Robust coincidence Bell inequalities for the noisy *n*-qutrit Greenberger-Horne-Zeilinger state

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The general form of the coincidence Bell inequalities for an arbitrary $(n, 2, 3)$ scenario [i.e., an n-party, two-setting, and three-dimensional system (qutrit) scenario] is presented. To detect the nonlocal properties of the noisy n-qutrit Greenberger-Horne-Zeilinger states, in this work we investigate the most robust (4,2,3)-scenario and (5,2,3)-scenario coincidence Bell inequalities. By deriving the most robust $(n-1)$-party coincidence inequalities, we have established two tight and inequivalent $(n, 2, 3)$ coincidence Bell inequalities with a visibility of 0.5 for $n = 4$ and, similarly, two tight and inequivalent $(n, 2, 3)$ coincidence Bell inequalities with a visibility of 0.488785 for $n = 5$. These inequalities are all tight. To our knowledge, up to now these inequalities have been the most robust Bell inequalities for the corresponding scenarios. Our results are useful for building the iteration formula of $(n, 2, 3)$-scenario coincidence Bell inequalities with the lowest critical visibility.

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I. INTRODUCTION

Bell nonlocality describes the fact that not all quantum-mechanical correlations can be predicted by local-hidden-variable (LHV) models [1,2]. The Bell inequality is at the heart of the study of nonlocality and is the most famous legacy of the late physicist John S. Bell [3,4]. For the simplest composite quantum system, namely, a system of two two-dimensional particles (or two qubits), the Clauser-Horne-Shimony-Holt (CHSH) inequality [5] has a more amenable form for experimental verification. Since then, Bell's arguments have been generalized to more complicated situations, either for a larger number of particles [6–8] or more measurement settings [9] or for two particles of dimension higher than 2 [10–13]. Bell inequalities have led to surprising insights into quantum information processing. Examples are the connection between Bell inequalities and communication complexity [14,15] and Bell inequalities that are useful for multiparty conference key agreement [16].

For the bipartite d-dimensional systems (two-qudit), Collins *et al.* presented a family of tight Bell inequalities, which is called the Collins-Gisin-Linden-Massar-Popescu (CGLMP) inequality [11] and belongs to the $(2, 2, d)$-scenario Bell inequality. Independently, in [17], Kaszlikowski *et al.* also introduced a tight coincidence Bell inequality for two qutrits which is equivalent to the CGLMP inequality for $d = 3$. However, when one moves to the multibody and high-dimensional cases, it is difficult to identify all tight Bell inequalities, i.e., facets of the convex polytope of local-realistic models, due to the computational complexity of characterizing the set of local correlations [18]. Interested in the robustness of the Greenberger-Horne-Zeilinger (GHZ) state $|\Psi\rangle$ to the mixture of white noise, the researchers in Ref. [12] presented a coincidence Bell inequality for three qutrits with the lowest critical visibility $v_c = 0.6$. When $v > v_c$, the noisy GHZ state $\rho = v|\Psi\rangle\langle\Psi| + (1 - v)\,\mathbb{I}/27$ violates the three-qutrit Bell inequality. Here $\mathbb{I}$ denotes the unit matrix for the corresponding quantum system, and $\mathbb{I}/27$ represents the white noise for the three-qutrit system. To the best of our knowledge, it is the most robust Bell inequality for the (3,2,3)-scenario at present. Moreover, the three-qutrit inequality is tight, and from the insight of iteration, it can reduce to the CGLMP inequality for two qutrits.

All Bell inequalities can be grouped in *equivalent* families by composing symmetry transformations [18], namely, relabeling the party index, the observable index, or the outcome index. For the noisy n-qutrit GHZ state $\rho = v|\Psi\rangle\langle\Psi| + (1 - v)\,\mathbb{I}/3^n$, the most robust Bell inequalities are parameterized by the lowest critical visibility v_c satisfying the condition that if $v > v_c$, then the state ρ violates the n-qutrit Bell inequality. Apparently, two equivalent Bell inequalities have the same visibility. In this paper, we focus on the coincidence Bell inequalities for an arbitrary $(n, 2, 3)$ scenario. For each observer, the two von Neumann measuring apparatuses are confined to the class of tritter measurements (or an unbiased six-port beam splitter), which can be used to maximally violate the coincidence Bell inequalities for the quantum systems of qutrits and are experimentally realizable [19,20]. First, we present the general form of coincidence Bell inequalities for the $(n, 2, 3)$ scenario. Then, with the constraint of being reduced to the most robust coincidence Bell inequality for three qutrits presented in [12], we construct two tight and inequivalent coincidence Bell inequalities for four qutrits with $v_c = 0.5$. Increasing the number of particles, we also propose two tight and inequivalent coincidence Bell inequalities for

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five qutrits with $v_c = 0.488785$, where the former one can degenerate into the first Bell inequality for four qutrits and the latter one can degenerate into the second Bell inequality for four qutrits. Moreover, we find that the two (4,2,3)-scenario inequalities cannot detect more resistance to noise for the generalized GHZ states than the resistance to noise for the GHZ state, while the two (5,2,3)-scenario inequalities can detect more. These results are useful for building the iteration formula of $(n, 2, 3)$-scenario coincidence Bell inequalities with the lowest critical visibility.

II. GENERAL FORM OF $(n, 2, 3)$-SCENARIO CONINCIDENCE BELL INEQUALITIES

Let us consider the $(n, 2, 3)$ scenario: There are n space-separated observers labeled by $X^{(0)}, X^{(1)}, \dots, X^{(n-1)}$; the ith observer $X^{(i)}$ carries out two settings referred to as $X_0^{(i)}$ and $X_1^{(i)}$, and the results of measurement $X_j^{(i)}$ are denoted by $x_j^{(i)}$ labeled by 0,1,2. Then $(2\times 3)^n$ probability terms $P(X_{j_0}^{(0)} = x_{j_0}, X_{j_1}^{(1)} = x_{j_1}, \dots, X_{j_{n-1}}^{(n-1)} = x_{j_{n-1}})$ are obtained. For simplicity, we focus on the *coincidence* probabilities using the same idea as in Ref. [12]; that is, the coincidence probability term $P(\sum_{i=0}^{n-1} X_{j_i}^{(i)} = r)$ is expressed in the form

$$P\left(\sum_{i=0}^{n-1} X_{j_i}^{(i)} = r\right) = \sum_{\mathrm{Mod}[\sum_{i=0}^{n-1} x_{j_i}^{(i)},3]=r} P\big(X_{j_0}^{(0)} = x_{j_0}, X_{j_1}^{(1)} = x_{j_1}, \dots, X_{j_{n-1}}^{(n-1)} = x_{j_1}\big), \quad (1)$$

where $r = 0, 1, 2$ and $\mathrm{Mod}[x, 3] = r$ means that the remainder of x module 3 is r. Now, we are ready to introduce the general coincidence Bell inequality for the $(n, 2, 3)$ scenario in probability form,

$$\mathcal{I}_n = \sum_j \sum_{r=0}^{2} \omega_{j,r} P\left(\sum_{i=0}^{n-1} X_{j_i}^{(i)} = r\right) \overset{\text{LHV}}{\leqslant} L, \quad (2)$$

where $j = (j_0, j_1, \dots, j_{n-1})$ goes through all 2^n possible measurements for the n observers, $\omega_{j,r}$s are the real weight coefficients, and L is the upper bound of the Bell function $\mathcal{I}_n$ in the LHV theory. When the number of particles is small, we shall follow the more acceptable notations for the body index without any doubt, such as A, B, C for the three-particle system.

All these inequalities can be grouped in families of equivalent inequalities. Two Bell inequalities are *equivalent* if we can transform one into the other by composing the following symmetry transformations [18]: relabeling of the party index, the observable index in each particle, and the outcome index for one observable. For instance, the symmetry transformations in the (2,2,3)-scenario contain party exchange (party symmetry), $P(A_i = a, B_j = b) \mapsto P(A_j = a, B_i = b)$; observable exchange (observable symmetry), $P(A_i = a, B_j = b) \mapsto P(A_{\bar{i}} = a, B_{\bar{j}} = b)$, where $\bar{0} = 1, \bar{1} = 0$; and relabeling of outcomes (outcome symmetry), $P(A_i = a, B_j = b) \mapsto P(A_i = \mathrm{Mod}[a + a_i, 3], B_j = \mathrm{Mod}[b + b_j, 3])$, where $a_i, b_j \in \{0, 1, 2\}$. In addition, a Bell inequality is called *symmetric* if it is invariant under any party-symmetric transformation.

Let us now consider the quantum system in the noisy n-qutrit GHZ state. The density matrix of the system is given by $\rho = v|\Psi\rangle\langle\Psi| + (1 - v)\,\mathbb{I}/3^n$, with $v \in [0, 1]$, and the pure state

$$|\Psi\rangle = \frac{1}{\sqrt{3}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n} + |2\rangle^{\otimes n}) \quad (3)$$

is the so-called n-qutrit GHZ state. Then inequality (2) can detect the nonlocality of ρ if and only if $v * NL_{|\Psi\rangle} + (1 - v) * NL_{\mathbb{I}} > L$, i.e.,

$$v > v_c = (L - NL_{\mathbb{I}})/(NL_{|\Psi\rangle} - NL_{\mathbb{I}}), \quad (4)$$

where $NL_{|\Psi\rangle}$ and $NL_{\mathbb{I}}$ are the maximal quantum violations of $\mathcal{I}_n$ for state $|\Psi\rangle$ and state $\mathbb{I}/3^n$, respectively. Hence, the parameter v_c exactly reflects the ability of inequality (2) to detect nonlocal ρ. It is just the meaning of the critical visibility. Apparently, two equivalent Bell inequalities have the same visibility. If a Bell inequality has a lower critical visibility, then for more noise $(\mathbb{I}/3^n)$ it still can detect the nonlocality of ρ, which means that it is *more robust*.

Example 1. For the (3,2,3)-scenario coincidence Bell inequality in [12], if the outcomes c_1 of C_1 are denoted $\mathrm{Mod}[c_1 - 1, 3]$, then we obtain a coincidence Bell inequality,

$$\begin{aligned}\mathcal{I}_3 = {} & P(A_0 + B_0 + C_0 = 0) + P(A_0 + B_1 + C_1 = 0) \\ & + P(A_1 + B_0 + C_1 = 0) + P(A_1 + B_1 + C_0 = 1) \\ & + 2P(A_1 + B_1 + C_1 = 2) - P(A_1 + B_0 + C_0 = 2) \\ & - P(A_0 + B_1 + C_0 = 2) - P(A_0 + B_0 + C_1 = 1) \overset{\text{LHV}}{\leqslant} 3,\end{aligned} \quad (5)$$

which is outcome symmetric with the inequality in [12]. If the measurement outcomes of C_0 and C_1 are both zero, then Bell inequality (5) is reduced to

$$\begin{aligned}\mathcal{I}_2 = {} & P(A_0 + B_0 = 0) + P(A_0 + B_1 = 0) \\ & + P(A_1 + B_0 = 0) + P(A_1 + B_1 = 1) \\ & + 2P(A_1 + B_1 = 2) - P(A_1 + B_0 = 2) \\ & - P(A_0 + B_1 = 2) - P(A_0 + B_0 = 1) \overset{\text{LHV}}{\leqslant} 3,\end{aligned} \quad (6)$$

which is a coincidence Bell inequality for the two-qutrit system. It is easy to see that (6) is outcome symmetric with the CGLMP inequality. For the Bell inequality (5), $L = 3$, when the measurements are restricted to unbiased symmetric six-port beam splitters [12], $NL_{|\Psi\rangle} = 13/3, NL_{\mathbb{I}} = 1$. Then $v_c = (L - NL_{\mathbb{I}})/(NL_{|\Psi\rangle} - NL_{\mathbb{I}}) = 0.6$. Suppose we would like to generalize the two-qutrit inequality (6) to three qutrits such that the (3,2,3)-scenario coincidence Bell inequality can reduce to inequality (6); then the Bell inequality (5) is the most robust one by far.

In this work, the observables are also confined to the unbiased symmetric $(n \times 2)$-port beam splitters. The action of these devices in the computational basis is as follows: First, a phase factor is applied depending on the initial state, i.e.,

$|j\rangle \mapsto e^{i\phi_j}|j\rangle$. Following this, a quantum Fourier transform (QFT) is performed, and the resulting state is measured in the computational basis. Therefore, any of these measurements is defined by a three-phase vector $\vec{\phi} = (\phi_0, \phi_1, \phi_2)$ and the corresponding unitary transformation $[U_{\rm QFT}U(\vec{\phi})]_{ij} = \frac{1}{\sqrt{3}}e^{i\frac{2\pi}{3}(i-1)(j-1)}e^{i\phi_{i-1}}$. Given a measurement apparatus for n parties specified by the three-phase vectors $\vec{\phi}^{(X^{(i)})} = (\phi_0^{(X^{(i)})}, \phi_1^{(X^{(i)})}, \phi_2^{(X^{(i)})})$ and an initial state $|\Phi\rangle \in (\mathbb{C}^3)^{\otimes n}$, the probability of obtaining the outcome $(x^{(0)}, x^{(1)}, \dots, x^{(n-1)})$ is

$$P(X^{(0)} = x^{(0)}, X^{(1)} = x^{(1)}, \dots, X^{(s_{n-1})} = x^{(n-1)}) = |\langle x^{(0)}x^{(1)}\cdots x^{(n-1)}| \otimes_{i=0}^{n-1} U_{\rm QFT}U(\vec{\phi}^{(X^{(i)})})|\Phi\rangle|^2. \quad (7)$$

The phase vectors can be changed by the observers; they represent the local macroscopic parameters available to them. For the coincidence terms appearing in the Bell inequality and the maximally entangled state (3), the direct calculation yields

$$P\left(\sum_{i=0}^{n-1} X_{j_i}^{(i)} = r\right) = \frac{1}{9}\left\{3 + 2\sum_{t=0}^{1}\sum_{s=t+1}^{2}\cos\left[\sum_{i=0}^{n-1}\left(\phi_s^{(X_{j_i}^{(i)})} - \phi_t^{(X_{j_i}^{(i)})}\right) + \frac{2(s-t)r\pi}{3}\right]\right\}. \quad (8)$$

III. MOST ROBUST (4,2,3)-SCENARIO COINCIDENCE BELL INEQUALITIES

For the (2,2,3)-scenario, inequality (6) is the most robust coincidence Bell inequality. When the number of particles increases by one, the (3,2,3)-scenario coincidence Bell inequality in Example 1 is the most robust one. Moreover, it has the popular iteration property; that is, (5) is reduced to inequality (6) when $c_0 = c_1 = 0$. In this section, we focus on presenting the most robust (4,2,3)-scenario coincidence Bell inequalities under the constraint of being reduced to the most robust (3,2,3)-scenario coincidence inequality (5).

Form inequality (2) we have the (4,2,3)-scenario coincidence Bell function as

$$\mathcal{I}_4 = \sum_{i,j,k,\ell=0,1}\sum_{r=0}^{2}\omega_{i,j,k,\ell,r}P(A_i + B_j + C_k + D_\ell = r). \quad (9)$$

For convenience, we introduce the coefficient matrix $W = (w_{i'+1,j'+1})$ associated with it, where $i' = i \times 2 + j$ is the decimal expression of ij and $j' = (k \times 2 + \ell) \times 3 + r$, with $k \times 2 + \ell$ being the decimal expression of $k\ell$, $w_{i',j'} = \omega_{i,j,k,\ell,r}$. We find two inequivalent and tight Bell inequalities with the lowest critical visibility of 0.5. The first one is described in the joint probability form as

$$\begin{aligned}\mathcal{I}_4^{(1)} = &-P(A_0 + B_0 + C_0 + D_0 = 2) - P(A_0 + B_0 + C_0 + D_1 = 1)\\ &- P(A_0 + B_0 + C_1 + D_1 = 1) - P(A_0 + B_1 + C_0 + D_0 = 2)\\ &- P(A_0 + B_1 + C_1 + D_0 = 2) - P(A_0 + B_1 + C_1 + D_1 = 1)\\ &+ P(A_1 + B_0 + C_0 + D_0 = 0) + P(A_1 + B_0 + C_0 + D_1 = 1)\\ &+ P(A_1 + B_0 + C_1 + D_1 = 0) + P(A_1 + B_1 + C_0 + D_0 = 1)\\ &+ P(A_1 + B_1 + C_1 + D_0 = 2) + P(A_1 + B_1 + C_1 + D_1 = 2) \overset{\rm LHV}{\leqslant} 2,\end{aligned} \quad (10)$$

whose coefficient matrix is

$$\left(\begin{array}{ccc|ccc|ccc|ccc} 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0\\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0\\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right). \quad (11)$$

It is upper bounded by $L = 2$ for the LHV theory, has maximal quantum violations $NL_{|\Psi\rangle} = 4$ and $NL_{\mathbb{I}} = 0$ for states $|\Psi\rangle$ and $\mathbb{I}/3^4$, respectively, and is reduced to (5) when $d_0 = d_1 = 0$. Therefore, $v_c = 1/2 = 0.5$. The second one's coefficient matrix is

$$\left(\begin{array}{ccc|ccc|ccc|ccc} 1 & -1 & 0 & 1 & 0 & -1 & 0 & -1 & 1 & 1 & -1 & 0\\ 1 & 0 & -1 & 0 & 1 & -1 & 1 & -1 & 0 & 1 & 0 & -1\\ 1 & 0 & -1 & 0 & 1 & -1 & 0 & 1 & -4 & 0 & -4 & 1\\ 0 & 1 & -1 & -1 & 1 & 0 & 0 & -4 & 1 & -4 & 0 & 1 \end{array}\right), \quad (12)$$

whose joint probability form is presented in Appendix A. It is upper bounded by $L = 2$ for the LHV theory, has maximal quantum violations $NL_{|\Psi\rangle} = 8$ and $NL_{\mathbb{I}} = -4$ for states $|\Psi\rangle$ and state $\mathbb{I}/3^4$, respectively, and is reduced to (5) when $d_0 = d_1 = 0$. Therefore, its critical visibility is $v_c = (2+4)/(8+4) = 0.5$, which is the same as that of the first (4,2,3)-scenario Bell inequality. In Appendix A, we prove the above statements. Apparently, they are inequivalent. For any deterministic outcome, $A_0 = a_0, A_1 = a_1, B_0 = b_0, B_1 = b_1, C_0 = c_0, C_1 = c_1, D_0 = d_0, D_1 = d_1$, we can associate it with a generator,

$$|a_0b_0c_0d_0\rangle \oplus |a_0b_0c_0d_1\rangle \oplus |a_0b_0c_1d_0\rangle \oplus |a_0b_0c_1d_1\rangle \oplus |a_0b_1c_0d_0\rangle \oplus |a_0b_1c_0d_1\rangle \oplus |a_0b_1c_1d_0\rangle \oplus |a_0b_1c_1d_1\rangle \oplus |a_1b_0c_0d_0\rangle \oplus |a_1b_0c_0d_1\rangle \oplus |a_1b_0c_1d_0\rangle \oplus |a_1b_0c_1d_1\rangle \oplus |a_1b_1c_0d_0\rangle \oplus |a_1b_1c_0d_1\rangle \oplus |a_1b_1c_1d_0\rangle \oplus |a_1b_1c_1d_1\rangle, \tag{13}$$

following Ref. [18], where $\oplus$ is the direct sum of vectors. Since an inequality is tight if and only if the number of linearly independent generators that saturate the inequality turns out to be equal to the number of linearly independent generators minus one [18], we can check the tightness of the two inequalities by numerical computation.

In summary, we have given two inequivalent and tight (4,2,3)-scenario coincidence Bell inequalities with the lowest critical visibility of 0.5 and the iteration property.

IV. MOST ROBUST (5,2,3)-SCENARIO COINCIDENCE BELL INEQUALITIES

Since the most robust (4,2,3)-scenario coincidence Bell inequalities in Sec. III are inequivalent, we need to research at least the two most robust (5,2,3)-scenario coincidence Bell inequalities, which can be reduced to the two inequivalent (4,2,3)-scenario inequalities.

Like for the coefficient matrix representation of the (4,2,3)-scenario coincidence Bell function, we associate the (5,2,3)-scenario coincidence Bell function $\mathcal{I}_5 = \sum_{i,j,k,\ell,t=0,1}\sum_{r=0}^{2}\omega_{i,j,k,\ell,t,r}P(A_i+B_j+C_k+D_\ell+E_t=r)$ with a coefficient matrix $W=(w_{i'+1,j'+1})$, where $i'=i\times 2^2+j\times 2+k$, $j'=(\ell\times 2+t)\times 3+r$, and $w_{i',j'}=\omega_{i,j,k,\ell,r}$. Fortunately, under the constraint of iteration, we find two inequivalent and tight coincidence Bell inequalities for the (5,2,3) scenario with the same lowest critical visibility of 0.488785. To avoid the complex expression, we give only their associated matrices here. In Appendix B, we provide their joint probability forms. The first one's associated matrix is

$$\left(\begin{array}{ccc|ccc|ccc|ccc}
0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{6} & 0 & -\frac{2}{3} & 0 & 0 & \frac{1}{6} & 0\\
0 & 0 & -\frac{1}{6} & 0 & 0 & \frac{1}{6} & 0 & \frac{1}{6} & 0 & 0 & -\frac{2}{3} & 0\\
0 & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0\\
0 & 0 & -\frac{1}{3} & 0 & 0 & -\frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{3} & 0\\
\frac{1}{3} & 0 & 0 & \frac{1}{6} & 0 & 0 & 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{2} & \frac{1}{3}\\
\frac{1}{6} & 0 & 0 & -\frac{1}{6} & 0 & 0 & 0 & -\frac{1}{2} & -\frac{1}{6} & 0 & 0 & -\frac{1}{3}\\
-\frac{1}{3} & 0 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & 0 & 0 & -\frac{1}{6} & 0 & 0 & \frac{1}{6}\\
0 & -\frac{1}{3} & \frac{1}{6} & -\frac{1}{3} & 0 & 0 & 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{3}
\end{array}\right). \tag{14}$$

The second one's associated matrix is

$$\left(\begin{array}{ccc|ccc|ccc|ccc}
0 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 & 0 & -\frac{1}{3}\\
0 & -\frac{1}{6} & 0 & 0 & 0 & \frac{1}{6} & 0 & -\frac{1}{3} & 0 & 0 & 0 & -\frac{1}{6}\\
0 & \frac{1}{6} & -\frac{1}{3} & 0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & -\frac{1}{6} & -\frac{1}{2}\\
0 & -\frac{1}{6} & -\frac{1}{6} & 0 & -\frac{1}{6} & 0 & 0 & -\frac{1}{3} & -\frac{1}{3} & 0 & \frac{1}{6} & 0\\
0 & -\frac{1}{6} & 0 & 0 & 0 & -\frac{1}{3} & 0 & \frac{1}{6} & 0 & 0 & 0 & -\frac{1}{6}\\
0 & \frac{1}{6} & 0 & 0 & 0 & -\frac{2}{3} & 0 & -\frac{2}{3} & 0 & 0 & 0 & \frac{1}{6}\\
0 & -\frac{1}{6} & -\frac{1}{6} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{6} & \frac{1}{6} & 0 & \frac{1}{6} & 0\\
0 & -\frac{1}{3} & \frac{1}{6} & 0 & -\frac{1}{3} & 0 & 0 & \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{2}
\end{array}\right). \tag{15}$$

For the first one, $L=-1/6, NL_{|\Psi\rangle}=0.879225$, and $NL_{\mathbb{I}}=-7/6$. Hence, $v_c=(-1/6+7/6)/(0.879225+7/6)\approx 0.488785$. For the second one, $L=0, NL_{\mathbb{I}}=-1$, and $NL_{|\Psi\rangle}=1.04589$. Hence, $v_c=(0+1)/(1.04589+1)\approx 0.488785$. Their tightness and inequivalence can be checked by numerical computation. Furthermore, the first one is reduced to (10), and the second one is reduced to (12) for $e_0=e_1=0$. In Appendix B, we present the details of the proof. Therefore, we have written two inequivalent and tight (5,2,3)-scenario coincidence Bell inequalities with the lowest critical visibility of 0.488785 and the iteration property.

For the above four Bell inequalities, it is interesting to ask a question: Are the generalized GHZ states

$$|\Psi(\theta_1,\theta_2)\rangle = \sin\theta_1\sin\theta_2|0\rangle^{\otimes n} + \sin\theta_1\cos\theta_2|1\rangle^{\otimes n} + \cos\theta_1|2\rangle^{\otimes n} \tag{16}$$

more robust against white noise than the maximally entangled GHZ state $|\Psi\rangle$? For the generalized GHZ states $|\Psi(\theta_1,\theta_2)\rangle$,

TABLE I. Answers to whether the generalized GHZ states are more robust against white noise than the maximally entangled GHZ state for the most robust $(n, 2, d)$-scenario coincidence Bell inequalities (Ineq.). A dash (−) denotes that we have no idea.

	$d = 2$	$d = 3$	$d = 4$	$d = 5$
$n = 2$	CHSH Ineq., No	CGLMP Ineq., Yes	CGLMP Ineq., Yes	CGLMP Ineq., Yes
$n = 3$	MABK Ineq., No	Example 1, Yes	(5) in Ref. [13], No	(10) in Ref. [13], Yes
$n = 4$	MABK Ineq., No	$\mathcal{I}_4^{(1)} \leqslant 2$, No; $\mathcal{I}_4^{(2)} \leqslant 2$, No	−	−
$n = 5$	MABK Ineq., No	$\mathcal{I}_5^{(1)} \leqslant -\frac{1}{6}$, Yes; $\mathcal{I}_5^{(2)} \leqslant 0$, Yes	−	−

the coincidence terms have the form

$$P\left(\sum_{i=0}^{n-1} X_{j_i}^{(i)} = r\right) = \frac{1}{3} + \frac{\sin^2\theta_1 \sin 2\theta_2 \cos\left(\frac{2r\pi}{3} + \varphi\right)}{3} + \frac{\sin 2\theta_1 \sin\theta_2 \cos\left(\frac{4r\pi}{3} + \varphi'\right)}{3} + \frac{\sin 2\theta_1 \cos\theta_2 \cos\left(\frac{2r\pi}{3} + \varphi' - \varphi\right)}{3}, \quad (17)$$

where $\varphi = \sum_{i=0}^{n-1} \phi_1^{(X_{j_i}^{(i)})}$, $\varphi' = \sum_{i=0}^{n-1} \phi_2^{(X_{j_i}^{(i)})}$, and we assume that the first phase of any three-phase vector is zero. Going through all quantum states $|\Psi(\theta_1, \theta_2)\rangle$, $\theta_1, \theta_2 \in [0, \pi/2]$ and the unbiased symmetric $(n \times 2)$-port beam splitters, the maximal quantum violations of $\mathcal{I}_4^{(1)}$ and $\mathcal{I}_4^{(2)}$ are the same as those for the maximally entangled GHZ state $|\Psi\rangle$, while the maximal violations of $\mathcal{I}_5^{(1)}$ and $\mathcal{I}_5^{(2)}$ are 0.879492 and 1.04616, respectively, which are slightly larger than those for the maximally entangled GHZ state $|\Psi\rangle$. Therefore, both answers are negative for the inequalities $\mathcal{I}_4^{(1)} \leqslant 2$ and $\mathcal{I}_4^{(2)} \leqslant 2$, and both answers are positive for the inequalities $\mathcal{I}_5^{(1)} \leqslant -1/6$ and $\mathcal{I}_5^{(2)} \leqslant 0$.

For the most robust $(n, 2, d)$-scenario coincidence Bell inequalities, are the generalized GHZ states more robust against white noise than the maximally entangled GHZ state? Since the $(n, 2, 2)$-scenario Mermin-Ardehali-Belinskiǐ-Klyshko (MABK) inequalities, the $(2, 2, d)$-scenario CGLMP inequalities, and the coincidence Bell inequalities (5) and (10) in Ref. [13] are the most robust coincidence Bell inequalities for the corresponding scenario, we summarize the answers in Table I, where a dash (−) denotes that we have no idea.

V. CONCLUSION AND DISCUSSION

In conclusion, the general form of the coincidence Bell inequality for any multiparty, three-dimensional system with two-measurements has been presented. Due to the limitation of our computers, we focus on the (4,2,3)-scenario and (5,2,3)-scenario coincidence Bell inequalities. Fortunately, we have found the Bell inequalities with the lowest critical visibility and the popular iteration property. If we pay attention to the variation of the critical visibility value v_c with respect to the number n of parties, then we may conjecture that v_c will decline as n increases. For the two (4,2,3)-scenario inequalities, the generalized GHZ states cannot violate them more than the GHZ state, while for the two (5,2,3)-scenario inequalities, the generalized GHZ states (some nonmaximally entangled states) can violate them a little more. These results may be very helpful for setting up the iteration formula of the most robust coincidence Bell inequalities for any $(n, 2, 3)$ scenario.

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APPENDIX A: THE TWO (4,2,3)-SCENARIO COINCIDENCE BELL INEQUALITIES

In this Appendix, we give the two (4,2,3)-scenario coincidence Bell inequalities with joint probability forms and prove that both of them have the critical visibility of 0.5.

The first (4,2,3)-scenario Bell inequality is

$$\begin{aligned}\mathcal{I}_4^{(1)} = &-P(A_0 + B_0 + C_0 + D_0 = 2) - P(A_0 + B_0 + C_0 + D_1 = 1) - P(A_0 + B_0 + C_1 + D_1 = 1)\\ &- P(A_0 + B_1 + C_0 + D_0 = 2) - P(A_0 + B_1 + C_1 + D_0 = 2) - P(A_0 + B_1 + C_1 + D_1 = 1)\\ &+ P(A_1 + B_0 + C_0 + D_0 = 0) + P(A_1 + B_0 + C_0 + D_1 = 1) + P(A_1 + B_0 + C_1 + D_1 = 0)\\ &+ P(A_1 + B_1 + C_0 + D_0 = 1) + P(A_1 + B_1 + C_1 + D_0 = 2) + P(A_1 + B_1 + C_1 + D_1 = 2) \overset{\text{LHV}}{\leqslant} 2.\end{aligned} \quad (A1)$$

We prove that it is reduced to inequality (5) for $d_0 = d_1 = 0$. Moreover, $L = 2$ is the upper bound of $\mathcal{I}_4^{(1)}$ in LHV theory. Traversing the class of tritter measurements, $\mathcal{I}_4^{(1)'}$s maximal quantum violations for the maximally entangled state $|\Psi\rangle$ and the maximally mixed state $\mathbb{I}/3^4$ are $NL_{|\Psi\rangle} = 4$ and $NL_{\mathbb{I}} = 0$, respectively, which imply $v_c = 0.5$.

Proof. First, we prove that in LHV theory (A1) holds for the cases of $d_0 = 0$, $d_1 = 0, 1, 2$.

Case (i). When $d_0 = d_1 = 0$, (A1) is reduced to

$$P(A_0 + B_0 + C_0 = 0) - P(A_0 + B_0 + C_1 = 1) - P(A_0 + B_1 + C_0 = 2) + P(A_0 + B_1 + C_1 = 0) - P(A_1 + B_0 + C_0 = 2) + P(A_1 + B_0 + C_1 = 0) + P(A_1 + B_1 + C_0 = 1) + 2P(A_1 + B_1 + C_1 = 2) \overset{\text{LHV}}{\leqslant} 3, \quad \text{(A2)}$$

which is just the (3,2,3)-scenario Bell inequality (5). Hence, when $a_i = b_j = c_k = d_l = 0\, \forall\, i, j, k, l = 0, 1, \mathcal{I}_4^{(1)}$ reaches a value of 2.

Case (ii). When $d_0 = 0, d_1 = 1$, (A1) is reduced to

$$P(A_0 + B_0 + C_0 = 1) - P(A_0 + B_0 + C_1 = 0) - P(A_0 + B_1 + C_0 = 2) + P(A_0 + B_1 + C_1 = 1) + 2P(A_1 + B_0 + C_0 = 0) + P(A_1 + B_0 + C_1 = 2) + P(A_1 + B_1 + C_0 = 1) - P(A_1 + B_1 + C_1 = 0) \overset{\text{LHV}}{\leqslant} 3. \quad \text{(A3)}$$

Composed of the observable symmetric transformations $B_0 \leftrightarrow B_1$ and $C_0 \leftrightarrow C_1$ and the outcome symmetric transformation $a_i \mapsto \text{Mod}[a_i + 2, 3]$, (A3) degenerates into the (3,2,3)-scenario Bell inequality (5).

Case (iii). When $d_0 = 0, d_1 = 2$, (A1) is reduced to

$$-2P(A_0 + B_0 + C_0 = 2) - P(A_0 + B_0 + C_1 = 2) - P(A_0 + B_1 + C_0 = 2) - 2P(A_0 + B_1 + C_1 = 2) - P(A_1 + B_0 + C_0 = 1) + P(A_1 + B_0 + C_1 = 1) + P(A_1 + B_1 + C_0 = 1) - P(A_1 + B_1 + C_1 = 1) \overset{\text{LHV}}{\leqslant} 0. \quad \text{(A4)}$$

Case (iii a). When $P(A_1 + B_0 + C_1 = 1) = P(A_1 + B_1 + C_0 = 1) = 1$, we have $2a_1 + b_0 + b_1 + c_0 + c_1 = 2$ and $b_1 + c_0 - b_0 - c_1 = 0$, and then $P(A_0 + B_0 + C_1 = 2) = P(A_0 + B_1 + C_0 = 2)$, and $P(A_1 + B_0 + C_0 = 1) = P(A_1 + B_1 + C_1 = 1)$. Hence, provided one of the above four probabilities equals 1, (A4) holds. Therefore, we need to consider only the case that any one of them equals zero. In this case,

$$\begin{aligned}
&a_1 + b_0 + c_1 = 1, a_1 + b_1 + c_0 = 1, a_0 + b_0 + c_1 = 0, a_0 + b_1 + c_0 = 0, a_1 + b_0 + c_0 = 0, a_1 + b_1 + c_1 = 2; \text{or}\\
&a_1 + b_0 + c_1 = 1, a_1 + b_1 + c_0 = 1, a_0 + b_0 + c_1 = 0, a_0 + b_1 + c_0 = 0, a_1 + b_0 + c_0 = 2, a_1 + b_1 + c_1 = 0; \text{or}\\
&a_1 + b_0 + c_1 = 1, a_1 + b_1 + c_0 = 1, a_0 + b_0 + c_1 = 1, a_0 + b_1 + c_0 = 1, a_1 + b_0 + c_0 = 0, a_1 + b_1 + c_1 = 2; \text{or}\\
&a_1 + b_0 + c_1 = 1, a_1 + b_1 + c_0 = 1, a_0 + b_0 + c_1 = 1, a_0 + b_1 + c_0 = 1, a_1 + b_0 + c_0 = 2, a_1 + b_1 + c_1 = 0.
\end{aligned} \quad \text{(A5)}$$

For the first and last cases, $P(A_0 + B_0 + C_0 = 2) = 1$. For the second and third cases, $P(A_0 + B_1 + C_1 = 2) = 1$. In other words, (A4) holds.

Case (iii b). In this case, we assume $P(A_1 + B_0 + C_1 = 1) = 1$ and $P(A_1 + B_1 + C_0 = 1) = 0$. If any one of $P(A_0 + B_0 + C_1 = 2), P(A_0 + B_1 + C_0 = 2), P(A_1 + B_0 + C_0 = 1)$, and $P(A_1 + B_1 + C_1 = 1)$ equals 1, then (A4) holds. Hence, we need to consider only that any one of them equals zero.

If $a_1 + b_1 + c_0 = 0$, then $2a_1 + b_0 + b_1 + c_0 + c_1 = 1$, and $b_1 + c_0 - b_0 - c_1 = 2$, which imply $a_0 + b_0 + c_1 = a_0 + b_1 + c_0 = 1$ and $a_1 + b_0 + c_0 = a_1 + b_1 + c_1 = 2$. These yield $P(A_0 + B_0 + C_0 = 2) = P(A_0 + B_1 + C_1 = 2) = 1$. Hence, (A4) holds.

If $a_1 + b_1 + c_0 = 2$, then $2a_1 + b_0 + b_1 + c_0 + c_1 = 0$, and $b_1 + c_0 - b_0 - c_1 = 1$, which imply $a_0 + b_0 + c_1 = 0, a_0 + b_1 + c_0 = 1, a_1 + b_0 + c_0 = 0$, and $a_1 + b_1 + c_1 = 0$. These yield $P(A_0 + B_0 + C_0 = 2) = 1$. Hence, (A4) holds.

Case (iii c). When $P(A_1 + B_0 + C_1 = 1) = 0$ and $P(A_1 + B_1 + C_0 = 1) = 1$, the proof is similar to that of case (iii b).

So far, we have proved that (A1) holds for the $d_0 = 0, d_1 = 0, 1, 2$ cases. For the arbitrary case $d_0 = d, d_1 = d'$, we can accomplish the proof by repeating that for the case with $d_0 = 0, d_1 = \text{Mod}[d - d', 3]$ and replacing the outcomes a_i by $\text{Mod}[a_i + d, 3]$ since the general term is the coincidence probability. Hence, the maximal value of $\mathcal{I}_4^{(1)}$ is 2 in LHV theory.

When the quantum system is in the maximally entangled GHZ state, we go through the class of tritter measurements and obtain $NL_{|\Psi\rangle} = 4$. Especially, $\mathcal{I}_4^{(1)} = 4$ is arrived at

$$\begin{aligned}
&\phi_0^{(A_0)} = \phi_0^{(A_1)} = \phi_0^{(B_0)} = \phi_0^{(B_1)} = \phi_0^{(C_0)} = \phi_0^{(C_1)} = \phi_0^{(D_0)} = \phi_0^{(D_1)} = \phi_2^{(D_1)} = 0, \quad \phi_1^{(A_1)} = \phi_1^{(A_0)} = -\frac{\pi}{4}, \quad \phi_2^{(A_0)} = -\frac{\pi}{12},\\
&\phi_2^{(A_1)} = \frac{2\pi}{3} + \phi_2^{(A_0)} = \frac{7\pi}{12}, \quad \phi_1^{(B_0)} = \phi_1^{(B_1)} = \phi_2^{(A_0)} = -\frac{\pi}{12}, \quad \phi_2^{(B_0)} = -\phi_2^{(A_1)} = -\frac{7\pi}{12}, \quad \phi_2^{(B_1)} = \frac{\pi}{3} - \phi_1^{(B_0)} = \frac{5\pi}{12},\\
&\phi_1^{(C_0)} = -\frac{\pi}{3} + \phi_2^{(A_0)} = -\frac{5\pi}{12}, \quad \phi_2^{(C_0)} = \frac{\pi}{3} - \phi_2^{(D_1)} = \frac{\pi}{3}, \quad \phi_1^{(C_1)} = \phi_2^{(A_1)} = \frac{7\pi}{12}, \quad \phi_2^{(C_1)} = \phi_2^{(C_0)} = \frac{\pi}{3},\\
&\phi_1^{(D_1)} = -\phi_2^{(A_0)} = \frac{\pi}{12}, \quad \phi_1^{(D_0)} = \frac{\pi}{3} + \phi_1^{(D_1)} = \frac{5\pi}{12}, \quad \phi_2^{(D_0)} = -\phi_2^{(C_0)} = -\frac{\pi}{3}.
\end{aligned} \quad \text{(A6)}$$

In this case,

$$\begin{aligned}
&P(A_0+B_0+C_0+D_0=2)=\tfrac{1}{9}, \quad P(A_0+B_0+C_0+D_1=1)=\tfrac{1}{9}, \quad P(A_0+B_0+C_1+D_1=1)=\tfrac{1}{9},\\
&P(A_0+B_1+C_0+D_0=2)=\tfrac{1}{9}, \quad P(A_0+B_1+C_1+D_0=2)=\tfrac{1}{9}, \quad P(A_0+B_1+C_1+D_1=1)=\tfrac{1}{9},\\
&P(A_1+B_0+C_0+D_0=0)=\tfrac{7}{9}, \quad P(A_1+B_0+C_0+D_1=1)=\tfrac{7}{9}, \quad P(A_1+B_0+C_1+D_1=0)=\tfrac{7}{9},\\
&P(A_1+B_1+C_0+D_0=1)=\tfrac{7}{9}, \quad P(A_1+B_1+C_1+D_0=2)=\tfrac{7}{9}, \quad P(A_1+B_1+C_1+D_1=2)=\tfrac{7}{9}.
\end{aligned} \tag{A7}$$

Apparently, $NL_{\mathbb{I}}=0$. Hence, $v_c=0.5$. ■

The second (4,2,3)-scenario Bell inequality with the probability form is

$$\begin{aligned}
\mathcal{I}_4^{(2)} = {}& P(A_0+B_0+C_0+D_0=0)-P(A_0+B_0+C_0+D_0=1)+P(A_0+B_0+C_0+D_1=0)\\
&-P(A_0+B_0+C_0+D_1=2)-P(A_0+B_0+C_1+D_0=1)+P(A_0+B_0+C_1+D_0=2)\\
&+P(A_0+B_0+C_1+D_1=0)-P(A_0+B_0+C_1+D_1=1)+P(A_0+B_1+C_0+D_0=0)\\
&-P(A_0+B_1+C_0+D_0=2)+P(A_0+B_1+C_0+D_1=1)-P(A_0+B_1+C_0+D_1=2)\\
&+P(A_0+B_1+C_1+D_0=0)-P(A_0+B_1+C_1+D_0=1)+P(A_0+B_1+C_1+D_1=0)\\
&-P(A_0+B_1+C_1+D_1=2)+P(A_1+B_0+C_0+D_0=0)-P(A_1+B_0+C_0+D_0=2)\\
&+P(A_1+B_0+C_0+D_1=1)-P(A_1+B_0+C_0+D_1=2)+P(A_1+B_0+C_1+D_0=1)\\
&-4P(A_1+B_0+C_1+D_0=2)-4P(A_1+B_0+C_1+D_1=1)+P(A_1+B_0+C_1+D_1=2)\\
&+P(A_1+B_1+C_0+D_0=1)-P(A_1+B_1+C_0+D_0=2)-P(A_1+B_1+C_0+D_1=0)\\
&+P(A_1+B_1+C_0+D_1=1)-4P(A_1+B_1+C_1+D_0=1)+P(A_1+B_1+C_1+D_0=2)\\
&-4P(A_1+B_1+C_1+D_1=0)+P(A_1+B_1+C_1+D_1=2)\overset{\text{LHV}}{\leqslant} 2.
\end{aligned} \tag{A8}$$

We prove that it is reduced to (5) for $d_0=d_1=0$, $L=2$, $NL_{|\Psi\rangle}=8$, and $NL_{\mathbb{I}}=-4$, which yield $v_c=0.5$.

Proof. By the proof of (A1), it is sufficient to prove that (A8) holds for the cases $d_0=0$, $d_1=0,1,2$.

Case (i). When $d_0=d_1=0$, (A8) is reduced to

$$\begin{aligned}
&P(A_0+B_0+C_0=0)-P(A_0+B_0+C_1=1)-P(A_0+B_1+C_0=2)+P(A_0+B_1+C_1=0)\\
&-P(A_1+B_0+C_0=2)+P(A_1+B_0+C_1=0)+P(A_1+B_1+C_0=1)+2P(A_1+B_1+C_1=2)\overset{\text{LHV}}{\leqslant} 3,
\end{aligned} \tag{A9}$$

which is just the (3,2,3)-scenario inequality (5). Hence, when $a_i=b_j=c_k=d_l=0\,\forall\, i,j,k,l=0,1$, $\mathcal{I}_4^{(2)}$ reaches a value of 2.

Case (ii). When $d_0=0$, $d_1=1$, (A8) is reduced to

$$\begin{aligned}
&-P(A_0+B_0+C_0=1)+P(A_0+B_0+C_1=2)+P(A_0+B_1+C_0=0)-P(A_0+B_1+C_1=1)\\
&+P(A_1+B_0+C_0=0)+2P(A_1+B_0+C_1=1)-P(A_1+B_1+C_0=2)+P(A_1+B_1+C_1=0)\overset{\text{LHV}}{\leqslant} 3.
\end{aligned} \tag{A10}$$

Composed of the observable symmetric transformation $B_0\leftrightarrow B_1$ and the outcome symmetric transformation $b_1\mapsto \mathrm{Mod}[b_1+1,3]$, (A10) becomes the (3,2,3)-scenario Bell inequality (5).

Case (iii). When $d_0=0$, $d_1=2$, (A8) is reduced to

$$-P(A_1+B_0+C_1=2)-P(A_1+B_1+C_1=1)\overset{\text{LHV}}{\leqslant} 0, \tag{A11}$$

which holds forever.

In order to arrive at the maximal quantum violation of (A8) for the state $|\Psi\rangle$, we can take

$$\begin{aligned}
&\phi_0^{(A_0)}=\phi_0^{(A_1)}=\phi_0^{(B_0)}=\phi_0^{(B_1)}=\phi_0^{(C_0)}=\phi_0^{(C_1)}=\phi_0^{(D_0)}=\phi_0^{(D_1)}=0,\\
&\phi_1^{(A_0)}=\frac{\pi}{2},\quad \phi_2^{(A_0)}=-\frac{\pi}{18},\quad \phi_1^{(A_1)}=\phi_1^{(A_0)}=\frac{\pi}{2},\quad \phi_2^{(A_1)}=\frac{2\pi}{3}+\phi_2^{(A_0)}=\frac{11\pi}{18},\\
&\phi_1^{(B_0)}=\frac{7\pi}{9},\quad \phi_2^{(B_0)}=-\phi_1^{(A_0)}=-\frac{\pi}{2},\quad \phi_1^{(B_1)}=\frac{\pi}{2}+\phi_2^{(A_0)}=\frac{4\pi}{9},\quad \phi_2^{(B_1)}=\frac{\pi}{3}-\phi_1^{(A_0)}=-\frac{\pi}{6},\\
&\phi_1^{(C_0)}=\phi_1^{(C_1)}=\frac{4\pi}{9},\quad \phi_2^{(C_0)}=\frac{\pi}{3}+\phi_2^{(A_0)}=\frac{5\pi}{18},\quad \phi_1^{(C_0)}=-\frac{4\pi}{3}+\phi_1^{(B_1)}=-\frac{8\pi}{9},\quad \phi_2^{(C_1)}=\phi_2^{(C_0)}=\frac{5\pi}{18},\\
&\phi_1^{(D_0)}=\frac{\pi}{3}+\phi_2^{(A_0)}=\frac{5\pi}{18}, \phi_2^{(D_0)}=\phi_2^{(A_0)}=-\frac{\pi}{18},\quad \phi_1^{(D_1)}=\phi_2^{(A_0)}=-\frac{\pi}{18},\quad \phi_2^{(D_1)}=\frac{\pi}{3}+\phi_2^{(D_0)}=\frac{5\pi}{18}.
\end{aligned} \tag{A12}$$

Then

$$P(A_1+B_0+C_1+D_0=1)=\tfrac{4}{9},\quad P(A_1+B_0+C_1+D_0=2)=\tfrac{1}{9},\quad P(A_1+B_0+C_1+D_1=1)=\tfrac{1}{9},$$
$$P(A_1+B_0+C_1+D_1=2)=\tfrac{4}{9},\quad P(A_1+B_1+C_1+D_0=1)=\tfrac{1}{9},\quad P(A_1+B_1+C_1+D_0=2)=\tfrac{4}{9},$$
$$P(A_1+B_1+C_1+D_1=0)=\tfrac{1}{9},\quad P(A_1+B_1+C_1+D_1=2)=\tfrac{4}{9}.$$

For other terms, if their coefficient is positive, then the joint probability term equals 7/9; if their coefficient is negative, then the joint probability term equals 1/9. Hence, $NL_{|\Psi\rangle}=8$. If the system is in the maximally mixed state, then $NL_{\mathbb{I}}=-4$. Therefore, $v_c=(2+4)/(8+4)=0.5$. ■

APPENDIX B: THE TWO (5,2,3)-SCENARIO COINCIDENCE BELL INEQUALITIES

In this Appendix, we give the two (5,2,3)-scenario coincidence Bell inequalities with the joint probability form and prove that both of them have the critical visibility of 0.488785.

The first (5,2,3)-scenario Bell inequality is

$$\begin{aligned}\mathcal{I}_5^{(1)}=&-\tfrac{1}{3}P(A_0+B_0+C_0+D_0+E_0=2)-\tfrac{1}{6}P(A_0+B_0+C_0+D_0+E_1=2)\\&-\tfrac{2}{3}P(A_0+B_0+C_0+D_1+E_0=1)+\tfrac{1}{6}P(A_0+B_0+C_0+D_1+E_1=1)-\tfrac{1}{6}P(A_0+B_0+C_1+D_0+E_0=2)\\&+\tfrac{1}{6}P(A_0+B_0+C_1+D_0+E_1=2)+\tfrac{1}{6}P(A_0+B_0+C_1+D_1+E_0=1)-\tfrac{2}{3}P(A_0+B_0+C_1+D_1+E_1=1)\\&-\tfrac{1}{6}P(A_0+B_1+C_0+D_0+E_0=2)-\tfrac{1}{3}P(A_0+B_1+C_0+D_0+E_1=2)+\tfrac{1}{6}P(A_0+B_1+C_0+D_1+E_0=1)\\&-\tfrac{1}{6}P(A_0+B_1+C_0+D_1+E_1=1)-\tfrac{1}{3}P(A_0+B_1+C_1+D_0+E_0=2)-\tfrac{1}{6}P(A_0+B_1+C_1+D_0+E_1=2)\\&-\tfrac{1}{6}P(A_0+B_1+C_1+D_1+E_0=1)-\tfrac{1}{3}P(A_0+B_1+C_1+D_1+E_1=1)+\tfrac{1}{3}P(A_1+B_0+C_0+D_0+E_0=0)\\&+\tfrac{1}{6}P(A_1+B_0+C_0+D_0+E_1=0)-\tfrac{1}{3}P(A_1+B_0+C_0+D_1+E_0=2)+\tfrac{1}{2}P(A_1+B_0+C_0+D_1+E_1=1)\\&+\tfrac{1}{3}P(A_1+B_0+C_0+D_1+E_1=2)+\tfrac{1}{6}P(A_1+B_0+C_1+D_0+E_0=0)-\tfrac{1}{6}P(A_1+B_0+C_1+D_0+E_1=0)\\&-\tfrac{1}{2}P(A_1+B_0+C_1+D_1+E_0=1)-\tfrac{1}{6}P(A_1+B_0+C_1+D_1+E_0=2)-\tfrac{1}{3}P(A_1+B_0+C_1+D_1+E_1=2)\\&-\tfrac{1}{3}P(A_1+B_1+C_0+D_0+E_0=0)+\tfrac{1}{6}P(A_1+B_1+C_0+D_0+E_1=1)-\tfrac{1}{3}P(A_1+B_1+C_0+D_0+E_1=2)\\&-\tfrac{1}{6}P(A_1+B_1+C_0+D_1+E_0=2)+\tfrac{1}{6}P(A_1+B_1+C_0+D_1+E_1=2)-\tfrac{1}{3}P(A_1+B_1+C_1+D_0+E_0=1)\\&+\tfrac{1}{6}P(A_1+B_1+C_1+D_0+E_0=2)-\tfrac{1}{3}P(A_1+B_1+C_1+D_0+E_1=0)+\tfrac{1}{6}P(A_1+B_1+C_1+D_1+E_0=2)\\&+\tfrac{1}{3}P(A_1+B_1+C_1+D_1+E_1=2)\overset{\text{LHV}}{\leqslant}-\tfrac{1}{6}.\end{aligned}\tag{B1}$$

We prove that it is reduced to (A1) for $e_0=e_1=0$, $L=-1/6$, $NL_{|\Psi\rangle}=0.879225$, and $NL_{\mathbb{I}}=-7/6$. Hence, $v_c\approx 0.488785$.

Proof. It is sufficient to prove that (B1) holds for the cases $e_0=0$, $e_1=0,1,2$.

Case (i). When $e_0=e_1=0$, (B1) is reduced to

$$\begin{aligned}&-P(A_0+B_0+C_0+D_0=2)-P(A_0+B_0+C_0+D_1=1)-P(A_0+B_0+C_1+D_1=1)\\&-P(A_0+B_1+C_0+D_0=2)-P(A_0+B_1+C_1+D_0=2)-P(A_0+B_1+C_1+D_1=1)\\&+P(A_1+B_0+C_0+D_0=0)+P(A_1+B_0+C_0+D_1=1)+P(A_1+B_0+C_1+D_1=0)\\&+P(A_1+B_1+C_0+D_0=1)+P(A_1+B_1+C_1+D_0=2)+P(A_1+B_1+C_1+D_1=2)\overset{\text{LHV}}{\leqslant}2,\end{aligned}\tag{B2}$$

which is just the (4,2,3)-scenario Bell inequality (A1). Then the Bell function reaches a value of $-1/6$ for any $a_i=b_j=c_k=d_l=e_s=0$.

Case (ii). When $e_0=0$, $e_1=1$, (B1) is reduced to

$$\begin{aligned}&P(A_0+B_0+C_0+D_0=0)-P(A_0+B_0+C_0+D_0=2)+P(A_0+B_0+C_0+D_1=0)\\&-4P(A_0+B_0+C_0+D_1=1)+P(A_0+B_0+C_1+D_0=1)-P(A_0+B_0+C_1+D_0=2)\\&-4P(A_0+B_0+C_1+D_1=0)+P(A_0+B_0+C_1+D_1=1)+P(A_0+B_1+C_0+D_0=0)\\&-P(A_0+B_1+C_0+D_0=1)-P(A_0+B_1+C_0+D_1=0)+P(A_0+B_1+C_0+D_1=1)\\&+P(A_0+B_1+C_1+D_0=0)-P(A_0+B_1+C_1+D_0=2)-P(A_0+B_1+C_1+D_1=0)\\&+P(A_0+B_1+C_1+D_1=2)+P(A_1+B_0+C_0+D_0=0)-P(A_1+B_0+C_0+D_0=1)\\&+P(A_1+B_0+C_0+D_1=0)-4P(A_1+B_0+C_0+D_1=2)+P(A_1+B_0+C_1+D_0=0)\end{aligned}$$

$$\begin{aligned}
&- P(A_1+B_0+C_1+D_0=2)+P(A_1+B_0+C_1+D_1=0)-4P(A_1+B_0+C_1+D_1=1)\\
&- P(A_1+B_1+C_0+D_0=1)+P(A_1+B_1+C_0+D_0=2)+P(A_1+B_1+C_0+D_1=1)\\
&- P(A_1+B_1+C_0+D_1=2)+P(A_1+B_1+C_1+D_0=0)-P(A_1+B_1+C_1+D_0=1)\\
&- P(A_1+B_1+C_1+D_1=0)+P(A_1+B_1+C_1+D_1=1)\overset{\text{LHV}}{\leqslant} 2. \qquad \text{(B3)}
\end{aligned}$$

Composed of the four symmetric transformations (1) $A \leftrightarrow B$, (2) $A_0 \leftrightarrow A_1$, (3) $C \leftrightarrow D$, and (4) $b_1 \mapsto b_1+1, c_0 \mapsto c_0+2, d_0 \mapsto d_0+1, d_1 \mapsto d_1+1$, (B3) becomes

$$\begin{aligned}
&+ P(A_0+B_0+C_0+D_0=0)-P(A_0+B_0+C_0+D_0=1)+P(A_0+B_0+C_0+D_1=0)\\
&- P(A_0+B_0+C_0+D_1=2)-P(A_0+B_0+C_1+D_0=1)+P(A_0+B_0+C_1+D_0=2)\\
&+ P(A_0+B_0+C_1+D_1=0)-P(A_0+B_0+C_1+D_1=1)+P(A_0+B_1+C_0+D_0=0)\\
&- P(A_0+B_1+C_0+D_0=2)+P(A_0+B_1+C_0+D_1=1)-P(A_0+B_1+C_0+D_1=2)\\
&+ P(A_0+B_1+C_1+D_0=0)-P(A_0+B_1+C_1+D_0=1)+P(A_0+B_1+C_1+D_1=0)\\
&- P(A_0+B_1+C_1+D_1=2)+P(A_1+B_0+C_0+D_0=0)-P(A_1+B_0+C_0+D_0=2)\\
&+ P(A_1+B_0+C_0+D_1=1)-P(A_1+B_0+C_0+D_1=2)+P(A_1+B_0+C_1+D_0=1)\\
&- 4P(A_1+B_0+C_1+D_0=2)-4P(A_1+B_0+C_1+D_1=1)+P(A_1+B_0+C_1+D_1=2)\\
&+ P(A_1+B_1+C_0+D_0=1)-P(A_1+B_1+C_0+D_0=2)-P(A_1+B_1+C_0+D_1=0)\\
&+ P(A_1+B_1+C_0+D_1=1)-4P(A_1+B_1+C_1+D_0=1)+P(A_1+B_1+C_1+D_0=2)\\
&- 4P(A_1+B_1+C_1+D_1=0)+P(A_1+B_1+C_1+D_1=2)\overset{\text{LHV}}{\leqslant} 2, \qquad \text{(B4)}
\end{aligned}$$

which is just the (4,2,3)-scenario Bell inequality (A8).

Case (iii). When $e_0=0, e_1=2$, (B1) is reduced to

$$\begin{aligned}
&P(A_0+B_0+C_0+D_0=1)-P(A_0+B_0+C_0+D_0=2)-4P(A_0+B_0+C_0+D_1=1)\\
&+ P(A_0+B_0+C_0+D_1=2)+P(A_0+B_0+C_1+D_0=0)-P(A_0+B_0+C_1+D_0=2)\\
&+ P(A_0+B_0+C_1+D_1=1)-4P(A_0+B_0+C_1+D_1=2)-P(A_0+B_1+C_0+D_0=0)\\
&+ P(A_0+B_1+C_0+D_0=1)+P(A_0+B_1+C_0+D_1=1)-P(A_0+B_1+C_0+D_1=2)\\
&+ P(A_0+B_1+C_1+D_0=1)-P(A_0+B_1+C_1+D_0=2)+P(A_0+B_1+C_1+D_1=0)\\
&- P(A_0+B_1+C_1+D_1=2)+P(A_1+B_0+C_0+D_0=0)-P(A_1+B_0+C_0+D_0=2)\\
&+ P(A_1+B_0+C_0+D_1=0)-P(A_1+B_0+C_0+D_1=1)+P(A_1+B_0+C_1+D_0=0)\\
&- P(A_1+B_0+C_1+D_0=1)-P(A_1+B_0+C_1+D_1=1)+P(A_1+B_0+C_1+D_1=2)\\
&- 4P(A_1+B_1+C_0+D_0=0)+P(A_1+B_1+C_0+D_0=2)+P(A_1+B_1+C_0+D_1=0)\\
&- P(A_1+B_1+C_0+D_1=2)-4P(A_1+B_1+C_1+D_0=1)+P(A_1+B_1+C_1+D_0=2)\\
&+ P(A_1+B_1+C_1+D_1=0)-P(A_1+B_1+C_1+D_1=1)\overset{\text{LHV}}{\leqslant} 2. \qquad \text{(B5)}
\end{aligned}$$

Case (iii a). When $d_0=0, d_1=0$, (B5) is reduced to

$$\begin{aligned}
&- P(A_0+B_0+C_0=1)-2P(A_0+B_0+C_1=2)+P(A_0+B_1+C_0=1)-P(A_0+B_1+C_1=2)\\
&+ P(A_1+B_0+C_0=0)-P(A_1+B_0+C_1=1)-P(A_1+B_1+C_0=0)-2P(A_1+B_1+C_1=1)\overset{\text{LHV}}{\leqslant} 0. \qquad \text{(B6)}
\end{aligned}$$

Composed of the three symmetric transformations (1) $A_0 \leftrightarrow A_1, B_0 \leftrightarrow B_1, C_0 \leftrightarrow C_1$, (2) $A \leftrightarrow C$, and (3) $c_0 \mapsto c_0+1$, (B5) becomes

$$\begin{aligned}
&- 2P(A_0+B_0+C_0=2)-P(A_0+B_0+C_1=2)-P(A_0+B_1+C_0=2)-2P(A_0+B_1+C_1=2)\\
&- P(A_1+B_0+C_0=1)+P(A_1+B_0+C_1=1)+P(A_1+B_1+C_0=1)-P(A_1+B_1+C_1=1)\overset{\text{LHV}}{\leqslant} 0, \qquad \text{(B7)}
\end{aligned}$$

which is just the inequality (A4)

Case (iii b). When $d_0 = 0, d_1 = 1$, (B5) is reduced to

$$-P(A_0+B_0+C_0=0)+P(A_0+B_0+C_0=1)+P(A_0+B_0+C_1=0)-P(A_0+B_0+C_1=1)$$
$$-P(A_1+B_1+C_0=0)+P(A_1+B_1+C_0=2)-P(A_1+B_1+C_1=1)+P(A_1+B_1+C_1=2)\overset{\text{LHV}}{\leqslant}2. \quad \text{(B8)}$$

If three probabilities with positive coefficients equal 1, then one joint probability with a negative coefficient equals 1. Hence (B8) holds. For example, if $P(A_0+B_0+C_0=1)=P(A_0+B_0+C_1=0)=P(A_1+B_1+C_0=2)=1$, then $a_0+b_0+c_0=1$, $a_0+b_0+c_1=0$, $a_1+b_1+c_0=2$, which yield $a_1+b_1+c_1=1$, i.e., $P(A_1+B_1+C_1=1)=1$.

Case (iii c). When $d_0 = 0, d_1 = 2$, (B5) is reduced to

$$-2P(A_0+B_0+C_0=2)-P(A_0+B_0+C_1=0)-P(A_0+B_1+C_0=0)+P(A_0+B_1+C_1=1)$$
$$-P(A_1+B_0+C_0=2)+P(A_1+B_0+C_1=0)-2P(A_1+B_1+C_0=0)-P(A_1+B_1+C_1=1)\overset{\text{LHV}}{\leqslant}0. \quad \text{(B9)}$$

Composed of the three symmetric transformations (1) $A\leftrightarrow C$, (2) $A\leftrightarrow C$, and (3) $a_1\mapsto a_1+1$, $b_1\mapsto b_1+2$, (B9) becomes

$$-2P(A_0+B_0+C_0=2)-P(A_0+B_0+C_1=2)-P(A_0+B_1+C_0=2)-2P(A_0+B_1+C_1=2)$$
$$-P(A_1+B_0+C_0=1)+P(A_1+B_0+C_1=1)+P(A_1+B_1+C_0=1)-P(A_1+B_1+C_1=1)\overset{\text{LHV}}{\leqslant}0, \quad \text{(B10)}$$

which is just the inequality (A4).

In order to arrive at the maximal quantum violation of (B1) for state $|\Psi\rangle$, we can take

$$\phi_0^{(A_0)}=\phi_0^{(A_1)}=\phi_0^{(B_0)}=\phi_0^{(B_1)}=\phi_0^{(C_0)}=\phi_0^{(C_1)}=\phi_0^{(D_0)}=\phi_0^{(D_1)}=\phi_0^{(E_0)}=\phi_0^{(E_1)}=0,$$
$$\phi_1^{(A_0)}=0,\quad \phi_2^{(A_0)}=\frac{\pi}{2},\quad \phi_1^{(A_1)}=\frac{\pi}{3},\quad \phi_2^{(A_1)}=\frac{\pi}{6},\quad \phi_1^{(B_0)}=0,\quad \phi_2^{(B_0)}=\frac{\pi}{3},\quad \phi_1^{(B_1)}=\frac{11\pi}{36},\quad \phi_2^{(B_1)}=\frac{\pi}{36},$$
$$\phi_1^{(C_0)}=0,\quad \phi_1^{(C_1)}=-\frac{\pi}{3}+\phi_2^{(C_0)},\quad \phi_2^{(C_1)}=\frac{\pi}{3},\quad \phi_1^{(D_0)}=-\frac{23\pi}{36},\quad \phi_2^{(D_0)}=-\frac{7\pi}{36}-\phi_2^{(C_0)},$$
$$\phi_1^{(D_1)}=\frac{\pi}{3},\quad \phi_2^{(D_1)}=\frac{5\pi}{6}-\phi_2^{(C_0)},$$
$$\phi_1^{(E_0)}=-\phi_1^{(C_1)}=\frac{\pi}{3}-\phi_2^{(C_0)},\quad \phi_2^{(E_0)}=\phi_1^{(C_1)}=-\frac{\pi}{3}+\phi_2^{(C_0)},\quad \phi_1^{(E_1)}=-\phi_1^{(B_0)}=0,\quad \phi_2^{(E_1)}=0. \quad \text{(B11)}$$

Then

$$\mathcal{I}_5^{(1)}=\frac{1}{54}\bigg[-47+32\cos\left(\frac{\pi}{36}-\phi_2^{(C_0)}\right)+36\cos\left(\frac{\pi}{3}+2\phi_2^{(C_0)}\right)+32\sin\frac{7\pi}{36}$$
$$+16\sin\left(\frac{5\pi}{36}-\phi_2^{(C_0)}\right)+8\sin\left(\frac{\pi}{6}+\phi_2^{(C_0)}\right)\bigg], \quad \text{(B12)}$$

which arrives at its maximal value of 0.879225 at $\phi_2^{(C_0)}=-0.429805$. Hence, $NL_{|\Psi\rangle}=0.879225$. If the system is in the maximally mixed state, then $NL_{\mathbb{I}}=-7/6$. Hence, $v_c=(-1/6+7/6)/(0.879225+7/6)\approx 0.488785$. ■

The second (5,2,3)-scenario Bell inequality with the joint probability form is

$$\begin{aligned}\mathcal{I}_5^{(2)}=&-\tfrac{1}{3}P(A_0+B_0+C_0+D_0+E_0=1)-\tfrac{1}{6}P(A_0+B_0+C_0+D_0+E_1=2)\\
&-\tfrac{1}{6}P(A_0+B_0+C_0+D_1+E_0=1)-\tfrac{1}{3}P(A_0+B_0+C_0+D_1+E_1=2)-\tfrac{1}{6}P(A_0+B_0+C_1+D_0+E_0=1)\\
&+\tfrac{1}{6}P(A_0+B_0+C_1+D_0+E_1=2)-\tfrac{1}{3}P(A_0+B_0+C_1+D_1+E_0=1)-\tfrac{1}{6}P(A_0+B_0+C_1+D_1+E_1=2)\\
&+\tfrac{1}{6}P(A_0+B_1+C_0+D_0+E_0=1)-\tfrac{1}{3}P(A_0+B_1+C_0+D_0+E_0=2)-\tfrac{1}{3}P(A_0+B_1+C_0+D_0+E_1=1)\\
&+\tfrac{1}{3}P(A_0+B_1+C_0+D_1+E_0=1)+\tfrac{1}{3}P(A_0+B_1+C_0+D_1+E_0=2)-\tfrac{1}{6}P(A_0+B_1+C_0+D_1+E_1=1)\\
&-\tfrac{1}{2}P(A_0+B_1+C_0+D_1+E_1=2)-\tfrac{1}{6}P(A_0+B_1+C_1+D_0+E_0=1)-\tfrac{1}{6}P(A_0+B_1+C_1+D_0+E_0=2)\\
&-\tfrac{1}{6}P(A_0+B_1+C_1+D_0+E_1=1)-\tfrac{1}{3}P(A_0+B_1+C_1+D_1+E_0=1)-\tfrac{1}{3}P(A_0+B_1+C_1+D_1+E_0=2)\\
&+\tfrac{1}{6}P(A_0+B_1+C_1+D_1+E_1=1)-\tfrac{1}{6}P(A_1+B_0+C_0+D_0+E_0=1)-\tfrac{1}{3}P(A_1+B_0+C_0+D_0+E_1=2)\\
&+\tfrac{1}{6}P(A_1+B_0+C_0+D_1+E_0=1)-\tfrac{1}{6}P(A_1+B_0+C_0+D_1+E_1=2)+\tfrac{1}{6}P(A_1+B_0+C_1+D_0+E_0=1)\\
&-\tfrac{2}{3}P(A_1+B_0+C_1+D_0+E_1=2)-\tfrac{2}{3}P(A_1+B_0+C_1+D_1+E_0=1)+\tfrac{1}{6}P(A_1+B_0+C_1+D_1+E_1=2)\\
&-\tfrac{1}{6}P(A_1+B_1+C_0+D_0+E_0=1)-\tfrac{1}{6}P(A_1+B_1+C_0+D_0+E_0=2)+\tfrac{1}{3}P(A_1+B_1+C_0+D_0+E_1=1)\end{aligned}$$

$$
\begin{aligned}
&+\tfrac{1}{6}P(A_1+B_1+C_0+D_1+E_0=1)+\tfrac{1}{6}P(A_1+B_1+C_0+D_1+E_0=2)+\tfrac{1}{6}P(A_1+B_1+C_0+D_1+E_1=1)\\
&-\tfrac{1}{3}P(A_1+B_1+C_1+D_0+E_0=1)+\tfrac{1}{6}P(A_1+B_1+C_1+D_0+E_0=2)-\tfrac{1}{3}P(A_1+B_1+C_1+D_0+E_1=1)\\
&+\tfrac{1}{3}P(A_1+B_1+C_1+D_1+E_0=1)+\tfrac{1}{3}P(A_1+B_1+C_1+D_1+E_0=2)+\tfrac{1}{3}P(A_1+B_1+C_1+D_1+E_1=1)\\
&+\tfrac{1}{2}P(A_1+B_1+C_1+D_1+E_1=2)\overset{\text{LHV}}{\leqslant}0. \qquad \text{(B13)}
\end{aligned}
$$

We prove that it is reduced to (A8) for $e_0=e_1=0$, $L=0$, $NL_{\mathbb{I}}=-1$, and $NL_{|\Psi\rangle}=1.04589$, which lead to $v_c\approx 0.488785$.

Proof. It is sufficient to prove that (B13) holds for the cases $e_0=0$, $e_1=0,1,2$.

Case (i). When $e_0=e_1=0$, (B13) is reduced to

$$
\begin{aligned}
&P(A_0+B_0+C_0+D_0=0)-P(A_0+B_0+C_0+D_0=1)+P(A_0+B_0+C_0+D_1=0)\\
&-P(A_0+B_0+C_0+D_1=2)-P(A_0+B_0+C_1+D_0=1)+P(A_0+B_0+C_1+D_0=2)\\
&+P(A_0+B_0+C_1+D_1=0)-P(A_0+B_0+C_1+D_1=1)+P(A_0+B_1+C_0+D_0=0)\\
&-P(A_0+B_1+C_0+D_0=2)+P(A_0+B_1+C_0+D_1=1)-P(A_0+B_1+C_0+D_1=2)\\
&+P(A_0+B_1+C_1+D_0=0)-P(A_0+B_1+C_1+D_0=1)+P(A_0+B_1+C_1+D_1=0)\\
&-P(A_0+B_1+C_1+D_1=2)+P(A_1+B_0+C_0+D_0=0)-P(A_1+B_0+C_0+D_0=2)\\
&+P(A_1+B_0+C_0+D_1=1)-P(A_1+B_0+C_0+D_1=2)+P(A_1+B_0+C_1+D_0=1)\\
&-4P(A_1+B_0+C_1+D_0=2)-4P(A_1+B_0+C_1+D_1=1)+P(A_1+B_0+C_1+D_1=2)\\
&+P(A_1+B_1+C_0+D_0=1)-P(A_1+B_1+C_0+D_0=2)-P(A_1+B_1+C_0+D_1=0)\\
&+P(A_1+B_1+C_0+D_1=1)-4P(A_1+B_1+C_1+D_0=1)+P(A_1+B_1+C_1+D_0=2)\\
&-4P(A_1+B_1+C_1+D_1=0)+P(A_1+B_1+C_1+D_1=2)\overset{\text{LHV}}{\leqslant}2, \qquad \text{(B14)}
\end{aligned}
$$

which is just the (4,2,3)-scenario Bell inequality (A8). Hence, when $a_i=b_j=c_k=d_l=e_s=0\,\forall\, i,j,k,l,s=0,1$, $\mathcal{I}_5^{(2)}$ reaches a value of zero.

Case (ii). When $e_0=0$, $e_1=1$, (B13) is reduced to

$$
\begin{aligned}
&-P(A_0+B_0+C_0+D_0=1)-P(A_0+B_0+C_0+D_1=1)-P(A_0+B_0+C_1+D_1=1)\\
&+P(A_0+B_1+C_0+D_0=1)+P(A_0+B_1+C_0+D_1=2)+P(A_0+B_1+C_1+D_1=0)\\
&-P(A_1+B_0+C_0+D_0=1)-P(A_1+B_0+C_1+D_0=1)-P(A_1+B_0+C_1+D_1=1)\\
&+P(A_1+B_1+C_0+D_0=0)+P(A_1+B_1+C_1+D_0=2)+P(A_1+B_1+C_1+D_1=1)\overset{\text{LHV}}{\leqslant}2. \qquad \text{(B15)}
\end{aligned}
$$

The proof of (B15) is similar to that of (B2).

Case (iii). When $e_0=0$, $e_1=2$, (B13) is reduced to

$$
\begin{aligned}
&-P(A_0+B_0+C_0+D_0=1)+P(A_0+B_0+C_0+D_0=2)-P(A_0+B_0+C_0+D_1=0)\\
&+P(A_0+B_0+C_0+D_1=2)+P(A_0+B_0+C_1+D_0=0)-P(A_0+B_0+C_1+D_0=1)\\
&-P(A_0+B_0+C_1+D_1=1)+P(A_0+B_0+C_1+D_1=2)+P(A_0+B_1+C_0+D_0=1)\\
&-4P(A_0+B_1+C_0+D_0=2)-4P(A_0+B_1+C_0+D_1=0)+P(A_0+B_1+C_0+D_1=1)\\
&+P(A_0+B_1+C_1+D_0=0)-P(A_0+B_1+C_1+D_0=2)+P(A_0+B_1+C_1+D_1=0)\\
&-P(A_0+B_1+C_1+D_1=1)-P(A_1+B_0+C_0+D_0=0)+P(A_1+B_0+C_0+D_0=2)\\
&-P(A_1+B_0+C_0+D_1=0)+P(A_1+B_0+C_0+D_1=1)-4P(A_1+B_0+C_1+D_0=0)\\
&+P(A_1+B_0+C_1+D_0=1)+P(A_1+B_0+C_1+D_1=0)-4P(A_1+B_0+C_1+D_1=1)\\
&-P(A_1+B_1+C_0+D_0=1)+P(A_1+B_1+C_0+D_0=2)-P(A_1+B_1+C_0+D_1=0)\\
&+P(A_1+B_1+C_0+D_1=2)+P(A_1+B_1+C_1+D_0=0)-P(A_1+B_1+C_1+D_0=1)\\
&-P(A_1+B_1+C_1+D_1=1)+P(A_1+B_1+C_1+D_1=2)\overset{\text{LHV}}{\leqslant}2. \qquad \text{(B16)}
\end{aligned}
$$

The proof of (B16) is similar to that of (B5).

In order to arrive at the maximal quantum violation of (B13) for state $|\Psi\rangle$, we can take

$$\phi_0^{(A_0)}=\phi_0^{(A_1)}=\phi_0^{(B_0)}=\phi_0^{(B_1)}=\phi_0^{(C_0)}=\phi_0^{(C_1)}=\phi_0^{(D_0)}=\phi_0^{(D_1)}=\phi_0^{(E_0)}=\phi_1^{(E_0)}=0,$$
$$\phi_1^{(A_0)}=-\frac{\pi}{3}-4\phi_2^{(D_1)}-6\phi_2^{(C_1)},\quad \phi_2^{(A_0)}=0,\quad \phi_1^{(A_1)}=0,\quad \phi_2^{(A_1)}=\frac{\pi}{6}+2\phi_2^{(D_1)}+3\phi_2^{(C_1)},$$
$$\phi_1^{(B_0)}=-\frac{\pi}{3}+4\phi_2^{(D_1)}+8\phi_2^{(C_1)},\quad \phi_2^{(B_0)}=-\frac{\pi}{4},\quad \phi_1^{(B_1)}=\frac{\pi}{3}+4\phi_2^{(D_1)}+8\phi_2^{(C_1)},\quad \phi_2^{(B_1)}=\frac{\pi}{12},$$
$$\phi_1^{(C_0)}=\frac{\pi}{6}+2\phi_2^{(D_1)}+2\phi_2^{(C_1)},\quad \phi_2^{(C_0)}=\frac{\pi}{6}+2\phi_2^{(D_1)}+4\phi_2^{(C_1)},\quad \phi_1^{(C_1)}=-\frac{\pi}{6}-2\phi_2^{(D_1)}-4\phi_2^{(C_1)},$$
$$\phi_1^{(D_0)}=-\frac{\pi}{6}-2\phi_2^{(D_1)}-4\phi_2^{(C_1)},\quad \phi_2^{(D_0)}=-\frac{\pi}{4}-2\phi_2^{(D_1)}-4\phi_2^{(C_1)},\quad \phi_1^{(D_1)}=\frac{\pi}{3}+4\phi_2^{(D_1)}+4\phi_2^{(C_1)},\quad \phi_2^{(D_1)}=0.331520,$$
$$\phi_1^{(E_0)}=0,\quad \phi_2^{(E_0)}=0,\quad \phi_1^{(E_1)}=\frac{5\pi}{6}-6\phi_2^{(D_1)}-8\phi_2^{(C_1)}. \tag{B17}$$

In this case,

$$\mathcal{I}_5^{(2)}=\frac{2}{27}\left[-14+8\cos\left(\frac{\pi}{12}+\phi_2^{(C_1)}+\phi_2^{(D_1)}\right)+8\cos\left(\frac{\pi}{6}+2\phi_2^{(C_1)}+3\phi_2^{(D_1)}\right)\right]$$
$$+\frac{2}{27}\left[2\cos\left(\frac{\pi}{4}+3\phi_2^{(C_1)}+4\phi_2^{(D_1)}\right)-4\sin\left(\frac{\pi}{12}-5\phi_2^{(C_1)}-7\phi_2^{(D_1)}\right)+9\sin\left(8\phi_2^{(C_1)}+6\phi_2^{(D_1)}\right)\right]+\frac{1}{3}, \tag{B18}$$

which arrives at its maximal value of 1.04589 at $\phi_2^{(C_1)}=-0.075739$, $\phi_2^{(D_1)}=0.331520$. Hence, $NL_{|\Psi\rangle}=1.04589$. If the system is in the maximally mixed state, then $NL_{\mathbb{I}}=-1$. Hence, $v_c=(0+1)/(1.04589+1)\approx 0.488785$. ■

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