Eigenstates of Maxwell's equations in multiconstituent microstructures

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The eigenstates of Maxwell's monochromatic equations in a multiconstituent composite medium are developed and used to expand a physical field that is created either by an externally given current density or by an incident external field. The local electric permittivity $\kappa(\mathbf{r})$, as well as the local magnetic permeability $\mu(\mathbf{r})$, are assumed to have uniform values in each constituent, but to differ in the different constituents.

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I. INTRODUCTION

Eigenstates of Maxwell's equations in a two-constituent composite medium were first introduced in 1979 for the quasistatic regime [1]. These were used to expand the local electric field $\mathbf{E}(\mathbf{r})$ in such systems and subsequently to calculate their macroscopic response [2], including the situation where a strong, uniform, static magnetic field is present [3]. This was extended to a similar treatment of static elasticity in such systems [4]. It was also extended to treat the full Maxwell equations [5]. These eigenstates were used to expand the local physical field created by a given current density $\mathbf{J}_{\text{ex}}$ anywhere in the system. In these articles only the dielectric constant ε and the electric conductivity σ were assumed to have different values in the two constituents, but the magnetic permeability μ was taken to be 1 everywhere. More recently, eigenstates of the full Maxwell equations in a two-constituent composite medium were defined for the case where ε, σ, and μ are all heterogeneous [6]. In the latter article the external field was expanded both when it results from a given external current density $\mathbf{J}_{\text{ex}}(\mathbf{r})$ and when it results from a given incident field $\mathbf{E}_0(\mathbf{r})$.

The eigenstates developed in this article are special monochromatic fields, electric as well as magnetic, with a given fixed angular frequency ω which is real. Thus the material moduli of the various constituents—$\varepsilon_i(\omega)$, $\sigma_i(\omega)$, $\mu_i(\omega)$—have well-defined values which can be measured experimentally. The eigenvalues of Maxwell's equations in any composite mixture of these constituents are special values of the constituent moduli or their ratios for which nontrivial solutions of those equations exist without any current or charge sources. This scheme differs from other schemes, such as the quasinormal modes, where the eigenvalues are special, nonphysical values of the frequency ω [7,8]. Those are usually complex quantities, therefore one needs to extend the functions $\varepsilon_i(\omega)$, $\sigma_i(\omega)$, $\mu_i(\omega)$ into the complex plane of ω. In our approach the unphysical nature only appears in the eigenvalues of ε_i, σ_i, μ_i. Also, the eigenstates are always solutions of a linear partial differential equation. By contrast, in the quasinormal mode scheme, where the eigenvalues are special, nonphysical, complex values of ω, and where ε_i, σ_i, μ_i depend on that ω, the calculation of the eigenstates becomes a nonlinear problem [7,8].

In this article, I continue to develop the approach described in Ref. [6] to the case of a multiconstituent composite. This can be applied to the case where the composite is an array of spherical inclusions of various sizes and physical properties (i.e., various different values of ε_i, σ_i, and μ_i) embedded in an otherwise uniform host constituent. As an example we treat in detail the case of a single metallic sphere which is excited by a point-source electric current density.

Because the eigenstates of an isolated spherical inclusion are known in essentially closed form, the calculation of the local electric field is also achieved in essentially closed form. That is quite different from calculations of the field in such a system using COMSOL or other numerical procedures—see, e.g., Ref. [9].

In Sec. II the basic theory of the eigenstates is presented. In Sec. III the expansion of a physical field in the eigenstates is presented. In Sec. IV the main conclusions resulting from this article are described. In the Appendix the eigenstates of an isolated sphere are developed. They are used in Sec. III to calculate the physical field produced by a current density source outside of the sphere. They are also used in that section to set up the calculation of the eigenstates of a collection of nonintersecting spherical inclusions.

II. THE BASIC THEORY

Maxwell's equations, which are first-order partial differential equations (PDEs) for the two vector fields $\mathbf{E}(\mathbf{r})$, $\mathbf{H}(\mathbf{r})$,

$$\nabla \cdot (\varepsilon \mathbf{E}) = 4\pi \rho, \tag{1}$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial(\mu \mathbf{H})}{\partial t}, \tag{2}$$

$$\nabla \cdot (\mu \mathbf{H}) = 0, \tag{3}$$

$$\nabla \times \mathbf{H} = \frac{1}{c}\frac{\partial(\varepsilon \mathbf{E})}{\partial t} + \frac{4\pi}{c}(\sigma \mathbf{E} + \mathbf{J}_{\text{ex}}), \tag{4}$$

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are first translated into a single second-order PDE for a monochromatic version of either $\mathbf{E}$ or $\mathbf{H}$, where the time dependence is always given by a multiplicative factor $e^{-i\omega t}$:

$$-\nabla\times(\nabla\times\mathbf{E})+k_n^2\mathbf{E} = k_n^2\sum_i u_i\theta_i\mathbf{E}-\sum_i v_i\nabla\times[\theta_i(\nabla\times\mathbf{E})]-\frac{4\pi i\omega\mu_n}{c^2}\mathbf{J}_{\rm ex}, \tag{5}$$

$$-\nabla\times(\nabla\times\mathbf{H})+k_n^2\mathbf{H}=k_n^2\sum_i w_i\theta_i\mathbf{H} -\sum_i t_i\nabla\times[\theta_i(\nabla\times\mathbf{H})] -\frac{4\pi i}{c\kappa_n}\nabla\times\left[\left(1+\sum_i t_i\theta_i\right)\mathbf{J}_{\rm ex}\right]. \tag{6}$$

Here, $\theta_i(\mathbf{r})=1$ for $\mathbf{r}$ inside the subvolume V_i of the i constituent and $\theta_i(\mathbf{r})=0$ elsewhere, while

$$\kappa_i\equiv\varepsilon_i+\frac{4\pi i\sigma_i}{\omega},\quad k_i^2=\frac{\omega^2}{c^2}\kappa_i\mu_i, \tag{7}$$

$$u_i=1-\frac{\kappa_i}{\kappa_n},\quad t_i=1-\frac{\kappa_n}{\kappa_i}=\frac{u_i}{u_i-1}, \tag{8}$$

$$v_i=1-\frac{\mu_n}{\mu_i},\quad w_i=1-\frac{\mu_i}{\mu_n}=\frac{v_i}{v_i-1}, \tag{9}$$

where ε_i, σ_i, and μ_i are the dielectric constant, electric conductivity, and magnetic permeability of that constituent. Note that u_n, t_n, v_n, and w_n all vanish. The local externally given current density $\mathbf{J}_{\rm ex}$ is assumed to differ from 0 only in the n constituent. The constituent moduli κ_i and μ_i are complex quantities, in general. If these constituents are real physical materials, then $\mathrm{Im}(\varepsilon_i)$, $\mathrm{Re}(\sigma_i)$, and $\mathrm{Im}(\mu_i)$ are all non-negative quantities. In that case these quantities represent energy dissipation. However, when eigenstates are being considered, these quantities can have negative values. The wave number k_n is given by

$$k_n^2\equiv\frac{\omega^2}{c^2}\kappa_n\mu_n,$$

i.e., it is the wave number in the n constituent.

The following equations define eigenstates of Eq. (5):

$$-\nabla\times\left(\nabla\times\mathbf{E}^{(u)}_{im}\right)+k_n^2\mathbf{E}^{(u)}_{im}=k_n^2u_{im}\theta_i\mathbf{E}^{(u)}_{im}, \tag{10}$$

$$-\nabla\times\left(\nabla\times\mathbf{E}^{(v)}_{im}\right)+k_n^2\mathbf{E}^{(v)}_{im}=-v_{im}\nabla\times\left[\theta_i\left(\nabla\times\mathbf{E}^{(v)}_{im}\right)\right]. \tag{11}$$

Clearly, these are also eigenstates of Eq. (6):

$$\mathbf{H}^{(w)}_{im}=\mathbf{E}^{(u)}_{im},\quad w_{im}=u_{im},\quad \mathbf{H}^{(t)}_{im}=\mathbf{E}^{(v)}_{im},\quad t_{im}=v_{im}. \tag{12}$$

The $\mathbf{E}^{(v)}_{im}$ and $\mathbf{H}^{(w)}_{im}$ eigenfunctions are related to each other by means of Eq. (4) where $\mathbf{J}_{\rm ex}=\mathbf{0}$,

$$\nabla\times\mathbf{E}^{(u)}_{im}\equiv\nabla\times\mathbf{H}^{(w)}_{im}=-\frac{i\omega}{c}\kappa_{im}\mathbf{E}^{(v)}_{im},$$

where κ_{im} is the value of κ in the eigenstate $\mathbf{E}^{(u)}_{im}$. Therefore, when $\mathbf{r}\in V_i$ we get

$$\mathbf{E}^{(v)}_{im}=-\frac{c}{i\omega\kappa_n(1-u_{im})}\left(\nabla\times\mathbf{E}^{(u)}_{im}\right). \tag{13}$$

All the eigenfunctions are divergence free everywhere while the physical fields $\mathbf{E}$ and $\mathbf{H}$ are divergence free only outside V_n. Therefore we can only expand them in terms of the eigenfunctions outside of V_n.

We will assume that only V_p extends out to infinity and that $\mathbf{J}_{\rm ex}$ does not. Therefore the local physical fields $\mathbf{E}$, $\mathbf{H}$, as well as the various eigenfunctions, all behave asymptotically as $\mathbf{a}e^{ikr}/r$ with $\mathbf{a}\perp\mathbf{r}$ when $r\to\infty$, where $k=k_n$ in all cases except for $\mathbf{E}^{(u)}_{pm}$ and $\mathbf{E}^{(v)}_{pm}$, when $k=k_n\sqrt{1-u_{pm}}$ and $k=k_n\sqrt{1-w_{pm}}$, respectively. From this asymptotic behavior it follows that [10]

$$\nabla\times\mathbf{E}=(\mathbf{n}\times\mathbf{a})\frac{ik}{r}e^{ikr}+O\left(\frac{1}{r^2}\right),\quad \mathbf{n}\equiv\frac{\mathbf{r}}{r}\perp\mathbf{a}. \tag{14}$$

We now consider the following integral,

$$\int dV\mathbf{E}\cdot\left[-\nabla\times(\nabla\times\mathbf{F})+k_n^2\mathbf{F}\right]. \tag{15}$$

This is easily transformed into the sum of a surface integral over the system envelope and a volume integral that is symmetric in the two vector fields, which behave asymptotically as $\mathbf{E}\propto\mathbf{a_E}e^{ik_{\mathbf{E}}r}/r$ and $\mathbf{F}\propto\mathbf{a_F}e^{ik_{\mathbf{F}}r}/r$:

$$\oint d\mathbf{S}\cdot[\mathbf{E}\times(\nabla\times\mathbf{F})] -\int dV\left[(\nabla\times\mathbf{E})\cdot(\nabla\times\mathbf{F})\right]-k_n^2(\mathbf{E}\cdot\mathbf{F})\big]. \tag{16}$$

In order that the volume integral converge and that the surface integral tend to 0 when the system envelope is sent to infinity, it is necessary that the wave numbers $k_{\mathbf{E}}$ and $k_{\mathbf{F}}$ have an imaginary part which is greater than 0. This requirement dictates which of the two square roots should be used in the definition

$$k\equiv\left(\frac{\omega^2}{c^2}\kappa\mu\right)^{1/2}.$$

If κ_i and μ_i are both real and both positive or both negative, then we need to regularize them by adding a small positive imaginary part $i\delta$ to the wave number k. This will result in a convergent behavior for e^{ikr}. At the end of the calculation we then need to let δ tend to 0 through positive values, i.e., $\delta\to0^+$. These ideas and regularizations apply to the physical field as well as to the eigenfunctions.

It follows that the integral in Eq. (15) equals the following integral,

$$\int dV\mathbf{F}\cdot\left[-\nabla\times(\nabla\times\mathbf{E})+k_n^2\mathbf{E}\right]. \tag{17}$$

Applying this result to the two eigenfunctions $\mathbf{E}^{(u)}_{il}$, $\mathbf{E}^{(u)}_{im}$ we find that

$$0=(u_{il}-u_{im})\int dV\theta_i\left(\mathbf{E}^{(u)}_{il}\cdot\mathbf{E}^{(u)}_{im}\right)\equiv(u_{il}-u_{im})\left\langle\mathbf{E}^{(u)}_{il}\middle|\mathbf{E}^{(u)}_{im}\right\rangle_i, \tag{18}$$

where we have defined the i-scalar product of $\mathbf{E}$ and $\mathbf{F}$ as

$$\langle \mathbf{E}|\mathbf{F}\rangle_i \equiv \int dV\theta_i(\mathbf{E}\cdot\mathbf{F}). \tag{19}$$

We now get that $\mathbf{E}_{il}^{(u)}$ and $\mathbf{E}_{im}^{(u)}$ are orthogonal inside V_i unless $u_{il}=u_{im}$. Applying the equality of the integrals of Eqs. (15) and (17) to the two eigenfunctions $\mathbf{E}_{il}^{(v)}$ and $\mathbf{E}_{im}^{(v)}$ we find that

$$0=(v_{il}-v_{im})\int dV\theta_i\big(\nabla\times\mathbf{E}_{il}^{(v)}\big)\cdot\big(\nabla\times\mathbf{E}_{im}^{(v)}\big). \tag{20}$$

From this it follows that $\mathbf{E}_{il}^{(v)}$ and $\mathbf{E}_{im}^{(v)}$ are i orthogonal in the sense that

$$\int dV\theta_i\big(\nabla\times\mathbf{E}_{il}^{(v)}\big)\cdot\big(\nabla\times\mathbf{E}_{im}^{(v)}\big)\equiv\big\langle\nabla\times\mathbf{E}_{il}^{(v)}\big|\nabla\times\mathbf{E}_{im}^{(v)}\big\rangle_i \tag{21}$$

vanishes unless $v_{il}=v_{im}$. The biorthogonality properties described by Eqs. (18) and (20) can be used to expand physical fields in terms of the eigenfunctions. We note that the scalar product defined by Eq. (19), when $\mathbf{F}\equiv\mathbf{E}$, is not necessarily a nonzero quantity. When this becomes a crucial property, as sometimes in the case of eigenfunctions, this must therefore be checked in each case.

We also note that in order to derive Eqs. (18) and (20) it was necessary to define scalar products of vector fields in Hilbert space by Eq. (19). The usual definition of such a scalar product would have $\mathbf{E}$ replaced by its complex conjugate $\mathbf{E}^*$. That is why the eigenstates form a biorthogonal set and not an orthogonal set.

We recall that both types of eigenfunctions are divergence free. By contrast, the physical fields are only divergence free where $\mathbf{J}_{\text{ex}}=\mathbf{0}$. Therefore they can only be expanded in series of the eigenfunctions inside V_i for $i\neq n$. These expansions will be represented as

$$\theta_i\mathbf{E}=\theta_i\sum_l A_{il}\mathbf{E}_{il}^{(u)}=\theta_i\sum_l B_{il}\mathbf{E}_{il}^{(v)},\quad i\neq n. \tag{22}$$

The expansion coefficients A_{il} are calculated by considering the following vanishing differences of integrals:

$$\begin{aligned}0&=\int dV\mathbf{E}_{im}^{(u)}\cdot\big[-\nabla\times(\nabla\times\mathbf{E})+k_n^2\mathbf{E}\big]-\int dV\mathbf{E}\cdot\big[-\nabla\times\big(\nabla\times\mathbf{E}_{im}^{(u)}\big)+k_n^2\mathbf{E}_{im}^{(u)}\big]\\&=k_n^2\sum_j u_j\int dV\theta_j\big(\mathbf{E}_{im}^{(u)}\cdot\mathbf{E}\big)-\sum_j v_j\int dV\mathbf{E}_{im}^{(u)}\cdot[\nabla\times\theta_j(\nabla\times\mathbf{E})]-k_n^2u_{im}\int dV\theta_i\big(\mathbf{E}\cdot\mathbf{E}_{im}^{(u)}\big)\\&\quad-\frac{4\pi\omega\mu_n}{c^2}\int dV\big(\mathbf{E}_{im}^{(u)}\cdot\mathbf{J}_{\text{ex}}\big).\end{aligned} \tag{23}$$

In order to continue we use the first expansion from Eq. (22) for the physical field $\mathbf{E}(\mathbf{r})$ to deal with terms such as

$$\theta_i\nabla\times[\theta_j(\nabla\times\mathbf{E}_{jl})] \tag{24}$$

in Eq. (23). Because we assume that there are no overlaps between different subvolumes V_i, therefore the last expression vanishes when $j\neq i$. However, when $j=i$ this expression becomes problematic. In order to circumvent any possible complications, we will take V_i to be slightly larger than V_j in all directions. At the end of the calculations we will take the limit $V_i\to V_j$. We will therefore be able to ignore θ_i in Eq. (24) and transform the second term on the right-hand side of Eq. (23) as follows:

$$\begin{aligned}\int dV\mathbf{E}_{im}^{(u)}\cdot[\nabla\times\theta_j(\nabla\times\mathbf{E})]&=-\int dV\nabla\cdot\big[\mathbf{E}_{im}^{(u)}\times\theta_j(\nabla\times\mathbf{E})\big]+\int dV\theta_j\big(\nabla\times\mathbf{E}_{im}^{(u)}\big)\cdot(\nabla\times\mathbf{E})\\&=-\oint\theta_j d\mathbf{S}\cdot\big[\mathbf{E}_{im}^{(u)}\times(\nabla\times\mathbf{E})\big]+\big\langle\nabla\times\mathbf{E}_{im}^{(u)}\big|\nabla\times\mathbf{E}\big\rangle_j.\end{aligned} \tag{25}$$

The surface integral is over the system envelope. It tends to zero when that envelope is sent to infinity. This leads from Eq. (23) to the following set of inhomogeneous linear algebraic equations for the A_{jl} coefficients:

$$0=k_n^2(u_i-u_{im})A_{im}\big\langle\mathbf{E}_{im}^{(u)}\big|\mathbf{E}_{im}^{(u)}\big\rangle_i+k_n^2\sum_{jl,j\neq i}u_jA_{jl}\big\langle\mathbf{E}_{im}^{(u)}\big|\mathbf{E}_{jl}^{(u)}\big\rangle_j-\sum_{jl}v_jA_{jl}\big\langle\nabla\times\mathbf{E}_{im}^{(u)}\big|\nabla\times\mathbf{E}_{jl}^{(u)}\big\rangle_j-\frac{4\pi i\omega\mu_n}{c^2}\big\langle\mathbf{E}_{im}^{(u)}\big|\mathbf{J}_{\text{ex}}\big\rangle_n. \tag{26}$$

A similar development leads to equations for the other expansion coefficients B_{jl}:

$$\begin{aligned}0&=\int dV\mathbf{E}_{im}^{(v)}\cdot\big[-\nabla\times(\nabla\times\mathbf{E})+k_n^2\mathbf{E}\big]-\int dV\mathbf{E}\cdot\big[-\nabla\times\big(\nabla\times\mathbf{E}_{im}^{(v)}\big)+k_n^2\mathbf{E}_{im}^{(v)}\big]\\&=k_n^2\sum_j u_j\int dV\theta_j\big(\mathbf{E}_{im}^{(v)}\cdot\mathbf{E}\big)-\sum_j v_j\int dV\mathbf{E}_{im}^{(v)}\cdot[\nabla\times\theta_j(\nabla\times\mathbf{E})]-\frac{4\pi\omega\mu_n}{c^2}\int dV\big(\mathbf{E}_{im}^{(v)}\cdot\mathbf{J}_{\text{ex}}\big)\\&\quad+v_{im}\int dV\mathbf{E}\cdot\big[\nabla\times\theta_i\big(\nabla\times\mathbf{E}_{im}^{(v)}\big)\big].\end{aligned} \tag{27}$$

Using the second expansion from Eq. (22) for the physical field $\mathbf{E}(\mathbf{r})$ we get the following set of inhomogeneous linear algebraic equations for the B_{jl} coefficients:

$$0=(v_{im}-v_i)B_{im}\langle\nabla\times\mathbf{E}_{im}^{(v)}|\nabla\times\mathbf{E}_{im}^{(v)}\rangle_i+k_n^2\sum_{jl}u_jB_{jl}\langle\mathbf{E}_{im}^{(v)}|\mathbf{E}_{jl}^{(v)}\rangle_j-\sum_{jl,j\neq i}v_jB_{jl}\langle\nabla\times\mathbf{E}_{im}^{(v)}|\nabla\times\mathbf{E}_{jl}^{(v)}\rangle_j-\frac{4\pi i\omega\mu_n}{c^2}\langle\mathbf{E}_{im}^{(v)}|\mathbf{J}_{\rm ex}\rangle_n. \tag{28}$$

Note that Eq. (26) remains valid even if $\langle\mathbf{E}_{im}^{(u)}|\mathbf{E}_{im}^{(u)}\rangle_i=0$. Likewise, Eq. (28) remains valid even if $\langle\nabla\times\mathbf{E}_{im}^{(v)}|\nabla\times\mathbf{E}_{im}^{(v)}\rangle_i=0$. This means that even if those eigenstates are not normalizable, the coefficients A_{im} and B_{im} can still be found, and used to expand the physical fields.

III. EXPANDING A PHYSICAL FIELD

Because u_n, v_n, t_n, and w_n all vanish, therefore all the θ_i-dependent terms in Eqs. (5) and (6) can be expanded in the various divergence-free eigenfunctions. For example,

$$k_n^2\sum_i u_i\theta_i\mathbf{E}=k_n^2\sum_{im}u_i\theta_iA_{im}\mathbf{E}_{im}^{(u)}$$
$$=\sum_{im}\frac{u_i}{u_{im}}A_{im}\big[-\nabla\times\big(\nabla\times\mathbf{E}_{im}^{(u)}\big)+k_n^2\mathbf{E}_{im}^{(u)}\big], \tag{29}$$

where we have used Eq. (10) to get the final result, which no longer includes the step functions $\theta_i(\mathbf{r})$. Similarly, we get that

$$-\sum_i v_i\nabla\times[\theta_i(\nabla\times\mathbf{E})]$$
$$=-\sum_{im}v_iB_{im}\nabla\times\big[\theta_i\big(\nabla\times\mathbf{E}_{im}^{(v)}\big)\big]$$
$$=\sum_{im}\frac{v_i}{v_{im}}B_{im}\big[-\nabla\times\big(\nabla\times\mathbf{E}_{im}^{(v)}\big)+k_n^2\mathbf{E}_{im}^{(v)}\big], \tag{30}$$

where we have used Eq. (11) to get the final result, which no longer includes the step functions $\theta_i(\mathbf{r})$.

Using these results to expand the terms in Eq. (5) which depend upon θ_i we finally get the following PDE for the physical field $\mathbf{E}$:

$$\big[-\nabla\times(\nabla\times)+k_n^2\big]$$
$$\cdot\left[\mathbf{E}-\sum_{im}\frac{u_i}{u_{im}}A_{im}\mathbf{E}_{im}^{(u)}-\sum_{im}\frac{v_i}{v_{im}}B_{im}\mathbf{E}_{im}^{(v)}\right]=-\frac{4\pi i\omega\mu_n}{c^2}\mathbf{J}_{\rm ex}. \tag{31}$$

The quantity inside the large square brackets is just the "incident field $\mathbf{E}_0$," which satisfies the following PDE:

$$-\nabla\times(\nabla\times\mathbf{E}_0)+k_n^2\mathbf{E}_0=-\frac{4\pi i\omega\mu_n}{c^2}\mathbf{J}_{\rm ex}. \tag{32}$$

Therefore the local physical field in the composite medium is given by

$$\mathbf{E}=\mathbf{E}_0+\sum_{im}\frac{u_i}{u_{im}}A_{im}\mathbf{E}_{im}^{(u)}+\sum_{im}\frac{v_i}{v_{im}}B_{im}\mathbf{E}_{im}^{(v)} \tag{33}$$

for all values of $\mathbf{r}$. Note that all the sums over i in Eqs. (29)–(31) and (33) in practice do not include the value $i=n$.

In order to use a given incident field $\mathbf{E}_0$ to calculate the local physical field consider the following vanishing difference of two integrals:

$$0=\int dV\mathbf{E}_{im}^{(u)}\cdot\big[-\nabla\times(\nabla\times\mathbf{E}_0)+k_n^2\mathbf{E}_0\big]$$
$$-\int dV\mathbf{E}_0\cdot\big[-\nabla\times\big(\nabla\times\mathbf{E}_{im}^{(u)}\big)+k_n^2\mathbf{E}_{im}^{(u)}\big]$$
$$=-\frac{4\pi i\omega\mu_n}{c^2}\langle\mathbf{E}_{im}^{(u)}|\mathbf{J}_{ex}\rangle_n-k_n^2u_{im}\langle\mathbf{E}_0|\mathbf{E}_{im}^{(u)}\rangle_i. \tag{34}$$

From this it follows that

$$-\frac{4\pi i\omega\mu_n}{c^2}\langle\mathbf{E}_{im}^{(u)}|\mathbf{J}_{\rm ex}\rangle_n=k_n^2u_{im}\langle\mathbf{E}_0|\mathbf{E}_{im}^{(u)}\rangle_i. \tag{35}$$

This can be substituted for the inhomogeneous term in Eq. (26) in order to calculate the A_{jl} expansion coefficients.

A similar treatment where $\mathbf{E}_{im}^{(v)}$ replaces $\mathbf{E}_{im}^{(u)}$ leads to

$$0=\int dV\mathbf{E}_{im}^{(v)}\cdot\big[-\nabla\times(\nabla\times\mathbf{E}_0)+k_n^2\mathbf{E}_0\big]$$
$$-\int dV\mathbf{E}_0\cdot\big[-\nabla\times\big(\nabla\times\mathbf{E}_{im}^{(v)}\big)+k_n^2\mathbf{E}_{im}^{(v)}\big]$$
$$=-\frac{4\pi i\omega\mu_n}{c^2}\langle\mathbf{E}_{im}^{(v)}|\mathbf{J}_{\rm ex}\rangle_n+v_{im}\langle\nabla\times\mathbf{E}_0|\nabla\times\mathbf{E}_{im}^{(v)}\rangle_i. \tag{36}$$

From this it follows that

$$-\frac{4\pi i\omega\mu_n}{c^2}\langle\mathbf{E}_{im}^{(v)}|\mathbf{J}_{\rm ex}\rangle_n=-v_{im}\langle\nabla\times\mathbf{E}_0|\nabla\times\mathbf{E}_{im}^{(v)}\rangle_i. \tag{37}$$

This can be substituted for the inhomogeneous term in Eq. (28) in order to calculate the B_{jl} expansion coefficients.

A. One or two isolated spherical inclusions

A simple example is the field created by a source current that is near an isolated sphere of volume V_1 centered at the origin. We assume that μ has the same value in the inclusions and in the host and that only κ differs in the two constituents. In order to calculate this field we specialize Eq. (26) to read

$$A_{1l}=\frac{1}{u_1-u_{1l}}\frac{4\pi i}{\omega\kappa_2}\frac{\langle\mathbf{E}_{1l}|\mathbf{J}_{ex}\rangle_2}{\langle\mathbf{E}_{1l}|\mathbf{E}_{1l}\rangle_1}. \tag{38}$$

We assume that $\mathbf{J}_{\rm ex}$ is a point source, namely

$$\mathbf{J}_{\rm ex}(\mathbf{r})=\mathbf{J}_0\delta^3(\mathbf{r}-\mathbf{r}_0).$$

It follows that, if the physical value of u_1 is very close to the eigenvalue u_{1l}, then only one term is important in the expansion of the physical field $\mathbf{E}(\mathbf{r})$ in Eq. (33):

$$\mathbf{E} \approx A_{1l}\frac{u_1}{u_{1l}}\mathbf{E}_{1l}(\mathbf{r}) = \frac{u_1}{u_{1l}}\frac{1}{u_1-u_{1l}}\frac{4\pi i}{\omega\kappa_2}\frac{[\mathbf{J}_0\cdot\mathbf{E}_{1l}(\mathbf{r}_0)]\mathbf{E}_{1l}(\mathbf{r})}{\langle\mathbf{E}_{1l}|\mathbf{E}_{1l}\rangle_1}.$$

When $|k_2a| \ll 1$, all the eigenvalues $u_{bl}^{(E)}$ and $u_{bl}^{(M)}$, $b \geqslant 1$, are large, real, and positive [see Eqs. (A6) and (A7)]. The parameters ε_1 and ε_2 are real and positive when both constituents are dielectrics. However, if the sphere is metallic, and if the electromagnetic (EM) frequency ω is much greater than the inverse relaxation time $1/\tau$ but much less than the plasma frequency ω_p, then ε_1 is real and can be negative. Therefore u_1 can also be real, $O(1)$, and positive. Consequently, usually the only eigenvalue that can be near u_1 is one of the $u_{0l}^{(M)}$, i.e., a quasistatic eigenvalue. The series representation of the physical field, Eq. (33), is always a useful expansion—it allows that field to be calculated as a sum over the eigenfunctions. Each of these is an electric multipole field of order l. By contrast, the incident field $\mathbf{E}_0$, which would be observed if the spherical inclusion were absent, is always an electric dipole field at large distances—see Ref. [10]. In this simple example all the eigenstates, as well as the local physical field, are obtained as closed-form expressions—see Eqs. (A19)–(A24) in the Appendix.

This will work also for a collection of nonoverlapping spheres, even when those are quite close to each other.

When there are just two such spheres, $i = 1, 2$ for the spheres, $i = 3$ for the homogeneous host medium, where $\mu_1 = \mu_2 = \mu_3$, then Eqs. (26) become

$$(u_{1m}-u_1)\langle\mathbf{E}_{1m}^{(u)}|\mathbf{E}_{1m}^{(u)}\rangle_1 A_{1m} = \sum_l u_2A_{2l}\langle\mathbf{E}_{1m}^{(u)}|\mathbf{E}_{2l}^{(u)}\rangle_2 - \frac{4\pi i}{\omega\kappa_3}[\mathbf{E}_{1m}^{(u)}(\mathbf{r}_0)\cdot\mathbf{J}_0], \quad (39)$$

$$(u_{2m}-u_2)\langle\mathbf{E}_{2m}^{(u)}|\mathbf{E}_{2m}^{(u)}\rangle_2 A_{2m} = \sum_l u_1A_{1l}\langle\mathbf{E}_{2m}^{(u)}|\mathbf{E}_{1l}^{(u)}\rangle_1 - \frac{4\pi i}{\omega\kappa_3}[\mathbf{E}_{2m}^{(u)}(\mathbf{r}_0)\cdot\mathbf{J}_0]. \quad (40)$$

Here, I have again assumed that

$$\mathbf{J}_{\text{ex}} = \mathbf{J}_0\delta^3(\mathbf{r}-\mathbf{r}_0), \quad |\mathbf{r}_0| > a,$$

where $\mathbf{r}_0 \in V_3$. From Eqs. (39) and (40) we obtain the following equations, where A_{1m} and A_{2m} appear separately:

$$(u_{1m}-u_1)\langle\mathbf{E}_{1m}^{(u)}|\mathbf{E}_{1m}^{(u)}\rangle_1 A_{1m} - \sum_{l,p} A_{1p}\frac{u_1u_2}{u_{2l}-u_2}\frac{\langle\mathbf{E}_{2l}^{(u)}|\mathbf{E}_{1p}^{(u)}\rangle_1\langle\mathbf{E}_{1m}^{(u)}|\mathbf{E}_{2l}^{(u)}\rangle_2}{\langle\mathbf{E}_{2l}^{(u)}|\mathbf{E}_{2l}^{(u)}\rangle_2}$$
$$= -\frac{4\pi i}{\omega\kappa_3}\left\{\left[(\mathbf{E}_{1m}^{(u)}(\mathbf{r}_0)\cdot\mathbf{J}_0\right] + \sum_l \frac{u_2}{u_{2l}-u_2}\frac{\langle\mathbf{E}_{1m}^{(u)}|\mathbf{E}_{2l}^{(u)}\rangle_2}{\langle\mathbf{E}_{2l}^{(u)}|\mathbf{E}_{2l}^{(u)}\rangle_2}\left[\mathbf{E}_{2l}^{(u)}(\mathbf{r}_0)\cdot\mathbf{J}_0\right]\right\}, \quad (41)$$

$$(u_{2m}-u_2)\langle\mathbf{E}_{2m}^{(u)}|\mathbf{E}_{2m}^{(u)}\rangle_2 A_{2m} - \sum_{l,p} A_{2p}\frac{u_1u_2}{u_{1l}-u_1}\frac{\langle\mathbf{E}_{2m}^{(u)}|\mathbf{E}_{1l}^{(u)}\rangle_1\langle\mathbf{E}_{1l}^{(u)}|\mathbf{E}_{2p}^{(u)}\rangle_2}{\langle\mathbf{E}_{1l}^{(u)}|\mathbf{E}_{1l}^{(u)}\rangle_1}$$
$$= -\frac{4\pi i}{\omega\kappa_3}\left\{\left[(\mathbf{E}_{2m}^{(u)}(\mathbf{r}_0)\cdot\mathbf{J}_0\right] + \sum_l \frac{u_1}{u_{1l}-u_1}\frac{\langle\mathbf{E}_{2m}^{(u)}|\mathbf{E}_{1l}^{(u)}\rangle_1}{\langle\mathbf{E}_{1l}^{(u)}|\mathbf{E}_{1l}^{(u)}\rangle_1}\left[\mathbf{E}_{1l}^{(u)}(\mathbf{r}_0)\cdot\mathbf{J}_0\right]\right\}. \quad (42)$$

Although these two systems of equations are somewhat simpler than Eqs. (39) and (40), since they can be solved separately, they are still difficult to handle. In particular, they do not allow a simple discussion of the case where u_1 or u_2 is close to one of the cluster eigenvalues.

A different technique for treating a collection of nonintersecting inclusions made of the same material is to first calculate the eigenstates $\mathbf{E}_\alpha^{(u)}(\mathbf{r})$ of the cluster. This can be done using the approach described in Eq. (44) of Ref. [6]: The eigenstates of the cluster are expanded in those of the isolated inclusions,

$$\theta_i\mathbf{E}_\alpha^{(u)} = \theta_i\sum_m A_{im}^{(\alpha)}\mathbf{E}_{im}^{(u)}.$$

The coefficients $A_{im}^{(\alpha)}$ satisfy the following equations,

$$\frac{1}{u_\alpha}A_{im}^{(\alpha)} = \sum_{jl} M_{im,jl}A_{jl}^{(\alpha)}, \quad (43)$$

$$M_{im,jl} = \frac{1}{u_{im}}\frac{\langle\mathbf{E}_{im}^{(u)}|\mathbf{E}_{jl}^{(u)}\rangle_j}{\langle\mathbf{E}_{im}^{(u)}|\mathbf{E}_{im}^{(u)}\rangle_i}, \quad (44)$$

where $\hat{M}$ is a symmetric matrix if $\langle\mathbf{E}_{im}^{(u)}|\mathbf{E}_{im}^{(u)}\rangle_i = 1$, i.e., $M_{jl,im} = M_{im,jl}$, which depends only on the isolated inclusion eigenstates. If the source $\mathbf{J}_{\text{ex}}(\mathbf{r})$ of the physical field is nonzero only in the host medium, then that field can be expanded inside the cluster of inclusions using the cluster eigenstates as in Eq. (22).

The overlap integral of eigenstates from two different identical spheres

$$\langle\mathbf{E}_{bm}^{(u)}|\mathbf{E}_{b'l}^{(u)}\rangle_b = \int dV\theta_b\left(\mathbf{E}_{bm}^{(u)}\cdot\mathbf{E}_{b'l}^{(u)}\right), \quad b \neq b',$$

can be found in Ref. [11]. Here, we only reproduce it for the case where both eigenstates are of the transverse electric field type. The suffixes b, b' characterize the locations of the particular spheres and the particular isolated sphere eigenstates in question.

In order to calculate the overlap integrals of eigenstates from two spheres, one of which is centered at the origin while the other is centered at an arbitrary point $\mathbf{b}$, we need to expand the latter state in terms of the vector spherical harmonics around the origin [see Eq. (48) in Ref. [11]]:

$$f_l(k|\mathbf{r}-\mathbf{b}|)\mathbf{Y}_{JlM}(\Omega_{\mathbf{r}-\mathbf{b}}) = \sum_{l'J'M'\lambda\mu} i^{l'-l+\lambda}(-1)^{1+l+l'-M'}[4\pi(2l+1)(2l'+1)(2J+1)(2J'+1)(2\lambda+1)]^{1/2} \times \begin{pmatrix}\lambda & J' & J\\ \mu & M' & -M\end{pmatrix}\begin{pmatrix}\lambda & l' & l\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}\lambda & J' & J\\ 1 & l & l'\end{Bmatrix} f_\lambda(k|\mathbf{b}|)Y_{\lambda\mu}(\Omega_{\mathbf{b}})j_{l'}(kr)\mathbf{Y}_{J'l'M'}(\Omega_{\mathbf{r}}) \times \text{ for } |\mathbf{b}|>a \text{ and } |\mathbf{r}-\mathbf{b}|>a. \quad (45)$$

Using this expansion, we get the following result [see Eq. (32) in Ref. [11]],

$$\langle \mathbf{E}^{(M)}_{blm}|\mathbf{E}^{(M)}_{b'l'm'}\rangle_{b'} = A^{(M)}_{bl}A^{(M)}_{b'l'}(-1)^{1+l-l'-m'}(4\pi)^{1/2}(2l+1)(2l'+1)I_l\left(a,u^{(M)}_{bl}\right) \times \sum_{\lambda\mu} i^\lambda(2\lambda+1)^{1/2}\begin{pmatrix}\lambda & l & l'\\ \mu & m & -m'\end{pmatrix}\begin{pmatrix}\lambda & l & l'\\ 0 & 0 & 0\end{pmatrix}\begin{Bmatrix}\lambda & l & l'\\ 1 & l' & l\end{Bmatrix} h^{(1)}_\lambda(k|\mathbf{b}'-\mathbf{b}|)Y_{\lambda\mu}(\Omega_{\mathbf{b}'-\mathbf{b}}), \quad (46)$$

where

$$I_l(a,u) \equiv \int_0^a dr\, r^2 j_l[kr(1-u)^{1/2}]j_l(kr) = -\frac{a^3}{kau}[j_l(x)j_{l-1}(ka) - (1-u)^{1/2}j_{l-1}(x)j_l(ka)]_{x=ka(1-u)^{1/2}},$$

and where $A^{(M)}_{bl}, A^{(M)}_{b'l'}$ are the normalization coefficients of the two eigenfunctions.

If the separation between any pair of spheres is much greater than the radii a, then the spheres can often be treated as isolated. The only case when this is unjustified is when the two isolated-sphere eigenstates are the same and the eigenvalues $u^{(F)}_{1m} = u^{(F)}_{2m}$ are very close to $u_1 = u_2$. In that case the isolated-sphere eigenvalue is split by the interaction and one of the split values can be very close to $u_1 = u_2$. This can lead to an enhanced response of the pair of spheres to the field source $\mathbf{J}_{\rm ex}$ which would depend sensitively on the sphere separation.

The two-sphere eigenstates were calculated in detail for the case where μ is homogeneous in Refs. [11,12]. They were then used to calculate the lifetimes of eigenstates in such systems, including both quasistatic and nonquasistatic eigenstates.

In a future publication the procedure outlined in this article will be applied to more complex microstructures, e.g., to a dense cluster of spherical inclusions, to a periodic array of such inclusions, and to cases where both $\kappa(\mathbf{r})$ and $\mu(\mathbf{r})$ are heterogeneous.

IV. CONCLUSIONS

Eigenstates of Maxwell's equations in a composite medium have been extended to deal with a multiconstituent structure where each constituent i has a subvolume V_i where it is a uniform material with its own values of electric permittivity κ_i and magnetic permeability μ_i. The eigenfunctions are vector functions $\mathbf{E}_{i\alpha}$ of the electric field which are divergence free in each constituent. The eigenvalues are essentially special values of κ_i and μ_i. The physical fields $\mathbf{E}(\mathbf{r})$ and $\mathbf{H}(\mathbf{r})$ were assumed to be produced by a given electric current density $\mathbf{J}_{\rm ex}(\mathbf{r})$ which is nonzero only in one constituent, denoted by the suffix n. Consequently $\nabla\cdot\mathbf{E} = \nabla\cdot\mathbf{H} = 0$ in all the subvolumes V_i, with the exception of V_n. Therefore the physical fields can be expanded in a series of eigenfunctions only inside V_i, $i \neq n$. This is sufficient for obtaining expressions for the physical fields that are valid everywhere, i.e., even inside V_n. I assumed that the eigenfunctions form a complete set inside all V_i, $i \neq n$. I showed how $\mathbf{E}(\mathbf{r})$ can be calculated in a collection of nonintersecting inclusions in terms of the eigenfunctions of the isolated inclusions. This was shown in detail for a collection of spherical inclusions, in particular for a single pair of spheres. This works even if the spheres are very close to each other and the constituent permittivities are very close to some of the eigenvalues.

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APPENDIX: EIGENFUNCTIONS OF AN ISOLATED SPHERE

This Appendix is based upon the Appendix of Ref. [6]. Our discussion will therefore be brief. More details appear in that article.

The inside of the sphere, of radius a, is called V_1, and its outside is called V_2. The sphere is taken to be centered at the origin. The $\mathbf{E}^{(u)}_{blm}$ eigenfunctions are of two types—transverse electric field and transverse magnetic field [these are eigenstates for the inside of the sphere]:

$$\begin{aligned}\mathbf{E}^{(uE)}_{blm}(\mathbf{r}) &\propto j_l(k_{bl}r)\mathbf{Y}_{llm}(\Omega), \quad r<a,\\ \mathbf{H}^{(uM)}_{blm}(\mathbf{r}) &\propto j_l(k_{bl}r)\mathbf{Y}_{llm}(\Omega), \quad r<a.\end{aligned} \quad (A1)$$

Here, $\mathbf{Y}_{Jlm}(\Omega)$ is a vector spherical harmonic (VSH), defined by

$$\mathbf{Y}_{JlM}(\Omega) = \sum_{mq} Y_{lm}(\Omega)\mathbf{e}_q(lm1q|l1JM), \quad (A2)$$

j_l is the regular spherical Bessel function, and

$$k^2_{bl} \equiv \frac{\omega^2}{c^2}\kappa_2\mu_2(1-u_{bl}),$$

where u_{bl} is the eigenvalue of $\mathbf{E}^{(u)}_{blm}$ in Eq. (10). The vectors $\mathbf{e}_q$, $q = -1, 0, 1$ are the complex spherical unit vectors

$$\mathbf{e}_0 \equiv \mathbf{e}_z, \quad \mathbf{e}_{\pm 1} \equiv \mp \frac{1}{\sqrt{2}}(\mathbf{e}_x \pm i\mathbf{e}_y)$$

and $(lm1q|l1JM)$ is a Clebsch-Gordan coefficient. Note that $\mathbf{Y}_{llm} = -i(\mathbf{r}\times\nabla)Y_{lm}/\sqrt{l(l+1)}$ is perpendicular to the position vector $\mathbf{r}$. When $r > a$, the above two eigenfunctions have $j_l(k_{bl}r)$ replaced by $h^{(1)}_l(k_2 r)$, where $h^{(1)}_l$ is the singular outgoing spherical Bessel function. The eigenfunctions must satisfy the usual continuity conditions on the tangential components of the electric and magnetic fields at the sphere surface. These are quite straightforward to apply in the case of the transverse electric field eigenfunctions, since $\mathbf{Y}_{llM}(\Omega) \perp \mathbf{r}$. In the case of the transverse magnetic field eigenfunctions the electric field is no longer transverse, in contrast with the magnetic field, since $\mathbf{E} \propto \frac{1}{\kappa}(\nabla \times \mathbf{H})$, from which it follows that [see Eq. (5.9.19) in Ref. [13]]

$$\mathbf{E} \propto \frac{1}{\kappa(\mathbf{r})}\nabla \times f_l(kr)\mathbf{Y}_{llm} = \frac{i}{\kappa(\mathbf{r})}\left[\left(\frac{l}{2l+1}\right)^{1/2}\left(\frac{d}{dr}-\frac{l}{r}\right)f_l(kr)\mathbf{Y}_{l\,l+1\,m} + \left(\frac{l+1}{2l+1}\right)^{1/2}\left(\frac{d}{dr}+\frac{l+1}{r}\right)f_l(kr)\mathbf{Y}_{l\,l-1\,m}\right]$$
$$= -\frac{ik}{\kappa(\mathbf{r})}\left[\left(\frac{l}{2l+1}\right)^{1/2} f_{l+1}(kr)\mathbf{Y}_{l\,l+1\,m} - \left(\frac{l+1}{2l+1}\right)^{1/2} f_{l-1}(kr)\mathbf{Y}_{l\,l-1\,m}\right], \tag{A3}$$

where f_l can be any spherical Bessel function. This is not perpendicular to $\mathbf{r}$. The component of this electric field which is perpendicular to $\mathbf{r}$ can be obtained by noting that, for any function $\phi(r)$, the following equality holds,

$$\mathbf{r}\times[\nabla\times\phi(r)\mathbf{Y}_{llm}(\Omega)] = -\frac{\partial}{\partial r}[r\phi(r)]\mathbf{Y}_{llm}(\Omega)].$$

The continuity requirements at the sphere surface lead to the following equations for the eigenvalues $u^{(E)}_{bl}$ and $u^{(M)}_{bl}$:

$$\left.\frac{y j_l'(y)}{j_l(y)}\right|_{y=k_2 a\sqrt{1-u^{(E)}_{bl}}} = \left.\frac{x h^{(1)\prime}_l(x)}{h^{(1)}_l(x)}\right|_{x=k_2 a}, \tag{A4}$$

$$\left[\frac{1}{y^2}+\frac{j_l'(y)}{y j_l(y)}\right]_{y=k_2 a\sqrt{1-u^{(M)}_{bl}}} = \left[\frac{1}{x^2}+\frac{h^{(1)\prime}_l(x)}{x h^{(1)}_l(x)}\right]_{x=k_2 a}. \tag{A5}$$

These are transcendental equations which can usually only be solved numerically. Only when the sphere is much smaller than the incident wavelength $2\pi/k_2$, i.e., $|k_2|a \ll 1$, can the small argument form of $h^{(1)}_l(x)$ be used to get the following closed-form results [$y_{bl}, b \geqslant 1$ is the bth zero of $j_l(y)$],

$$\left(\frac{k^{(E)}_{bl}}{k_2}\right)^2 \equiv 1 - u^{(E)}_{bl} \approx \frac{y^2_{b\,l-1}}{k_2^2 a^2} = O\left(\frac{1}{|k_2|^2 a^2}\right) \gg 1, \tag{A6}$$

where

$$j_{l-1}\left(k^{(E)}_{bl}a\right) \approx \frac{(k_2 a)^2}{(2l-1)y_{b\,l-1}} j_l(y_{b\,l-1});$$

$$\left(\frac{k^{(M)}_{bl}}{k_2}\right)^2 \equiv 1 - u^{(M)}_{bl} \approx \frac{y^2_{bl}}{k_2^2 a^2} = O\left(\frac{1}{|k_2|^2 a^2}\right) \gg 1, \tag{A7}$$

where

$$j_l\left(k^{(M)}_{bl}a\right) \approx -\frac{(k_2 a)^2}{l y_{bl}} j_{l-1}(y_{bl});$$

$$\left(\frac{k^{(M)}_{0l}}{k_2}\right)^2 \equiv 1 - u^{(M)}_{0l} = -\frac{l+1}{l} + O(|k_2|^2 a^2). \tag{A8}$$

From these results it follows that, when $|k_2|a \ll 1$, then $u^{(E)}_{bl} \gg 1$ and $u^{(M)}_{bl} \gg 1$ for all $b \geqslant 1$ while $u^{(M)}_{0l} \approx (2l+1)/l = O(1)$.

Because of the spherical symmetry, the eigenvalues $u^{(E)}_{bl}, u^{(M)}_{bl}$ have much degeneracy—they are independent of m. Also, we can reorganize the biorthogonality properties of the eigenstates as follows [5]: First, we now define the scalar product in the more standard way as

$$\langle \mathbf{E}|\mathbf{F}\rangle_i \equiv \int dV \theta_i (\mathbf{E}^* \cdot \mathbf{F}) = \langle \mathbf{F}|\mathbf{E}\rangle_i^*.$$

Second, the left eigenfunction that is conjugate to any of the right eigenfunctions is defined by

$$\mathcal{C}[f_{nl}(r)\mathbf{Y}_{llm}(\Omega)] \equiv f^*_{nl}(r)\mathbf{Y}_{llm}(\Omega),$$
$$\mathcal{C}[\nabla\times f_{nl}(r)\mathbf{Y}_{llm}(\Omega)] \equiv \nabla\times f^*_{nl}(r)\mathbf{Y}_{llm}(\Omega),$$

i.e., only the radial part is complex conjugated. It follows that

$$\mathcal{C}\left\{\frac{1}{\kappa(r)}[\nabla\times f_l(kr)\mathbf{Y}_{llm}(\Omega)]\right\}^* = -i\frac{k}{\kappa(r)}\left[\left(\frac{l}{2l+1}\right)^{1/2} f_{l+1}(kr)\mathbf{Y}^*_{l\,l+1\,m} - \left(\frac{l+1}{2l+1}\right)^{1/2} f_{l-1}(kr)\mathbf{Y}^*_{l\,l-1\,m}\right]. \tag{A9}$$

Using Eqs. (A3) and (A9) with $f_l(kr) = j_l(k^{(M)}_{bl}r)$ when $r < a$ and $f_l(kr) = h^{(1)}_l(k_2 r)$ when $r > a$ for the transverse magnetic field eigenfunctions $\mathbf{E}^{(uM)}_{1blm}$, which will henceforth be denoted as $\mathbf{E}^{(M)}_{blm}$, we now get

$$\mathbf{E}^{(M)}_{blm} \equiv \frac{ic}{\omega\kappa(\mathbf{r})}\left(\nabla\times\mathbf{H}^{(M)}_{blm}\right) = \frac{k^{(M)}_{bl}}{\kappa^{(M)}_{bl}}A^{(M)}_{bl}\left[\left(\frac{l}{2l+1}\right)^{1/2} j_{l+1}\left(k^{(M)}_{bl}r\right)\mathbf{Y}_{ll+1\,m}(\Omega) - \left(\frac{l+1}{2l+1}\right)^{1/2} j_{l-1}\left(k^{(M)}_{bl}r\right)\mathbf{Y}_{ll-1\,m}(\Omega)\right], \quad r < a,$$

$$\mathbf{E}^{(M)}_{blm} = \frac{k_2}{\kappa_2}B^{(M)}_{bl}\left[\left(\frac{l}{2l+1}\right)^{1/2} h^{(1)}_{l+1}(k_2 r)\mathbf{Y}_{ll+1\,m}(\Omega) - \left(\frac{l+1}{2l+1}\right)^{1/2} h^{(1)}_{l-1}(k_2 r)\mathbf{Y}_{ll-1\,m}(\Omega)\right], \quad r > a.$$

The normalization properties of the V_1 eigenfunctions become

$$\begin{aligned} 1 &= \int_{r<a} dV \left[\mathcal{C}\mathbf{E}^{(E)}_{blm}\right]^* \cdot \mathbf{E}^{(E)}_{blm} \\ &= \left[A^{(E)}_{bl}\right]^2 \int_{r<a} dV \left[j_l\left(k^{(E)}_{bl} r\right)\right]^2 |\mathbf{Y}_{llm}|^2 \approx \left[A^{(E)}_{bl}\right]^2 \int_{r<a} r^2\, dr \left[j_l\left(k^{(E)}_{bl} r\right)\right]^2 \\ &= \left[A^{(E)}_{bl}\right]^2 \left\{\frac{r^3}{2}\left(\left[j_l\left(k^{(E)}_{bl} r\right)\right]^2 - j_{l-1}\left(k^{(E)}_{bl} r\right) j_{l+1}\left(k^{(E)}_{bl} r\right)\right)\right\}_0^a \approx \left[A^{(E)}_{bl}\right]^2 \frac{a^3}{2} j_l^2(y_{bl-1}), \end{aligned} \tag{A10}$$

$$\begin{aligned} 1 &= \int_{r<a} dV \left[\mathcal{C}\mathbf{E}^{(M)}_{blm}\right]^* \cdot \mathbf{E}^{(M)}_{blm} \\ &= \left(\frac{k^{(M)}_{bl}}{\kappa^{(M)}_{bl}}\right)^2 \int_{r<a} dV \left[A^{(M)}_{bl}\right]^2 \left[j_{l+1}\left(k^{(M)}_{bl} r\right)\left(\frac{l}{2l+1}\right)^{1/2} \mathbf{Y}^*_{l\,l+1\,m}(\Omega) - j_{l-1}\left(k^{(M)}_{bl} r\right)\left(\frac{l+1}{2l+1}\right)^{1/2} \mathbf{Y}^*_{l\,l-1\,m}(\Omega)\right] \\ &\times \left[j_{l+1}\left(k^{(M)}_{bl} r\right)\left(\frac{l}{2l+1}\right)^{1/2} \mathbf{Y}_{l\,l+1\,m}(\Omega) - j_{l-1}\left(k^{(M)}_{bl} r\right)\left(\frac{l+1}{2l+1}\right)^{1/2} \mathbf{Y}_{l\,l-1\,m}(\Omega)\right] \\ &= \left(\frac{k^{(M)}_{bl}}{\kappa^{(M)}_{bl}}\right)^2 \left[A^{(M)}_{bl}\right]^2 \int_{r<a} r^2 dr \left\{ \frac{l}{2l+1}\left[j_{l+1}\left(k^{(M)}_{bl} r\right)\right]^2 + \frac{l+1}{2l+1}\left[j_{l-1}\left(k^{(M)}_{bl} r\right)\right]^2 \right\} \\ &= \left(\frac{k^{(M)}_{bl}}{\kappa^{(M)}_{bl}}\right)^2 \left[A^{(M)}_{bl}\right]^2 \frac{a^3}{2} \left[\frac{l}{2l+1}\left(j^2_{l+1}\left(k^{(M)}_{bl} a\right) - j_l\left(k^{(M)}_{bl} a\right) j_{l+2}\left(k^{(M)}_{bl} a\right)\right) \right. \\ &\left. + \frac{l+1}{2l+1}\left(j^2_{l-1}\left(k^{(M)}_{bl} a\right) - j_{l-2}\left(k^{(M)}_{bl} a\right) j_l\left(k^{(M)}_{bl} a\right)\right)\right] \approx \frac{a}{2}\left[A^{(M)}_{bl}\right]^2 \frac{(k_2 a)^4 j^2_{l+1}(y_{bl})}{\kappa_2^2 y_{bl}^2}, \end{aligned} \tag{A11}$$

$$\begin{aligned} 1 &= \int_{r<a} dV \left[\mathcal{C}\mathbf{E}^{(M)}_{0lm}\right]^* \cdot \mathbf{E}^{(M)}_{0lm} = \left(\frac{k^{(M)}_{0l}}{\kappa^{(M)}_{0l}}\right)^2 \frac{\left(A^{(M)}_{0l}\right)^2}{2l+1} \int_{r<a} r^2 dr \left[l\, j^2_{l+1}\left(k^{(M)}_{0l} r\right) + (l+1) j^2_{l-1}\left(k^{(M)}_{0l} r\right)\right] \\ &\approx \frac{a(l+1)\left[A^{(M)}_{0l}\right]^2}{\kappa_2^2} \left(-\frac{l+1}{l}\right)^{l-2} \frac{(k_2 a)^{2l}}{[(2l+1)!!]^2}. \end{aligned} \tag{A12}$$

These expressions determine the normalization coefficients $A^{(E)}_{bl}$ and $A^{(M)}_{bl}$. These eigenfunctions will be used to expand the physical field when that field is divergence free.

In this way the following results are obtained for the normalization coefficients of the $\mathbf{E}^{(E)}_{blm}$ and $\mathbf{E}^{(M)}_{blm}$ eigenfunctions when $|k_2|a \ll 1$:

$$A^{(E)}_{bl} \approx \left(\frac{2}{a^3}\right)^{1/2} \frac{1}{j_l(y_{b\,l-1})}, \tag{A13}$$

$$B^{(E)}_{bl} \approx i\left(\frac{2}{a^3}\right)^{1/2} \frac{(k_2 a)^{l+3}}{(2l-1)!!}, \tag{A14}$$

$$A^{(M)}_{bl} \approx \left(\frac{2}{a}\right)^{1/2} \frac{y_{bl}\kappa_2}{(k_2 a)^2 j_{l+1}(y_{bl})}, \tag{A15}$$

$$B^{(M)}_{bl} \approx -i\left(\frac{2}{a}\right)^{1/2} \frac{(k_2 a)^{l+1}\kappa_2}{(2l-1)!!}, \tag{A16}$$

$$A^{(M)}_{0l} \approx \frac{1}{\sqrt{a(l+1)}}\left(-\frac{l}{l+1}\right)^{\frac{l-2}{2}} \frac{\kappa_2 (2l+1)!!}{(k_2 a)^l}, \tag{A17}$$

$$B^{(M)}_{0l} \approx -i\kappa_2\left(\frac{l+1}{a}\right)^{1/2} \frac{(k_2 a)^{l+2}}{(2l+1)!!}. \tag{A18}$$

Note that the sign used when calculating the square root of $(A^{(F)}_{bl})^2$ to get $A^{(F)}_{bl}$ was chosen arbitrarily. This does not affect the final results for the physical fields. From these results we can write the following closed-form expressions for the eigenfunctions when $|k_2|a \ll 1$:

$$\mathbf{E}^{(E)}_{blm} \approx \left(\frac{2}{a^3}\right)^{1/2} \frac{j_l(r y_{b\,l-1}/a)}{j_l(y_{b\,l-1})} \mathbf{Y}_{llm}, \quad r < a, \tag{A19}$$

$$\mathbf{E}^{(E)}_{blm} \approx i\left(\frac{2}{a^3}\right)^{1/2} \frac{(k_2 a)^{l+3}}{(2l-1)!!} h^{(1)}_l(k_2 r)\mathbf{Y}_{llm}, \quad r > a, \tag{A20}$$

$$\mathbf{E}^{(M)}_{blm} \approx \left(\frac{2}{(2l+1)a^3}\right)^{1/2} \frac{1}{j_{l-1}(y_{bl})}[l^{1/2} j_{l+1}(ry_{bl}/a)\mathbf{Y}_{ll+1m}(\Omega) - (l+1)^{1/2} j_{l+1}(ry_{bl}/a)\mathbf{Y}_{ll-1m}(\Omega)], \;\; r < a, \tag{A21}$$

$$\mathbf{E}^{(M)}_{blm} \approx -i\left(\frac{2(2l+1)}{a^3}\right)^{1/2} \frac{(k_2a)^{l+2}}{(2l+1)!!}\left[l^{1/2} h^{(1)}_{l+1}(k_2r)\mathbf{Y}_{ll+1m}(\Omega) - (l+1)^{1/2} h^{(1)}_{l-1}(k_2r)\mathbf{Y}_{ll-1m}(\Omega)\right], \;\; r > a, \tag{A22}$$

$$\mathbf{E}^{(M)}_{0lm} \approx \left(\frac{2l+1}{(l+1)a^3}\right)^{1/2}\left(-\frac{l}{l+1}\right)^{\frac{l-1}{2}} \frac{(2l-1)!!}{(k_2a)^{l-1}}$$
$$\times [l^{1/2} j_{l+1}[ik_2r\sqrt{(l+1)/l}]\mathbf{Y}_{ll+1m}(\Omega) - (l+1)^{1/2} j_{l-1}[ik_2r\sqrt{(l+1)/l}]\mathbf{Y}_{ll-1m}(\Omega)], \;\; r < a, \tag{A23}$$

$$\mathbf{E}^{(M)}_{0lm} \approx -i\left(\frac{l+1}{a^3(2l+1)}\right)^{1/2} \frac{(k_2a)^{l+3}}{(2l+1)!!}\left[l^{1/2} h^{(1)}_{l+1}(k_2r)\mathbf{Y}_{ll+1m}(\Omega) - (l+1)^{1/2} h^{(1)}_{l-1}(k_2r)\mathbf{Y}_{ll-1m}(\Omega)\right], \;\; r > a. \tag{A24}$$

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