properties of $\mathbb{U}(t)$], which when Fourier transformed back, we get $\bar{\mathbb{P}}^{\Lambda}_{SI}[\boldsymbol{\Upsilon}, t]$. Using thus obtained solution, $\bar{\mathbb{P}}^{\Lambda}_{SI}[\boldsymbol{\Upsilon}, t]$, $\mathbb{P}^{\Lambda}_{SI}[\boldsymbol{\Upsilon}, t]$ is given as

$$\mathbb{P}^{\Lambda}_{SI}[\boldsymbol{\Upsilon}, t] = \int \mathcal{D}[\boldsymbol{\Upsilon}']\mathbb{G}[\boldsymbol{\Upsilon}, t|\boldsymbol{\Upsilon}', 0]\mathbb{P}[\boldsymbol{\Upsilon}', 0], \tag{A16}$$

with the Green's function or the propagator given by

$$\mathbb{G}[\boldsymbol{\Upsilon}, t|\boldsymbol{\Upsilon}', 0] = \frac{1}{\pi^M}\frac{e^{(1/2)\mathbf{Tr}[\Gamma]t}}{\sqrt{\mathbf{Det}[\mathbb{U}_{21}(t)\boldsymbol{\sigma}_x]}} e^{-(1/2)\{\boldsymbol{\Upsilon}^T[\mathbb{U}_{12}(t)\mathbb{U}_{22}(t)^{-1}]\boldsymbol{\Upsilon} + [\boldsymbol{\Upsilon} - \mathbb{U}_{22}(t)\boldsymbol{\Upsilon}']^T[\mathbb{U}_{21}(t)\mathbb{U}_{22}(t)^T]^{-1}[\boldsymbol{\Upsilon} - \mathbb{U}_{22}(t)\boldsymbol{\Upsilon}']\}}, \tag{A17}$$

and $\int\mathcal{D}[\boldsymbol{\Upsilon}'] = \int_{-\infty}^{+\infty} d[\mathrm{Re}(\gamma_1')\int_{-\infty}^{+\infty} d[\mathrm{Im}(\gamma_1')]\cdots\int_{-\infty}^{+\infty} d[\mathrm{Re}(\gamma_M')\int_{-\infty}^{+\infty} d[\mathrm{Im}(\gamma_M')]$.

b. Method II

The formal solution of the parabolic partial differential equation [89]

$$\frac{\partial}{\partial t}\mathbb{P}^{\Lambda}_{SI}[\boldsymbol{\Upsilon}, t] = \frac{1}{2}\left[\begin{pmatrix} \boldsymbol{\Upsilon} \\ \boldsymbol{\nabla}_{\boldsymbol{\Upsilon}} \end{pmatrix}^T \mathbb{H}\begin{pmatrix} \boldsymbol{\Upsilon} \\ \boldsymbol{\nabla}_{\boldsymbol{\Upsilon}} \end{pmatrix} + \mathbf{Tr}[\Gamma]\right]\mathbb{P}^{\Lambda}_{SI}[\boldsymbol{\Upsilon}, t] \tag{A18}$$

is

$$\mathbb{P}^{\Lambda}_{SI}[\boldsymbol{\Upsilon}, t] = e^{(1/2)\left[\begin{pmatrix} \boldsymbol{\Upsilon} \\ \boldsymbol{\nabla}_{\boldsymbol{\Upsilon}} \end{pmatrix}^T \mathbb{H}\begin{pmatrix} \boldsymbol{\Upsilon} \\ \boldsymbol{\nabla}_{\boldsymbol{\Upsilon}} \end{pmatrix} + \mathbf{Tr}[\Gamma]\right]t}\mathbb{P}_S[\boldsymbol{\Upsilon}, 0]. \tag{A19}$$

The exponential operator in the above equation can be put in a more manageable form using the Wei-Norman method [90,91] inspired technique [92] as

$$e^{(1/2)\left[\begin{pmatrix} \boldsymbol{\Upsilon} \\ \boldsymbol{\nabla}_{\boldsymbol{\Upsilon}} \end{pmatrix}^T \mathbb{H}\begin{pmatrix} \boldsymbol{\Upsilon} \\ \boldsymbol{\nabla}_{\boldsymbol{\Upsilon}} \end{pmatrix}\right]t} = \frac{1}{\sqrt{\mathbf{Det}[\mathbb{U}_{22}(t)]}} e^{-(1/2)\boldsymbol{\Upsilon}^T[\mathbb{U}_{12}(t)\mathbb{U}_{22}(t)^{-1}]\boldsymbol{\Upsilon}} e^{-\boldsymbol{\Upsilon}^T[\ln\mathbb{U}_{22}(t)^T]\boldsymbol{\nabla}_{\boldsymbol{\Upsilon}}} e^{(1/2)\boldsymbol{\nabla}^T_{\boldsymbol{\Upsilon}}[\mathbb{U}_{22}(t)^{-1}\mathbb{U}_{21}(t)]\boldsymbol{\nabla}_{\boldsymbol{\Upsilon}}}, \tag{A20}$$

where $\mathbb{U}_{xy}(t)$ are the same as defined previously. Using this, $\mathbb{P}^{\Lambda}_{SI}[\boldsymbol{\Upsilon}, t]$ can be expressed [89] in the same form as given previously in Eq. (A16) along with Eq. (A17).

4. Details of Sec. IV A

Using the explicit expressions for $\mathbb{U}_{11}(t)$ and $\mathbb{U}_{12}(t)$ in Eq. (10) with $\boldsymbol{\chi} = \chi_1$ and $\boldsymbol{\lambda} = \lambda_1$ gives

$$\tilde{\mathcal{Z}}[\chi_1, \lambda_1, t] = \frac{e^{\Gamma_1 t/2}}{\sqrt{\mathbf{Det}\left[\cosh\left[\frac{\Gamma_1 t}{2}\right]I + \sinh\left[\frac{\Gamma_1 t}{2}\right]\frac{\Xi_{1S}[\chi_1, \lambda_1]}{\frac{\Gamma_1}{2}}\right]}}, \tag{A21}$$

where $\Xi_{\alpha S}[\chi_\alpha, \lambda_\alpha]$ (here $\alpha = 1$) is given as

$$\Xi_{\alpha S}[\chi_\alpha, \lambda_\alpha] = \frac{\Gamma_\alpha}{2} I - \Gamma_\alpha \times \bigg\{ e^{i\epsilon\lambda_\alpha\sigma_z}\bigg[\sigma_x D_\alpha - \frac{1}{2}I\bigg] e^{-i\epsilon\lambda_\alpha\sigma_z}\bigg[\sigma_x D_S + \frac{1}{2}I\bigg]\big(e^{i\epsilon\chi_\alpha} - 1\big) + e^{i\epsilon\lambda_\alpha\sigma_z}\bigg[\sigma_x D_\alpha + \frac{1}{2}I\bigg] e^{-i\epsilon\lambda_\alpha\sigma_z}\bigg[\sigma_x D_S - \frac{1}{2}I\bigg]\big(e^{-i\epsilon\chi_\alpha} - 1\big)\bigg\}, \tag{A22}$$

with $D_\alpha = -i\sigma_y e^{-iS_\alpha\sigma_y}[n_\alpha(\epsilon\sigma_z) + \frac{1}{2}I]e^{iS_\alpha\sigma_y}$, $\epsilon = \epsilon_1$, $S_\alpha = Z_\alpha\sigma_z e^{i\sigma_z\phi_\alpha}$, and $n_\alpha(x) = (e^{\beta_\alpha x} - 1)^{-1} \equiv n_\alpha$ (for $\alpha = S$ and 1). Substituting this expression for Ξ_{1S} in Eq. (A21) and upon simplification gives Eq. (15).

5. Details of Sec. IV B

Using the explicit expressions for $\mathbb{U}_{11}(t)$ and $\mathbb{U}_{12}(t)$ in Eq. (10), we get an expression for $\tilde{\mathcal{Z}}[\boldsymbol{\chi}, \boldsymbol{\lambda}, t]$ given as

$$\tilde{\mathcal{Z}}[\boldsymbol{\chi}, \boldsymbol{\lambda}, t] = e^{[(\Gamma_1+\Gamma_2)/2]t}\left[1 - \mathbb{X}_{--}[\boldsymbol{\chi}, \boldsymbol{\lambda}] + \begin{bmatrix} \cosh(\Lambda_-[\boldsymbol{\chi}, \boldsymbol{\lambda}]t) \\ \frac{1}{\Lambda_-[\boldsymbol{\chi}, \boldsymbol{\lambda}]}\sinh(\Lambda_-[\boldsymbol{\chi}, \boldsymbol{\lambda}]t)\end{bmatrix}^T \mathbb{X}[\boldsymbol{\chi}, \boldsymbol{\lambda}] \begin{bmatrix} \cosh(\Lambda_+[\boldsymbol{\chi}, \boldsymbol{\lambda}]t) \\ \frac{1}{\Lambda_+[\boldsymbol{\chi}, \boldsymbol{\lambda}]}\sinh(\Lambda_+[\boldsymbol{\chi}, \boldsymbol{\lambda}]t)\end{bmatrix}\right]^{-1/2}, \tag{A23}$$

where $\boldsymbol{\chi} = (\chi_1 \quad \chi_2)^T$, $\boldsymbol{\lambda} = (\lambda_1 \quad \lambda_2)^T$, and

$$\Lambda_\mp[\boldsymbol{\chi}, \boldsymbol{\lambda}] = \sqrt{\left[\frac{\mathbf{Tr}[\Xi_{12}[\boldsymbol{\chi}, \boldsymbol{\lambda}]]}{2}\right] \pm \sqrt{\left[\frac{\mathbf{Tr}[\Xi_{12}[\boldsymbol{\chi}, \boldsymbol{\lambda}]]}{2}\right]^2 - \mathbf{Det}[\Xi_{12}[\boldsymbol{\chi}, \boldsymbol{\lambda}]]}}, \tag{A24}$$

with

$$\Xi_{\alpha\alpha'}[\boldsymbol{\chi}, \boldsymbol{\lambda}] = \left(\frac{\Gamma_\alpha + \Gamma_{\alpha'}}{2}\right)^2 I - \Gamma_\alpha\Gamma_{\alpha'}\bigg\{ e^{i\epsilon\lambda_\alpha\sigma_z}\bigg[\sigma_x D_\alpha - \frac{1}{2}I\bigg] e^{-i\epsilon(\lambda_\alpha - \lambda_{\alpha'})\sigma_z}\bigg[\sigma_x D_{\alpha'} + \frac{1}{2}I\bigg] e^{-i\epsilon\lambda_{\alpha'}\sigma_z}\big(e^{i\epsilon(\chi_\alpha - \chi_{\alpha'})} - 1\big) + e^{i\epsilon\lambda_\alpha\sigma_z}\bigg[\sigma_x D_\alpha + \frac{1}{2}I\bigg] e^{-i\epsilon(\lambda_\alpha - \lambda_{\alpha'})\sigma_z}\bigg[\sigma_x D_{\alpha'} - \frac{1}{2}I\bigg] e^{-i\epsilon\lambda_{\alpha'}\sigma_z}\big(e^{-i\epsilon(\chi_\alpha - \chi_{\alpha'})} - 1\big)\bigg\} \tag{A25}$$

and $\mathbb{X}[\boldsymbol{\chi}, \boldsymbol{\lambda}] = \begin{pmatrix} \mathbb{X}_{--}[\boldsymbol{\chi}, \boldsymbol{\lambda}] & \mathbb{X}_{-+}[\boldsymbol{\chi}, \boldsymbol{\lambda}] \\ \mathbb{X}_{+-}[\boldsymbol{\chi}, \boldsymbol{\lambda}] & \mathbb{X}_{++}[\boldsymbol{\chi}, \boldsymbol{\lambda}] \end{pmatrix}$ is given as

$$\mathbb{X}_{--}[\boldsymbol{\chi}, \boldsymbol{\lambda}] = \frac{1}{2} - \frac{1}{2}\frac{1}{\left(\Lambda_-[\boldsymbol{\chi}, \boldsymbol{\lambda}]^2 - \Lambda_+[\boldsymbol{\chi}, \boldsymbol{\lambda}]^2\right)^2}\mathbf{Det}\left[\sum_{\substack{\alpha,\alpha'=1,2\\ \alpha\neq\alpha'}}\left\{\left(\frac{\Lambda_-[\boldsymbol{\chi}, \boldsymbol{\lambda}]^2 + \Lambda_+[\boldsymbol{\chi}, \boldsymbol{\lambda}]^2}{2}\right)I - \Xi_{\alpha\alpha'}[\boldsymbol{\chi}, \boldsymbol{\lambda}]\right\} + \Xi_{12S}[\boldsymbol{\chi}, \boldsymbol{\lambda}]\right],$$

$$\mathbb{X}_{\mp\pm}[\boldsymbol{\chi}, \boldsymbol{\lambda}] = \frac{1}{2}\mathbf{Tr}\left[\sum_{\alpha=1,2}\Xi_{\alpha S}[\chi_\alpha, \lambda_\alpha]\right] \pm \frac{1}{\left(\Lambda_-[\boldsymbol{\chi}, \boldsymbol{\lambda}]^2 - \Lambda_+[\boldsymbol{\chi}, \boldsymbol{\lambda}]^2\right)}\mathbf{Tr}\left[\sum_{\substack{\alpha,\alpha'=1,2\\ \alpha\neq\alpha'}}\left\{\left(\frac{\Lambda_-[\boldsymbol{\chi}, \boldsymbol{\lambda}]^2 + \Lambda_+[\boldsymbol{\chi}, \boldsymbol{\lambda}]^2}{2}\right)I - \Xi_{\alpha\alpha'}[\boldsymbol{\chi}, \boldsymbol{\lambda}]\right\}\Xi_{\alpha S}[\chi_\alpha, \lambda_\alpha]\right],$$

and

$$\mathbb{X}_{++}[\boldsymbol{\chi}, \boldsymbol{\lambda}] = \mathbf{Det}\left[\sum_{\alpha=1,2}\Xi_{\alpha S}[\chi_\alpha, \lambda_\alpha]\right] + \left(\frac{\Lambda_-[\boldsymbol{\chi}, \boldsymbol{\lambda}]^2 + \Lambda_+[\boldsymbol{\chi}, \boldsymbol{\lambda}]^2}{2}\right)(1 - \mathbb{X}_{--}[\boldsymbol{\chi}, \boldsymbol{\lambda}]), \tag{A26}$$

with

$$\Xi_{12S}[\boldsymbol{\chi}, \boldsymbol{\lambda}] = \Gamma_1\Gamma_2[e^{i\epsilon\lambda_1\sigma_z}\sigma_x D_1 e^{-i\epsilon\lambda_1\sigma_z}, e^{i\epsilon\lambda_2\sigma_z}\sigma_x D_2 e^{-i\epsilon\lambda_2\sigma_z}] \times \bigg\{\bigg[\sigma_x D_S - \frac{1}{2}I\bigg][(e^{-i\epsilon\chi_1} - 1) - (e^{-i\epsilon\chi_2} - 1)][(e^{i\epsilon\chi_1} - 1) + (e^{i\epsilon\chi_2} - 1)] - \bigg[\sigma_x D_S + \frac{1}{2}I\bigg][(e^{i\epsilon\chi_1} - 1) - (e^{i\epsilon\chi_2} - 1)][(e^{-i\epsilon\chi_1} - 1) + (e^{-i\epsilon\chi_2} - 1)]\bigg\} \tag{A27}$$

and $\Xi_{\alpha S}[\chi_\alpha, \lambda_\alpha]$ is given in Eq. (A22).

In the above equations, $D_\alpha = -i\sigma_y e^{-iS_\alpha\sigma_y}[n_\alpha(\epsilon\sigma_z) + \frac{1}{2}I]e^{iS_\alpha\sigma_y}$, $\epsilon = \epsilon_1$, $S_\alpha = Z_\alpha\sigma_z e^{i\sigma_z\phi_\alpha}$, and $n_\alpha(x) = (e^{\beta_\alpha x} - 1)^{-1} \equiv n_\alpha$ (for $\alpha = S,\ 1,\ \text{and}\ 2$).

$\tilde{\mathcal{Z}}[\boldsymbol{\chi}, \boldsymbol{\lambda}, t]$ is a periodic function of both λ_1 and λ_2 with period $\frac{2\pi}{\epsilon}$. Hence the joint moment-generating function $\mathcal{Z}[\boldsymbol{\chi}, t] = \int_{\boldsymbol{\lambda}\in\mathbb{R}^2} \frac{d^2\boldsymbol{\lambda}}{(2\pi)^2} \tilde{\mathcal{Z}}[\boldsymbol{\chi}, \boldsymbol{\lambda}, t]$ diverges. To make $\mathcal{Z}[\boldsymbol{\chi}, t]$ a proper moment-generating function, we introduce two cutoffs in $\boldsymbol{\lambda}$ integrals, renormalize $\mathcal{Z}[\boldsymbol{\chi}, t]$ by $\mathcal{Z}[\mathbf{0}, t]$, and send the cutoffs to infinity to obtain the following expression:

$$\mathcal{Z}[\boldsymbol{\chi}, t] = \left(\frac{\epsilon}{2\pi}\right)^2 \int\limits_{\boldsymbol{\lambda}\in[-\pi/\epsilon, +\pi/\epsilon]^2} d^2\boldsymbol{\lambda}\, \tilde{\mathcal{Z}}[\boldsymbol{\chi}, \boldsymbol{\lambda}, t]. \tag{A28}$$

Furthermore, by doing the change of variables $\lambda_{1/2} \to \lambda_{1/2} - (\frac{\phi_{1/2}-\phi_S}{2\epsilon})$ and using the periodic property of $\tilde{\mathcal{Z}}[\boldsymbol{\chi}, \boldsymbol{\lambda}, t]$ with respect to $\lambda_{1/2}$, it can be shown that the squeezing phases of the initial states of the system (ϕ_S) and both the reservoirs (ϕ_1 and ϕ_2) do not affect the statistics of the energy flow from the system into the reservoirs.

The marginal moment-generating function corresponding to Δe_s (defined in the main text), $\mathcal{Z}_s[\chi_s, t] = \bar{\mathcal{Z}}[0, \chi_s, t]$, is obtained as

$$\mathcal{Z}_s[\chi_s, t] = \left(\frac{\epsilon}{2\pi}\right)^2 \int\limits_{\boldsymbol{\lambda}\in[-\pi/\epsilon, +\pi/\epsilon]^2} d^2\boldsymbol{\lambda}\, \tilde{\mathcal{Z}}_s[\chi_s, \boldsymbol{\lambda}, t], \tag{A29}$$

with

$$\tilde{\mathcal{Z}}_s[\chi_s, \boldsymbol{\lambda}, t] = \frac{e^{[(\Gamma_1+\Gamma_2)/2]t}}{\sqrt{\mathbf{Det}\left\{\cosh\left[\left(\frac{\Gamma_1+\Gamma_2}{2}\right)t\right]I + \sinh\left[\left(\frac{\Gamma_1+\Gamma_2}{2}\right)t\right]\frac{\Xi_{RS}[\chi_s,\boldsymbol{\lambda}]}{\left(\frac{\Gamma_1+\Gamma_2}{2}\right)}\right\}}}, \tag{A30}$$

where

$$\begin{aligned}\Xi_{RS}[\chi_s, \boldsymbol{\lambda}] = {}& \left[\sum_{\alpha=1}^{2}\frac{\Gamma_\alpha}{2}\right]I - \left\{\sum_{\alpha=1}^{2}\Gamma_\alpha\left[e^{i\epsilon\lambda_\alpha\sigma_z}\left(\sigma_x D_\alpha - \frac{1}{2}I\right)e^{-i\epsilon\lambda_\alpha\sigma_z}\right]\left[\sigma_x D_S + \frac{1}{2}I\right](e^{i\epsilon\chi_s} - 1)\right.\\ &\left. + \sum_{\alpha=1}^{2}\Gamma_\alpha\left[e^{i\epsilon\lambda_\alpha\sigma_z}\left(\sigma_x D_\alpha + \frac{1}{2}I\right)e^{-i\epsilon\lambda_\alpha\sigma_z}\right]\left[\sigma_x D_S - \frac{1}{2}I\right](e^{-i\epsilon\chi_s} - 1)\right\}.\end{aligned} \tag{A31}$$

The expression for $\tilde{\mathcal{Z}}_s[\chi_s, \boldsymbol{\lambda}, t]$ given above in Eq. (A30), apart from its dependence on λ_1 and λ_2, has a similar mathematical structure as for the moment-generating function for energy transfer in the presence of a single reservoir. This indicates that the dynamical behavior of the statistics of the system's energy loss to reservoirs is similar to the case of a single reservoir. Further, from Eq. (A30), it is clear that $\lim_{t\to\infty}\mathcal{Z}_s[\chi_s, t]$ is finite, indicating that the statistics of Δe_s also becomes independent of time in the long-time limit. This indicates that the fluctuations of energy flow out of the system saturate with time as the system reaches steady state.

The marginal distribution of the energy flow from the reservoir "2" into the reservoir "1" (Δe_r) (defined in the main text), i.e., $P[\Delta e_r, t]$ in the $t \to \infty$, becomes simpler in the long-time limit.

In the long-time limit ($t \to \infty$), the moment-generating function corresponding to the energy flow (Δe_r), defined as $\mathcal{Z}_r[\chi_r, t] = \bar{\mathcal{Z}}[\chi_r, 0, t]$, is obtained by substituting the leading term of Eq. (A23) into Eq. (A28). This is given as

$$\begin{aligned}\mathcal{Z}_r[\chi_r, t] = \left(\frac{\epsilon}{2\pi}\right)^2 \int\limits_{\boldsymbol{\lambda}\in[-\pi/\epsilon, +\pi/\epsilon]^2} d^2\boldsymbol{\lambda}\, 2&\left\{\left[\mathbb{X}_{--}[\boldsymbol{\chi}, \boldsymbol{\lambda}] + \frac{\mathbb{X}_{-+}[\boldsymbol{\chi}, \boldsymbol{\lambda}]}{\Lambda_+[\boldsymbol{\chi}, \boldsymbol{\lambda}]} + \frac{\mathbb{X}_{+-}[\boldsymbol{\chi}, \boldsymbol{\lambda}]}{\Lambda_-[\boldsymbol{\chi}, \boldsymbol{\lambda}]} + \frac{\mathbb{X}_{++}[\boldsymbol{\chi}, \boldsymbol{\lambda}]}{\Lambda_+[\boldsymbol{\chi}, \boldsymbol{\lambda}]\Lambda_-[\boldsymbol{\chi}, \boldsymbol{\lambda}]}\right]^{-1/2}\right.\\ &\left.\left. \times\, e^{([(\Gamma_1+\Gamma_2)/2] - [(\Lambda_+[\boldsymbol{\chi},\boldsymbol{\lambda}] + \Lambda_-[\boldsymbol{\chi},\boldsymbol{\lambda}])/2])t}\right\}\right|_{\chi_{1/2}\to\pm(\chi_r/2)}.\end{aligned} \tag{A32}$$

As noted already, the squeezing phases can be gauged to zero by shifting the integration variables $\boldsymbol{\lambda}$ in Eq. (A32), and hence we can set $\phi_S = \phi_1 = \phi_2 = 0$.

For performing $\boldsymbol{\lambda}$ integrals, it is convenient to change the integration variables to $\lambda = \lambda_1 - \lambda_2$ and $\bar{\lambda} = \frac{\lambda_1+\lambda_2}{2}$. Although $\Lambda_\pm[\boldsymbol{\chi}, \boldsymbol{\lambda}]$ depends only on λ [this can be seen from Eq. (A24) along with Eq. (A25)], $\mathbb{X}_{\pm\pm}[\boldsymbol{\chi}, \boldsymbol{\lambda}]$ depend on both λ and $\bar{\lambda}$. However, when the system's initial state is not squeezed, i.e., $Z_S = 0$, $\mathbb{X}_{\pm\pm}[\boldsymbol{\chi}, \boldsymbol{\lambda}]$ becomes independent of $\bar{\lambda}$. This is because the simultaneous measurements of both the reservoirs' energies (in the weak-coupling limit) is equivalent to measuring the system's energy and the difference of energies of the two reservoirs. The $\bar{\lambda}$ dependence, which accounts for the noncommutativity of the initial system's energy measurement with the initial system's density matrix, drops out as the system's initial state commutes with the initial energy projective measurement for this case. We focus on the statistics at steady state where the system's initial state does not play a role. Therefore, for simplification purposes, we consider the case where the system's initial state is a thermal state. For this case, $\bar{\lambda}$ in Eq. (A32) can be integrated out, leaving only the λ integral behind, which, in the long-time limit, is performed in the saddle-point approximation. The saddle point of the exponent in Eq. (A32) is found at $\lambda = 0$. This finally gives the steady-state scaled cumulant-generating function given in Eq. (24) along with Eq. (25).

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