Giant diamagnetism of a quantum charged particle after inversion of the magnetic field

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We consider a quantum charged particle moving in the xy plane under the action of a uniform perpendicular time-dependent magnetic field, in the presence of a parabolic binding potential. The time-dependent probability distribution of the magnetic moment is calculated analytically and numerically in the case of initial thermodynamic equilibrium state. In the high-temperature regime, the initial distribution is almost symmetric, resulting in a tiny mean diamagnetism. However, if the magnetic field eventually changes its sign, the fragile balance between the diamagnetic and paramagnetic "wings" of the probability distribution becomes broken, resulting in a giant mean diamagnetic moment, exceeding the initial one by several orders of magnitude. The final mean value of the magnetic moment is inversely proportional to the strength of the binding potential, and it does not depend on the Planck constant in the high-temperature regime. Strong fluctuations of the magnetic moment (described in terms of the variance) exist in all temperature regimes.

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I. INTRODUCTION

Magnetic properties of quantum charged particles in the thermodynamic equilibrium state have been studied by many researchers since 1930 [1–12]. One of the main results is the Darwin formula for the mean magnetic moment of a spinless free particle with mass M and charge e in the homogeneous magnetic field B [2]:

$$\mathcal{M} = \mu[(\mu B\beta)^{-1} - \coth(\mu B\beta)], \tag{1}$$

where $\mu = e\hbar/(2Mc)$ is the Bohr magneton and β the inverse temperature (we use the Gauss system of units). In particular, $\mathcal{M} = -\mu$ in the low-temperature limit $\mu B\beta \gg 1$, whereas $\mathcal{M} = -\mu^2 B\beta/3$ (i.e., $|\mathcal{M}| \ll \mu$) in the high-temperature limit $\mu B\beta \ll 1$. Recently [13], we have obtained a new explanation of these results, considering the magnetic moment probability distribution in the *equilibrium* state. It was shown that the small high-temperature diamagnetism is a consequence of *quasi* total compensation of the diamagnetic and paramagnetic "wings" of the magnetic moment probability distribution. But what can happen if the system goes *out of equilibrium*?

Many authors studied the evolution of quantum states of a charged particle in time-dependent magnetic fields [14–23]. Continuing those studies, it was discovered recently [24–26] that the evolution of the initial equilibrium state in the time-dependent magnetic field can result in a strong amplification of the diamagnetic moment, especially in the high-temperature case, under the magnetic field inversion. The aim of the present paper is to study this unexpected effect in more detail, paying special attention to the evolution of the magnetic moment probability distribution.

II. EIGENSTATES OF THE MAGNETIC MOMENT OPERATOR

To introduce the magnetic moment operator, we use the definition of the classical magnetic moment [27,28]

$$\mathbf{M} = \frac{1}{2c}\int dV[\mathbf{r}\times\mathbf{j}]. \tag{2}$$

Then, using the expression for the quantum probability current density in the presence of the magnetic field,

$$\mathbf{j} = ie\hbar(\psi\nabla\psi^* - \psi^*\nabla\psi)/(2M) - e^2\boldsymbol{A}\psi^*\psi/(Mc),$$

one can write vector (2) as the mean value of vector operator

$$\hat{\mathbf{M}} = (\hat{\mathbf{r}}\times\hat{\boldsymbol{\pi}})e/(2Mc). \tag{3}$$

Here, $\boldsymbol{A}$ is the magnetic field vector potential and vector $\boldsymbol{\pi} = \boldsymbol{p} - e\boldsymbol{A}/c$ is the *kinetic* momentum. Vector $\boldsymbol{p}$ is the *canonical* momentum, whose quantum operator is $\hat{\boldsymbol{p}} = -i\hbar\nabla$. The form (3) of the magnetic moment operator was justified from different points of view by many authors in [6,8,29–34].

We consider the case of a homogeneous magnetic field directed along the z axis. We are interested in the projector of vector (3) on the z axis, using the symbol $\hat{\mathcal{M}}$ for this projection. Using the "circular" gauge of the vector potential, $\boldsymbol{A} = B(-y, x)/2$, we can write

$$\hat{\mathcal{M}} = [\hat{x}\hat{p}_y - \hat{y}\hat{p}_x - M\omega(\hat{x}^2 + \hat{y}^2)]e/(2Mc) \equiv \mu\hat{\Lambda}. \tag{4}$$

Here, $\omega = eB/(2Mc)$ is the Larmor frequency, so that $\mu B = \hbar\omega$. The spectrum of the dimensionless operator $\hat{\Lambda}$ is continuous. Eigenfunctions of this operator were found in

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[13] in the momentum representation in terms of the Bessel functions:

$$\Psi_{\Lambda,m}(p,\varphi) = (|\kappa|/4\pi)^{1/2} e^{im\varphi} J_{|m|}(\gamma_m p), \tag{5}$$

$$\int_0^{2\pi} d\varphi \int_0^\infty p\,dp \Psi^*_{\Lambda,m}\Psi_{\Lambda',m'} = \delta_{mm'}\delta(\Lambda-\Lambda'),$$

$$m = 0, \pm1, \pm2, \dots, \quad \gamma_m = \sqrt{\kappa(m-\Lambda)}, \quad \kappa = (M\hbar\omega)^{-1}. \tag{6}$$

It is important to remember that frequency ω can be positive or negative (as well as parameter κ). For each value of discrete parameter m, the coefficient γ_m must be real (then, it can be always chosen positive). The continuous parameter Λ must satisfy the restrictions

$$-\infty < \Lambda \leqslant m \text{ if } \omega > 0; \quad m \leqslant \Lambda < \infty \text{ if } \omega < 0. \tag{7}$$

III. EVOLUTION OF THE WAVE FUNCTIONS

In the nonrelativistic case considered in this paper, it is sufficient to consider the motion in the plane xy perpendicular to the z axis. To avoid the problems with normalization and degeneracy of energy levels in the stationary state, we assume, following Darwin [2], that the particle motion is confined by means of the isotropic harmonic potential $V(x,y) = Mg^2(x^2+y^2)/2$. Therefore, the motion in the xy plane is governed by the Hamiltonian

$$\hat{H} = \hat{\boldsymbol{\pi}}^2/(2M) + Mg^2\big(x^2+y^2\big)/2. \tag{8}$$

We discard the effects of spin, since they are independent from the orbital motion effects within the nonrelativistic approximation.

The stationary Schrödinger equation $\hat{H}\psi = E\psi$ with Hamiltonian (8) and $\boldsymbol{A} = B(-y,x)/2$ was solved in the coordinate representation in polar coordinates by Fock [35]. The Fourier transform of that solution, resulting in the wave function in the momentum representation, was calculated in [13]:

$$\psi_{n_r m}(p,\varphi) = (-i)^{|m|}(-1)^{n_r}\sqrt{\frac{\kappa_g n_r!}{\pi(n_r+|m|)!}}(\kappa_g p^2)^{|m|/2} \times L_{n_r}^{(|m|)}(\kappa_g p^2)\exp(-\kappa_g p^2/2 + im\varphi). \tag{9}$$

Here, function $L_n^{(\alpha)}(z)$ is the generalized Laguerre polynomial, defined as [36,37]

$$L_n^{(\alpha)}(z) = \frac{1}{n!}e^z z^{-\alpha}\frac{d^n}{dz^n}(e^{-z}z^{\alpha+n}),$$

$$\omega_g = \sqrt{\omega^2+g^2}, \quad \kappa_g = (M\hbar\omega_g)^{-1}, \quad p = \sqrt{p_x^2+p_y^2}.$$

The energy spectrum is given by the formula

$$E_{n_r m} = \hbar\omega_g(1+|m|+2n_r) - \hbar\omega m, \tag{10}$$

where $m = 0, \pm1, \pm2, \dots$, $n_r = 0, 1, 2, \dots$. Note that solution (9) is valid even for $g^2 < 0$, provided the frequency ω_g is real (we assume that $\omega_g > 0$).

What happens with the energy eigenstates (9) if the frequency ω depends on time after some instant t_*? The answer was given by Malkin *et al.* [15]. It appears that the initial function (9) takes the following form at $t \geqslant t_*$ (see the Appendix for details):

$$\Psi_{n_r m}(p,\varphi;t) = \sqrt{\frac{n_r![K_g(t)]^{1+|m|}p^{2|m|}}{\pi(n_r+|m|)!}}L_{n_r}^{(|m|)}[K_g(t)p^2] \times \exp\left(im\varphi - \frac{i\varepsilon p^2}{2M\hbar\dot{\varepsilon}} + i\chi(t)\right). \tag{11}$$

Here, $K_g(t) = [M\hbar|\dot{\varepsilon}(t)|^2]^{-1}$, where the complex function $\varepsilon(t)$ is the solution to the classical equation of the harmonic oscillator with a time-dependent frequency,

$$\ddot{\varepsilon} + \omega_g^2(t)\varepsilon = 0, \tag{12}$$

satisfying the initial conditions

$$\varepsilon(t_*) = [\omega_g(t_*)]^{-1/2}, \quad \dot{\varepsilon}(t_*) = i[\omega_g(t_*)]^{1/2}. \tag{13}$$

The explicit form of phase factor $\chi(t)$ is not important for our purposes.

IV. EVOLUTION OF THE MAGNETIC MOMENT PROBABILITY DISTRIBUTION FROM THE INITIAL THERMAL STATES

When the magnetic field depends on time, it is reasonable to consider the *instantaneous* magnetic moment operator (4) and its eigenfunctions (5) with the time-dependent Larmor frequency $\omega(t)$. Then, the instantaneous probability distribution of the magnetic moment (normalized by the Bohr magneton μ) in the time-dependent state $\Psi_{n_r m}(t)$ is given by the function $\mathcal{P}_{n_r m}(\Lambda;t) = |\langle\Psi_{\Lambda,m}(t)|\Psi_{n_r m}(t)\rangle|^2$. Note that the quantum numbers m coincide in the bra and ket vectors in this formula, because $\langle\Psi_{\Lambda,m'}(t)|\Psi_{n_r m}(t)\rangle = 0$ if $m' \neq m$. Since function (11) has the same structure as (9), the result has the same form as that obtained in [13]:

$$\mathcal{P}_{n_r m}(\Lambda;t) = \frac{n_r!|q(t)|[\xi_m(\Lambda;q(t))]^{|m|}}{(n_r+|m|)!}\exp[-\xi_m(\Lambda;q(t))] \times \left(L_{n_r}^{(|m|)}[\xi_m(\Lambda;q(t))]\right)^2. \tag{14}$$

We see that the probability distribution depends on time through the time-dependent functions

$$q(t) = [\omega(t)|\varepsilon(t)|^2]^{-1}, \quad \xi_m(\Lambda;q(t)) = q(t)(m-\Lambda). \tag{15}$$

Therefore, we use hereafter the notation $\mathcal{P}_{n_r m}(\Lambda;q(t))$.

In the stationary case, we have $|\varepsilon(t)|^2 = \omega_g^{-1}$, so $q(t)$ goes to the constant coefficient $q = \omega_g/\omega$ used in paper [13]. Formula (14) holds provided $\xi_m(\Lambda;q(t)) \geqslant 0$, i.e., under the restrictions (7). Otherwise, $\mathcal{P}_{n_r m}(\Lambda;q(t)) = 0$. The normalization $\int_{-\infty}^{\infty}\mathcal{P}_{n_r m}(\Lambda;q(t))d\Lambda = 1$ is nothing but the standard normalization of the Laguerre polynomials (see formula 8.980 in [37]). It is crucial that time-dependent parameter $q(t)$ can be positive or negative, depending on the sign of frequency $\omega(t)$.

The energy eigenstate is a specific quantum state, which can be created in experiments with some difficulties. A more realistic initial state seems the equilibrium one. In this case, the (normalized) total time-dependent magnetic moment probability density at $t > t_*$ can be calculated as

$$\mathcal{P}(\Lambda; q(t)) = \sum_{n_r,m} \mathcal{P}_{n_r m}(\Lambda; q(t)) \exp(-\beta E_{n_r m})/\mathcal{Z}(\beta), \quad (16)$$

where $\mathcal{Z}(\beta) = \sum_{n_r,m} \exp(-\beta E_{n_r m})$ is the statistical sum, which can be easily calculated [2], due to the linear nature of the spectrum (10) (see also [6,38] for other approaches):

$$(2\mathcal{Z})^{-1} = \cosh(\eta_g) - \cosh(\eta), \quad (17)$$

$$\eta = \hbar\beta\omega_i, \quad \eta_g = \hbar\beta\sqrt{\omega_i^2 + g^2}. \quad (18)$$

We assume that the initial Larmor frequency ω_i is positive. Making the same calculations as in [13], we obtain

$$\mathcal{P}(\Lambda; q(t)) = \sum_m \mathcal{P}_m(\Lambda; q(t)), \quad (19)$$

$$\mathcal{P}_m(\Lambda; q(t)) = |q(t)| G \exp[m\eta - \xi_m(\Lambda; q(t)) \coth(\eta_g)] \times I_{|m|}\left[\frac{\xi_m(\Lambda; q(t))}{\sinh(\eta_g)}\right], \quad (20)$$

$$G = [\cosh(\eta_g) - \cosh(\eta)]/\sinh(\eta_g). \quad (21)$$

Formula (20) holds for $\xi_m(\Lambda; q(t)) \geqslant 0$, otherwise $\mathcal{P}_m(\Lambda; q(t)) = 0$.

At zero temperature, $\beta = \eta = \eta_g = \infty$ and $G = 1$. In this case, the only contribution to the probability density is from the ground state with $n_r = m = 0$:

$$\mathcal{P}(\Lambda; q(t)) = \begin{cases} |q(t)| \exp(-|q(t)\Lambda|), & q(t)\Lambda \leqslant 0 \\ 0, & q(t)\Lambda > 0. \end{cases} \quad (22)$$

Hence, $\langle\Lambda\rangle(t) = \int_{-\infty}^{\infty} \Lambda \mathcal{P}(\Lambda; q(t)) d\Lambda = -q^{-1}(t)$. The distribution width is characterized by the variance $\sigma_\Lambda \equiv \langle\Lambda^2\rangle - \langle\Lambda\rangle^2$. Since $\langle\Lambda^2\rangle = 2q^{-2}(t)$ for the distribution (22), the variance is rather big in this case: $\sigma_\Lambda(t) = q^{-2}(t) = [\langle\Lambda\rangle(t)]^2$.

V. MEAN VALUES

The mean value

$$\langle\Lambda\rangle(t) = \int_{-\infty}^{\infty} \Lambda \mathcal{P}(\Lambda; q(t)) d\Lambda = \sum_{m=-\infty}^{\infty} \langle\Lambda\rangle_m(t)$$

can be calculated in the same way as was done in [13] in the stationary case. The result is as follows:

$$\langle\Lambda\rangle(t) = F_- - \omega(t)|\varepsilon(t)|^2[F_+ + \coth(\eta_g)],$$
$$F_\pm = \frac{\sinh^2(\eta_+) \pm \sinh^2(\eta_-)}{2\sinh(\eta_+)\sinh(\eta_-)\sinh(\eta_g)}, \quad \eta_\pm = \frac{1}{2}(\eta_g \pm \eta). \quad (23)$$

This expression is another form of the result obtained in [24] with the aid of the density matrix formalism. The separation of $\langle\Lambda\rangle(t)$ in the constant and time-dependent parts in Eq. (23) corresponds to the form (4) of the magnetic moment operator: the constant part is due to the conserved *canonical* angular momentum $\hat{x}\hat{p}_y - \hat{y}\hat{p}_x$, whereas the second term describes the contribution of the quantity $\omega(t)\langle\hat{x}^2 + \hat{y}^2\rangle$ which depends on time.

Note that $\eta_- \to 0$ when $g \to 0$. This means that the limit $g \to 0$ should be taken with some care, only at the final stage, because of the term $\sinh(\eta_-)$ in the denominators of time-independent coefficients $F_\pm$. For example, the zero-temperature limit $\beta \to \infty$ should be taken for $g \neq 0$, assuming that $\eta_- \to \infty$. Then, $F_+ = F_- = 0$, so that

$$\langle\Lambda\rangle(t)_{\beta=\infty} = -q^{-1}(t) = -\omega(t)|\varepsilon(t)|^2, \quad (24)$$

in accordance with the distribution (22). In the case of constant frequency, when $\omega(t)|\varepsilon(t)|^2 = \omega/\omega_g$, formula (23) goes to the Darwin formula $\langle\Lambda\rangle = \eta^{-1} - \coth(\eta)$ if $g \to 0$.

For a quasifree particle ($g \ll \omega_i$), we have $\eta_- \ll \eta_+$ and $F_+ \approx F_- \approx [2\sinh(\eta_-)]^{-1}$. A simple approximate formula in this case reads

$$\langle\Lambda\rangle(t) \approx \frac{1 - q^{-1}(t)}{2\sinh(\eta_-)} - q^{-1}(t)\coth(\eta). \quad (25)$$

Hence, the mean magnetic moment can be increased immensely if $\eta_- \ll 1$ and the factor $q(t)$ is not close to unity. This does not happen in the adiabatic regime, when

$$\varepsilon_{ad}(t) \approx [\omega(t)]^{-1/2} \exp\left[i\int^t \omega(\tau)d\tau\right]. \quad (26)$$

However, the simple formula (26) fails when the frequency $\omega(t)$ goes to zero and changes the sign. Then, $q(t) < 0$ and the difference $1 - q^{-1}(t)$ cannot be small in the asymptotic regime with $\omega_f < 0$. Consequently, a giant amplification of diamagnetism can happen when $\omega_f < 0$ and $\eta_- \ll 1$, due to the small denominator $\sinh(\eta_-)$ in Eq. (25).

VI. SOME EXACT FORMULAS FOR TIME-DEPENDENT FUNCTIONS

We suppose that the frequency $\omega(t)$ goes asymptotically to some finite value ω_f. In this limit, the function $\omega_g(t)$ tends to the constant value $\Omega = \sqrt{\omega_f^2 + g^2}$, which is always positive, while ω_f can be either positive or negative. The asymptotic solution to Eq. (12) has the form

$$\varepsilon(t) = \Omega^{-1/2}[u_+ e^{i\Omega t} + u_- e^{-i\Omega t}]. \quad (27)$$

Constant complex coefficients $u_\pm$ obey the condition

$$|u_+|^2 - |u_-|^2 = 1. \quad (28)$$

This relation follows from the preservation in time of the Wronskian between two linear independent solutions $\varepsilon(t)$ and $\varepsilon^*(t)$ and the initial conditions (13):

$$\dot{\varepsilon}(t)\varepsilon^*(t) - \dot{\varepsilon}^*(t)\varepsilon(t) \equiv 2i. \quad (29)$$

Hence, the function $q^{-1}(t) = \omega(t)|\varepsilon(t)|^2$ in the asymptotic regime has the form

$$q^{-1}(t) = \frac{\omega_f}{\Omega}[|u_+|^2 + |u_-|^2 + 2\mathrm{Re}(u_+ u_-^* e^{2i\Omega t})]. \quad (30)$$

This function oscillates with frequency 2Ω between the extreme values

$$q_{\rm ext}^{-1} = (\omega_f/\Omega)(|u_+| \pm |u_-|)^2. \quad (31)$$

Several concrete functions $\omega(t)$ admitting exact solutions to Eq. (12) were considered in Refs. [24,26]. We confine ourselves here to the special example of the Epstein-Eckart profiles,

$$\omega(t) = \frac{\omega_f \exp(\gamma t) + \omega_i}{\exp(\gamma t) + 1}, \quad -\infty < t < \infty, \quad \gamma > 0. \quad (32)$$

Moreover, we consider the case of the exact field inversion, $\omega_f = -\omega_i$, when the coefficient u_- has the most simple analytic form [24],

$$u_- = \frac{i\cos(\pi\sqrt{1/4 - 4\tilde{\Omega}^2})}{\sinh(2\pi\tilde{\Omega})}, \qquad \tilde{\Omega} = \Omega/\gamma. \quad (33)$$

For $\gamma \gg \Omega$ (the "sudden jump" of the frequency) we have $u_- \approx 0$ and $q \approx -1$ (assuming $g \ll \omega_i$). In this case, according to Eq. (24), the mean magnetic moment simply changes its sign in the zero-temperature case (still meaning the diamagnetism, because the magnetic field changed the sign as well). However, a great amplification of the diamagnetism can be observed for nonzero initial temperatures, under the condition $\eta_- \ll 1$ (i.e., when the product βg^2 is small):

$$\langle\Lambda\rangle \approx \eta_-^{-1} \approx 4\omega_i/(\hbar\beta g^2) = \Lambda_*. \quad (34)$$

In the opposite limit case of $\gamma \ll \Omega \approx \omega_i$ (a very slow evolution) we have $|u_-| = 1$. Then, the mean value $\langle\Lambda\rangle$ oscillates between the values $\Lambda_*(2 \pm \sqrt{2})$. It is remarkable that the mean magnetic moment in this quasifree high-temperature case does not depend on the Planck constant:

$$\mu\Lambda_* = B_i k_B T[e/(Mcg)]^2. \quad (35)$$

The origin of this formula is discussed in Sec. VIII. For the electron in the magnetic field $B \sim 1$ T, the condition $\eta \ll 1$ means the absolute temperature $T \gg 1$ K.

VII. THE DISTRIBUTION FUNCTIONS IN THE GIANT DIAMAGNETISM CASE

It seems interesting to see the magnetic moment distribution functions resulting in the giant diamagnetism. Unfortunately, the series (19) with functions (20) cannot be calculated analytically. Therefore, we had to perform numerical calculations. The summation for each value of Λ was performed with 13 000 terms in the series (19), according to the restriction $(m - \Lambda)/\omega(t) \geqslant 0$, which guaranteed the convergence of the series with the machine precision. The modified Bessel functions were generated via recurrence relations either for $I_n(z)$ or $e^{-z}I_n(z)$ (when $z \gg 1$), using the Miller's algorithm to avoid overflows. The numerical value of the normalization integral $\int_{-\infty}^{\infty}\mathcal{P}(\Lambda; q(t))d\Lambda$ was more than 0.9993 in all cases.

Figures 1–3 show the function $\mathcal{P}(\Lambda; q(t))$ in the logarithmic scale (with details in the usual scale) for $\eta = 1/10$ (high-temperature case) and $\eta_g = 1.1\eta$ (i.e., $\eta_-^{-1} = 200$). Figure 1 corresponds to the initial equilibrium distribution with $q = \omega_g/\omega_i = 1.1$. In this case, the distribution is almost symmetric, with a tiny asymmetry at the origin, resulting in the small mean value $\langle\Lambda\rangle_{\text{eq}} \approx -0.005$. Figures 2 and 3 show $\mathcal{P}(\Lambda; q(t))$ for $q = -(3 + 2\sqrt{2})$ and $q = -(3 - 2\sqrt{2})$ [the extreme values (31) for the slow evolution to the exactly inverted magnetic field with $\omega_f/\Omega = -1$ and $|u_-|^2 = 1$'.

We see two striking features in all figures. The first one is the "sawtooth fine structure" of distributions for small values

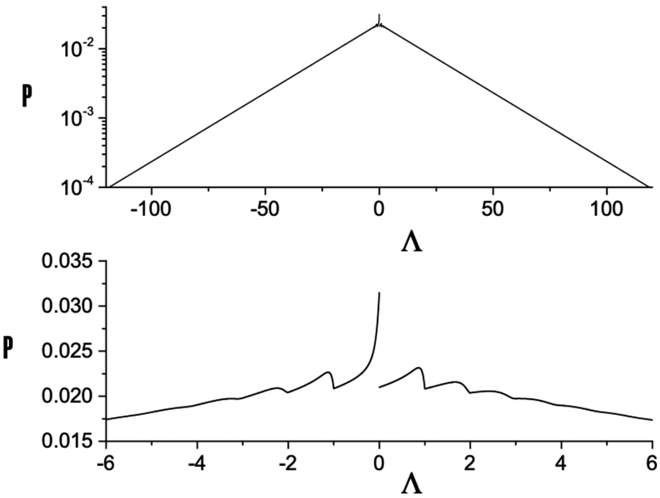

FIG. 1. Function $\mathcal{P}(\Lambda; q(t))$ for $\eta = 0.1$, $\eta_g = 0.11$, and $q = 1.1$ (the initial equilibrium distribution). Top: the behavior in the large interval of values of Λ, using the logarithmic scale for $\mathcal{P}$. Bottom: the details for small values of Λ.

of Λ and the discontinuity at $\Lambda = 0$ (where the probability density attains the maximal value). This discontinuity is explained by the appearance or disappearance of the term $\mathcal{P}_0(0; q(t))$ in the series (19), when the variable Λ passes through the point $\Lambda = 0$. Since $I_0(0) = 1$, the discontinuity at $\Lambda = 0$ equals $|q(t)|G \approx |q(t)|(\eta_g - \eta)$, in an excellent agreement with all plots. When Λ passes through integer values $\Lambda = m \neq 0$, the new Bessel function I_m enters the game (or goes out). Since $I_m(0) = 0$ for $m \neq 0$, function $\mathcal{P}(\Lambda; q(t))$ remains continuous at these points, but its derivative changes abruptly. This explains the "sawtooth" structure, which becomes less and less visible with the increase of $|\Lambda|$.

The second striking feature is the practically ideal exponential form of $\mathcal{P}(\Lambda)$ in the large scale, excluding a small region near the origin. In this (logarithmic) scale, we see

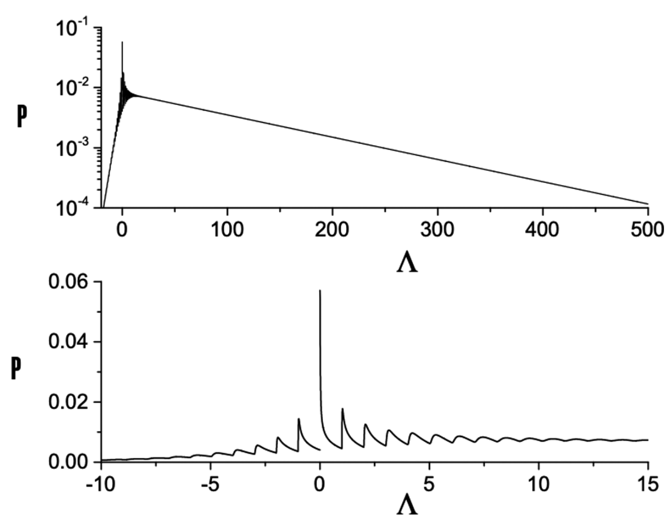

FIG. 2. Function $\mathcal{P}(\Lambda; q(t))$ for $\eta = 0.1$ and $\eta_g = 0.11$ at the instant when $q(t) = -(3 + 2\sqrt{2})$. In this case, $\langle\Lambda\rangle \approx 113.2$. Top: the behavior in the large interval of values of Λ, using the logarithmic scale for $\mathcal{P}$. Bottom: the details for small values of Λ.

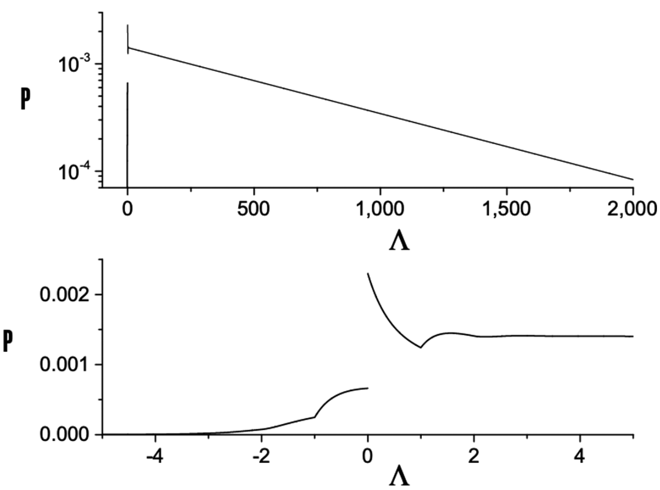

FIG. 3. Function $\mathcal{P}(\Lambda; q(t))$ for $\eta = 0.1$ and $\eta_g = 0.11$ at the instant when $q(t) = -(3 - 2\sqrt{2})$. In this case, $\langle\Lambda\rangle \approx 705.7$. Top: the behavior in the large interval of values of Λ, using the logarithmic scale for $\mathcal{P}$. Bottom: the details for small values of Λ.

two linear wings of the distribution, which changes from the almost symmetric form in the high-temperature equilibrium state to strongly asymmetric forms at zero temperature (22) or after the field inversion in the high-temperature initial state. The equilibrium case was discussed in Ref. [13]. Here we analyze the distributions after the field inversion, shown in Figs. 2 and 3. Neglecting a small contribution of the left "wing" (for $\Lambda < 0$), the distribution for $\Lambda > 0$ can be approximated by the simple function $\tilde{\mathcal{P}}(\Lambda) = \langle\Lambda\rangle^{-1} \exp(-\Lambda/\langle\Lambda\rangle)$. The mean value $\langle\Lambda\rangle$ can be taken either from the exact formula (23) or its simplified version (25). Note, however, that the exact mean values given for each figure were obtained numerically, using the exact distribution function $\mathcal{P}(\Lambda; q(t))$ shown in the figures. The main advantage of the approximate function $\tilde{\mathcal{P}}(\Lambda)$ is the possibility to calculate easily the dispersion of the distribution, which is very big: $\sigma_\Lambda = \langle\Lambda\rangle^2$. It is worth remembering that the function $\mathcal{P}(\Lambda; q(t))$ oscillates in time with the frequency $2|\omega_f|$ between the extreme distributions shown in Figs. 2 and 3. However, the form of the distribution in the large scale remains the same: only the value $\langle\Lambda\rangle$ in function $\tilde{\mathcal{P}}(\Lambda)$ changes (i.e., the inclination of straight wings in the logarithmic scale).

VIII. CONCLUSION

We have demonstrated that a giant mean diamagnetic moment can arise when the magnetic field changes its sign, especially for the initial high-temperature thermodynamic equilibrium state. This effect can be understood if one takes into account two factors. The first one is the structure of the magnetic moment operator (4), which contains two parts. One part is proportional to the canonical angular momentum, and its mean value is preserved in the case under study due to the symmetric geometry. But the mean value of the second part, $M\omega(\hat{x}^2 + \hat{y}^2)$, depends on the extension of the particle wave packet in the coordinate space. And here the second factor enters the game, namely, the "weakly quantum" (or "quasiclassical") nature of the high-temperature equilibrium state of a charged particle in the magnetic field. Indeed, using the results of Ref. [25], it can be shown that the spatial extension of the equilibrium wave packet is very large under the condition $\hbar\beta g \ll 1$ (remember that we assume that $g \ll \omega_i$):

$$\langle\hat{x}^2\rangle_{\rm eq} = \langle\hat{y}^2\rangle_{\rm eq} = \frac{k_B T}{M g^2}\{1 + O([\hbar\beta g^2/\omega_i]^2)\}. \tag{36}$$

The principal term in the right-hand side of this relation is in total accord with the classical equipartition theorem for the harmonic oscillator with frequency g. Initially, the huge value $M\omega_i\langle\hat{x}^2 + \hat{y}^2\rangle_{\rm eq}$ is *almost* balanced by the first term in Eq. (4), resulting in the tiny Landau-Darwin diamagnetism. However, this balance is very fragile, and it can be broken when the magnetic field depends on time. The total breakdown happens when the magnetic field changes its sign. This can be seen in the most distinct form in the case of sudden jump of the Larmor frequency to the value $\omega_f = -\omega_i$. Immediately after the "jump," the state of the system does not change. This implies that the mean value $\langle\hat{x}^2 + \hat{y}^2\rangle$ remains the same. But now this mean value should be multiplied by $-\omega_i$ (instead of ω_i) in the expression for the mean magnetic moment. Hence, two terms of Eq. (4) do not cancel each other after the jump. On the contrary, the new magnetic moment is the double value of the second term (with an opposite sign). One can verify that this double value coincides exactly with the right-hand side of Eq. (35). If the evolution of the magnetic field is not so fast, the final values of the mean magnetic moment can be different. However, the results of Sec. VI show that the order of magnitude of these values is the same. Consequently, the reason of the giant diamagnetism is the immense spatial extension of the initial equilibrium (quasiclassical) wave packet and the breakdown of the fragile balance between two parts of the magnetic moment operator under the magnetic field inversion.

The magnitude of the asymptotic value of the mean magnetic moment depends on the details of field evolution, but in all regimes—from the "instant jump" to the adiabatic evolution—it is inversely proportional to the strength of the binding potential. This potential is chosen in the parabolic form in order to obtain explicit exact analytical solutions to the Schrödinger equation (following many studies, starting from Darwin's paper [2]). Such a potential is necessary in order to take into account the fact that the true motion is always confined in some region of space. In more realistic models, including the explicit presence of boundaries, the asymptotic mean value of magnetic moment must depend on the size of container (instead of parameter g). However, using the reasoning of the preceding paragraph, we may expect that the effect of giant amplification will exist even under more realistic assumptions (provided the size of container is large enough). Of course, studies of more realistic models would be very interesting. Note in addition that the parabolic binding potential is frequently used in the models of "artificial atoms" or quantum dots in semiconductors [39,40].

It would be interesting to try to verify the effect of giant amplification of the mean magnetic moment in experiments with single electrons or ions in traps or quantum dots, when the magnetic field changes its sign.

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APPENDIX: DERIVATION OF THE TIME-DEPENDENT SOLUTIONS

An elegant method of solving the time-dependent Schrödinger equation with time-dependent parameters was suggested in 1969 by Lewis and Riesenfeld [14]. The crucial idea is to find the time-dependent operator—integral of motion $\hat{I}(t)$, satisfying the equation

$$i\hbar\partial\hat{I}/\partial t = [\hat{H}(t), \hat{I}(t)]. \quad \text{(A1)}$$

Then, any eigenstate of $\hat{I}(t)$ satisfies the Schrödinger equation $i\hbar\partial\psi/\partial t = \hat{H}(t)\psi(t)$ automatically. Of course, this method is not universal, because one has to guess the structure of operator $\hat{I}(t)$. However, it works quite well for systems with quadratic Hamiltonians, when $\hat{I}(t)$ can be looked for as some quadratic form of the coordinates and momenta. A more simple and more efficient method was proposed by Malkin *et al.* [15], who showed that it is sufficient to find the solutions to (A1) as *linear* combinations of coordinates and momenta operators. In particular, in the case of Hamiltonian (8), the following linear integrals of motion exist [15]:

$$\hat{A}(t) = \frac{\Phi(t)}{2\sqrt{M\hbar}}[\varepsilon(t)(\hat{p}_x + i\hat{p}_y) - M\dot{\varepsilon}(t)(\hat{x} + i\hat{y})], \quad \text{(A2)}$$

$$\hat{B}(t) = \frac{\Phi^*(t)}{2\sqrt{M\hbar}}[\varepsilon(t)(\hat{p}_y + i\hat{p}_x) - M\dot{\varepsilon}(t)(\hat{y} + i\hat{x})], \quad \text{(A3)}$$

where $\Phi(t) = \exp[i\int^t \omega(\tau)d\tau]$ and $\varepsilon(t)$ is any solution to Eq. (12). Choosing the *complex* solution with the Wronskian (29), we obtain the time-independent commutation relations

$$[\hat{A}(t), \hat{A}^\dagger(t)] = [\hat{B}(t), \hat{B}^\dagger(t)] = \hat{1},$$
$$[\hat{A}(t), \hat{B}(t)] = [\hat{A}(t), \hat{B}^\dagger(t)] = 0.$$

Then, one can construct the set of *coherent* states as common eigenstates of operators $\hat{A}(t)$ and $\hat{B}(t)$. In turn, these coherent states are generating states of the *generalized Fock states*. The stationary coherent states in the magnetic field are obtained for $\varepsilon(t) = \omega_g^{-1/2}\exp(i\omega_g t)$ [5,41]. Comparing this formula with the general structure of operators $\hat{A}(t)$ and $\hat{B}(t)$, one can conclude that the evolution of stationary solutions under the time-dependent magnetic field can be obtained, roughly speaking, by means of replacement of the frequency ω in the expressions for the stationary wave function with the fraction $-i\dot{\varepsilon}(t)/\varepsilon(t)$, provided the solution $\varepsilon(t)$ is determined by Eq. (12) and the initial conditions (13). Following this line, one can obtain the following generalization of the Fock's solution [35] to the case of a time-dependent magnetic field:

$$\tilde{\Psi}_{n_r m}(r, \varphi; t) = \sqrt{\frac{\tilde{K}_g n_r!(\tilde{K}_g r^2)^{|m|}}{\pi(n_r + |m|)!}} L_{n_r}^{(|m|)}(\tilde{K}_g r^2) \times \exp\left(i\frac{M\dot{\varepsilon}}{2\hbar\varepsilon}r^2 + im\varphi + i\tilde{\chi}(t)\right). \quad \text{(A4)}$$

Here, $\tilde{K}_g(t) = M/[\hbar|\varepsilon(t)|^2]$. Since the phase $\tilde{\chi}(t)$ is not important for our purposes, we do not bring its explicit (rather complicated) expression. The time-dependent wave function in the momentum representation can be calculated in the same manner as in the stationary case. The identity (29) is important in these calculations. The result is given by Eq. (11) of the main text.

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