Quantum decoherence of a two-level system in colored environments

Longli Zheng[1] and Yonggang Peng[2,*]
[1]*Taishan College & School of Physics, Shandong University, Jinan 250100, China*
[2]*School of Physics, Shandong University, Jinan 250100, China*

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In this paper, we investigate quantum decoherence dynamics of a two-level system in colored environments described by stochastic processes. We present the extension of the theoretical framework of the stochastic Liouville approach, and we employ the approach in studying the decoherence dynamics of the two-level system in typical colored environments: the Ornstein-Uhlenbeck environment, a simple fluctuating bottleneck environment, and the dichotomic (two-state) environment. The analytical expressions of the decoherence factor of a two-level system in colored environments are obtained by introducing the marginal average operator, whose eigenvalues and eigenfunctions provide different viewpoints from which to understand the behavior of the decoherence factor. In addition, the investigations provide us a potential way to manipulate the decoherence of the quantum system in colored environments.

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I. INTRODUCTION

Quantum coherence, as one of the important phenomena of quantum physics, plays an essential role in a variety of physical processes. Specially, with the advance of quantum technologies, the creation and evolution of quantum coherence play a central role in a range of quantum community, such as quantum information processing, quantum simulation, the coherent control, etc. [1–11]. However, the coupling between the quantum system and the environment always destroys the coherence. That is, the quantum system is susceptible to decoherence by the perturbation of the environment. In recent years, the quantum decoherence of open quantum systems has drawn increasing attention in a wide variety of fields because of its potential applications, such as quantum computing, quantum measurements and quantum information science, and potential effects in the harvesting and transport of electronic energy in photobiological systems [6–12].

The dynamical decoherence of open quantum systems has been widely investigated based on the assumption of equilibrium and a white noise environment [13–26]. Some stochastic processes have been employed to model the effect of an equilibrium environment in a large variety of physical systems for a wide class of problems, such as low-frequency noise in Josephson, solid-state, and superconducting qubits [13–16], and intermittent fluorescence of single molecules and nanocrystals [27–29].

The environmental effect on the quantum system is usually described by the stochastic processes [30–34], which could let us note the stochastic environments using the same name as the stochastic processes. With the development of experimental technologies, researchers have found that the nonequilibrium environment and the colored environment are becoming more important for some physical situations. The dynamical decoherence of quantum systems living in a nonequilibrium environment is interesting to scientists [35–39]. A nonequilibrium environment could induce non-Markovian quantum dynamics and speedup the quantum evolution [38,39]. In addition, the light-induced coherent electronic process in biological systems is too short to consider the thermal environment as an equilibrium [35,36,40].

Some typical colored environments of quantum systems are ideal to use in investigations. For example, the $1/f$ noise and the time-independent colored noise process have been employed to investigate coherence properties of optical fields, the shot effect of transport in random media, stochastic resonance, and so on [41–43]. It is a challenge to explore the quantum decoherence dynamics of a quantum system within a colored environment. Here, by using a colored environment, we employ the usual implication. Namely, its correlation time is of the same timescale or longer than the internal dynamical time and cannot be approximated by the Dirac δ function [17,44–50]. Some analytical and numerical approaches have been developed, such as the path-integral method under a noninteracting blip approximation, the variational method based on unitary transformation, the quantum Monte Carlo method, etc. [51–53]. The decoherence dynamics has been analytically investigated for some typical nonequilibrium environments and colored environments-a thermal nonequlibrium environment described by a Fourier series with random coefficients [35–38].

In this paper, we investigate the dynamics of the quantum decoherence of a two-level system in colored environments. The dephasing of the two-level system is represented by a stochastic process. Based on the two-level system, the decoherence factor satisfies a Kubo-oscillator-type stochastic differential equation. We derive the analytical solutions of the off-diagonal density element $\rho_{21}(t)$ of a two-level system interacting with the environments: the Wiener environment

*ygpeng@sdu.edu.cn

(Gaussian white noise), the Ornstein-Uhlenbeck environment, a simple fluctuating bottleneck environment, and the dichotomic (two-state) environment. Based on our analytical solutions, we present a potential way to manipulate the decoherence of a quantum system under the perturbation of the environment. In addition, we present the extension of the stochastic Liouville methodology suggested by Kubo and Zwanzig [54,55]. We investigate the decoherence dynamics of a two-level system within colored environments using the extended stochastic Liouville methodology. A marginal average operator $\mathbb{H}$ is introduced; its eigenvalues and eigenfunctions provide us different viewpoints from which to understand the behavior of the decoherence dynamics of a quantum system in colored environments.

The organization of the paper is as follows. In Sec. II we introduce the general model and the stochastic model of the environments. The dynamical decoherence and the analytical expressions of decoherence factors are derived for the environments: the Wiener environment, the Ornstein-Uhlenbeck environment, a simple fluctuating bottleneck environment, and the dichotomic (two-state) environment. In Sec. III, we discuss the decoherence dynamics of a quantum system in colored environments. The numerical results of decoherence dynamics are shown for different environments according to the analytical expressions. In Sec. IV we give the conclusions drawn from the present study.

II. MODEL AND GENERAL THEORETICAL FRAMEWORK

A. General considering of the stochastic Liouville methodology

The general form of a stochastic differential equation can be written as follows [33,34]:

$$\dot{u} = F[u, t; X(t)], \tag{1}$$

where $u = \{u_1, u_2, \ldots, u_n\}$ and $F = \{F_1, F_2, \ldots, F_n\}$ could be vectors, and $X(t)$ stands for one or more random functions where the stochastic properties are known. Generally, F is linear in u and does not explicitly depend on time in natural science; that is, Eq. (1) takes the form

$$\dot{u}_\nu = \sum_\mu A_{\nu,\mu}(X(t))u_\mu, \tag{2}$$

where u_j are the elements of the vector u, and $A_{\nu,\mu}$ are the elements of the coefficient matrix A.

To solve the stochastic differential Eq. (2), we here present the extension of the stochastic Liouville methodology [54,55]. In this work, we suppose the transition probability density $\Pi(x, t)$ [generally, the transition probability density is denoted as $\Pi(x, t|x_0, t_0)$; for simplicity, we denote it as $\Pi(x, t)$] of the stochastic process $X(t)$, assumed to be a Markovian process, can be written as

$$\dot{\Pi}(x, t) = \mathbb{W}\Pi(x, t), \tag{3}$$

where $\mathbb{W}$ is an operator, and x is supposed to be the sample of $X(t)$ at time t.

Based on the stochastic Liouville equation suggested by Kubo [56], the joint probability density $\wp(u, x, t)$ can be written as [33,57]

$$\frac{\partial}{\partial t}\wp(u, x, t) = -\sum_{\nu,\mu} A_{\nu,\mu}(x)\frac{\partial}{\partial u_\nu}u_\mu\wp(u, x, t) + \mathbb{W}\wp(u, x, t). \tag{4}$$

By defining the marginal averages of u_ν [33,57],

$$\bar{u}_\nu(t) = \int u_\nu\wp(u, x, t)d^n u, \tag{5}$$

where $d^n u$ stands for integral to all elements in vector u. We can, after multiplying Eq. (4) with u_ν and integrating, derive

$$\begin{aligned}\frac{\partial\bar{u}_\nu(x, t)}{\partial t} &= \sum_\mu A_{\nu,\mu}(x)\bar{u}_\mu(x, t) + \mathbb{W}\bar{u}_\nu(x, t)\\ &\equiv \mathbb{H}\bar{u}_\nu(x, t),\end{aligned} \tag{6}$$

where $\mathbb{H} = \sum_\mu A_{\nu,\mu}(x)\sum_\nu \delta_{\nu,\mu} + \mathbb{W}$. We here define $\mathbb{H}$ as the marginal average operator. This type of differential equation can be effectively solved by the eigenfunction expansion approach, which has been employed to solve the Schrödinger equation with a time-independent Hamiltonian.

Typically, for the case that the marginal average operator $\mathbb{H}$ is time independent and non-Hermitian, there are different ways to deal with a dynamical equation with a non-Hermitian operator [58–61]. Here, we employ the biorthogonal bases method.

The eigenvalue equation of $\mathbb{H}$ reads [60,61]

$$\mathbb{H}g_n(x) = \lambda_n g_n(x), \tag{7}$$

where λ_n and $g_n(x)$ are the eigenvalues and the eigenfunctions, respectively. The eigenfunctions satisfy

$$\int dx\tilde{g}_m(x)g_n(x) = \delta_{n,m}, \tag{8}$$

where $\tilde{g}_n(x)$ is the left eigenfunction (adjoint function) for the non-Hermitian operator $\mathbb{H}$, the Hermitian conjugate of $g_n(x)$ for the Hermitian operator $\mathbb{H}$.

Correspondingly, the marginal average $\bar{u}_\nu(x, t)$ can generally be expressed as

$$\bar{u}_\nu(x, t) = \sum_m c_{\nu,m}(t)g_m(x). \tag{9}$$

By multiplying $\tilde{g}_n(x)$ on both sides, and integrating with respect to x, one has

$$c_{\nu,n}(t) = \int \tilde{g}_n(x)\bar{u}_\nu(x, t)dx. \tag{10}$$

Immediately, one could obtain the coefficient $c_{\nu,n}(t)$ by

$$\dot{c}_{\nu,n}(t) = \lambda_n c_{\nu,n}(t). \tag{11}$$

The marginal average $\bar{u}_\nu(x, t)$ can be written as

$$\bar{u}_\nu(x, t) = \sum_n c_{\nu,n}(0)e^{\lambda_n t}g_n(x), \tag{12}$$

where the coefficients $c_{\nu,n}(0)$ are determined by the initial distributions of the marginal average $f(x) = \bar{u}_{\nu,n}(x, 0)$, namely,

$$c_{\nu,n}(0) = \int \tilde{g}_n(x)f(x)dx. \tag{13}$$

Finally, $\langle u_\nu(t)\rangle$ can be obtained as

$$\langle u_\nu(t)\rangle = \sum_n c_{\nu,n}(0)\varphi_n(t). \tag{14}$$

In Eq. (14), we define the constituent elements $\varphi_n(t)$:

$$\varphi_n(t) = \int e^{\lambda_n t} g_n(x)dx. \tag{15}$$

Equation (14) presents that two essential parts could influence the dynamical behaviors of $\langle u_\nu(t)\rangle$: the constituent elements $\varphi_n(t)$ and their partitions $c_{\nu,n}(0)$, which are determined by the initial conditions. This provides us a potential way to manipulate the nature of $\langle u_\nu(t)\rangle$ by choosing the different partitions $c_{\nu,n}(0)$ of the constituent elements $\varphi_n(t)$. Once the partitions $c_{\nu,n}(0)$ are determined by choosing the initial conditions, the constituent elements $\varphi_n(t)$ could indicate the behaviors of $\langle u_\nu(t)\rangle$.

B. Model

In the process of a quantum system evolving, the environmental effects result in the decoherence of the quantum system. The system we consider is described by the density operator $\hat{\rho}(t)$ with an energy gap

$$\Delta E = \hbar\omega(t) = \hbar\omega_0 + \hbar\xi(t), \tag{16}$$

where $\omega_0 = \langle\omega(t)\rangle$ is the average frequency, and $\xi(t) = \omega(t) - \omega_0$, the fluctuation of the difference frequency, is a stochastic process with zero average. The evolution of the coherence $\rho_{21}(t)$ can be written as

$$\begin{aligned}\rho_{21}(t) &= \left\langle \exp\left(i\int_0^t \xi(s)ds\right)\right\rangle e^{i\omega_0 t}\rho_{21}(0)\\ &\equiv \langle\mathcal{F}(t)\rangle e^{i\omega_0 t}\rho_{21}(0),\end{aligned} \tag{17}$$

where $\mathcal{F}(t) = \exp(i\int_0^t \xi(s)ds)$ is the decoherence factor.

We notify that the decoherence factor $\mathcal{F}(t)$ can, after taking the differentiation of $\mathcal{F}(t)$ with respect to time t, be written as [56]

$$\frac{d\mathcal{F}(t)}{dt} = i\xi(t)\mathcal{F}(t). \tag{18}$$

Equation (18) is the Kubo-oscillator-type stochastic differential equation, which is first employed to study the line shape by Kubo. It was also employed to investigate the line broadening, paramagnetic resonance, rate processes with dynamical disorder, etc. [30,31,54,55].

The Kubo-oscillator-type stochastic differential equation can be solved by employing the extension stochastic Liouville approach presented Sec. II A. In this work, we suppose that time-dependent stochastic variable $\xi(t)$ originates from the random variable x, i.e., $\xi(t) \equiv \xi(x(t))$. Learning from Eq. (5), the marginal average of the decoherence factor can be defined as

$$\bar{\mathcal{F}}(x,t) = \int \mathcal{F}_\xi(\mathcal{F},x,t)d\mathcal{F}, \tag{19}$$

and the average of the decoherence factor can be obtained as

$$\langle\mathcal{F}(t)\rangle = \int \bar{\mathcal{F}}(x,t)dx. \tag{20}$$

Also, the marginal average of the decoherence factor $\bar{\mathcal{F}}$ satisfies equations of the type of Eq. (6).

C. The analytical solutions for typical environments

In this section we consider the analytical solutions for the Wiener environment and three typical colored environments: the Ornstein-Uhlenbeck environment, a simple fluctuating bottleneck environment, and the dichotomic (two-state) environment.

1. The Wiener environment

We first consider the Wiener environment, namely, $\xi(t) = x(t)$ is a Wiener process. The Wiener environment is generally considered as Gaussian white noise with zero mean stochastic process. Here, we consider the Wiener environment as a special case to demonstrate the analytical solution of the stochastic Liouville equation.

The transition probability satisfies the following equation [33]:

$$\frac{\partial\Pi(x,t)}{\partial t} = \frac{q}{2}\frac{\partial^2\Pi(x,t)}{\partial x^2}, \tag{21}$$

where $q/2$ is the correlation intensity of the noise, and the corresponding operator $\mathbb{W} = \frac{q}{2}\frac{\partial^2}{\partial x^2}$.

The marginal average equation reads

$$\frac{\partial\bar{\mathcal{F}}}{\partial t} = \mathbb{H}\bar{\mathcal{F}}, \tag{22}$$

where the marginal average operator $\mathbb{H}$ is

$$\mathbb{H} = ix + \frac{q}{2}\frac{\partial^2}{\partial x^2}. \tag{23}$$

The corresponding eigenvalue equation reads

$$\mathbb{H}\psi_\lambda(x) = \lambda\psi_\lambda(x), \tag{24}$$

where λ is the eigenvalue and $\psi_\lambda(x)$ is the eigenfunction. Considering the boundary condition, the eigenfunction can be expressed by the first Airy function [62,63]:

$$\psi_\lambda(x) = N\mathrm{Ai}\left[\sqrt[3]{\frac{2}{q}}\left(-xe^{i\pi/6} - \lambda e^{2i\pi/3}\right)\right], \tag{25}$$

where N is the normalization constant, and the eigenvalue λ could be complex.

The initial distribution $f(x)$ can be expanded in the corresponding eigenfunctions [63]:

$$f(x) = \int c(\lambda)\psi_\lambda(x)d\lambda, \tag{26}$$

with

$$c(\lambda) = \int \psi_\lambda(x)f(x)dx. \tag{27}$$

Finally, one could obtain

$$\bar{\mathcal{F}}(x,t) = \int c(\lambda)\psi_\lambda(x)e^{\lambda t}d\lambda, \tag{28}$$

and the averaged decoherence factor

$$\langle \mathcal{F}(t)\rangle = \int \bar{\mathcal{F}}(x,t)dx = \int c(\lambda)\varphi(\lambda,t)d\lambda, \tag{29}$$

with the constituent elements

$$\varphi(\lambda,t) = e^{\lambda t}\int \psi_\lambda(x)dx. \tag{30}$$

2. The Ornstein-Uhlenbeck environment

In this subsection, we consider the Ornstein-Uhlenbeck environment, namely, $\xi(t) = x(t)$ is an Ornstein-Uhlenbeck process. The transition probability $\Pi(x,t)$ satisfies [34]

$$\frac{\partial \Pi(x,t)}{\partial t} = \gamma \frac{\partial}{\partial x} x\Pi(x,t) + \frac{q}{2}\frac{\partial^2 \Pi(x,t)}{\partial x^2}, \tag{31}$$

where q is related to the spectral strength of the Ornstein-Uhlenbeck process, and $\gamma = 1/\tau_c$ serves to vary the color of the Ornstein-Uhlenbeck process, with τ_c being the correlation time. The corresponding marginal average operator is

$$\mathbb{H} = ix + \gamma\frac{\partial}{\partial x}x + \frac{q}{2}\frac{\partial^2}{\partial x^2}. \tag{32}$$

To obtain the eigenvalues and eigenfunctions of the operator $\mathbb{H}$, the following transformation is introduced [34]:

$$\bar{\mathcal{F}} = e^{-\frac{\gamma x^2}{2q}}\psi, \tag{33}$$

and then, the marginal average Eq. (6) can be rewritten as

$$\frac{\partial \psi}{\partial t} = e^{\frac{\gamma x^2}{2q}}\mathbb{H}e^{-\frac{\gamma x^2}{2q}}\psi = \mathbb{H}'\psi, \tag{34}$$

where the modified marginal average operator $\mathbb{H}'$ is

$$\mathbb{H}' = \frac{q}{2}\frac{\partial^2}{\partial x^2} - \frac{\gamma^2}{2q}\left(x - \frac{qi}{\gamma^2}\right)^2 + \left(\frac{\gamma}{2} - \frac{q}{2\gamma^2}\right), \tag{35}$$

namely, this is the quantum harmonic oscillator to within a sign change and an additive constant. From standard quantum mechanics [32,34], the eigenvalues and eigenfunctions of the operator $\mathbb{H}'$ are well known:

$$\lambda_n = -n\gamma - \frac{q}{2\gamma^2}, \qquad \psi_n(x) = \left(\frac{\gamma}{\pi q}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}H_n(\zeta)e^{-\zeta^2/2}, \tag{36}$$

where $\zeta = \sqrt{\frac{\gamma}{q}}(x - \frac{qi}{\gamma^2})$, and $H_n(\zeta)$ is the nth Hermite polynomial of ζ.

Hence, the marginal average $\bar{\mathcal{F}}$ can be obtained as

$$\bar{\mathcal{F}}(x,t) = \sum_n c_n e^{\lambda_n t}\psi_n(x)e^{-\frac{\gamma x^2}{2q}}. \tag{37}$$

The coefficient c_n can be obtained once the initial condition of $\bar{\mathcal{F}}(x,0) = f(x)$ is specified:

$$c_n = \left(\frac{\gamma}{\pi q}\right)^{1/4}\frac{1}{\sqrt{2^n n!}}e^{\frac{q}{2\gamma^3}}\int_{-\infty}^{+\infty} H_n(\zeta^*)e^{-\frac{ix}{\gamma}}f(x)dx. \tag{38}$$

By using Eq. (20), the decoherence factor $\langle \mathcal{F}(t)\rangle$ can be obtained as

$$\langle \mathcal{F}(t)\rangle = \int \bar{\mathcal{F}}(x,t)dx = \sum_n c_n e^{\lambda_n t}\int \psi_n(x)e^{-\frac{\gamma x^2}{2q}}dx. \tag{39}$$

Considering the expression of $\psi_n(x)$, we introduce

$$\sigma_n = \int_{-\infty}^{+\infty} H_n(\zeta)e^{-\zeta^2/2}e^{-\frac{\gamma x^2}{2q}}dx = \int_{-\infty}^{+\infty} e^{(-\frac{\gamma}{q}x^2 + \frac{i}{\gamma}x)}H_n(\zeta)dx. \tag{40}$$

After some algebra, one could have

$$\sigma_n = \left(-i\sqrt{\frac{q}{\gamma^3}}\right)^n\sqrt{\frac{\pi q}{\gamma}}e^{-\frac{q}{4\gamma^3}}. \tag{41}$$

Finally, we have the analytical expression of the averaged decoherence factor:

$$\langle \mathcal{F}(t)\rangle = \left(\frac{\gamma}{\pi q}\right)^{1/4}\sum_n c_n\frac{\sigma_n}{\sqrt{2^n n!}}e^{\lambda_n t} \equiv \sum_n c_n\varphi_n(t), \tag{42}$$

where the constituent elements

$$\varphi_n(t) = \left(\frac{\gamma}{\pi q}\right)^{1/4}\frac{\sigma_n}{\sqrt{2^n n!}}e^{\lambda_n t}.$$

3. Simple fluctuating bottleneck environment

We now consider the case that the fluctuation of the difference frequency $\xi(t)$ is modeled as a simple fluctuating bottleneck process; namely, we suppose

$$\xi(t) = \kappa x^2(t), \tag{43}$$

where κ is a constant, $x(t)$ is the Ornstein-Uhlenbeck process, and $x \geqslant 0$. The transition probability of the stochastic process $x(t)$ is given by Eq. (31).

In this process, the equation of the marginal average, Eq. (6), after considering the marginal transform of Eq. (33), can be written as

$$\frac{\partial \psi}{\partial t} = \mathbb{H}'\psi, \tag{44}$$

with the modified marginal average operator $\mathbb{H}'$ being

$$\mathbb{H}' = \frac{q}{2}\frac{\partial^2}{\partial x^2} - \left(\frac{\gamma^2}{2q} - i\kappa\right)x^2 + \frac{\gamma}{2}, \tag{45}$$

where the parameters q and γ have meanings similar to those in Eq. (31). Correspondingly, the eigenvalues and eigenfunctions of the operator $\mathbb{H}'$ are as follows:

$$\lambda_n = -\left(n + \frac{1}{2}\right)\sqrt{\gamma^2 - 2\kappa qi} + \frac{\gamma}{2}, \qquad \psi_n(x) = \left(\frac{\gamma^2 - 2\kappa qi}{\pi^2 q^2}\right)^{1/8}\frac{1}{\sqrt{2^{n-1}n!}}H_n(\alpha x)e^{-\alpha^2 x^2/2}, \tag{46}$$

where $\alpha = (\frac{\gamma^2}{q^2} - \frac{2\kappa i}{q})^{1/4}$.

Again, the marginal average of the coherence factor can be expressed as

$$\bar{\mathcal{F}}(x,t)=\sum_n c_n e^{\lambda_n t}\psi_n(x)e^{-\frac{\gamma x^2}{2q}}, \tag{47}$$

where

$$c_n=\left(\frac{\gamma^2+2\kappa q i}{\pi^2 q^2}\right)^{1/8}\frac{1}{\sqrt{2^{n-1}n!}} \times\int_0^{+\infty}H_n(\alpha^* x)e^{\frac{1}{2}(\frac{\gamma}{q}-\alpha^{*2})x^2}f(x)dx, \tag{48}$$

and $f(x)$ is the initial condition of $\bar{\mathcal{F}}$, i.e., $\bar{\mathcal{F}}(x,0)=f(x)$.

The marginal average $\bar{\mathcal{F}}$ can be continuous analytically in the whole space as an even function, namely,

$$\bar{\mathcal{F}}(x)=\bar{\mathcal{F}}(-x),\quad \text{for } x<0. \tag{49}$$

Then, the decoherence factor can be expressed as

$$\bar{\mathcal{F}}(x,t)=\sum_{\text{even } n} c_n e^{\lambda_n t}\psi_n(x)e^{-\frac{\gamma x^2}{2q}}. \tag{50}$$

Let

$$\sigma_n=\int_0^{+\infty}p(x)H_n(\alpha x)dx,\quad n=0,2,4,\ldots, \tag{51}$$

with $p(x)=\exp[-\frac{1}{2}(\alpha^2+\frac{\gamma}{q})x^2]$.

One could have

$$\sigma_{n+2}=2(n+1)\frac{q\alpha^2-\gamma}{q\alpha^2+\gamma}\sigma_n,\quad n=0,2,4,\ldots, \tag{52}$$

with the initial value

$$\sigma_0=\int_0^{+\infty}p(x)dx=\frac{1}{2}\sqrt{\frac{2\pi q}{q\alpha^2+\gamma}}. \tag{53}$$

Then σ_n can be written as

$$\sigma_n=2^{n-1}\Gamma\left(\frac{n+1}{2}\right)\left(\frac{q\alpha^2-\gamma}{q\alpha^2+\gamma}\right)^{n/2}\sqrt{\frac{2q}{q\alpha^2+\gamma}}, \tag{54}$$

where $\Gamma(n)$ is Euler's Gamma function, and n's are even integers.

Finally, by using Eq. (20), the decoherence factor is

$$\begin{aligned}\langle\mathcal{F}(t)\rangle&=\left(\frac{\gamma^2-2\kappa q i}{\pi^2 q^2}\right)^{1/8}\sum_{\text{even } n}c_n\frac{\sigma_n}{\sqrt{2^{n-1}n!}}e^{\lambda_n t}\\&=\sum_{\text{even } n}c_n\varphi_n(t),\end{aligned} \tag{55}$$

where the constituent elements

$$\varphi_n(t)=\left(\frac{\gamma^2-2\kappa q i}{\pi^2 q^2}\right)^{1/8}\frac{\sigma_n}{\sqrt{2^{n-1}n!}}e^{\lambda_n t}.$$

4. The dichotomic environment

In this subsection, we consider the dichotomic environment, which can be modeled by the dichotomic process. The dichotomic process is frequently exploited by the non-Gaussian and nonequilibrium processes. It is a two-state noise process: $\xi(t)=\{-\omega,+\omega\}$, $\omega>0$. Namely, $\xi(t)$ jumps between $-\omega$ and $+\omega$ with the rate constants $k_\pm$:

$$+\omega \underset{k_-}{\overset{k_+}{\rightleftarrows}} -\omega. \tag{56}$$

The transition probabilities satisfy the following equations:

$$\begin{aligned}\frac{\partial\Pi_+(t)}{\partial t}&=-k_-\Pi_+ + k_+\Pi_-,\\ \frac{\partial\Pi_-(t)}{\partial t}&=k_-\Pi_+ - k_+\Pi_-.\end{aligned} \tag{57}$$

The equation of the marginal average of the coherence factor can be written as follows:

$$\frac{\partial}{\partial t}\begin{pmatrix}\bar{\mathcal{F}}_+\\ \bar{\mathcal{F}}_-\end{pmatrix}=\mathbb{H}\begin{pmatrix}\bar{\mathcal{F}}_+\\ \bar{\mathcal{F}}_-\end{pmatrix}, \tag{58}$$

with the marginal average operator $\mathbb{H}$ being

$$\mathbb{H}=i\omega\begin{pmatrix}1&0\\0&-1\end{pmatrix}+\begin{pmatrix}-k_-&k_+\\k_-&-k_+\end{pmatrix}. \tag{59}$$

The solutions read

$$\begin{pmatrix}\bar{\mathcal{F}}_+\\ \bar{\mathcal{F}}_-\end{pmatrix}=a\boldsymbol{\eta}_1 e^{\lambda_1 t}+b\boldsymbol{\eta}_2 e^{\lambda_2 t}, \tag{60}$$

where a and b are the arbitrary constants, and λ_1, λ_2 and $\boldsymbol{\eta}_1$, $\boldsymbol{\eta}_2$ are eigenvalues and eigenvectors of $\mathbb{H}$, respectively. They are written as follows:

$$\begin{aligned}\lambda_{1,2}&=-k\pm(\alpha+i\beta),\\ \boldsymbol{\eta}_{1,2}&=N_{1,2}\begin{pmatrix}-k_+\\ k+i\omega\mp(\alpha+i\beta)\end{pmatrix},\end{aligned} \tag{61}$$

where $k=(k_++k_-)/2$, $\alpha,\beta\geqslant 0$ are real numbers and satisfy $(\alpha+i\beta)^2=k^2-\omega^2+i\omega(k_+-k_-)$, and $N_{1,2}$ are normalization constants.

Finally, the averaged decoherence factor can be written as

$$\langle\mathcal{F}(t)\rangle=a(i\omega+\lambda_2)e^{\lambda_1 t}+b(i\omega+\lambda_1)e^{\lambda_2 t}, \tag{62}$$

with the initial condition $\langle\mathcal{F}(0)\rangle=1$.

Now we discuss some interesting behaviors of the averaged decoherence factor. First, when $k_+=k_-=k>\omega$, we know that $\alpha=\sqrt{k^2-\omega^2}$ and $\beta=0$. The averaged decoherence factor of Eq. (62) can be expressed as

$$\begin{aligned}\langle\mathcal{F}(t)\rangle=&-e^{-kt}[a(k+\alpha)e^{\alpha t}+b(k-\alpha)e^{-\alpha t}]\\&+i\omega e^{-kt}[ae^{\alpha t}+be^{-\alpha t}],\\=&\big[a(k+\alpha)e^{-(k-\alpha)t}+b(k-\alpha)e^{-(k+\alpha)t}\big]\\&+i\omega\big[ae^{-(k-\alpha)t}+be^{-(k+\alpha)t}\big].\end{aligned} \tag{63}$$

According to the definition of the parameter $\alpha=\sqrt{k^2-\omega^2}$, one could obtain $\alpha\leqslant k$ for $\omega\geqslant 0$. That is, the real and imaginary parts of the averaged decoherence factor demonstrate two exponential decay behaviors with the evolution time t.

Second, for $k_+=k_-=k<\omega$, we have $\alpha=0$ and $\beta=\sqrt{\omega^2-k^2}$. In this case, the averaged decoherence factor can be written as

$$\begin{aligned}\langle\mathcal{F}(t)\rangle=e^{-kt}[&a(i\omega-i\beta-k)e^{i\beta t}\\&+b(i\omega+i\beta-k)e^{-i\beta t}].\end{aligned} \tag{64}$$

Hence, $\langle \mathcal{F}(t) \rangle$ would rotate in the complex plane with a decreasing modulus.

Third, we notice that if $k_+ > k_-$ and $a < b$, then $\langle \mathcal{F}(t) \rangle$ would change its rotating direction, from clockwise to anticlockwise. According to the expression

$$\langle \mathcal{F}(t) \rangle = e^{-kt}[(i\omega - k)(ae^{\alpha t}e^{i\beta t} + be^{-\alpha t}e^{-i\beta t}) + (\alpha + i\beta)(-ae^{\alpha t}e^{i\beta t} + be^{-\alpha t}e^{-i\beta t})], \quad (65)$$

the orientation rotation behaviors are determined by the terms $\pm ae^{\alpha t}e^{i\beta t} + be^{-\alpha t}e^{-i\beta t}$. At the beginning ($t$ is small), we have $ae^{\alpha t} < be^{-\alpha t}$ because $a < b$. So the rotation is mainly driven by $e^{-i\beta t}$. However, for long enough time, we have $ae^{\alpha t} > be^{-\alpha t}$. Then the term $e^{i\beta t}$ becomes the main driven part.

III. NUMERICAL RESULTS AND DISCUSSIONS

In this section we present the numerical results for the decoherence factors discussed in the previous section. Although the analytical expressions of the decoherence factors for the four typical models are given, to clearly demonstrate the behaviors of the decoherence factors or the constituent elements (since the factors consist of the constituent elements), we give the numerical results according to the analytical expressions.

1. The Wiener environment

The Wiener process is generally employed to study the Brownian motion and the diffusion processes. The diffusion processes are widely found in physics, chemistry, biophysics, and other natural science fields. The averaged decoherence factor of Eq. (42) can be expressed as

$$\langle \mathcal{F}(t) \rangle = \int c(\lambda)\varphi(\lambda, t)d\lambda. \quad (66)$$

Equation (66) shows that the decoherence factor is influenced by two parts: the constituent element $\varphi(\lambda, t)$ and its corresponding partition $c(\lambda)$. A typical case for this environment is $c(\lambda) = \delta(\lambda - \lambda_0)$, and the decoherence factor $\langle \mathcal{F}(t) \rangle$ is reduced to a typical constituent element $\varphi(\lambda_0, t)$. To understand the behavior of $\langle \mathcal{F}(t) \rangle$, we demonstrate several typical constituent elements $\varphi(\lambda, t)$ in Fig. 1. The absolute value, the real part, and the imaginary part of the constituent elements $\varphi(\lambda, t)$ are plotted. The yellow solid lines, the red dash-dotted lines, and the blue dashed lines correspond to absolute, real, and imaginary values of the constituent elements $\varphi(\lambda, t)$; Figs. 1(a)–1(c) correspond to the parameters $\lambda = 1, \lambda = i$, and $\lambda = 1 + i$, respectively. The real part of $\varphi(\lambda, t)$ determines the decay behavior, and the imaginary part determines the oscillation behavior. Generally speaking, the greater the real part is, the faster $\varphi(\lambda, t)$ decays, and the imaginary part of λ is greater, the stronger the $\varphi(\lambda, t)$ oscillationis.

The decoherence factor $\langle \mathcal{F}(t) \rangle$ could be composed of more than one constituent element $\varphi(\lambda, t)$, with the eigenvalue λ being a complex number. Here, we consider that $\langle \mathcal{F}(t) \rangle$ is composed of two constituent elements, $\varphi(\lambda_1, t)$ and $\varphi(\lambda_2, t)$, and we suppose

$$\varphi(\lambda, t) \sim e^{-\lambda t}, \quad (67)$$

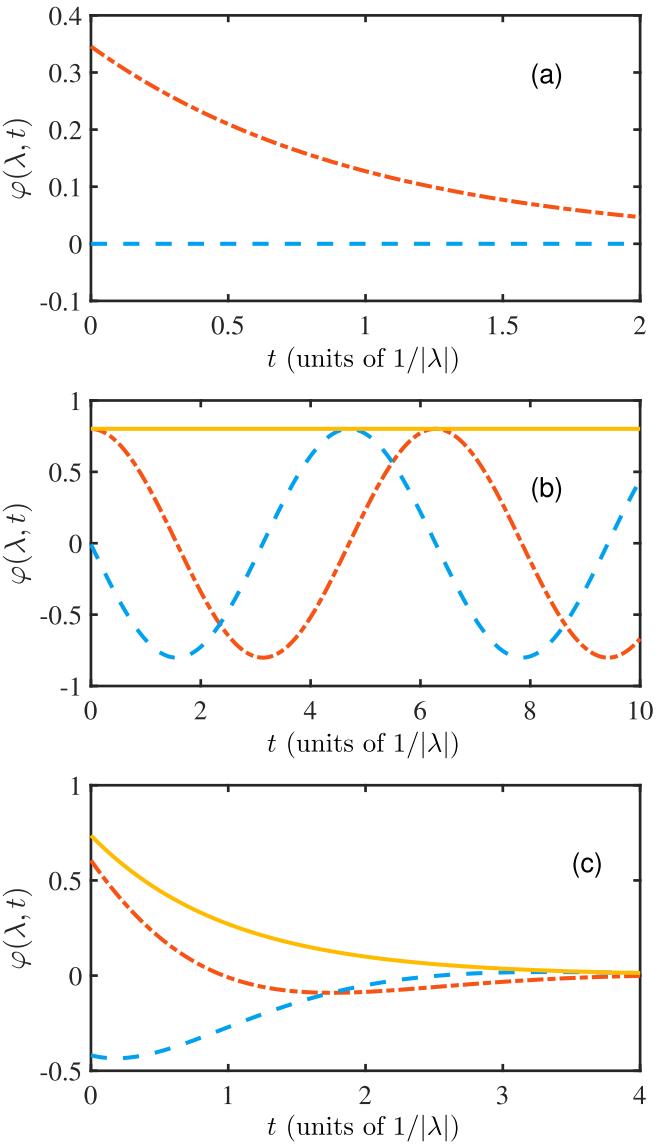

FIG. 1. The constituent element $\varphi(\lambda, t)$ of the Wiener environment versus time t. Panels (a)–(c) correspond to $\lambda = 1$, $\lambda = i$, and $\lambda = 1 + i$, respectively. The yellow solid lines, the red dash-dotted lines, and the blue dashed lines represent the absolute values, the real parts, and the imaginary parts, respectively.

all others are contributed to the proportional constant $c(\lambda)$. The eigenvalues λ_1 and λ_2 can be written as

$$\lambda_1 = a + ib, \quad \lambda_2 = c + id, \quad (68)$$

where a, b, c, and d are real numbers. If $c(\lambda_1) = c(\lambda_2)$, one could obtain

$$\langle \mathcal{F}(t) \rangle \sim e^{-(a+ib)t} + e^{-(c+id)t} \quad (69)$$

and the corresponding absolute value

$$|\langle \mathcal{F}(t) \rangle|^2 \sim 2e^{-2(a+c)t}\{1 + \cos[(b - d)t]\}. \quad (70)$$

Equation (70) demonstrates the decoherence factor is not exponentially decaying with respect to the evolution time, but rather is oscillating. This behavior is connected to the

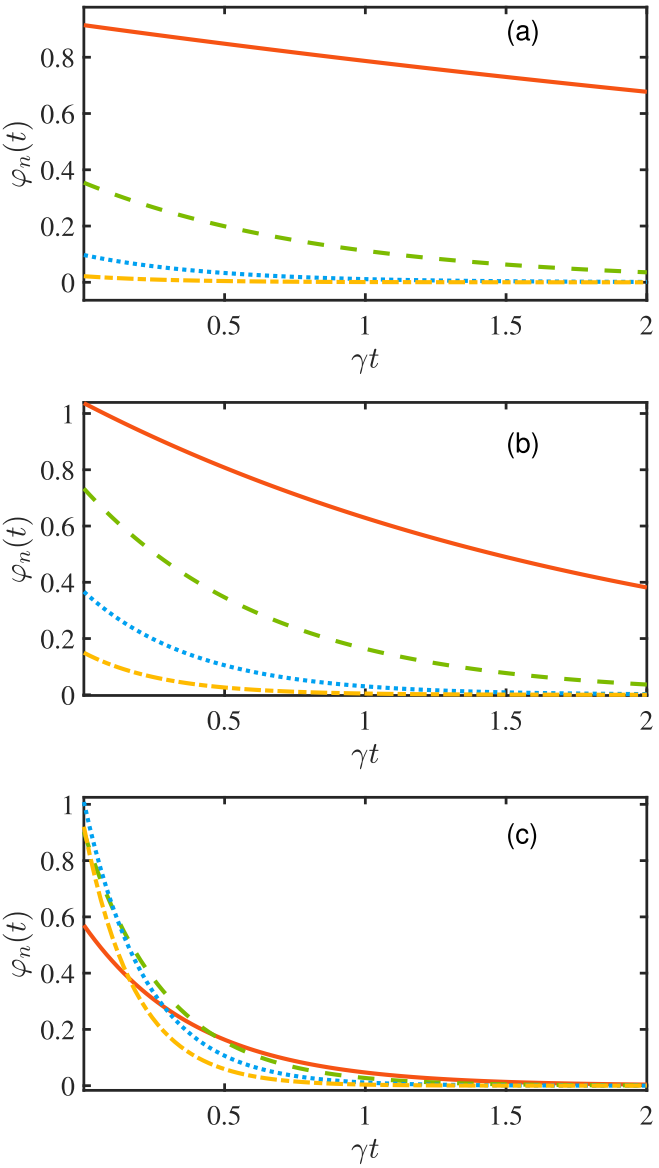

FIG. 2. The results of $\varphi_n(t)$ for the Ornstein-Uhlenbeck process. Panels (a)–(c) correspond to $q/\gamma^3 = 0.3$, 1.0, and 5.0, respectively. The red solid lines, the green dashed lines, the blue dotted lines, and the yellow dash-dotted lines correspond to $n = 0$, 1, 3, and 4, respectively.

non-Markovian behavior of the dynamical evolution of the quantum system [37].

2. *The Ornstein-Uhlenbeck environment*

The Ornstein-Uhlenbeck process has been widely employed to investigate different physical problems [64–67]. We here present the numerical results of the quantum decoherent dynamics. In our numerical results, we use the scaled parameters, namely, $t \to \gamma t$, and the constituent element $\varphi_n(t) \to \sqrt{\gamma}\varphi_n(t)$, etc. Correspondingly, the averaged decoherence factor of Eq. (42) can be expressed as

$$\langle \mathcal{F}(t) \rangle = \sum_n c_n \varphi_n(t). \tag{71}$$

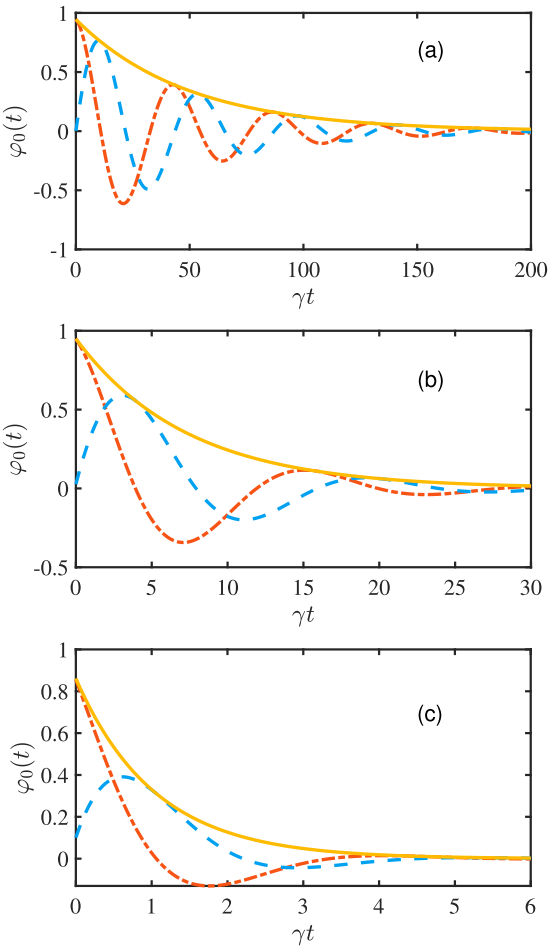

FIG. 3. The result of $\varphi_0(t)$ for the fluctuating bottleneck. Panels (a)–(c) correspond to $\kappa/\gamma = 0.3$, 1.0, and 8.0, respectively. The blue dash-dotted lines represent the real part, the red dashed lines represent the imaginary part, and the yellow solid lines are the modulus of the decoherence factor. The parameter $q/\gamma = 1$.

In Fig. 2, we present the results of the first four constituent elements $\varphi_n(t)$ ($n = 0$, 1, 2, and 3) for $q/\gamma = 0.3$, 1.0, and 5.0 in panels (a)–(c), respectively. Figure 2 shows that the constituent elements $\varphi_n(t)$ have a long coherence time for small q. From Eqs. (36) and (42), we know the dynamical decay of $\varphi_n(t)$ is proportional to $e^{\lambda_n t}$. This demonstrates that the "low constituent element" (namely, the small n) has a long coherence time as shown in Fig. 2. Conversely, the "high constituent element" has a short coherence time; i.e., the "high constituent elements" dynamical decay fast.

Expression (36) shows the eigenvalues of the marginal average operator $\mathbb{H}$ in the Ornstein-Uhlenbeck environment are real. The single constituent element $\varphi_n(t)$ shows exponential decay behavior, and the absolute value of the averaged decoherence factor $|\langle \mathcal{F}(t) \rangle|$, which is composed of several constituent elements $\varphi_n(t)$, could show the multi-exponential-decay behavior.

3. *Simple fluctuating bottleneck environment*

We now give the numerical results of the dynamical decoherence for the quantum system under the environment

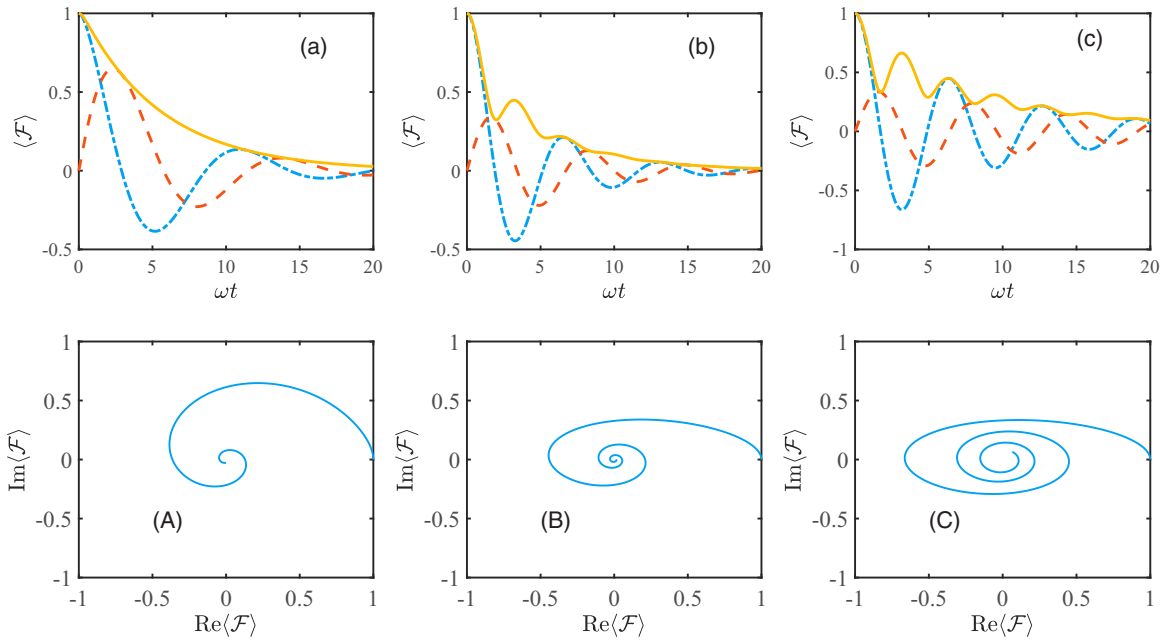

FIG. 4. The decoherence factor as a function of time. The results of panels (a)–(c) correspond to the parameters $(k_+/\omega, k_-/\omega) = (3.0, 1.0)$, (0.4, 0.2), and (0.2, 0.1), respectively. The blue dash-dotted lines represent the real part, the red dashed lines represent the imaginary part, and the yellow solid lines are the modulus of decoherence factor. Panels (A)–(C) are the trajectories of the decoherence factor in space of $\mathrm{Re}\langle\mathcal{F}\rangle \sim \mathrm{Im}\langle\mathcal{F}\rangle$, which correspond to panels (a)–(c), respectively.

of the fluctuating bottleneck process. The process has been employed to investigate some interesting physical processes, such as multi-time-correlation functions, single molecule kinetics, etc. [64–68]. Also, in this model, we use the scaled parameter $t \to \gamma t$. In this case, the decoherence factor can be written in a form similar to that of the case of the Ornstein-Uhlenbeck process of Eq. (71). However, the coefficients in Eq. (55) are complex, and the real and imaginary parts of the constituent elements have rich behaviors.

In Fig. 3, we present the results of the constituent element $\varphi_0(t)$ for the cases of $\kappa = 0.3$, 1.0, and 8.0 in panels (a)–(c), respectively. As shown in the figure, the oscillations of the real and imaginary parts of $\varphi_0(t)$ fade as the parameter κ increases. At the same time, their coherence time is becoming short. Physically, the quantum system under the strong effects of the environment decays fast, or the strong stochastic environment makes the quantum system obviously decoherent. This is clearly shown in Fig. 3.

4. The dichotomic environment

To present the numerical results, we also use the scaled parameters in the dichotomic process: $t \to \omega t$ and $k_\pm \to k_\pm/\omega$. The initial condition is taken as the stationary distributions. Considering the constituent elements φ_n, behaviors are demonstrated in Sec. II C 4. In this subsection, we give the behaviors of the decoherence factor obviously. In Fig. 4, we present the numerical results of the decoherence factor and the real and imaginary parts of the decoherence factor versus the evolution time. The trajectories of the decoherence factor in space of $\mathrm{Re}\langle\mathcal{F}\rangle \sim \mathrm{Im}\langle\mathcal{F}\rangle$ are also shown.

As shown in the figure, the oscillations appear for the decoherence factor in the process of dynamical decaying as the time goes on. These oscillations of the decoherence factor indicate the non-Markovian behaviors induced by the colored dichotomic environment. Our results show that the slower the jump between the two states of the environment the more obvious the induced non-Markovian behaviors are. Physically, if the environment jumps fast [as shown in Fig. 4(a)], the physical specifics would be wiped out by its fast-jumping process. Therefore, the decoherence factor decays monotonically. But, the system could "keep" some details of the behaviors of the environment if the environment jumps slowly. Namely, this can induce the non-Markovian nature of the quantum system as shown in Fig. 4(c).

In the space of $\mathrm{Re}\langle\mathcal{F}\rangle \sim \mathrm{Im}\langle\mathcal{F}\rangle$, $|\langle\mathcal{F}(t)\rangle|$ is the "distance" from the space point $(\mathrm{Re}\langle\mathcal{F}\rangle, \mathrm{Im}\langle\mathcal{F}\rangle)$ to the ending point, and the changing rate of the distance is related to the non-Markovian behavior in the dynamical evolution of the quantum system [37].

IV. CONCLUSIONS

We investigated the dynamical decoherence of a two-level quantum system in the Wiener environment and three typical colored environments: the Ornstein-Uhlenbeck environment, a simple fluctuating bottleneck environment, and the dichotomic environment, respectively. We have derived the analytical expressions of the averaged decoherence factor for the these typical stochastic environments.

We introduce the operator of the marginal average $\mathbb{H}$, and correspondingly, the product of the eigenfunctions $g_n(y)$ of $\mathbb{H}$ and the exponential function of the eigenvalues $e^{\lambda_n t}$ are defined as the constituent elements of the decoherence factor. We have shown that the decoherence behavior of a two-level quantum system can be controlled by manipulating the

constituent elements of the decoherence factor. The partitions of the constituent elements of the decoherence factor can be determined *via* the initial distributions of the environments.

The eigenvalues of the marginal average operator $\mathbb{H}$ are generally complex, which results in the absolute value of the decoherence factor $|\langle\mathcal{F}(t)\rangle|$ oscillating and decaying with respect to the evolution time. The oscillating and decaying behaviors of the decoherence factor $|\langle\mathcal{F}(t)\rangle|$ reflect the dynamical evolution of the quantum system connected to non-Markovian behaviors. The strength of the "exchanging information" of the different constituent elements $\varphi_n(t)$ and $\varphi_m(t)$ is determined by the partitions $\sqrt{c_n c_m}$, which provides a way to control the non-Markovian behaviors.

The possible role played by the colored environment in some dynamical evolution could be investigated, such as quantum dynamical speedup induced by a colored/nonequilibrium environment, the nature of quantum speed limits and the geometric phase of a quantum system under a colored environment, etc. In addition, by introducing the frequency shift and the decoherence rate, the dynamical evolution of the reduced density matrix of a quantum system can be described by a time-local master equation. Hence, some quantum effects induced by the colored environment can be discussed.

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