Irreversible entropy production rate in a parametrically driven-dissipative system: The role of self-correlation between noncommuting observables

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In this paper, we explore the Wigner entropy production rate in the stationary state of a two-mode Gaussian system. The interacting modes dissipate into different local thermal baths. Also, one of the bosonic modes evolves into the squeezed-thermal state because of the parametric amplification process. Using the Heisenberg-Langevin approach combined with the quantum phase-space formulation, we get an analytical expression for the steady-state Wigner entropy production rate. It contains two key terms. The first one is an Onsager-like expression that describes heat flow within the system. The second term results from vacuum fluctuations of the baths. Analyses show that self-correlation between the quadratures of the parametrically amplified mode pushes the mode toward the thermal squeezed state. It increases vacuum entropy production of the total system and reduces the heat current between the modes. The results imply that, unlike in previous proposals, squeezing can constrain the efficiency of actual nonequilibrium heat engines by irreversible flows.

I. INTRODUCTION

The entropy production rate, which is related to the second law of thermodynamics, plays a fundamental role in nonequilibrium classical and quantum thermodynamics. It provides a good framework to describe and quantify the irreversibility of physical phenomena. For a system in contact with a bath, the variation of entropy per unit time can be expressed as

$$\frac{dS(t)}{dt} = \Pi(t) - \Phi(t), \tag{1}$$

where S is the entropy of the system, $\Phi(t)$ is the flow of entropy between the system and its bath (per unit time), and $\Pi(t)$ is the entropy production rate. When a system reaches a nonequilibrium steady state, $dS(t)/dt$ vanishes, and we have $\Pi_s = \Phi_s > 0$. If the stationary state is a thermal equilibrium state, then $\Pi_s = \Phi_s = 0$.

So far, the theory of entropy production has been developed and formulated in different contexts. In the classical domain, the most popular approaches are based on Onsager's theory[1–3], classical master equations [4,5], and Fokker-Planck equations [6–10]. The extension of these approaches to the quantum domain leads to new formulations based on the quantum master equations [11–14], quantum trajectories [15], and fluctuation theorems [16,17].

More recently, considerable attention has been directed toward the theoretical and experimental characterization of entropy production in nonequilibrium bosonic systems based on quantum phase-space distributions and Fokker-Planck equations [18–28].

In this framework, one can express entropy production in terms of means and variances of independent variables and can identify irreversible quasiprobability currents in phase space for single-mode Gaussian systems in contact with a single reservoir [19,22] and multimode Gaussian systems connected to multiple reservoirs [18,23,24,28]. Identifying the quasiprobability currents in phase space provides physical interpretations of the irreversibility in microscopic systems at the quantum level [22]. It can keep the Wigner entropy production rate finite for a system in contact with a thermal reservoir at zero temperature [21,23]. Also, the formalism works very well for describing irreversibility in open quantum systems exposed to different nonequilibrium reservoirs, such as squeezed and dephasing reservoirs [19].

This paper is motivated by the above studies and recent proposals [29–39] to use squeezed thermal states as a resource to increase the efficiency of heat engines beyond the Carnot limit. A squeezed-thermal reservoir can be obtained by applying a unitary transformation U_{sq} to a thermal reservoir $\rho_{th-sq} = U_{sq}\rho_{th}U_{sq}^{\dagger}$. However, any squeezing operation increases the energy of the thermal bath. It is believed that the second law of thermodynamics remains valid in these machines if the required energy is accounted for.

On the other hand, obtaining Carnot efficiency and surpassing it have limited significance for practical applications since the reversible process has to be infinitely slow and the corresponding power is zero. In practical applications, when the cycling process takes place in a finite time, one should recognize the sources of irreversibility within the system. The system considered in this study is a step toward understanding the limits that thermodynamics imposes on the efficiency of squeezed-thermal machines for practical applications in the quantum regime.

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The system under study is composed of two harmonic oscillators which are linearly coupled and dissipate into different local thermal baths. One oscillator is parametrically driven. Hence, the degenerate parametric amplification process which is responsible for generating squeezed-thermal states happens within the system. By applying a squeezing transformation to this system, one can diagonalize the Hamiltonian and obtain a unitarily equivalent system comprising two linearly coupled harmonic oscillators dissipating into a thermal bath and a squeezed-thermal bath. Identifying the sources of irreversibility for each of these systems and comparing them are steps toward determining the required energy for constructing a nonequilibrium squeezed bath and calculating the practical efficiencies of nonequilibrium squeezed-thermal machines.

Using the quantum phase-space method, we obtain an analytical expression for the Wigner entropy production of the system. It has been shown that, in composite quantum systems, the entropy production is related to the internal correlation between subsystems [20,23,25]. We show that the parametric amplification process not only modifies the internal correlation between the subsystems but also produces an additional correlation between noncommuting observables (q and p) of the parametrically driven oscillator. This self-correlation term, which is responsible for the squeezing of the mode, also contributes to the entropy production. To analyze the role of squeezing, we decompose the Wigner entropy production into two terms. The first term is an Onsager-like expression that describes heat flow between the subsystems, and the second one describes the irreversibility due to vacuum fluctuations of both thermal baths. The second contribution is always nonzero, even when both reservoirs are in the vacuum state.

We demonstrate that the self-correlation between quadratures (q and p) of the driven oscillator produces a heat current from cold to hot bath, reducing the effective heat current between them. Also, analyzing the vacuum part of the entropy production allows us to discriminate the local and nonlocal effects of the baths on the oscillators. To support our analytical results, we calculate vacuum entropy production and its components in two different regimes. In both regimes, self-correlation increases the local contribution of the vacuum entropy production, the internal correlation increases the nonlocal contribution of the reservoirs, and they compete with one another.

This paper is structured as follows: In Sec. II, by introducing the physical model, we derive the quantum Langevin equations of motion for the system operators. In Sec. III, we derive an analytical expression for steady-state Wigner entropy production in terms of self-correlation and internal correlation terms of the covariance matrix. Then, we identify different sources of irreversibility within the system. In Sec. IV, we support our analytical results with numerical calculations to show the effect of squeezing on the Wigner entropy production in two different regimes. Section V contains the conclusions of the study.

II. THE MODEL

The system consists of two coupled harmonic quantum oscillators. Each oscillator dissipates into its local thermal
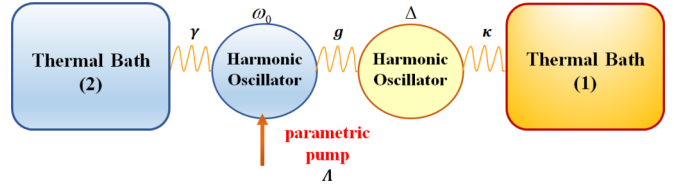

FIG. 1. Schematic of the system.

bath. The oscillations of one of the harmonic oscillators are parametrically amplified by an external force (Fig. 1). The unitary dynamics of the system can be described by the following Hamiltonian:

$$H = \Delta \hat{a}^\dagger \hat{a} + \omega_0 \hat{b}^\dagger \hat{b} + \Lambda(\hat{b}^2 + \hat{b}^{\dagger 2}) - g(\hat{a} + \hat{a}^\dagger)(\hat{b} + \hat{b}^\dagger), \quad (2)$$

where a ($a^\dagger$) and b ($b^\dagger$) are the annihilation (creation) operators of the harmonic oscillators with frequency Δ and ω_0, respectively. The first two terms of Eq. (2) describe the free energy of harmonic oscillators, the third term describes the parametric amplification of the harmonic oscillator with frequency ω and strength Λ, and the last term describes the coupling between the harmonic oscillators with coupling constant g. The first harmonic oscillator is locally connected to thermal bath 1 and dissipates at rate κ, and the second one is locally connected to thermal bath 2 and dissipates at rate γ.

In practice, the system can be an optomechanical system that contains a $\chi^{(2)}$ medium [40] or an optomechanical system with a Duffing nonlinear mechanical oscillator [41]. In these systems, the fluctuations of the operators of the subsystems (cavity-field and mechanical oscillators) around the mean-field values play the role of the Gaussian modes [18,20,27].

Considering the conventional picture of system-bath coupling in the form

$$H_{SB} = i\hbar \int_{-\infty}^{\infty} d\omega \kappa(\omega) [B^\dagger(\omega) c - c^\dagger B(\omega)] \quad (3)$$

to describe the dissipation of each mode at the level of standard input-output theory [42,43], we can get the damping of each mode. We use the Heisenberg-Langevin approach to describe the dynamics of the system [43]. In a recent paper [44], excellent agreement was found between the Langevin-Heisenberg approach and the global Lindblad master equation in such systems. According to this approach, including dissipation caused by system-bath couplings and the corresponding noises, quantum Langevin equations of motion are given by

$$\dot{\hat{a}} = -(i\Delta + \kappa)\hat{a} + ig(\hat{b} + \hat{b}^\dagger) + \sqrt{2\kappa}\, a_{in}, \quad (4a)$$

$$\dot{\hat{b}} = -(i\omega_0 + \gamma)\hat{b} - 2i\Lambda\hat{b}^\dagger + ig(\hat{a} + \hat{a}^\dagger) + \sqrt{2\gamma}\, b_{in}, \quad (4b)$$

where a_{in} and b_{in} are the input thermal noise operators characterized by the following Markovian correlation functions:

$$\langle a_{in}(t) a_{in}(t') \rangle = \langle a_{in}^\dagger(t) a_{in}^\dagger(t') \rangle = 0, \quad (5a)$$

$$\langle a_{in}(t) a_{in}^\dagger(t') \rangle = (\bar{n}_1 + 1)\delta(t - t'), \quad (5b)$$

$$\langle b_{in}(t) b_{in}(t') \rangle = \langle b_{in}^\dagger(t) b_{in}^\dagger(t') \rangle = 0, \quad (5c)$$

$$\langle b_{in}(t) b_{in}^\dagger(t') \rangle = (\bar{n}_2 + 1)\delta(t - t'), \quad (5d)$$

where the thermal occupation numbers corresponding to reservoirs at temperature T_i $(i = 1, 2)$ are given by

$$\bar{n}_1 = (e^{\hbar\Delta/k_B T_1} - 1)^{-1}, \quad \bar{n}_2 = (e^{\hbar\omega_0/k_B T_2} - 1)^{-1}. \quad (6)$$

Defining $x = (\hat{a} + \hat{a}^\dagger)/\sqrt{2}$ and $y = (\hat{a} - \hat{a}^\dagger)/i\sqrt{2}$ as the quadratures of the first mode, $q = (\hat{b} + \hat{b}^\dagger)/\sqrt{2}$ and $p = (\hat{b} - \hat{b}^\dagger)/i\sqrt{2}$ as the quadratures of the second mode (and, similarly, their corresponding input-noise quadratures), and coupling strength $G = 2g$, quantum Langevin equations for the quadratures can be written in compact matrix form:

$$\dot{u}(t) = Au(t) + f(t), \quad (7)$$

where $u(t) = (x, y, q, p)^T$ is the vector of the operators, $f(t) = (\sqrt{2\kappa}x_{in}, \sqrt{2\kappa}y_{in}, \sqrt{2\gamma}q_{in}, \sqrt{2\gamma}p_{in})^T$ is the vector of the noises, and the drift matrix A is given by

$$A = \begin{pmatrix} -\kappa & \Delta & 0 & 0 \\ -\Delta & -\kappa & G & 0 \\ 0 & 0 & -\gamma & \omega \\ G & 0 & -\Omega & -\gamma \end{pmatrix}, \quad (8)$$

where $\omega \equiv \omega_0 - 2\Lambda$ and $\Omega \equiv \omega_0 + 2\Lambda$. The formal solution of Eq. (7) is $u(t) = M(t)u(0) + \int_0^t ds M(s)f(t-s)$, where $M(t) = \exp\{At\}$. The system is stable and reaches a steady state if all eigenvalues of the drift matrix A have negative real parts. For $\Delta, \omega > 0$, applying the Routh-Hurwitz criterion [45] gives the following stability condition:

$$\eta := \delta^2\Gamma^2 - G^2\Delta\omega > 0, \quad (9)$$

where η is the stability parameter, $\delta = \sqrt{\Delta^2 + \kappa^2}$, and $\Gamma = \sqrt{\gamma^2 + \omega\Omega}$.

III. CORRELATION MATRIX AND WIGNER ENTROPY PRODUCTION

Since the dynamics of the system is linear and the quantum noise terms are Gaussian, the steady state of the system is a continuous-variable Gaussian state which can be characterized by the 4×4 correlation matrix (CM) σ with corresponding components $\sigma_{i,j} = \langle u_i(\infty)u_j(\infty) + u_j(\infty)u_i(\infty)\rangle/2$ (with the assumption that the first moments are zero). When the system is stable, each component of the CM is given by

$$\sigma_{i,j} = \Sigma_{k,l}\int_0^\infty ds \int_0^\infty ds' M_{ik}(s)M_{jl}(s')\phi_{kl}(s-s'), \quad (10)$$

where $\phi(s-s')$ is the matrix of the stationary noise correlation functions. It should be noted that the uncertainty relations among canonical operators $([u_i, u_j] = i\Omega_{ij})$ impose a constraint on the quantum CMs, corresponding to the inequality $\sigma + i\mathbf{\Omega}/2 \succeq 0$, where $\mathbf{\Omega}$ is the symplectic matrix [46].

When the stability condition is satisfied, $M(\infty) = 0$, and one gets the Lyapunov equation for the steady-state correlation matrix as

$$A\sigma + \sigma A^T = -D, \quad (11)$$

where $D = \text{diag}\{\kappa(2\bar{n}_1+1), \kappa(2\bar{n}_1+1), \gamma(2\bar{n}_2+1), \gamma(2\bar{n}_2+1)\}$ is the diffusion matrix. Equation (11), which is linear in σ, is exactly solvable in a straightforward manner, but the general exact expression is too cumbersome and will not be presented here.

The Gaussian nature of the system allows us to relate the Wigner entropy of the system to the covariance matrix σ by

$$w_\sigma(u) = \frac{1}{(2\pi)^n\sqrt{\det\sigma}}e^{-\frac{1}{2}u^T\sigma^{-1}u}, \quad (12)$$

which is always positive and allows us to identify the Wigner function as a quasiprobability distribution in phase space (n is the number of bosonic modes). It provides a fully equivalent description of the density matrix. These features make the Wigner entropy, introduced in Ref. [19] as

$$S_w = -\int du\, w_\sigma(u)\ln w_\sigma(u), \quad (13)$$

a perfectly suitable framework for quantification of the irreversibility in the current system. Reference [47] showed that Wigner entropy can be related to the Rényi-2 entropy and satisfies the strong subadditivity inequality. The link between general Rényi-α entropies and the thermodynamic properties of quantum systems has also been of interest in recent years [48,49].

The dynamics of the system can be equivalently expressed in terms of the Fokker-Planck equation for the Wigner function in Eq. (12). Therefore, following the approach of Refs. [10,18,20], the entropy production rate Π_s and entropy flux rate ϕ_s in the steady state take the following form:

$$\Pi_s = -\phi_s = \text{Tr}(2A^{irr}D^{-1}A^{irr}\sigma^s + A^{irr})$$
$$= 2\kappa\left(\frac{\sigma_{11}^s + \sigma_{22}^s}{N_1} - 1\right) + 2\gamma\left(\frac{\sigma_{33}^s + \sigma_{44}^s}{N_2} - 1\right), \quad (14)$$

where $A^{irr} = \text{diag}\{-\kappa, -\kappa, -\gamma, -\gamma\}$ and $N_i = (1 + 2\bar{n}_i)$ $(i = 1, 2)$. When the system is in the equilibrium state, we have $\sigma_{11}^s + \sigma_{22}^s = N_1$ and $\sigma_{33}^s + \sigma_{44}^s = N_2$, and hence, $\Pi_s = 0$. In this view, Eq. (14) can be expressed as $\Pi_s = \sum_{i=1,2}\Pi_i^s$ and is a suitable quantifier of the distance between the bath-imposed equilibrium state of each oscillator and its steady state [20,23,26]. However, we shall show that Π_s is not always capable of detecting self-correlation between quadratures of each subsystem.

It should be noted that according to the Lyapunov equation (11), in the steady state the diagonal terms (σ_{ii}) and off-diagonal terms $(\sigma_{ij\neq i} = \sigma_{ji\neq i})$ of the CM are not independent. In fact, it is easy to show that

$$\sigma_{11}^s = \frac{N_1}{2} + \frac{\Delta}{\kappa}\sigma_{12}^s, \quad (15a)$$

$$\sigma_{22}^s = \frac{N_1}{2} - \frac{\Delta}{\kappa}\sigma_{12}^s + \frac{G}{\kappa}\sigma_{23}^s, \quad (15b)$$

$$\sigma_{33}^s = \frac{N_2}{2} + \frac{\omega}{\gamma}\sigma_{34}^s, \quad (15c)$$

$$\sigma_{44}^s = \frac{N_2}{2} - \frac{\Omega}{\gamma}\sigma_{34}^s + \frac{G}{\gamma}\sigma_{14}^s, \quad (15d)$$

and hence, the internal correlation between the quadratures of the two modes (σ_{14}^s and σ_{23}^s) and the self-correlation between quadratures of each mode (σ_{34}^s and σ_{12}^s) are concealed in the full expression of the diagonal elements. In addition, the correlation terms result in a steady state with reduced uncertainty in one quadrature and enhanced uncertainty in the complementary conjugated quadrature for each mode. This

implies that, in general, the steady state of each bosonic mode is a thermal squeezed state.

Hence, the entropy production can be related to off-diagonal elements of the CM as

$$\Pi_s = \frac{2G}{N_2}\sigma_{14}^s + \frac{2G}{N_1}\sigma_{23}^s - \frac{2(\Omega - \omega)}{N_2}\sigma_{34}^s. \quad (16)$$

Equation (16) quantitatively links the steady-state entropy production rate to the correlation functions of the dynamical variables. The first two terms of Π_s, which are nonzero only for $G \neq 0$, represent the effect of coupling between the two modes, and the correlation between their quadratures and the last term, which is nonzero only for $\Omega \neq \omega$, represents the effect of the self-correlation between the quadratures of the parametrically amplified mode. As can be seen, the self-correlation between the quadratures of the first mode σ_{12}^s does not contribute to Π_s. Therefore, Π_s is incapable of detecting self-correlation between the quadratures of both modes.

It should be noted that all correlation terms in Π_s are functions of the internal coupling constant G, the strength of the parametric drive Λ, the mean occupation numbers of reservoirs N_1 and N_2, and dissipation rates κ and γ. The exact analytical expressions for these terms are too cumbersome and not tractable. Therefore, we leave the details for the Appendix. For the purpose of illustration, we present here the correlation terms (σ_{ij}) and steady-state entropy production in compact forms. The correlation terms can be compactly cast as

$$\sigma_{14}^s = \sigma_{41}^s = (a_{11}N_1 + a_{12}N_2)/2, \quad (17a)$$

$$\sigma_{23}^s = \sigma_{32}^s = (a_{21}N_1 + a_{22}N_2)/2, \quad (17b)$$

$$\sigma_{34}^s = \sigma_{43}^s = (a_{31}N_1 + a_{32}N_2)/2. \quad (17c)$$

Therefore, we can express Π_s as

$$\Pi_s = \Pi_0 + \Pi_1, \quad (18)$$

where

$$\Pi_0 = G(a_{12} + a_{11} + a_{22} + a_{21}) - 4\Lambda(a_{32} + a_{31}), \quad (19a)$$

$$\Pi_1 = \left(\frac{Ga_{11} - 4\Lambda a_{31}}{N_2} - \frac{Ga_{22}}{N_2}\right)(N_1 - N_2). \quad (19b)$$

Equations (18) and (19) are the main results of this study. In the decomposition of Π_s, Π_0 can be regarded as the vacuum part of the entropy production, which is not dependent on the mean excitation number of the baths and is not zero even for zero-temperature baths ($N_1 = N_2 = 1$).

For $N_1 = N_2 = 1$, we can express entropy production as $\Pi_0 = \sum_{k=1}^{3} j_k$, where

$$j_1 = \left\langle \frac{d}{dt}a^\dagger a \right\rangle = i\langle[H, a^\dagger a]\rangle = 2G\sigma_{14}^s, \quad (20)$$

$$j_2 = \left\langle \frac{d}{dt}b^\dagger b \right\rangle = i\langle[H, b^\dagger b]\rangle$$
$$= 2G\sigma_{23}^s - (\Omega - \omega)\sigma_{34}^s, \quad (21)$$

$$j_3 = -(\Omega - \omega)\sigma_{34}^s. \quad (22)$$

In these expressions, j_1 and j_2 are known as excitation currents due to node-node coupling in the lattices [23]. In the current system, the counterrotating terms of the interaction

Hamiltonian $(a^\dagger b^\dagger + \text{H.c.})$ do not conserve the number of excitations in the system. Hence, the unitary internal dynamics maintains a stationary current within the system, which is nonzero for $N_1 = N_2 = 1$ and $\Lambda = 0$. In other words, these two currents are related to intersystem correlations σ_{14} and σ_{23} that can be quantified by mutual information.

To explain the additional current j_3, we should note that any squeezed state is characterized by the mean excitation number $\langle b^\dagger b \rangle_s$ and two-excitation correlation functions $\langle b^2 \rangle_s$ and $\langle b^{\dagger 2} \rangle_s$. j_3 is related to the second property of the squeezed states.

Also, Π_1 is the part of the entropy production that arises from the temperature difference between the two baths (more exactly, the inverse Bose-Einstein occupations $N_i = 2\bar{n}_i + 1$). The expression for Π_1 is very similar to what is obtained in Ref. [23] for entropy production of a general linear network of harmonic nodes which are coupled to multiple heat baths at different temperatures. In this study, we have an additional term for the entropy production that arises from vacuum fluctuations of the baths. Also, the heat current between the two baths is affected by the parametric amplification strength Λ.

We can reexpress Π_1 as

$$\Pi_1 = (J_{12} + J_{12}')\left(\frac{1}{N_2} - \frac{1}{N_1}\right), \quad (23)$$

where $J_{12} = Ga_{11}(N_1 - N_2)$ is the energy current due to the temperature gradient between the baths. The additional current $J_{12}' = -(\Omega - \omega_m)(N_1 - N_2)a_{31}$ arises from temperature gradient and frequency gradient $(\Omega - \omega)$. As can be seen in the drift matrix of Eq. (8), due to the parametric amplification process, the quadratures of bosonic mode 2 oscillate at different frequencies ω and Ω, and this difference produces "parametric current" J_{12}' within the system. Since a_{11} and a_{31} are always positive (see the Appendix for details), the two currents have opposite signs. Hence, for $N_1 > N_2$, we have $J_{12} > 0$, which describes the heat flow from the hot thermal bath to the cold thermal bath. Also, the parametric current J_{12}' is the current in the opposite direction from the cold thermal bath to the hot thermal bath. Also, since $Ga_{11} > (\Omega - \omega)a_{31}$ (see the Appendix for details), the total energy current ($J = J_{12} + J_{12}'$) has the same sign as $N_1 - N_2$, and Π_1 is always positive.

It is easy to show that for $\Lambda = 0$ and in the classical regime where $\hbar\omega \ll k_B T_2$ and $\Delta \approx \omega$, Π_1 takes the following familiar form:

$$\Pi_1 \approx J_0\left(\frac{1}{T_2} - \frac{1}{T_1}\right), \quad (24)$$

which is the Onsager entropy production between two systems kept at different temperatures [1]. The heat current in this regime is given by $J_0 = \frac{\hbar\omega Ga_0(N_1 - N_2)}{2k_B}$ (the exact expression for $a_0 > 0$ is given in the Appendix).

It is obvious that for $N_i \gg 1$ ($i = 1, 2$), the Onsager part of the entropy production Π_1 is more important than the vacuum part Π_0. And for $N_2 = N_1$, Π_0 is the only part of the entropy production.

To get more insights into the vacuum entropy production Π_0, we decompose Π_0 into four terms,

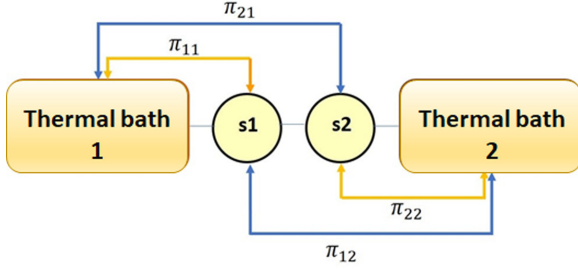$$\Pi_0 = \pi_{11} + \pi_{22} + \pi_{12} + \pi_{21}, \quad (25)$$

FIG. 2. Different components of the vacuum entropy production. The entropy production of each subsystem is affected by local and nonlocal thermal baths.

where π_{ii} are the parts of the vacuum entropy production due to the coupling between mode i ($i = 1, 2$) and its local bath i and $\pi_{ij\neq i}$ are the parts of the vacuum entropy production due to the coupling between mode i ($i = 1, 2$) and its nonlocal bath $j \neq i$. These terms are given by

$$\pi_{12} = Ga_{22}, \quad \pi_{2,1} = Ga_{11} - 4\Lambda a_{31}, \quad (26a)$$

$$\pi_{11} = Ga_{21}, \quad \pi_{2,2} = Ga_{12} - 4\Lambda a_{32}. \quad (26b)$$

To explain the naming of these terms, we should go back to Eqs. (15) and (17) to express the mean excitation number of each oscillator in terms of input-noise correlations. According to these equations, the mean excitation number of each mode in the steady state (N_i^s) is given by

$$N_1^s = \sigma_{11}^s + \sigma_{22}^s$$
$$= \left(1 + \frac{Ga_{21}}{2\kappa}\right)N_1 + \frac{Ga_{22}}{2\kappa}N_2, \quad (27a)$$

$$N_2^s = \sigma_{33}^s + \sigma_{44}^s$$
$$= \left(1 + \frac{Ga_{12} - 4\Lambda a_{32}}{2\gamma}\right)N_2 + \frac{Ga_{11} - 4\Lambda a_{31}}{2\gamma}N_1, \quad (27b)$$

and hence, the mean excitation number of mode i is affected by the input-noise correlations of its local thermal bath (N_i) and input-noise correlations of its nonlocal thermal bath ($N_{j\neq i}$). Therefore, Ga_{21} quantifies the effect of local bath 1 on the mean excitation number of mode 1, Ga_{22} quantifies the effect of local bath 2 on the mean excitation number of mode 1, and so on (Fig. 2).

It is easy to show that $Ga_{11} - 4\Lambda a_{31} = Ga_{22} > 0$, and hence, we have $\pi_{12} = \pi_{21} > 0$ (see the Appendix for more details). On the other hand, π_{11} and π_{22} are not equal in general, and depending on the parameter values can be positive, negative, or zero.

Another interesting point that should be noted here is that for $N_1 \neq N_2$, we have $J_{12} + J'_{12} = \pi_{2,1}(N_1 - N_2)$; that is, the heat current is related to only the coupling between mode i and its nonlocal thermal bath $j \neq i$ in the steady state.

IV. NUMERICAL CALCULATION OF THE STEADY-STATE VACUUM ENTROPY PRODUCTION

To get additional insights into the analytical results of the previous section, we present some numerical results for Π_0 and its components in different regimes.
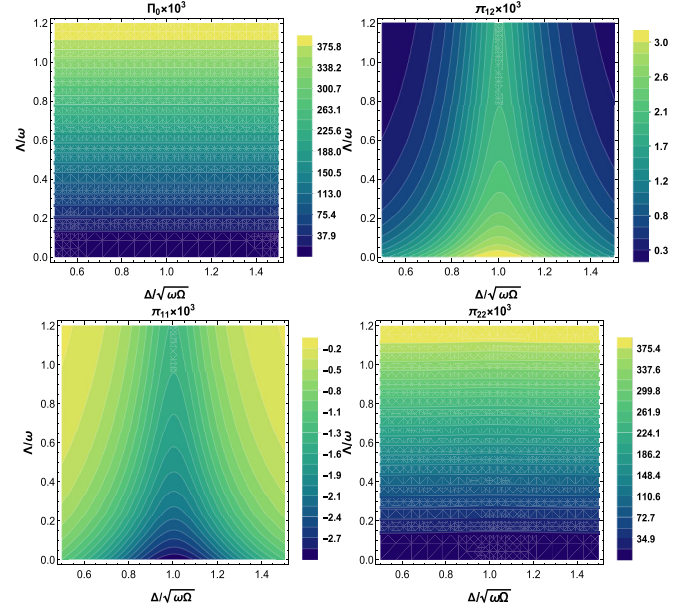

FIG. 3. Vacuum entropy production rate and its components for $N_m = N_f$ versus $\Delta/\sqrt{\Omega\omega}$ and Λ/ω. The parameters are $\kappa = \gamma = 0.2\omega$, $G = 0.05\omega$, $\omega = 1$.

In Fig. 3, Π_0 and its components are plotted as a function of $\Delta/\sqrt{\Omega\omega}$ and Λ/ω for $G < \kappa, \omega$. As can be seen, π_{12} maximizes at the resonance condition $\Delta = \sqrt{\Omega\omega}$. This is the frequency at which the beam-splitter-like term of the interaction Hamiltonian $g(a^\dagger b + ab^\dagger)$ is pronounced and the subsystems exchange excitations (note that this is also the frequency at which the total heat current maximizes).

In this regime, π_{11} makes a very small negative contribution to the total vacuum entropy production. However, π_{22} makes the main positive contribution to the total vacuum entropy production. It is easy to show that in this regime π_{22} is approximately given by

$$\pi_{22} \simeq -4\Lambda a_{32} \simeq \frac{8\gamma\Lambda^2}{\Gamma^2}, \quad (28)$$

which is always positive. The positivity of π_{22} in this range is related to the negativity of the self-correlation term $\sigma_{34}^s \simeq a_{32}/2$ and squeezing of subsystem 2 in q [note that according to Eq. (10) we have $\sigma_{33}^s < 1/2$ in this regime]. Therefore, self-correlation between the quadratures of the second mode is responsible for pushing the second mode away from the thermal equilibrium state toward the squeezed state and increasing the vacuum entropy production of the total system.

Figure 4 shows the vacuum entropy production and its components for $\kappa < G < \omega$ for Δ close to the resonance frequency. It shows that in the range of parameters in which π_{12} is maximum ($\Delta \simeq \sqrt{\Omega\omega}$ and $\Lambda < 0.2\omega$), π_{11} and π_{22} are negative. This implies that the contribution of each bath to the vacuum entropy production is twofold in this range of parameters. First, it pulls its local oscillator toward the equilibrium state (by decreasing π_{ii}, $i = 1, 2$) and pushes the other one away from the thermal equilibrium state (by increasing $\pi_{ij\neq i}$). In other words, due to the coupling between the subsystems, each subsystem is pushed away from thermal equilibrium, and its local bath pulls it toward an equilibrium state. As Λ gets
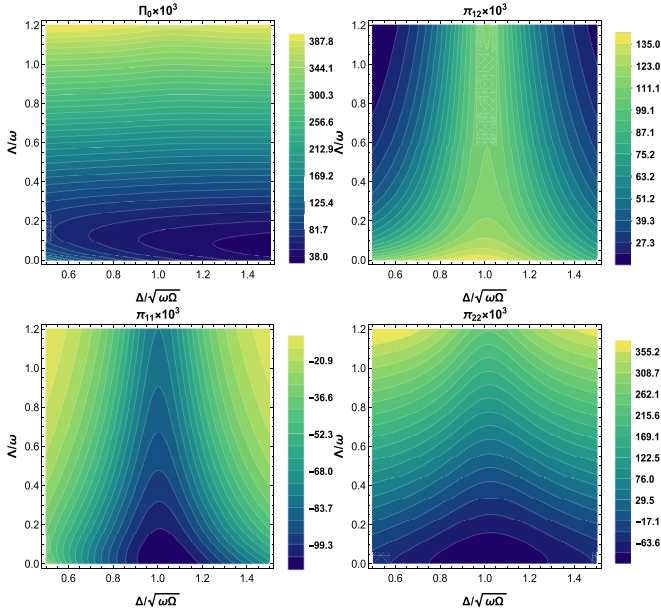
FIG. 4. Vacuum entropy production rate and its components versus $\Delta/\sqrt{\Omega\omega}$ and Λ/ω close to the resonance frequency. The parameters are $\kappa = \gamma = 0.2\omega$, $G = 0.5\omega$, $\omega = 1$.

larger, π_{12} becomes smaller, and π_{22} becomes larger. Therefore, self-correlation and internal correlation compete with one another at resonance. Comparing the maximum values of Π_0 in Figs. 3 and 4, it can be seen that for large values of Λ, the role of coupling strength is very small.

Also, in both regimes, π_0 is nonzero for $\Lambda = 0$ and $\Delta = \omega$. In this regime (resonance condition, $\Lambda = 0$ and $\gamma = \kappa$), π_0 and its components are given by

$$\pi_0 = \frac{G^2 \Gamma_1^2 \gamma}{d_0}, \tag{29}$$

$$\pi_{12} = \frac{G^2 \Gamma_1^2 \gamma \left(4 d_0 + 4\gamma^2 \Gamma_1^2 + G^2 \omega^2\right)}{4 \left(G^2 \omega^2 + 4\gamma^2 \Gamma_1^2\right) d_0}, \tag{30}$$

$$\pi_{11} = \pi_{22} = \frac{G^2 \Gamma_1^2 \gamma \omega^2 \left(5 G^2 - 4\Gamma_1^2\right)}{4 \left(G^2 \omega^2 + 4\gamma^2 \Gamma_1^2\right) d_0}, \tag{31}$$

where $d_0 = \Gamma_1^4 - G^2 \omega^2 > 0$ as a result of the stability condition and $\Gamma_1 = \sqrt{\omega^2 + \gamma^2}$. As expected, π_{12} is always positive, and π_{11} ($=\pi_{22}$) is always negative. Therefore, the composite system remains in the nonequilibrium state because of the interaction. It is predictable that vacuum entropy production can be suppressed when the interaction Hamiltonian is subjected to the constraints of thermal operations ($[H_0, H_{\text{int}}] \neq 0$) [50–56]. A detailed discussion of this issue will be given elsewhere.

V. CONCLUSIONS

We have studied the entropy production rate for a composite Gaussian system. The modes are exposed to the effects of local thermal baths. One of the coupled modes is driven into the thermal squeezed state through the parametric amplification process. Our analytical expression for steady-state Wigner entropy production shows that it depends on the

internal correlations between the modes and self-correlation between noncommuting operators of the parametrically driven mode. The self-correlation squeezes the driven mode, produces a heat current from the cold bath to the hot bath, and reduces the total heat current between the reservoirs at different temperatures. Also, we have demonstrated that entropy production is nonzero, even when the heat flux between the baths is zero (i.e., when they have identical thermal occupation numbers or when they are in the vacuum state). By recognizing the local and nonlocal effects of the baths on each mode, we have shown that self-correlation enhances the local contribution of one bath and internal correlation increases the nonlocal contributions of both baths to the total entropy production and they compete with one another in the steady state. These results imply that, unlike previous claims [29–39], squeezing may limit the efficiency of nonequilibrium powerful heat engines in two ways. It reduces the heat current between the baths and increases the effect of vacuum fluctuations of the baths on the total entropy production.

It should be noted that, in the current study, the interaction Hamiltonian is not subjected to the constraints of thermal operations. It is predictable that considering the constraints of thermal operation reduces the vacuum entropy production and irreversible flows, but we leave this for future studies.

APPENDIX

As stated, the correlation terms can be compactly cast as

$$\sigma_{14}^s = \sigma_{41}^s = (a_{11} N_1 + a_{12} N_2)/2, \tag{A1a}$$

$$\sigma_{23}^s = \sigma_{32}^s = (a_{21} N_1 + a_{22} N_2)/2, \tag{A1b}$$

$$\sigma_{34}^s = \sigma_{43}^s = (a_{31} N_1 + a_{32} N_2)/2, \tag{A1c}$$

where the coefficients are given by

$$a_{11} = \frac{G\kappa\gamma}{d_1} [\delta^2 \Gamma^2 u_1 + \eta(\Gamma^2 \kappa_1 + \delta^2 \kappa_2)], \tag{A2a}$$

$$a_{12} = \frac{G\Delta\gamma}{\omega d_1} \left\{ \delta^2 \Gamma_1^2 \gamma u_1 - \eta[\kappa\kappa_1(\omega^2 + \omega\Omega) + \gamma(u + \kappa_1 \omega^2)] \right\}, \tag{A2b}$$

$$a_{21} = \frac{G\kappa\omega}{\Delta d_1} [\delta^2 \Gamma^2 \kappa u_1 - \eta(\delta^2 \kappa_2^2 + \kappa\gamma \Gamma^2)], \tag{A2c}$$

$$a_{22} = a_{11} - \frac{G\kappa\gamma}{d_1} (\omega\Omega - \omega^2)[\delta^2 u_1 + \eta(\gamma + \kappa)], \tag{A2d}$$

$$a_{31} = \frac{G^2 \kappa\gamma}{d_1} [\delta^2 u_1 + \eta(\gamma + \kappa)]\omega, \tag{A2e}$$

$$\begin{aligned} a_{32} = \frac{\gamma}{\omega d_1} \Big\{ & \delta^4 \Gamma_1^2 \gamma u_1 + \eta^2 (\gamma + \kappa)\kappa_1 \\ & - \eta\gamma\delta^2 [2u + \kappa_1 \omega^2 - \kappa(\delta^2 + \gamma\kappa_1)] \\ & - \eta\gamma\kappa(\omega\Omega - \omega^2)(\omega\Omega + \kappa_1^2) \\ & - \eta\omega\Omega\delta^2 (\gamma^2 + \kappa_1 \kappa) \Big\}, \end{aligned} \tag{A2f}$$

and we have defined

$$d_1 = 2\eta[-\eta(\gamma + \kappa)^2 + u_1(\gamma\delta^2 + \kappa\Gamma^2)], \tag{A3a}$$

$$u = \gamma^3 + 4\gamma^2\kappa + \delta^2\kappa + 4\gamma\kappa^2, \tag{A3b}$$

$$u_1 = u + \gamma\omega\Omega, \tag{A3c}$$

$$\delta = \sqrt{\Delta^2 + \kappa^2}, \tag{A3d}$$

$$\Gamma = \sqrt{\gamma^2 + \omega\Omega}, \tag{A3e}$$

$$\Gamma_1 = \sqrt{\gamma^2 + \omega^2}, \tag{A3f}$$

$$\kappa_1 = (\gamma + 2\kappa), \tag{A3g}$$

$$\kappa_2 = (2\gamma + \kappa). \tag{A3h}$$

Using the definition of η, we can express d_1 as

$$d_1 = 2\eta\{\kappa\gamma[(\gamma + \kappa)^4 + 2(\gamma + \kappa)^2(\Delta^2 + \omega\Omega) + (\Delta^2 - \omega\Omega)^2] + G^2\Delta\omega(\gamma + \kappa)^2\}, \tag{A4}$$

which is obviously positive (note that η is the stability parameter and is positive). Also, according to Eqs. (A2 a)–(A2 f), a_{11} and a_{31} are always positive, and we have

$$Ga_{22} = Ga_{11} - (\Omega - \omega)a_{31}. \tag{A5}$$

Hence, for $\Lambda = 0$, we have $a_{22} = a_{11} = a_0$, where

$$a_0 = \frac{G\kappa\gamma_m[\delta^2\Gamma_1^2 u_1 + \eta_1(\Gamma_1^2\kappa_1 + \delta^2\kappa_2)]}{2\eta_1[-\eta_1(\gamma + \kappa)^2 + u_1(\gamma\delta^2 + \kappa\Gamma_1^2)]}, \tag{A6}$$

where $\eta_1 = \delta^2\Gamma_1^2 - G^2\Delta\omega$ is the stability parameter for the system when $\Lambda = 0$.

For $\Lambda \neq 0$, the relation between correlation terms in Eq. (A5) leads to Eq. (23) for Π_1, where the sum of the currents, which is given by

$$J_{12} + J'_{12} = \frac{G^2\kappa\gamma_m}{d_1}(N_1 - N_2)\{\delta^2\Gamma_1^2 u_1 + \eta[\Gamma^2\kappa + \delta^2\kappa_2 + \Gamma_1^2(\kappa + \gamma)]\}, \tag{A7}$$

has the same sign as $N_1 - N_2$.

According to Eq. (26), the elements of Π_0 are given by

$$\pi_{12} = Ga_{22}, \quad \pi_{2,1} = Ga_{11} - 4\Lambda a_{31}, \tag{A8a}$$

$$\pi_{11} = Ga_{21}, \quad \pi_{2,2} = Ga_{12} - 4\Lambda a_{32}. \tag{A8b}$$

Using Eq. (A2) for a_{ij}, we can reexpress the elements of the vacuum entropy production as

$$\pi_{12} = \frac{G^2\kappa\gamma}{d_1}[\Gamma_1^2\delta^2 u_1 + \eta\kappa(\Gamma^2 + \Gamma_1^2 + \delta^2) + \eta\gamma(2\delta^2 + \Gamma_1^2)], \tag{A9}$$

$$\pi_{11} = \frac{G^2\kappa\omega}{\Delta d_1}[\delta^2\Gamma^2\kappa u_1 - \eta(\delta^2\kappa_2^2 + \kappa\gamma\Gamma^2)], \tag{A10}$$

$$\pi_{22} = \Pi^{(G=0)} + Ga_{12} + \frac{\gamma(\Omega - \omega)\Gamma_1^2}{2\Gamma^2\omega} - \frac{\gamma(\Omega - \omega)}{d_1\omega}y_1, \tag{A11}$$

where

$$y_1 = \{u_1\gamma\Gamma_1^2\delta^4 + \eta^2\kappa(\gamma + \kappa) + \eta\kappa\gamma\Gamma_1^2(\kappa_1^2 + \Omega\omega) - \eta\delta^2[\Gamma_1^2\gamma\kappa_1 + \Gamma^2\kappa(\gamma + \kappa)]\} \tag{A12}$$

and $\pi_{12} = \pi_{21}$ as a result of Eq. (A5).

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