


Zitterbewegung of massless particlesAlexander J. Silenko *

*Bogoliubov Laboratory of Theoretical Physics, Joint Institute for Nuclear Research, Dubna 141980, Russia;
Institute of Modern Physics, Chinese Academy of Sciences, Lanzhou 730000, China;
and Research Institute for Nuclear Problems, Belarusian State University, Minsk 220030, Belarus*



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Zitterbewegung of massless particles with an arbitrary spin is analyzed in various representations. Dynamics of the group velocity of a massless particle as a whole and of the corresponding radius vector is determined. This radius vector defines any fixed point of the envelope of the moving wave packet characterizing the particle and its group velocity differs from the group velocities of any points of the wavefront. We consider free massless scalar and Dirac particles, the photon, and massive and massless particles with an arbitrary spin and describe them in different representations. For particles with an arbitrary spin, the generalized Feshbach-Villars representation and the Foldy-Wouthuysen one are used. *Zitterbewegung* takes place in any representation except for the Foldy-Wouthuysen one. Formulas describing the motion of a “trembling” free particle are the same in any representation. In the Foldy-Wouthuysen representation, the operators of the velocity and momentum of a free particle are proportional and *Zitterbewegung* does not take place. Since the radius vector (position) and velocity operators are the quantum-mechanical counterparts of the classical position and velocity just in the Foldy-Wouthuysen representation, *Zitterbewegung* is not observable. The same conclusion has been previously made for free massive particles. For relativistic massive particles with spins 0, 1/2, 1 and massless particles with spins 0, 1/2 in arbitrarily strong electromagnetic fields, independent of the external fields *Zitterbewegung* does not appear in the Foldy-Wouthuysen representation either. This conclusion is made for leading terms in the Hamiltonian proportional to the zero and first powers of the Planck constant and for such terms proportional to \hbar^2 which describe contact interactions.

DOI: [10.1103/PhysRevA.105.062211](https://doi.org/10.1103/PhysRevA.105.062211)**I. INTRODUCTION**

Zitterbewegung is one of the most important and widely discussed problems of quantum mechanics (QM). It is a well-known effect consisting in a superfast trembling motion of a free particle. This effect has been first described by Schrödinger [1] for relativistic electrons in a free space as a result of an interference between positive and negative energy states. This effect is also known for a scalar particle [2–4] and for a massive Proca one [4,5]. There are plenty of works devoted to *Zitterbewegung*. In the past, Schrödinger’s interpretation of *Zitterbewegung* was generally accepted [6–8]. Evidently, *Zitterbewegung* appears due to mixing states with positive and negative total energies. The Foldy-Wouthuysen (FW) transformation eliminates this mixing and, therefore, *Zitterbewegung*. Since just operators and equations of motion in the FW representation are counterparts of corresponding classical variables and equations (see Ref. [12] and references therein), *Zitterbewegung* is not an observable physical effect. We focus our attention on papers presenting a correct analysis of the observability of this effect for Dirac and scalar particles. The correct conclusions about the origin and observability of this effect have been made in Refs. [3–5,9–14]. However, *Zitterbewegung* of massless particles has not been thoroughly

studied. A probable cause is a demonstration by Newton and Wigner [15] and Wightman [16] for the photon that it cannot be strictly localized according to natural criteria. However, the group velocity of massless particles and the corresponding radius vector can be determined and a rigorous consideration of their *Zitterbewegung* can be fulfilled. Such a consideration is a goal of the present study. It has been concluded in some precedent investigations that *Zitterbewegung* of free photons exists [17,18]. The existence of this effect is denied in Refs. [19–21] but it is noted that *Zitterbewegung* exists and can in principle be observed for interacting photons. These conclusions have been based on the heuristic forms of the Dirac-like consideration [17,18] and on the Dirac-like quantum-mechanical equation (without the FW transformation) [20]. The possibilities of a rigorous quantum-mechanical determination of the appropriate position operator and the FW transformation of the Dirac-like quantum-mechanical equation have not been realized in precedent investigations of *Zitterbewegung* of the photon.

We also mention the precedent studies of *Zitterbewegung* of massless and massive particles in Refs. [22,23] and of the anyon in Refs. [23,24].

The paper is organized as follows. In the next section, we explain previously obtained results for *Zitterbewegung* of massive fermions and bosons. Distinguishing features of a quantum-mechanical description of massless particles are considered in Sec. III. In Sec. IV, we study *Zitterbewegung*

*alsilenko@mail.ru

of massless scalar and Dirac particles. Quantum mechanics of the free photon is presented in Sec. V, and Sec. VI is devoted to the analysis of its *Zitterbewegung*. In Sec. VII, we consider *Zitterbewegung* of particles with an arbitrary spin in the generalized Feshbach-Villars (GFV) representation. The problem of *Zitterbewegung* in external fields is analyzed in Sec. VIII. Finally, we summarize the obtained results in Sec. IX.

The system of units $\hbar = 1, c = 1$ and the standard denotations of the Dirac matrices (see, e.g., Ref. [25]) are used. Hereinafter, $\mathbf{p} \equiv -i\hbar\nabla$ denotes the momentum operator.

II. PREVIOUSLY OBTAINED RESULTS FOR MASSIVE FERMIONS AND BOSONS

In our short review of previously obtained results, we follow Ref. [4]. The Dirac Hamiltonian for a free spin-1/2 particle is given by

$$\mathcal{H}_D = \beta m + \boldsymbol{\alpha} \cdot \mathbf{p}, \quad (1)$$

and the Dirac velocity operator has the form

$$\mathbf{v}_D \equiv \frac{d\mathbf{r}}{dt} = i[\mathcal{H}_D, \mathbf{r}] = \boldsymbol{\alpha}. \quad (2)$$

The operator \mathbf{v}_D is time-dependent:

$$\frac{d\mathbf{v}_D}{dt} = i[\mathcal{H}_D, \mathbf{v}_D] = i\{\boldsymbol{\alpha}, \mathcal{H}_D\} - 2i\boldsymbol{\alpha}\mathcal{H}_D = 2i(\mathbf{p} - \boldsymbol{\alpha}\mathcal{H}_D). \quad (3)$$

The problem is usually considered in the Heisenberg picture:

$$\mathbf{v}_D(t) = e^{i\mathcal{H}_D t} \boldsymbol{\alpha} e^{-i\mathcal{H}_D t}. \quad (4)$$

In the Schrödinger picture, the result is the same. We suppose that the eigenvalues of the momentum and Hamiltonian operators are \mathbf{p} and H , respectively. In this case, Eq. (3) can be presented in terms of the Dirac velocity operator:

$$\frac{d\mathbf{v}_D}{dt} = 2i(\mathbf{p} - \mathbf{v}_D H). \quad (5)$$

Its integration shows that the Dirac velocity oscillates:

$$\mathbf{v}_D(t) = \left[\mathbf{v}_D(0) - \frac{\mathbf{p}}{H} \right] e^{-2iHt} + \frac{\mathbf{p}}{H}. \quad (6)$$

The evolution of the Dirac position operator obtained from this equation is given by

$$\mathbf{r}_D(t) = \mathbf{r}_D(0) + \frac{\mathbf{p}t}{H} + \frac{i}{2H} \left[\mathbf{v}_D(0) - \frac{\mathbf{p}}{H} \right] (e^{-2iHt} - 1). \quad (7)$$

A similar result has been obtained for a free scalar (spin-0) particle (see Ref. [3] and references therein). In this case, the initial Feshbach-Villars (FV) Hamiltonian reads [26]

$$\mathcal{H}_{FV} = \rho_3 m + (\rho_3 + i\rho_2) \frac{\mathbf{p}^2}{2m}, \quad (8)$$

where ρ_i ($i = 1, 2, 3$) are the Pauli matrices. The velocity operator in the FV representation is equal to

$$\mathbf{v}_{FV} = (\rho_3 + i\rho_2) \frac{\mathbf{p}}{m}. \quad (9)$$

The corresponding acceleration operator is defined by the equation similar to Eq. (3) [3]:

$$\begin{aligned} \frac{d\mathbf{v}_{FV}}{dt} &= i[\mathcal{H}_{FV}, \mathbf{v}_{FV}] = i\{\mathbf{v}_{FV}, \mathcal{H}_{FV}\} - 2i\mathbf{v}_{FV}\mathcal{H}_{FV} \\ &= 2i(\mathbf{p} - \mathbf{v}_{FV}\mathcal{H}_{FV}). \end{aligned} \quad (10)$$

It is supposed that the eigenvalues of the momentum and Hamiltonian operators are \mathbf{p} and H , respectively. As a result, the final equations of dynamics of the free scalar particle [3] are equivalent to the corresponding equations for the Dirac particle:

$$\mathbf{v}_{FV}(t) = \left[\mathbf{v}_{FV}(0) - \frac{\mathbf{p}}{H} \right] e^{-2iHt} + \frac{\mathbf{p}}{H}, \quad (11)$$

$$\mathbf{r}_{FV}(t) = \mathbf{r}_{FV}(0) + \frac{\mathbf{p}t}{H} + \frac{i}{2H} \left[\mathbf{v}_{FV}(0) - \frac{\mathbf{p}}{H} \right] (e^{-2iHt} - 1). \quad (12)$$

Zitterbewegung can also be considered in the GFV representation allowing one to describe not only massive scalar particles but also massless ones. This representation which has been previously used in Refs. [27–30] is, in fact, an infinite set of representations. It has been obtained in Ref. [4] that equations of motion in the GFV representation are equivalent to Eqs. (11) and (12).

Amazingly, the same physical situation takes place for Proca (spin-1) particles. Massive Proca particles have been investigated in Refs. [4,5]. *Zitterbewegung* takes place in the Sakata-Taketani representation [31,32] which does not exist for massless particles. The final equations of motion of a free Proca particle are equivalent to the corresponding equations for the Dirac and scalar particles [4,5]:

$$\mathbf{v}_{ST}(t) = \left[\mathbf{v}_{ST}(0) - \frac{\mathbf{p}}{H} \right] e^{-2iHt} + \frac{\mathbf{p}}{H}, \quad (13)$$

$$\mathbf{r}_{ST}(t) = \mathbf{r}_{ST}(0) + \frac{\mathbf{p}t}{H} + \frac{i}{2H} \left[\mathbf{v}_{ST}(0) - \frac{\mathbf{p}}{H} \right] (e^{-2iHt} - 1). \quad (14)$$

In connection with this equivalence, we can mention the existence of bosonic symmetries of the standard Dirac equation [33–39].

However, a coincidence of results obtained for particles with different spins in various representations does not mean that the effect of *Zitterbewegung* is observable. It has been pointed out in Ref. [40] that the transition to the FW representation establishes the proportionality of the operators \mathbf{p} and \mathbf{v} which should take place for free particles with any spin. In the FW representation, the acceleration vanishes and the Dirac Hamiltonian takes the form [41]

$$\mathcal{H}_{FW} = \beta\sqrt{m^2 + \mathbf{p}^2}, \quad \mathbf{p} \equiv -i\hbar \frac{\partial}{\partial \mathbf{r}}. \quad (15)$$

The velocity operator is given by

$$\mathbf{v}_{FW} = \beta \frac{\mathbf{p}}{\sqrt{m^2 + \mathbf{p}^2}} = \frac{\mathbf{p}}{\mathcal{H}_{FW}}. \quad (16)$$

As a result, $d\mathbf{v}_{FW}/(dt) = 0$ and *Zitterbewegung* does not take place. Similar relations have been obtained for massive particles with the spins 0 and 1 (see Ref. [4] and references therein).

The FW representation is the only one in which relativistic QM takes a Schrödinger form and expectation values of all operators correspond to respective classical variables (see Refs. [12,42–45] and references therein). Therefore, this representation is very convenient for checking an observability of physical effects. *Zitterbewegung* takes place *only* for operators which are not the quantum-mechanical counterparts of the

classical position and velocity. In particular, its appearance in the Dirac representation is caused by a significant difference between physical meanings of the Dirac and FW position operators (see Ref. [12] and references therein).

III. DISTINGUISHING FEATURES OF A QUANTUM-MECHANICAL DESCRIPTION OF MASSLESS PARTICLES

It should also be taken into account that particles can be in localized and delocalized states (see Ref. [46]). Photons being quanta of electromagnetic waves are usually delocalized and are commonly described in the framework of the wave theory. This theory determines *local* velocities of the wave field. Certainly, phase and group velocities are different. A local phase velocity is defined by the phase front $\Phi(\mathbf{r})$, $v_p = \omega/|\nabla\Phi(\mathbf{r})|$, where $\omega = ck$ is the angular frequency [47,48]. Another frequently used formula for the local phase velocity has been obtained in Ref. [49] (see also Ref. [50]):

$$v_p = c \left[1 + \frac{\nabla^2 \mathcal{A}(\mathbf{r})}{k^2 \mathcal{A}(\mathbf{r})} \right]^{-1/2}. \quad (17)$$

The local group velocity is given by $v_g = |\partial_\omega \nabla\Phi(\mathbf{r})|^{-1}$ [47,48] (see also Ref. [51] for details). The both local velocities can be subluminal and superluminal depending on a region. Certainly, they characterize important properties of light beams. For example, the local phase velocity defines an electron acceleration in a laser beam [52–54]. The distribution of the local phase velocity has been measured in Ref. [55].

The quantum-mechanical approach substantially differs. It is known that the photon and other massless particles cannot be spatially localized [15,16,56,57]. In any case, this statement is valid for plane-wave states. Conditions admitting a localization of massless particles have been considered in Ref. [58]. Examples of light fields localized in two dimensions are Hermite-Gaussian, Laguerre-Gaussian, and Gaussian beams [59]. Amazingly, the two-dimensional spatial localization of structured light beams results in nonzero *effective* masses of their quanta [60].

As a result, the position operator $\mathbf{r} = (x, y, z)$ does not characterize the coordinates of massless particles. To determine a physical sense of this operator, we can use the classical Hamilton equation for the velocity:

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{\partial \mathcal{H}}{\partial \mathbf{p}}, \quad (18)$$

where \mathcal{H} is the Hamiltonian. In QM, this equation remains valid as an operator equation and $\partial \mathcal{H}/(\partial \mathbf{p}) = i[\mathcal{H}, \mathbf{r}]$. For massless particles, \mathcal{H} and \mathbf{p} correspond to $\hbar\omega$ and $\hbar\mathbf{k}$, respectively, and $\partial \mathcal{H}/(\partial \mathbf{p})$ is the group velocity. The phase velocity is equal to $v_{ph} = \omega/k$. Thus, \mathbf{r} is a radius vector of a *moving point characterizing any fixed point of the envelope of the wave packet* and \mathbf{v} is the velocity of its motion. The latter quantity defines the group velocity of a massless particle *as a whole*. Despite great achievements of the wave theory, it cannot rigorously describe the motion of field quanta because any field quantum is extended over the entire space. In particular, a rigorous analysis of a subluminality of Gaussian and other structured light beams has been recently carried out just in the framework of QM [60].

The descriptions of massless and massive particles in QM are substantially different. In the general case, basic equations for massless particles cannot be obtained from the corresponding equations for massive ones either by the substitution $m = 0$ or in the limit of $m \rightarrow 0$. First of all, the definition of the spin should be radically changed. For massive particles, the conventional three-component spin (pseudo)vector \mathbf{s} is defined in the particle rest frame (see Ref. [12] and references therein). However, such a frame does not exist for massless particles. For such particles, the spin can be introduced but its projection onto the momentum direction can have only two values, minimum and maximum. Thus, the helicity of a massless particle is equal to (see, e.g., Ref. [25])

$$h = \frac{\mathbf{s} \cdot \mathbf{p}}{p} = \pm s, \quad (19)$$

where s is the spin quantum number. Two partial waves describing states with $h = \pm s$ can be coherent when these states have the same energy. When the helicity is fixed, this polarization is circular. When the two partial waves describing states with the opposite helicity are coherent, the particle polarization is different (e.g., it can be linear).

It can be easily shown that the Dirac-Pauli spin algebra is applicable for a description of massless Dirac fermions. Squaring Eq. (19) results in

$$(\mathbf{s} \cdot \mathbf{p})^2 = \frac{1}{4} p^2. \quad (20)$$

This equation is satisfied with the Pauli matrix $\boldsymbol{\sigma} = 2\mathbf{s}$. Thus, the polarization of massless Dirac fermions can be defined by the conventional Pauli and Dirac matrices. As a result, the Dirac equation with $m = 0$ can be used for a description of massless fermions. Of course, the operators $\boldsymbol{\sigma}$ and $\boldsymbol{\Sigma}$ characterize the polarization and helicity of massless particles but do not define the spin in the particle rest frame.

The situation is different for particles with higher spins including the photon ($s = 1$) and the graviton ($s = 2$). The number of components of wave functions is defined by the number of independent spin components ($2s + 1$ for massive particles). The quantum-mechanical description is equally applicable to states with a positive and a negative total energy. For massive particles, the minimum number of components of wave functions is therefore equal to $2(2s + 1)$. Such wave functions are bispinors for Dirac particles and bispinor-like wave functions for particles with other spins. In particular, the bispinor-like wave functions have been successfully used for massive spin-1 particles [4,32,61,62].

Some *initial* equations describing spinning massive particles may be applicable to massless ones. However, the condition (19) defining the helicity should be satisfied in the latter case. This condition defines two admissible longitudinal spin projections and reduces the number of independent spinor-like wave functions to four. In addition, wave functions of massless particles lose the probabilistic interpretation and should be considered as a distribution of a particle field strength. We can conclude that Hamiltonian equations for massive and massless particles substantially differ and massless particles should be considered separately. This conclusion is valid even for $s = 0, 1/2$ while the Klein-Gordon and Dirac equations also cover massless particles. For massless scalar particles, the well-known FV transformation [26] becomes in-

applicable and the GFV one [27] should be used. For massless Dirac particles, one should take into account the condition (19). However, one of properties of the Dirac-Pauli spin algebra is a possibility to present any spin polarization as a coherent superposition of two basic states with the helicity $h = \pm 1/2$. Therefore, the case of massless Dirac fermions is not very special and can basically be obtained from previous results for massive spin-1/2 particles. Of course, a difference between photons and massive spin-1 particles is much more substantial.

Zitterbewegung is a rather important quantum-mechanical problem and massless particles occupy a significant place in QM. However, the problem of *Zitterbewegung* of massless particles was not appropriately studied in previous investigations. In particular, the existence of *Zitterbewegung* of free photons has been claimed [17,18]. It has been noted in Refs. [19–21] that *Zitterbewegung* does not exist for free photons but takes place for interacting ones and is observable in this case. We consider *Zitterbewegung* of photons in Secs. VI and VIII.

Therefore, the problem of *Zitterbewegung* of massless particles is still unsolved. Its detailed study fulfilled in the present paper is rather important. In particular, the unsolved issue of great interest is *Zitterbewegung* in external fields.

IV. ZITTERBEWEGUNG OF MASSLESS SCALAR AND DIRAC PARTICLES

In this section, we show an absence of *Zitterbewegung* of massless scalar and Dirac particles in the FW representation and its existence in some other representations. We apply the conventional commutative spatial coordinates. The use of noncommutative coordinates (i.e., noncommutative geometry) has been considered, e.g., in Ref. [12].

Certainly, we can utilize the general equation for the Hamiltonian in the GFV representation derived in Ref. [27]. For a massless scalar particle, the initial Klein-Gordon equation has the form

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\psi = 0, \quad (21)$$

and the GFV Hamiltonian is given by

$$\mathcal{H}_{\text{GFV}} = \rho_3 \frac{\mathbf{p}^2 + N^2}{2N} + i\rho_2 \frac{\mathbf{p}^2 - N^2}{2N}, \quad (22)$$

where N is an arbitrary real nonzero parameter. Hereinafter, ρ_1 , ρ_2 , and ρ_3 are the Pauli matrices:

$$\rho_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \rho_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (23)$$

For a scalar particle, the normalization of the two-component wave function in the FV representation [26]

$$\Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \quad (24)$$

is given by

$$\int \Psi^\dagger \rho_3 \Psi dV = 1.$$

Any GFV Hamiltonian is pseudo-Hermitian (more exactly, ρ_3 -pseudo-Hermitian): $\mathcal{H}_{\text{GFV}}^\dagger = \rho_3 \mathcal{H}_{\text{GFV}} \rho_3 = \mathcal{H}_{\text{GFV}}$. For massless and massive particles, the normalization of GFV and FV wave functions is the same:

$$\int \Psi_{\text{GFV}}^\dagger \rho_3 \Psi_{\text{GFV}} dV = 1.$$

The velocity operator in the GFV representation is equal to

$$\mathbf{v}_{\text{GFV}} = (\rho_3 + i\rho_2) \frac{\mathbf{p}}{N}. \quad (25)$$

The corresponding acceleration operator reads

$$\begin{aligned} \frac{d\mathbf{v}_{\text{GFV}}}{dt} &= i[\mathcal{H}_{\text{GFV}}, \mathbf{v}_{\text{GFV}}] = i\{\mathbf{v}_{\text{GFV}}, \mathcal{H}_{\text{GFV}}\} - 2i\mathbf{v}_{\text{GFV}}\mathcal{H}_{\text{GFV}} \\ &= 2i(\mathbf{p} - \mathbf{v}_{\text{GFV}}\mathcal{H}_{\text{GFV}}). \end{aligned} \quad (26)$$

The dynamics of the massless scalar particle is independent of N and is defined by the following equations:

$$\begin{aligned} \mathbf{v}_{\text{GFV}}(t) &= \left[\mathbf{v}_{\text{GFV}}(0) - \frac{\mathbf{p}}{H} \right] e^{-2iHt} + \frac{\mathbf{p}}{H}, \quad (27) \\ \mathbf{r}_{\text{GFV}}(t) &= \mathbf{r}_{\text{GFV}}(0) + \frac{\mathbf{p}t}{H} + \frac{i}{2H} \left[\mathbf{v}_{\text{GFV}}(0) - \frac{\mathbf{p}}{H} \right] (e^{-2iHt} - 1). \end{aligned} \quad (28)$$

These equations are the same as the equations previously derived for massive particles.

For a massless Dirac particle, we use the initial Dirac equation (1), suppose that $m = 0$, and repeat all calculations presented in Sec. II. Dynamic equations are the same as Eqs. (6) and (7) and show the existence of *Zitterbewegung*.

As well as for massive particles, *Zitterbewegung* does not appear in the FW representation. The use of this representation for spinning massless particles should be commented.

For a massless Dirac particle, the FW transformation operator is given by (cf. Refs. [41,42,63])

$$U_{\text{FW}} = \frac{p + \boldsymbol{\gamma} \cdot \mathbf{p}}{\sqrt{2}p}. \quad (29)$$

The FW Hamiltonian reads

$$\mathcal{H}_{\text{FW}} = \beta\sqrt{\mathbf{p}^2} = \beta p, \quad (30)$$

and the velocity operator is given by

$$\mathbf{v}_{\text{FW}} = \beta \frac{\mathbf{p}}{p} = \frac{\mathbf{p}}{\mathcal{H}_{\text{FW}}}. \quad (31)$$

Thus, the velocity and momentum are proportional. We can repeat the conclusion [10,11] that *Zitterbewegung* is the result of the interference between positive and negative energy states. In the FW representation, it disappears not only for massive [9–11] but also for massless Dirac particles.

V. QUANTUM MECHANICS OF THE PHOTON

While the quantum theory of radiation is well known [64] and QM of the photon has a long history (see Refs. [65–67] and references therein), some important results, in particular, the FW transformation for the photon [68], have been obtained comparatively recently. Photon (light) beams are also extensively studied in optics. In optics, the wave function of the photon, Ψ , is not a wave function in the same sense as for the

electron and determines the relative amplitude of the electric field [65,69,70]. The full description of an electromagnetic field including its interaction with matter is based on the quantum field theory (see Refs. [71,72]). However, the propagation of light in a free space can be adequately described with the Riemann-Silberstein vector

$$\mathbf{F} = \frac{1}{\sqrt{2}}(\mathbf{E} + i\mathbf{B}).$$

It allows one to reduce the Maxwell equations and to present them in the form [73,74]

$$i\hbar \frac{\partial \mathbf{F}}{\partial t} = c(\mathbf{S} \cdot \mathbf{p})\mathbf{F}, \quad (32)$$

where $\mathbf{S} = (S_1, S_2, S_3)$ is a vector in which the components are the conventional spin-1 matrices [61]:

$$\begin{aligned} S_1 &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, & S_2 &= \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \\ S_3 &= \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \end{aligned} \quad (33)$$

This definition is not unique. One can use any other spin matrices satisfying the properties

$$[S_i, S_j] = ie_{ijk}S_k,$$

$$S_i S_j S_k + S_k S_j S_i = \delta_{ij}S_k + \delta_{jk}S_i, \quad \mathbf{S}^2 = 2\mathcal{I}, \quad (34)$$

where \mathcal{I} is the unit 3×3 matrix. The spin matrices act on three components of \mathbf{F} . The equation (32) is similar to the Weyl equation for a massless Dirac particle [74]. When the six-component wave function is defined by [66–68]

$$\Psi = \frac{1}{\sqrt{2}} \begin{pmatrix} \boldsymbol{\phi} \\ \boldsymbol{\chi} \end{pmatrix} \equiv \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{E} \\ i\mathbf{B} \end{pmatrix}, \quad (35)$$

the Dirac-like equation for the free electromagnetic field can be obtained [66–68]:

$$i\hbar \frac{\partial \Psi}{\partial t} = \boldsymbol{\alpha} \cdot \mathbf{p} \Psi, \quad \boldsymbol{\alpha} = \begin{pmatrix} 0 & \mathbf{S} \\ \mathbf{S} & 0 \end{pmatrix}. \quad (36)$$

In this equation, $\mathcal{H} = \boldsymbol{\alpha} \cdot \mathbf{p}$ is the Dirac-like Hamiltonian. Only Eq. (36) has been taken into account in Ref. [20].

The additional condition of an orthogonality of the momentum direction to the fields \mathbf{E} and \mathbf{B} should also be taken into account. It follows from the Maxwell equations in the free space that $\mathbf{p} \cdot \boldsymbol{\phi} = \mathbf{p} \cdot \boldsymbol{\chi} = 0$. As a result, the number of independent components of Ψ reduces to four.

The FW transformation of Eq. (36) has been carried out only in 2014 by Barnett [68]. It is instructive to mention that any operator \mathbf{V} satisfies the relation

$$i\mathbf{V} \times \mathbf{G} = (\mathbf{S} \cdot \mathbf{V})\mathbf{G},$$

where \mathbf{G} is equal to \mathbf{E} or \mathbf{B} . The FW transformation operator is equivalent to the operator (29) for the massless Dirac particle but has six components:

$$U_{\text{FW}} = \frac{p + \beta \boldsymbol{\alpha} \cdot \mathbf{p}}{\sqrt{2}p}, \quad \beta = \begin{pmatrix} \mathcal{I} & 0 \\ 0 & -\mathcal{I} \end{pmatrix}. \quad (37)$$

As a result, the FW wave function of the photon, $\Psi_{\text{FW}} = U_{\text{FW}}\Psi$, is defined by

$$\begin{aligned} \Psi_{\text{FW}}^{(+)} &= \begin{pmatrix} \boldsymbol{\phi}_{\text{FW}}^{(+)} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{E} \\ 0 \end{pmatrix} \quad \text{if } H = |\mathbf{p}| > 0, \\ \Psi_{\text{FW}}^{(-)} &= \begin{pmatrix} 0 \\ i\boldsymbol{\chi}_{\text{FW}}^{(-)} \end{pmatrix} = \begin{pmatrix} 0 \\ i\mathbf{B} \end{pmatrix} \quad \text{if } H = -|\mathbf{p}| < 0. \end{aligned} \quad (38)$$

Here \mathbf{p} and H are eigenvalues of the momentum and Hamiltonian operators, respectively. Negative values of H define photon states with a negative total energy. In Eq. (38), the fields \mathbf{E} and \mathbf{B} also characterize states with a positive and a negative total energy, respectively. It is convenient to present the fields in the matrix form, $\mathbf{E} = \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$.

For massless particles including the photon, the physical meaning of negative-energy states is mostly the same as for massive particles. The problem of these states cannot be satisfactorily resolved in the context of relativistic QM [75] but the negative-energy states are rather important for quantum electrodynamics and quantum field theory [76]. Such states can be used for a description of virtual particles and a state with a negative energy can characterize a virtual photon. We should note that masses of particles with negative energies are also negative [77]. Some of the most important forces can be considered as an exchange of virtual particles. This possibility exists for all major forces like electromagnetic, weak, strong, and gravitational ones. In particular, the electromagnetic interaction of a charged body can be interpreted as an exchange of virtual photons. Virtual photons play a pivotal role in quantum physics (see, e.g., Refs. [19–21,78–82]).

In the FW representation, $\mathbf{p} \cdot \boldsymbol{\phi} = 0$ if $H = |\mathbf{p}| > 0$ and $\mathbf{p} \cdot \boldsymbol{\chi} = 0$ if $H = -|\mathbf{p}| < 0$. Therefore, the number of nonzero and independent components of Ψ_{FW} reduces to two. It has been proven in Ref. [68] that

$$\begin{aligned} \mathcal{P}^2 \Psi &\equiv [\mathbf{p}^2 - (\boldsymbol{\alpha} \cdot \mathbf{p})^2] \Psi = [\mathbf{p}^2 - (\boldsymbol{\Sigma} \cdot \mathbf{p})^2] \Psi = 0, \\ \boldsymbol{\Sigma} &= \begin{pmatrix} \mathbf{S} & 0 \\ 0 & \mathbf{S} \end{pmatrix}. \end{aligned} \quad (39)$$

As a result of derivations [68], the FW Hamiltonian is defined by

$$i\hbar \frac{\partial \Psi_{\text{FW}}}{\partial t} = \mathcal{H}_{\text{FW}} \Psi_{\text{FW}}, \quad \mathcal{H}_{\text{FW}} = \beta p. \quad (40)$$

Quantum mechanics of the photon agrees with the general condition (19) for the helicity. For the photon, $s = \mathbf{S}$ and $s = \boldsymbol{\Sigma}$ when one uses two-component and four-component wave functions, respectively, and $s = 1$. Squaring Eq. (19) leads to the relation $(\boldsymbol{\Sigma} \cdot \mathbf{p})^2 = p^2$ which is equivalent to Eq. (39) (see Ref. [68] and references therein).

Equation (38) does not mean that the electric and magnetic fields defines the FW wave functions of real and virtual photons, respectively. In the electromagnetic wave, these fields are not independent and are connected by the relations $\mathbf{B} = \mathbf{n} \times \mathbf{E}$, $\mathbf{E} = -\mathbf{n} \times \mathbf{B}$, $\mathbf{n} = \mathbf{p}/|\mathbf{p}|$. In addition, one can introduce the transformed spinor-like wave functions $\boldsymbol{\phi}'_{\text{FW}} = \mathbf{n} \times \boldsymbol{\phi}_{\text{FW}}^{(+)}$, $\boldsymbol{\chi}'_{\text{FW}} = \mathbf{n} \times \boldsymbol{\chi}_{\text{FW}}^{(-)}$ with the same FW Hamiltonian (40). As a result, the transformed FW wave functions take the

form

$$\begin{aligned}\Psi_{\text{FW}}^{(+)} &= \begin{pmatrix} \phi_{\text{FW}}^{(+)} \\ 0 \end{pmatrix} = \begin{pmatrix} \mathbf{B} \\ 0 \end{pmatrix} \quad \text{if } H = |\mathbf{p}| > 0, \\ \Psi_{\text{FW}}^{(-)} &= \begin{pmatrix} 0 \\ i\chi_{\text{FW}}^{(-)} \end{pmatrix} = \begin{pmatrix} 0 \\ -i\mathbf{E} \end{pmatrix} \quad \text{if } H = -|\mathbf{p}| < 0.\end{aligned}\quad (41)$$

Equations (39) and (41) show that the both fields can be equivalently used for a definition of real and virtual photons.

To complete QM of the photon, we need to determine an equation of the second order in the temporal and spatial derivatives (Klein-Gordon-like equation). The quantum-mechanical description of the photon in the GFV representation will be carried out in Sec. VII. The Klein-Gordon-like (KGL) equation can be easily obtained by squaring the FW Hamiltonian equation (40) and has the form

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\psi = 0. \quad (42)$$

The three-component wave function ψ coincides with the upper three-component spinor-like part of Ψ_{FW} and is equal to \mathbf{E} for positive-energy states of the photon. For negative-energy states, $\psi = i\mathbf{B}$. The number of independent components of ψ reduces to two due to the orthogonality condition $\mathbf{p} \cdot \psi = 0$.

It can also be shown by squaring Eq. (32) and taking into account Eqs. (19) and (39) that the Riemann-Silberstein vector is also a solution of the KGL equation, $\psi = \mathbf{F} = \frac{1}{\sqrt{2}}(\mathbf{E} + i\mathbf{B})$, for states with both positive and negative total energies.

We note the nonequivalence of the Dirac-like and FW wave functions, Ψ and Ψ_{FW} . The former function defines both fields, \mathbf{E} and \mathbf{B} , reproduces the Maxwell equations, and establishes the connection between states with a positive and a negative total energy. While the FW wave function perfectly describes the light field, it does not possess these properties. However, just the wave function defined only by the electric field strength is used in optics.

VI. ZITTERBEWEGUNG OF THE FREE PHOTON

The use of the Dirac-like equation (36) and the KGL equation (42) allows us to study *Zitterbewegung* of the photon. The velocity and acceleration operators determined from Eq. (36) are given by

$$\begin{aligned}\mathbf{v} &\equiv \frac{d\mathbf{r}}{dt} = i[\mathcal{H}, \mathbf{r}] = \boldsymbol{\alpha}, \\ \frac{d\mathbf{v}}{dt} &= i[\mathcal{H}, \mathbf{v}] = i\{\boldsymbol{\alpha}, \mathcal{H}\} - 2i\boldsymbol{\alpha}\mathcal{H} = 2i(\mathbf{p} - \boldsymbol{\alpha}\mathcal{H})\end{aligned}\quad (43)$$

and are equivalent to the corresponding equations (2) and (3) for Dirac particles. Certainly, the derivation for the photon leads to the final formulas which are also equivalent to the corresponding ones for Dirac particles:

$$\mathbf{r}(t) = \mathbf{r}(0) + \frac{\mathbf{p}t}{H} + \frac{i}{2H} \left[\mathbf{v}(0) - \frac{\mathbf{p}}{H} \right] (e^{-2iHt} - 1), \quad (44)$$

$$\mathbf{v}(t) = \left[\mathbf{v}(0) - \frac{\mathbf{p}}{H} \right] e^{-2iHt} + \frac{\mathbf{p}}{H}. \quad (45)$$

To study *Zitterbewegung*, we can also use the GFV transformation of the KGL equation (42). In this case, we introduce

the spinor-like wave functions (cf. Ref. [27])

$$\psi = \phi + \chi, \quad i\frac{\partial\psi}{\partial t} = N(\phi - \chi). \quad (46)$$

Multiplying the second equation by $i\partial/(\partial t)$ allows one to obtain the GFV Hamiltonian [27]:

$$\mathcal{H}_{\text{GFV}} = \rho_3 \frac{\mathbf{p}^2 + N^2}{2N} + i\rho_2 \frac{\mathbf{p}^2 - N^2}{2N}. \quad (47)$$

We should mention that this Hamiltonian is also proportional to the 3×3 unit matrix which is omitted ($\mathcal{H}_{\text{GFV}}\mathcal{I} \rightarrow \mathcal{H}_{\text{GFV}}$). This is its only difference with the Hamiltonian (22). As a result, the final equations describing *Zitterbewegung* coincide with Eqs. (27) and (28) for the scalar particle and with Eqs. (44) and (45) in the Dirac-like representation.

Thus, *Zitterbewegung* of the photon in the Dirac-like and GFV representations is determined by the same equations which also coincide with all corresponding equations for other massless and massive particles presented in Secs. II–IV. Nevertheless, all noted equations have been obtained in representations different from the FW representation. The transition to the latter representation resulting in the Hamiltonian (40) eliminates *Zitterbewegung*. In this representation, the velocity operator has the form [cf. Eqs. (16) and (31)]

$$\mathbf{v}_{\text{FW}} = \beta \frac{\mathbf{p}}{\sqrt{m^2 + \mathbf{p}^2}} = \frac{\mathbf{p}}{\mathcal{H}_{\text{FW}}}. \quad (48)$$

Since the FW position and velocity operators are the quantum-mechanical counterparts of the corresponding classical variables (see Ref. [12] and references therein), the observability of *Zitterbewegung* should be determined just in the FW representation. As a result of proportionality of the velocity and momentum operators, *Zitterbewegung* of the free photon is unobservable. Such *Zitterbewegung* should be considered as a purely mathematical effect and does not exist as a real physical one. These our conclusions contradict to the conclusions made in Refs. [17,18].

VII. ZITTERBEWEGUNG OF PARTICLES WITH AN ARBITRARY SPIN IN THE GENERALIZED FESHBACH-VILLARS REPRESENTATION

Zitterbewegung can also be studied for particles with an arbitrary spin. It has been shown by Weinberg [83] that particles with an arbitrary spin satisfy the Klein-Gordon (more precisely, Klein-Gordon-like) equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi = 0, \quad (49)$$

where the wave function ψ has $2s + 1$ components. The number of components of this wave function is defined by the number of independent spin components. Evidently, Eq. (49) is equally applicable to states with a positive and a negative total energy. Any passage to the Hamiltonian formalism joins such states and needs an introduction of the $2(2s + 1)$ -component wave function Ψ .

Equation (49) is also applicable to massless particles. However, the condition (19) defining the helicity should be satisfied in this case. This condition defines two admissible

longitudinal spin projections and reduces the number of independent components of ψ to two.

We can carry out the GFV transformation [4,27]

$$\psi = \phi + \chi, \quad i\frac{\partial\psi}{\partial t} = N(\phi - \chi) \quad (50)$$

for particles with an arbitrary spin.

After multiplying the last relation by $i\partial/(\partial t)$, Eq. (50) can be presented in the matrix form [4,27]

$$i\frac{\partial\Psi_{\text{GFV}}}{\partial t} = \mathcal{H}_{\text{GFV}}\Psi_{\text{GFV}}, \quad \Psi_{\text{GFV}} = \begin{pmatrix} \phi \\ \chi \end{pmatrix},$$

$$\mathcal{H}_{\text{GFV}} = \rho_3 \frac{\mathbf{p}^2 + m^2 + N^2}{2N} + i\rho_2 \frac{\mathbf{p}^2 + m^2 - N^2}{2N}. \quad (51)$$

Here \mathcal{H}_{GFV} and Ψ_{GFV} are the Hamiltonian and the wave function in the GFV representation. Evidently, this representation connects the states with a positive and a negative total energy with each other. We note the difference between the ST and GFV Hamiltonians for massive spin-1 particles and underline the applicability of the GFV representation for both bosons and fermions. The same derivations as in Ref. [4] show that equations of motions in this representation coincide with the corresponding equations [4] for scalar particles and with Eqs. (27) and (28). Thus, the description of particles with an arbitrary spin in the GFV representation demonstrates the presence of *Zitterbewegung* and the perfect similarity of equations of motion for particles with any spin in any representation (except for the FW one). The derivation is the same for massive and massless particles. In the latter case, all obtained formulas remain valid provided that $m = 0$.

Like in other cases, *Zitterbewegung* does not take place in the FW representation. The FW transformation of the Hamiltonian (51) is exact. The general form of the FW transformation operator has been obtained in Ref. [42]. In the considered case, this operator reduces to [27]

$$U_{\text{GFV}\rightarrow\text{FW}} = \frac{\epsilon + N + \rho_1(\epsilon - N)}{2\sqrt{\epsilon N}}, \quad \epsilon = \sqrt{m^2 + \mathbf{p}^2}. \quad (52)$$

Since this operator is ρ_3 -pseudounitary [27], the operator of the inverse transformation is defined by

$$U_{\text{GFV}\rightarrow\text{FW}}^\dagger = \rho_3 U_{\text{GFV}\rightarrow\text{FW}}^{-1} \rho_3,$$

$$U_{\text{GFV}\rightarrow\text{FW}}^{-1} = U_{\text{FW}\rightarrow\text{GFV}} = \rho_3 U_{\text{GFV}\rightarrow\text{FW}}^\dagger \rho_3, \quad (53)$$

$$U_{\text{FW}\rightarrow\text{GFV}} = \frac{\epsilon + N - \rho_1(\epsilon - N)}{2\sqrt{\epsilon N}}.$$

For the photon, $m = 0$ and the GFV wave function $\Psi_{\text{GFV}} = U_{\text{FW}\rightarrow\text{GFV}}\Psi_{\text{FW}}$ is equal to

$$\Psi_{\text{GFV}}^{(+)} = \frac{\mathbf{E}}{2\sqrt{pN}} \begin{pmatrix} N+p \\ N-p \end{pmatrix}, \quad \Psi_{\text{GFV}}^{(-)} = \frac{i\mathbf{B}}{2\sqrt{pN}} \begin{pmatrix} N-p \\ N+p \end{pmatrix}. \quad (54)$$

for states with a positive and a negative total energy, respectively. The FW wave function corresponding to these states is defined by Eq. (38).

In the general case, we can determine the FW Hamiltonian and the corresponding wave function with the use of the

operator (52):

$$\mathcal{H}_{\text{FW}} = \beta\epsilon,$$

$$\Psi_{\text{FW}} = \Psi_{\text{FW}}^{(+)} = \frac{2\sqrt{\epsilon N}}{\epsilon + N} \begin{pmatrix} \phi \\ 0 \end{pmatrix} \quad \text{if } \mathcal{H}_{\text{FW}} = |\epsilon| > 0, \quad (55)$$

$$\Psi_{\text{FW}} = \Psi_{\text{FW}}^{(-)} = \frac{2\sqrt{\epsilon N}}{\epsilon + N} \begin{pmatrix} 0 \\ \chi \end{pmatrix} \quad \text{if } \mathcal{H}_{\text{FW}} = -|\epsilon| < 0.$$

The connection between the FW wave function and the initial GFV one is similar to that between the FW and Dirac wave functions [84] but the coupling factors are different in the two cases.

For massive and massless particles, the FW velocity operator is defined by

$$\mathbf{v}_{\text{FW}} = i[\mathcal{H}_{\text{FW}}, \mathbf{r}_{\text{FW}}] = \beta \frac{\mathbf{p}}{\sqrt{m^2 + \mathbf{p}^2}} = \frac{\mathbf{p}}{\mathcal{H}_{\text{FW}}}. \quad (56)$$

This equation is valid for particles with any spin. For massless particles, $\mathbf{v}_{\text{FW}} = \beta c \mathbf{p}/|\mathbf{p}|$.

Therefore, $d\mathbf{v}_{\text{FW}}/dt = 0$ and *Zitterbewegung* does not take place. Since the FW representation is the only representation in which relativistic QM takes a Schrödinger form and expectation values of all operators correspond to respective classical variables, *Zitterbewegung* takes place *only* for operators which are not the quantum-mechanical counterparts of the classical position and velocity. As a result, *Zitterbewegung* is not observable. This is the same conclusion which has been made in the previous studies [3–5,9–14].

VIII. ZITTERBEWEGUNG IN EXTERNAL FIELDS

The analysis of *Zitterbewegung* made in the above-mentioned previous publications is not exhaustive because only free particles were studied. However, the problem of *Zitterbewegung* of particles in external fields is also very important. In particular, it has been stated in Refs. [6,85,86] that just *Zitterbewegung* causes the electrostatic contact (Darwin) interaction.

We will consider *Zitterbewegung* of particles with spins 0, 1/2, and 1 in external fields. Both massless and massive particles will be taken into consideration. This is necessary because a consideration of massive particles simplifies making a conclusion on an existence of *Zitterbewegung* in external fields. Furthermore, we cannot include the massless photon into our consideration because its electromagnetic interactions can be described by quantum field theory but not by QM.

We use the method of the relativistic FW transformation elaborated in Refs. [42,44,63,87]. It is applicable for a particle in arbitrarily strong external fields (without taking into account effects of quantum field theory). We can mention that the Darwin interaction manifests itself even in the weak-field approximation. In this approximation, interactions with external fields do not significantly affect large *Zitterbewegung* in the Dirac and GFV representations and total *Zitterbewegung* is mostly defined by the equations for free particles (see Secs. II, IV, VI, and VII). The problem of existence of *Zitterbewegung* in external fields is nontrivial because *Zitterbewegung* is absent for free particles in the FW representation (see Secs. II, IV, VI, and VII) but the Darwin interac-

tion is usually explained by *Zitterbewegung*. To solve this problem, we use the previous derivations of relativistic FW Hamiltonians for scalar [27] and spinning [32,88] particles in electromagnetic fields.

Certainly, the FW transformation is perturbative in the presence of external fields and an exact FW separation into positive- and negative-energy subspaces is not possible in a nonperturbative way. However, the FW transformation method [42,44,63,87] applied also in Ref. [27] gives exact and compact relativistic expressions for all terms proportional to the zero and first powers of the Planck constant and only for such terms proportional to \hbar^2 which describe contact interactions (including the Darwin one). Hereinafter, we will reproduce only such expressions and will omit other terms. We should add that terms proportional to \hbar describe spin

interactions. While the precision of the used method of the FW transformation has been specified only in Ref. [63], previous results obtained by this method are also exact for the above-mentioned terms. The considered method of the FW transformation cannot be applied for a calculation of perturbations of next orders.

The relativistic FW Hamiltonian for a scalar particle in electromagnetic fields reads [27]

$$\mathcal{H}_{\text{FW}} = \rho_3 \epsilon' + e\Phi, \quad \epsilon' = \sqrt{m^2 + \boldsymbol{\pi}^2}. \quad (57)$$

We underline the absence of terms describing contact interactions.

The corresponding FW Hamiltonian for a spin-1/2 particle with magnetic and electric dipole moments (MDM and EDM) is given by [42,88]

$$\mathcal{H}_{\text{FW}} = \mathcal{H}_{\text{FW}}^{(\text{MDM})} + \mathcal{H}_{\text{FW}}^{(\text{EDM})}, \quad (58)$$

$$\begin{aligned} \mathcal{H}_{\text{FW}}^{(\text{MDM})} = & \beta \epsilon' + e\Phi + \frac{1}{4} \left\{ \left(\frac{\mu_0 m}{\epsilon' + m} + \mu' \right) \frac{1}{\epsilon'}, [\boldsymbol{\Sigma} \cdot (\boldsymbol{\pi} \times \mathbf{E} - \mathbf{E} \times \boldsymbol{\pi}) - \hbar \nabla \cdot \mathbf{E}] \right\} \\ & - \frac{1}{2} \left\{ \left(\frac{\mu_0 m}{\epsilon'} + \mu' \right), \boldsymbol{\Pi} \cdot \mathbf{B} \right\} + \beta \frac{\mu'}{4} \left\{ \frac{1}{\epsilon'(\epsilon' + m)}, [(\mathbf{B} \cdot \boldsymbol{\pi})(\boldsymbol{\Sigma} \cdot \boldsymbol{\pi}) + (\boldsymbol{\Sigma} \cdot \boldsymbol{\pi})(\boldsymbol{\pi} \cdot \mathbf{B}) + 2\pi \hbar (\boldsymbol{\pi} \cdot \mathbf{j} + \mathbf{j} \cdot \boldsymbol{\pi})] \right\}, \end{aligned} \quad (59)$$

$$\mathcal{H}_{\text{FW}}^{(\text{EDM})} = -d \boldsymbol{\Pi} \cdot \mathbf{E} + \frac{d}{4} \left\{ \frac{1}{\epsilon'(\epsilon' + m)}, [(\mathbf{E} \cdot \boldsymbol{\pi})(\boldsymbol{\Pi} \cdot \boldsymbol{\pi}) + (\boldsymbol{\Pi} \cdot \boldsymbol{\pi})(\boldsymbol{\pi} \cdot \mathbf{E})] \right\} - \frac{d}{4} \left\{ \frac{1}{\epsilon'}, (\boldsymbol{\Sigma} \cdot [\boldsymbol{\pi} \times \mathbf{B}] - \boldsymbol{\Sigma} \cdot [\mathbf{B} \times \boldsymbol{\pi}]) \right\} \quad (60)$$

Here $\mu_0 = e\hbar/(2mc)$ is the Dirac magnetic moment, μ' is the anomalous magnetic moment ($\mu = \mu_0 + \mu' = g e s \hbar / (2mc)$, where g is the g factor), $\epsilon' = \sqrt{m^2 + \boldsymbol{\pi}^2}$, $\boldsymbol{\pi} = \mathbf{p} - e\mathbf{A}$, $\boldsymbol{\Sigma}$, and $\boldsymbol{\Pi}$ are Dirac matrices, d is the electric dipole moment, and

$$\mathbf{j} = \frac{1}{4\pi} \left(c \nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} \right)$$

is the density of external electric current. The term in Eq. (59) proportional to $\nabla \cdot \mathbf{E}$ defines the Darwin (contact) interaction. While we take into account in Eq. (60) terms proportional to \hbar^2 and describing contact interactions with external charges and currents, such terms are zero due to the Maxwell equations

$$\nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}.$$

Terms proportional to the second and higher powers of \hbar and quadratic/bilinear in \mathbf{E} and \mathbf{B} are neglected.

For a spin-1 particle with a magnetic moment, taking into account quadrupole interactions [61,89] leads to the following FW Hamiltonian [89]:

$$\begin{aligned} \mathcal{H}_{\text{FW}} = & \rho_3 \epsilon' + e\Phi + \frac{e}{4m} \left\{ \left(\frac{g-2}{2} + \frac{m}{\epsilon' + m} \right) \frac{1}{\epsilon'}, \left[\mathbf{S} \cdot (\boldsymbol{\pi} \times \mathbf{E} - \mathbf{E} \times \boldsymbol{\pi}) \right] \right\} \\ & - \rho_3 \left[\left(g-2 + \frac{2m}{\epsilon'} \right), \mathbf{S} \cdot \mathbf{B} \right] + \rho_3 \left[\frac{g-2}{2\epsilon'(\epsilon' + m)}, (\mathbf{S} \cdot \boldsymbol{\pi}, \boldsymbol{\pi} \cdot \mathbf{B}) \right] \\ & + \frac{e(g-1)}{4m^2} \left\{ \left[\mathbf{S} \cdot \nabla - \frac{1}{\epsilon'(\epsilon' + m)} (\mathbf{S} \cdot \boldsymbol{\pi})(\boldsymbol{\pi} \cdot \nabla) \right], \left[\mathbf{S} \cdot \mathbf{E} - \frac{1}{\epsilon'(\epsilon' + m)} (\mathbf{S} \cdot \boldsymbol{\pi})(\boldsymbol{\pi} \cdot \mathbf{E}) \right] \right\} \\ & + \frac{e}{8m^2} \left\{ \frac{1}{\epsilon'(\epsilon' + m)} \left(g-1 + \frac{m}{\epsilon' + m} \right), [\mathbf{S} \cdot (\boldsymbol{\pi} \times \nabla), \mathbf{S} \cdot (\boldsymbol{\pi} \times \mathbf{E})] \right\} \\ & - \frac{e(g-1)}{2m^2} \nabla \cdot \mathbf{E} + \frac{e}{4m^2} \left[\frac{1}{\epsilon'^2} \left(g-1 + \frac{m^2}{4\epsilon'^2} \right), (\boldsymbol{\pi} \cdot \nabla)(\boldsymbol{\pi} \cdot \mathbf{E}) \right], \end{aligned} \quad (61)$$

where the g factor is used and the Planck constant is omitted. In this equation, a noncommutativity of operators is (partially) neglected. We should add that Eqs. (57) and (58)–(60) are applicable for massless particles while Eq. (61) describes only massive ones.

The presented equations perfectly agree with corresponding classical equations describing the momentum and spin dynamics [12,90]. The agreement takes also place for a spin-1 particle with MDM and EDM [12,32].

Equations (57)–(61) allow us to check the above-mentioned statement [6,85,86] that the electrostatic contact interaction (defined in the FW representation) is connected with *Zitterbewegung*. However, this statement cannot be right. The electrostatic contact interaction does not exist for scalar particles and exists for spinning ones while *Zitterbewegung* appears for any particles in representations different from the FW representation. An absence of observable *Zitterbewegung* for scalar particles and a similarity of Eqs. (57)–(60) for massless and massive particles show that it not caused by the particle localization (established in Ref. [15]) either. In fact, the electrostatic contact interaction is conditioned by the particle spin. Its essential difference for spin-1/2 and spin-1 particles confirms this important and unobvious statement. We can add that a spinning particle is not perfectly pointlike. In particular, its center of charge and center of mass do not coincide [91–96].

The effective root-mean-square radii do not depend on external fields and are the same as in vacuum.

We repeat that an exact FW separation into positive- and negative-energy subspaces is not possible in a nonperturbative way and our conclusions cannot be extended to perturbations of next orders.

In this section, we have analyzed only *Zitterbewegung* which appears even for free particles (like a nonzero root-mean-square radius of the electron) but manifests itself only for interacting ones. None of leading terms in equations of motion describes such *Zitterbewegung*. However, it is evident that external fields can lead to a trembling motion (e.g., *Zitterbewegung*) of particles which is observable *in these fields* but does not have any analog for particles in free space. For example, a charge undergoes *Zitterbewegung* in a field of a coherent electromagnetic wave. It has been established in Refs. [19–21] that *Zitterbewegung* takes place for *real* photons interacting with *virtual* longitudinal and scalar ones and this effect is observable. The effect cannot occur in the absence of virtual photons. It is described in the framework of quantum field theory and cannot be considered as a quantum-mechanical effect.

IX. DISCUSSION AND SUMMARY

We have analyzed *Zitterbewegung* of massless particles with an arbitrary spin in various representations. While *Zitterbewegung* of massless particles in the free space has been extensively studied, the quantum-mechanical descriptions of massless and massive particles substantially differ. In the general case, basic equations for massless particles cannot be obtained from the corresponding equations for massive ones and the definition of the spin should be substantially changed. For massive particles, the conventional three-component spin (pseudo)vector s is defined in the particle rest frame (see Ref. [12] and references therein). However, such a frame does not exist for massless particles. For such particles, the helicity has only two admissible values, $+s$ and $-s$. The spin defines the polarization and helicity of massless particles and cannot be defined in the particle rest frame. If $s \neq 0$, then the bispinor-like wave functions have only four independent components. Therefore, Hamiltonian equations for massive and massless particles substantially differ and massless particles

should be considered separately. For this reason, the results obtained are not obvious.

Photons and other field quanta are extended over the entire space and the radius vector $\mathbf{r} = (x, y, z)$ cannot define their coordinates. Therefore, \mathbf{r} is a radius vector of a moving point characterizing any fixed point of the envelope of the wave packet characterizing the particle and v is the velocity of its motion. The latter quantity defines the group velocity of the massless particle *as a whole* but not that of any point of the wavefront. We have considered massless scalar and Dirac particles, the photon, and massive and massless particles with an arbitrary spin. We have described them in different representations. For particles with an arbitrary spin, the GFV and FW representations have been used. In all cases, *Zitterbewegung* takes place in any representation except for the FW one. In such cases, formulas describing the particle motion are the same in any representation. In the FW representation, the operators of the velocity and momentum are proportional and *Zitterbewegung* does not take place. The radius vector (position) and velocity operators are the quantum-mechanical counterparts of the classical position and velocity just in the FW representation. Therefore, *Zitterbewegung* of free particles is not observable. This conclusion agrees with the conclusions made in the previous studies [3–5,9–14] for massive particles. However, it disagrees with the corresponding conclusions reached in many other investigations including precedent papers devoted to *Zitterbewegung* of the free photon [17,18].

The very important problem is *Zitterbewegung* of particles in external fields. *Zitterbewegung* has been noted in Refs. [6,85,86] as a reason of the electrostatic contact (Darwin) interaction being a manifestation of effective root-mean-square radii of spinning particles. However, our analysis given in Sec. VIII shows that this interaction cannot be caused by *Zitterbewegung* and is conditioned by the particle spin. We underline that the effective root-mean-square radii do not depend on external fields and are the same as in vacuum. Since an exact FW transformation is not possible in a nonperturbative way, our conclusions are restricted by the applicability of the used transformation method and cannot be extended to perturbations of next orders.

We have analyzed only *Zitterbewegung* which appears even for free particles (like a nonzero root-mean-square radius of the electron) but manifests itself only for interacting ones. None of leading terms in equations of motion describes such *Zitterbewegung*. However, it is evident that external fields can lead to a trembling motion (e.g., *Zitterbewegung*) of particles which is observable *in these fields* but does not have any analog for particles in free space. For example, a charge undergoes *Zitterbewegung* in a field of a coherent electromagnetic wave. It has been established in Refs. [19–21] that *Zitterbewegung* takes place for *real* photons interacting with *virtual* longitudinal and scalar ones and this effect is observable. The effect cannot occur in the absence of virtual photons. It is described in the framework of quantum field theory and cannot be considered as a quantum-mechanical effect.

Thus, *Zitterbewegung* of particles can appear due to interactions with external fields and effects of quantum field theory (see Refs. [19–21]). In such cases, it can be observable.

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