Control power of high-dimensional controlled dense coding

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We quantitatively analyze and evaluate the control power in a high-dimensional noisy quantum channel for transmitting classical information. We calculate the control power of the controlled dense coding scheme in a four-dimensional extended Greenberger-Horne-Zeilinger (GHZ) class state channel. Different from controlled teleportation, there is no tradeoff between the control power and the classical capacity of the optimal four-dimensional GHZ state quantum channel. Only when the channel is noisy and the sender is allowed to gain some control authority can the tradeoff between the control power and the classical capacity be activated, and the overall control power for transmitting classical information will be enhanced by reducing the classical capacity of the high-dimensional quantum channel. This noise induced characteristic is very different from that of transmitting quantum information, and it may provide new ideas and methods to develop the resource theory of quantum channels.

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I. INTRODUCTION

Quantum networks [1–5] are critical to the realization of quantum communication [6–15]. By means of sharing entangled states [16] over the nodes of the network, quantum networks could enable large-scale secure communication, remote quantum control [17–20], and distributed quantum computing and simulation [21,22]. Controlled quantum communication [23–25] is an important branch of quantum networks. Controlled quantum communication adds a controller to the original two-party quantum communication, and the controller can be an independent third party or the sender of the information. Without the cooperation of the controller, the receiver will not be able to ensure that all information of the sender is vouchsafed. Therefore, controlled quantum communication can allow the client to permit or restrict successful quantum state transfer in the network to ensure the security of the client's secret information.

Quantum networks with controlled third parties were first discussed for controlled quantum teleportation (CQT) [23], which was proposed by Karlsson and Bourennane in 1998. CQT is useful in various contexts in quantum communication, including in quantum networks and cryptographic conferences [26–29]. In 2004, Yonezawa *et al.* experimentally demonstrated controlled quantum teleportation by using continuous variable optics [30], and the fidelity reached 64%. In 2019, Barasiński *et al.* experimentally demonstrated CQT on linear optics of discrete variables [31], and the fidelity was as high as 83%, which far exceeds the classical limitation. Recently, many works have been devoted to the analysis of controller authority in CQT schemes and return the control power (CP) P [32–37]. The CP is a quantity to define the authority of the controller, and it is estimated as the difference in conditioned and nonconditioned fidelity of the CQT [33]. Conditioned and nonconditioned fidelity are two quantities of the fidelity of teleportation of a quantum state with and without the permission of the controller, respectively. In CQT schemes, the controller's authority, as well as the fidelity of the final state of teleportation, should be ensured.

Different from quantum teleportation, which transmits quantum information, quantum dense coding [10,15,38] provides an absolutely secure technique that transmits private classical information between legitimate parties. If a maximally entangled state is initially shared by the sender and receiver, quantum dense coding allows one to transmit 2 bits of classical information by sending only one qubit. It has been studied theoretically [39–43] and experimentally [44]. Controlled dense coding (CDC) was first proposed by Hao *et al.* in 2001 [38]. Subsequently, many CDC schemes for different states have been proposed [45–48]. In 2017 [49], Oh *et al.* quantitatively analyzed the minimal control power for CDC using three-qubit entangled-state channels, and the control power was defined by the channel capacity. Thus far, the control power of CDC has been mainly investigated in two-dimensional multiparticle entangled quantum channels; however, the control power for transmitting classical information by using high-dimensional quantum channels has not been discussed in detail.

In this paper, we calculate the control power of the CDC scheme in a four-dimensional extended Greenberger-Horne-Zeilinger (GHZ) class state channel. Different from controlled teleportation, there is no tradeoff between the control power and the classical capacity of the optimal four-dimensional GHZ state quantum channel. In the optimal channel, even if the classical capacity is intentionally reduced, the control power will not change for controlled transmission of classical information as the subspace of a perfect channel is symmetric. When the channel is noisy and allows the sender to gain some control authority, because noise destroys the symmetry

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of the channel, the tradeoff between the control power and the classical capacity can be activated. The overall control power for transmitting classical information will be enhanced by reducing the classical capacity of a high-dimensional quantum channel. Specifically, when using standard Bell-state basis measurement for decoding, the maximum control power can be increased by 0.866 bits, and when the control power is calculated by the channel capacity the maximum control power can be increased by 0.544 16 bit. This noise-induced characteristic has not been discussed in previous studies and it is very different from that of transmitting quantum information. Therefore, our paper provides a feasible approach for obtaining higher control power, and may provide new ideas and methods to develop the resource theory of quantum channels.

II. CP OF OPTIMAL CHANNELS

We first construct a high-dimensional controlled dense coding scenario. The sender Alice, the receiver Bob, and the controller Charlie initially share a standard four-dimensional GHZ state $|\Psi_0\rangle_{ABC} = \frac{1}{2}(|000\rangle + |111\rangle + |222\rangle + |333\rangle)_{ABC}$, and the four-dimensional qudits A, B, and C belong to Alice, Bob, and Charlie, respectively. Alice encodes classical information by performing a four-dimensional local operation on her qudit and then sends it to Bob. At the same time, Charlie measures his qudit on an optimal basis and broadcasts the measurement outcome to Alice and Bob. After receiving Alice's qudit, Bob performs a two-qudit measurement, such as a four-dimensional Bell-state analysis (BSA) measurement, so that he can recover the classical information that Alice wants to transmit with Charlie's measurement result. If Charlie does not disclose his measurement results to Alice and Bob, Bob cannot recover all the classical information sent by Alice, and the classical capacity that Alice and Bob achieve will be decreased. Therefore, one can define Charlie's control power as the difference between the classical capacity of the quantum channel with and without Charlie's participation [49]. That is,

$$P = C_{Cp} - C_{Cnp} \tag{1}$$

where C_{Cp} is the capacity of the channel when Charlie cooperates with Bob, and C_{Cnp} is the capacity of the channel when Charlie does not cooperate with Bob. In the optimal channel, $C_{Cp} = 2\log_2 d$ bits in the d-dimensional dense coding scheme, and C_{Cnp} is the number of correct bits transmitted by the quantum channel without Charlie's participation. Therefore, the control power P is the number of error bits obtained by Bob without cooperation of Charlie.

A. CP for the original CDC scheme

In this four-dimensional controlled dense coding scenario, let us calculate the control power for transmitting 4 bits of classical information in an optimal quantum channel. A four-dimensional qudit Q can be denoted as a system composed of two qubits Q_1 and Q_2. The state is correspondingly given by

$$\begin{aligned} &|00\rangle_{Q_1Q_2} \to |0\rangle_Q, |01\rangle_{Q_1Q_2} \to |1\rangle_Q, \\ &|10\rangle_{Q_1Q_2} \to |2\rangle_Q, |11\rangle_{Q_1Q_2} \to |3\rangle_Q. \end{aligned} \tag{2}$$

One can construct a standard four-dimensional GHZ state channel by two parallel two-dimensional quantum channels with two standard GHZ states. The quantum channel $|\Psi_0\rangle_{ABC}$ can be rewritten in the following form:

$$\begin{aligned} |\Psi_0\rangle_{ABC} &= |\psi_0\rangle_{A_1B_1C_1} \otimes |\psi_0\rangle_{A_2B_2C_2} \\ &= \tfrac{1}{2}(|000\rangle + |111\rangle)_{A_1B_1C_1} \otimes (|000\rangle + |111\rangle)_{A_2B_2C_2} \\ &= \tfrac{1}{2}(|++\rangle_{C_1C_2}|\phi^+\rangle_{A_1B_1} \otimes |\phi^+\rangle_{A_2B_2} \\ &\quad + |-+\rangle_{C_1C_2}|\phi^-\rangle_{A_1B_1} \otimes |\phi^+\rangle_{A_2B_2} \\ &\quad + |+-\rangle_{C_1C_2}|\phi^+\rangle_{A_1B_1} \otimes |\phi^-\rangle_{A_2B_2} \\ &\quad + |--\rangle_{C_1C_2}|\phi^-\rangle_{A_1B_1} \otimes |\phi^-\rangle_{A_2B_2}) \end{aligned} \tag{3}$$

where $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$ and $|\phi^\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$. Alice encodes 4 bits $n_1n_2n_3n_4$ of classical information by performing a four-dimensional local operation $U_{n_1n_2n_3n_4}$ on her qudit and then sends it to Bob. The corresponding relationship between Alice's operations and $n_1n_2n_3n_4$ she wants to transmit and the corresponding Bell states $|\Psi\rangle_{n_1n_2n_3n_4}$ Bob receives is shown in Table I. In addition, the corresponding Bell states Bob receives are $|\Psi\rangle_{n_1n_2n_3n_4} = U_{n_1n_2n_3n_4}|\Psi_0\rangle_{ABC}$.

TABLE I. The corresponding relationship between Alice's operations $U_{n_1n_2n_3n_4}$ and the 4 bits $(n_1n_2n_3n_4)$ of classical information she wants to transmit. Here, $I_z = |0\rangle\langle 0| + |1\rangle\langle 1|$ is the identity matrix. $\sigma_z^{A_1(A_2)} = |0\rangle_{A_1(A_2)}\langle 0| - |1\rangle_{A_1(A_2)}\langle 1|$ is the phase operation performed on a qubit A_1 (A_2). $\sigma_x = \sum_{i=0}^{3} |i+1\rangle\langle i|$ (mod 4) is the qudit-flip operation performed on the four-dimensional qudit A which is composed of two qubits A_1 and A_2. The superscript 1, 2, or 3 of the operator σ_x represents the number of times that the operator σ_x acts on qudit A.

$U_{n_1n_2n_3n_4}$	$n_1n_2n_3n_4$	$U_{n_1n_2n_3n_4}$	$n_1n_2n_3n_4$
$I_{A_1}I_{A_2}$	0000	σ_x^2	1000
$\sigma_z^{A_1}I_{A_2}$	0101	$\sigma_z^{A_1}\sigma_x^2$	1101
$I_{A_1}\sigma_z^{A_2}$	1010	$\sigma_z^{A_2}\sigma_x^2$	0010
$\sigma_z^{A_1}\sigma_z^{A_2}$	1111	$\sigma_z^{A_1}\sigma_z^{A_2}\sigma_x^2$	0111
σ_x^1	0100	σ_x^3	1100
$\sigma_z^{A_1}\sigma_x^1$	0001	$\sigma_z^{A_1}\sigma_x^3$	1001
$\sigma_z^{A_2}\sigma_x^1$	1110	$\sigma_z^{A_2}\sigma_x^3$	0110
$\sigma_z^{A_1}\sigma_z^{A_2}\sigma_x^1$	1011	$\sigma_z^{A_1}\sigma_z^{A_2}\sigma_x^3$	0011

If qubits C_1 and C_2 are both measured on the $|\pm\rangle$ basis, the state of the whole system becomes

$$\begin{aligned} |\Psi_0\rangle_{ABC} = \tfrac{1}{2}[&|\Phi\rangle_{0000}|{++}\rangle_{C_1C_2} + |\Phi\rangle_{0101}|{-+}\rangle_{C_1C_2} \\ &+ |\Phi\rangle_{1010}|{+-}\rangle_{C_1C_2} + |\Phi\rangle_{1111}|{--}\rangle_{C_1C_2}]. \end{aligned} \tag{4}$$

Alice uses $|\Psi_0\rangle_{ABC}$ to transfer the 4-bit information group $n_1n_2n_3n_4$. After Alice performs the local operations, the state $|\Psi_0\rangle_{ABC}$ becomes

$$\begin{aligned} \tfrac{1}{2}(&|\Psi_{n_1n_2n_3n_4}\rangle_{AB}|{++}\rangle_C + |\Psi_{n_1\overline{n_2}n_3\overline{n_4}}\rangle_{AB}|{-+}\rangle_C \\ &+ |\Psi_{\overline{n_1}n_2\overline{n_3}n_4}\rangle_{AB}|{+-}\rangle_C + |\Psi_{\overline{n_1n_2n_3n_4}}\rangle_{AB}|{--}\rangle_C) \end{aligned} \tag{5}$$

where $\overline{n_i} = 1 - n_i$.

If Charlie cooperates with Bob, the last three terms in Eq. (5) can be amended to the first term, and Bob can correctly infer the 4 bits of information transmitted by Alice. In this case, $C_{Cp} = 4$ bits, which represents the capacity of the channel when Charlie cooperates with Bob.

TABLE II. The corresponding relationship between Alice's DROs and the DUOs Bob performed.

Alice's DROs	Bob's DUOs
$\vert 3\rangle \to \vert 0\rangle$	$\vert 03\rangle \to \vert 33\rangle,\ \vert 02\rangle \to \vert 32\rangle,\ \vert 13\rangle \to \vert 03\rangle,\ \vert 01\rangle \to \vert 31\rangle,\ \vert 12\rangle \to \vert 02\rangle,\ \vert 23\rangle \to \vert 13\rangle$
$\vert 3\rangle \to \vert 1\rangle$	$\vert 13\rangle \to \vert 33\rangle,\ \vert 02\rangle \to \vert 32\rangle,\ \vert 23\rangle \to \vert 03\rangle,\ \vert 01\rangle \to \vert 31\rangle,\ \vert 12\rangle \to \vert 02\rangle,\ \vert 03\rangle \to \vert 13\rangle$
$\vert 3\rangle \to \vert 2\rangle$	$\vert 23\rangle \to \vert 33\rangle,\ \vert 02\rangle \to \vert 32\rangle,\ \vert 01\rangle \to \vert 31\rangle,\ \vert 12\rangle \to \vert 02\rangle$

If Charlie does not cooperate with Bob, there are always some items in Eq. (5) that will be wrong. Bob only has a 25% probability of obtaining 4 bits of correct information, he has a 50% probability of obtaining only 2 bits of correct information, and then Bob will not be able to obtain any correct information with 25% probability. The effective number of correct bits that Bob can obtain in this case is

$$C_{Cnp} = 4 \text{ bits} \times 25\% + 2 \text{ bits} \times 50\% = 2 \text{ bits} \quad (6)$$

and, as the total number of the bits transmitted is 4 bits, the average number of error bits obtained by Bob is 2 bits. The corresponding control power, defined as the difference between the capacity of Charlie with and without participation, is $P = C_{Cp} - C_{Cnp} = 2$ bits. Therefore, the control power P is equal to the number of error bits obtained by Bob without cooperation of Charlie.

B. CP for the decentralized CDC scheme

Alice can reduce the bit number of classical information transmitted by reducing the dimensionality of the qudit belonging to her. For example, Alice performs phase encoding first and then reduces the dimensionality of the particles in her hand. She can perform one of the following dimensionality reduction operations (DROs) on the qudit A: $|3\rangle \to |0\rangle$, $|3\rangle \to |1\rangle$, or $|3\rangle \to |2\rangle$; she then performs a three-dimensional bit flip operation for encoding on the qudit. At the same time, Charlie measures his qudit on an optimal basis and broadcasts the measurement outcome to Alice and Bob. After receiving Alice's qudit, Bob performs an appropriate dimensionality upgrading operation (DUO) according to Alice's dimensionality reduction operation and then performs a four-dimensional BSA measurement so that he can recover the classical information that Alice wants to transmit with Charlie's measurement result. The corresponding relationship between Alice's dimensionality reduction operations and the dimensionality upgrading operations Bob performed is shown in Table II.

When both Alice and Charlie cooperate with Bob, Bob can obtain the $\log_2 12$ bits of classic information delivered by Alice. If Alice and Charlie do not disclose their information to Bob, Bob cannot recover all the classical information sent by Alice; that is, Alice is assigned a part of the control authority. We call this scheme as a decentralized CDC scheme.

In this case, Bob only has a probability of 1/3 on average to perform the right dimensionality upgrading operation on his qudit, and the whole state of the three-qudit channel system becomes the form described in Eq. (5). The average number of error bits obtained by Bob is 2 bits. Otherwise, Bob has a probability of 2/3 on average to perform the incorrect dimensionality upgrading operation, and the average number of error bits obtained by Bob is also 2 bits. For example, when Alice wants to transmit 0000 to Bob, if Bob performs the incorrect dimensionality upgrading operation, the complete state of the three-qudit channel system becomes

$$\begin{aligned}
&\tfrac{1}{2}(|000\rangle + |111\rangle + |222\rangle + |133\rangle)_{ABC} \\
&\quad = \tfrac{1}{8}[|{+}{+}\rangle_C(3|\Phi_{0000}\rangle_{AB} + |\Phi_{0101}\rangle_{AB} + |\Phi_{1010}\rangle_{AB} - |\Phi_{1111}\rangle_{AB} + |\Phi_{1000}\rangle_{AB} - |\Phi_{1101}\rangle_{AB} - |\Phi_{0010}\rangle_{AB} + |\Phi_{0111}\rangle_{AB}) \\
&\qquad + |{+}{-}\rangle_C(|\Phi_{0000}\rangle_{AB} + 3|\Phi_{0101}\rangle_{AB} - |\Phi_{1010}\rangle_{AB} + |\Phi_{1111}\rangle_{AB} - |\Phi_{1000}\rangle_{AB} + |\Phi_{1101}\rangle_{AB} + |\Phi_{0010}\rangle_{AB} - |\Phi_{0111}\rangle_{AB}) \\
&\qquad + |{-}{+}\rangle_C(|\Phi_{0000}\rangle_{AB} - |\Phi_{0101}\rangle_{AB} + 3|\Phi_{1010}\rangle_{AB} + |\Phi_{1111}\rangle_{AB} - |\Phi_{1000}\rangle_{AB} + |\Phi_{1101}\rangle_{AB} + |\Phi_{0010}\rangle_{AB} - |\Phi_{0111}\rangle_{AB}) \\
&\qquad + |{-}{-}\rangle_C(-|\Phi_{0000}\rangle_{AB} + |\Phi_{0101}\rangle_{AB} + |\Phi_{1010}\rangle_{AB} + 3|\Phi_{1111}\rangle_{AB} + |\Phi_{1000}\rangle_{AB} - |\Phi_{1101}\rangle_{AB} - |\Phi_{0010}\rangle_{AB} + |\Phi_{0111}\rangle_{AB})]
\end{aligned}$$

or (7)

$$\begin{aligned}
&\tfrac{1}{2}(|000\rangle + |111\rangle + |222\rangle + |033\rangle)_{ABC} \\
&\quad = \tfrac{1}{8}[|{+}{+}\rangle_C(3|\Phi_{0000}\rangle_{AB} + |\Phi_{0101}\rangle_{AB} + |\Phi_{1010}\rangle_{AB} - |\Phi_{1111}\rangle_{AB} + |\Phi_{0100}\rangle_{AB} - |\Phi_{0001}\rangle_{AB} - |\Phi_{1110}\rangle_{AB} + |\Phi_{1011}\rangle_{AB}) \\
&\qquad + |{+}{-}\rangle_C(|\Phi_{0000}\rangle_{AB} + 3|\Phi_{0101}\rangle_{AB} - |\Phi_{1010}\rangle_{AB} + |\Phi_{1111}\rangle_{AB} - |\Phi_{0100}\rangle_{AB} + |\Phi_{0001}\rangle_{AB} + |\Phi_{1110}\rangle_{AB} - |\Phi_{1011}\rangle_{AB}) \\
&\qquad + |{-}{+}\rangle_C(|\Phi_{0000}\rangle_{AB} - |\Phi_{0101}\rangle_{AB} + 3|\Phi_{1010}\rangle_{AB} + |\Phi_{1111}\rangle_{AB} - |\Phi_{0100}\rangle_{AB} + |\Phi_{0001}\rangle_{AB} + |\Phi_{1110}\rangle_{AB} - |\Phi_{1011}\rangle_{AB}) \\
&\qquad + |{-}{-}\rangle_C(-|\Phi_{0000}\rangle_{AB} + |\Phi_{0101}\rangle_{AB} + |\Phi_{1010}\rangle_{AB} + 3|\Phi_{1111}\rangle_{AB} + |\Phi_{0100}\rangle_{AB} - |\Phi_{0001}\rangle_{AB} - |\Phi_{1110}\rangle_{AB} + |\Phi_{1011}\rangle_{AB})]. \quad (8)
\end{aligned}$$

In either case, the average number of error bits Bob obtains is 2 bits. Combining the above three situations, the control power for transmitting $\log_2 12$ bits of classical information is the same as that for transmitting 4 bits of classical information. One can see that in the perfect channel, even if the classical capacity is intentionally reduced by the sender, the control power will not be improved due to the symmetry of control. The manifestation of Charlie's control power leads

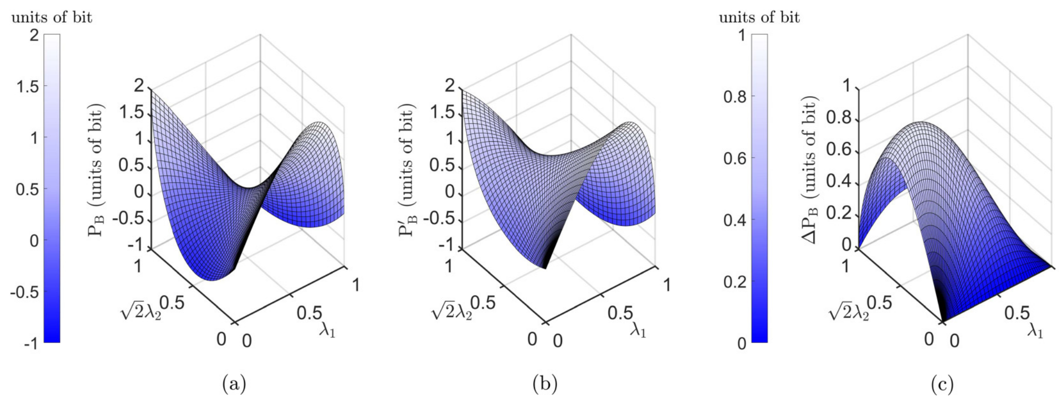

FIG. 1. Control power of four-dimensional channels as a function of the channel parameters to λ_1 and λ_2. (a) P_B is the control power for the original CDC scheme in the case of $\lambda_2 = \lambda_4$. (b) P_B' is the control power for the decentralized CDC scheme in the case of $\lambda_2 = \lambda_4$. (c) The change $\Delta P_B = P_B' - P_B$ in the case of $\lambda_2 = \lambda_4$.

to uncertainty of the subspace in the subsystem shared by Alice and Bob, and Alice's control power leads to uncertainty of measurement basis in the subspace. These uncertainties will lead to errors in decoding. The subspace of a perfect channel is symmetric, and the error caused by the uncertainty of the measurement basis is the same in each subspace and will not induce a higher average number of error bits. If this symmetry is broken, the control power may be improved. Therefore, there is no tradeoff between the control power and the classical capacity of the four-dimensional standard GHZ state quantum channel.

III. CP OF NOISY CHANNELS

In a noisy channel, maximally entangled states usually become partially entangled states. The standard two-dimensional GHZ state may become an extended GHZ-class state channel, which is defined as $|\psi_{e-\mathrm{GHZ}}\rangle_{ABC} = \lambda_1|000\rangle + \lambda_2|110\rangle + \lambda_3|111\rangle + \lambda_4|001\rangle$. The coefficient satisfies $\lambda_i \geqslant 0$ and $\sum_i \lambda_i^2 = 1$, $\{i = 1, 2, 3, 4\}$. The form of the state of two parallel extended GHZ-class state channels is shown as

$$\begin{aligned}|\psi_{4-e\mathrm{GHZ}}\rangle_{ABC} &= \lambda_1^2|000\rangle + \lambda_1\lambda_2|110\rangle + \lambda_1\lambda_3|111\rangle + \lambda_1\lambda_4|001\rangle \\ &+ \lambda_1\lambda_2|220\rangle + \lambda_2^2|330\rangle + \lambda_2\lambda_3|331\rangle + \lambda_2\lambda_4|221\rangle \\ &+ \lambda_1\lambda_3|222\rangle + \lambda_2\lambda_3|332\rangle + \lambda_3^2|333\rangle + \lambda_3\lambda_4|223\rangle \\ &+ \lambda_1\lambda_4|002\rangle + \lambda_2\lambda_4|112\rangle + \lambda_3\lambda_4|113\rangle + \lambda_4^2|003\rangle \end{aligned} \quad (9)$$

where the symmetry of the channel is broken, and the control power of this state will be improved by using a decentralized CDC scheme.

In the original CDC scheme, with the channel in the form of Eq. (9) and Charlie's assistance, by using the standard four-dimensional BSA measurement, the average number of error bits obtained by Bob is $E_0 = 2(\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_3 + \lambda_3^2 - 2\lambda_2\lambda_4 + \lambda_4^2)$. Without Charlie's assistance, the average number of error bits obtained by Bob is $E_1 = 2(\lambda_1 - \lambda_2)^2 + 2(\lambda_3 - \lambda_4)^2$. The corresponding control power of Charlie is $P_B = E_1 - E_0$. In the decentralized CDC scheme, with the same channel state as Eq. (9) but without Alice's and Charlie's assistance, the average number of error bits obtained by Bob is $E_1' = 2/3[3\lambda_1^4 - 6\lambda_1^3\lambda_2 + 3\lambda_2^4 + 3\lambda_3^4 - 2\lambda_3^3\lambda_4 + 6\lambda_3^2\lambda_4^2 - 6\lambda_3\lambda_4^3 + 3\lambda_4^4 + 6\lambda_1^2(\lambda_2^2 + \lambda_3^2 - \lambda_3\lambda_4 + \lambda_4^2) - 2\lambda_1\lambda_2(\lambda_2^2 + \lambda_3^2 + 3\lambda_4^2) + \lambda_2^2(6\lambda_3^2 - 2\lambda_3\lambda_4 + 6\lambda_4^2)]$. Therefore, the corresponding control power of Charlie for the decentralized CDC scheme is $P_B' = E_1' - E_0$.

To clearly illustrate the performance of CP in noisy channels, we numerically calculated P_B and P_B' in the case of $\lambda_2 = \lambda_4$. The results are shown in Figs. 1(a) and 1(b). The maximum values of P_B and P_B' occur at the same two points, (1) $\lambda_1 = \frac{1}{\sqrt{2}}$, $\lambda_2 = 0$ and (2) $\lambda_1 = 0$, $\lambda_2 = \frac{1}{\sqrt{2}}$, which also correspond to the cases of the optimal states. That is, under the action of noise, the control power of the channel is decreasing. However, only when the channel is noisy can the tradeoff between the control power and the classical capacity be activated.

To present this noise induced characteristic, we also give the change $\Delta P_B = P_B' - P_B$ in control power as a function of the parameters λ_1 and λ_2, as shown in Fig. 1(c). For the optimal state cases, or the nonoptimal state case of $\lambda_2 = \lambda_4 = 0$, the corresponding $\Delta P_B = 0$, otherwise $\Delta P_B > 0$. That is, the control power can be enhanced by reducing the dimension of the classical information in a suitably noisy quantum channel. The maximum value $\Delta P_B^{\max} = \sqrt{3}/2 = 0.866$ bit occurs at the point $\lambda_1 = 1/4$ and $\lambda_2 = \sqrt{3}/4$.

We notice that the control power has negative values in some areas of Fig. 1. This is an unreasonable result caused by Bob's inappropriate decoding operation. For example, when $\lambda_1 = 0$, $\lambda_2 = \lambda_4 = 0.3$, and $\lambda_3 = 0.906$, the channel state collapses to $[|+\rangle_{C_1}(1.206|00\rangle + 0.3|11\rangle)_{A_1B_1} - |-\rangle_{C_1}(0.606|00\rangle + 0.3|11\rangle)_{A_1B_1}]$ (non-normalized). Obviously, by using this state as a channel, the decoding strategy with Charlie's assistance (described in Sec. II) will cause more errors; conversely, decoding the classical information directly according to Bob's measurement results without Charlie's assistance will lead to a lower error rate. Therefore, P_B and P_B' are smaller than zero. To avoid this unreasonable situation, we calculate the CP using the quantum mutual information theorem [50,51] in which the entanglement-assisted classical capacity is defined as the maximum asymptotic rate of reliable bit transmission with the help of unlimited prior entanglement between the sender and receiver.

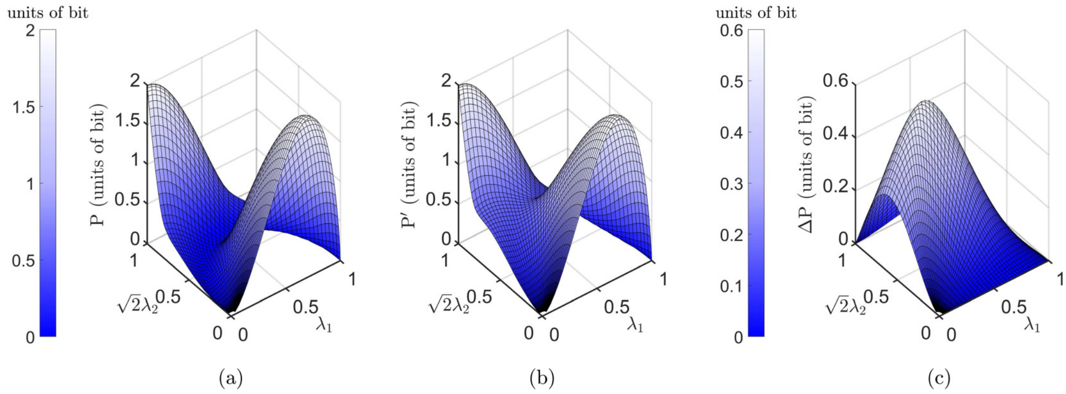

FIG. 2. Control power of four-dimensional channels as a function of the channel parameters to λ_1 and λ_2. (a) P is the control power for the original CDC scheme in the case of $\lambda_2 = \lambda_4$. (b) P' is the control power for the decentralized CDC scheme in the case of $\lambda_2 = \lambda_4$. (c) The change $\Delta P = P - P'$ in the case of $\lambda_2 = \lambda_4$.

By using the quantum mutual information, the dense coding channel capacity without Charlie's assistance is given by [50,51]

$$C(\rho_{AB}) = \log_2 d_A + S(\rho_B) - S(\rho_{AB}) \tag{10}$$

where ρ_{ABC} is the density matrix of the channel's state, reduced density matrix $\rho_{AB} = \mathrm{Tr}_C(\rho_{ABC})$, and $\rho_B = \mathrm{Tr}_A(\rho_{AB})$. d_A is the dimension of the qudit belonging to sender Alice, and in the original CDC scheme $d_A = 4$. $S(\rho) = -\mathrm{Tr}(\rho \log_2 \rho)$ is the von Neumann entropy.

To maximize the channel capacity between Alice and Bob, Charlie measures his qubits on the best basis of maximizing the average channel capacity. With Charlie's assistance, the classical channel capacity that Alice and Bob can achieve using CDC is [49]

$$C(\rho_{ABC}) = \max_U \sum_{i=0}^{N} \langle i|U\rho_C U^\dagger|i\rangle C\big(\rho_{AB}^{C=i}\big) \tag{11}$$

where $C(\rho_{AB}^{C=i})$ represents the channel capacity between Alice and Bob when the result of Charlie's measurement is i. The matrix U is the measurement basis selected by Charlie. Through the weighted summation of the different measurement results of Charlie, it is found that when U is a Hadamard operation the channel capacity can be maximized. In the noisy channels, the control power can be achieved by

$$P = C(\rho_{ABC}) - C(\rho_{AB}). \tag{12}$$

In the decentralized CDC scheme, since Alice participates in the control of the channel, the expression of the control power needs to be redefined as

$$P' = C'(\rho_{ABC}) - C'(\rho_{AB}) \tag{13}$$

where $C'(\rho_{ABC})$ indicates the channel capacity with Alice's and Charlie's assistance and $d_A = 3$. $C'(\rho_{AB})$ represents the channel capacity without Alice's and Charlie's assistance, and

$$C'(\rho_{AB}) = \frac{1}{3}\sum_{j=0}^{2}[\log_2 3 + S(\mathrm{Tr}_A(|j\rangle_A\langle 3|\rho_{AB}|3\rangle_A\langle j|) - S(|j\rangle_A\langle 3|\rho_{AB}|3\rangle_A\langle j|)]. \tag{14}$$

For comparison, we numerically calculated P and P' in the case of $\lambda_2 = \lambda_4$. The results are shown in Fig. 2, and we notice that the control power has no negative value in Fig. 2. The maximum values of $P = 2$ bits and $P' = 2$ bits also occur at the same optimal-state points: (1) $\lambda_1 = \frac{1}{\sqrt{2}}$, $\lambda_2 = 0$ and (2) $\lambda_1 = 0$, $\lambda_2 = \frac{1}{\sqrt{2}}$. We also give the change $\Delta P = P' - P$ in control power as a function of the parameters λ_1 and λ_2, as shown in Fig. 2(c). One can see that $\Delta P \geqslant 0$. When $P' = P = 2$ bits or $\lambda_2 = \lambda_4 = 0$, the equal sign in this formula holds. That is, the control power can be enhanced by reducing the dimension of the classical information in a suitably noisy quantum channel. The maximum value $\Delta P^{\max} = 0.544\,16$ bit $< \Delta P_B^{\max}$ occurs at $\lambda_1 = 0.365\,33$ and $\lambda_2 = \lambda_4 = 0.488\,77$. Even if the calculation method of CP is changed, the amplification effect caused by noise still exists.

The increase of control power shown here is due to decentralization operation. In the decentralized CDC scheme, Alice's dimensionality reduction operations will not change the fidelity of the quantum channel and will not induce the superior noise resistance [52,53] of subspace coding in high-dimensional systems. However, noise will destroy the symmetry of the channel, and the error caused by the uncertainty of the measurement basis is different in each subspace and it induces a nontrivial performance in the promotion of control power.

IV. COMPARISON BETWEEN CT AND CDC

Through the above analysis, we can see that the control power can be enhanced by reducing the dimension of the classical information in a suitably high-dimensional quantum channel. However, this tradeoff between the control power and the classical capacity is very different from that between the control power and the quantum capacity, especially in optimal-state channels. To illustrate this difference, we take the controlled teleportation scheme as an example and compare it with the controlled dense coding scheme.

In the controlled teleportation scheme for the optimal-state channel [37], CP is defined as $P_q = 1 - f_{NC}$, in which f_{NC} is the unconditioned teleportation fidelity without the controller's permission. By using the standard d-dimensional GHZ state to controlled teleport a d-dimensional qudit, the control power is $P_{q_1} = 1 - \frac{2}{d+1}$. If using a 2^N-dimensional GHZ state to transmit d-dimensional quantum information ($2^N > d > 2^{N-1}$), the control power can be improved to $P_{q_2} = 1 - \frac{d}{2^N}\frac{2}{d+1}$ with local operations [37].

For comparison purposes, one can define the CP of the CDC scheme as the controlled error rate, which is the error rate at Bob's side in the case without controller assistance in optimal-state channels. By using a standard d-dimensional GHZ state to control the transmission of $2\log_2 d$ bits of classical information, $P_{c_1} = 0.5$ when $d = 2^N$, which corresponds to a random guess for complete classical information units; otherwise, $P_{c_1} < 0.5$, which corresponds to a random guess for incomplete classical information units. For example, by using a three-dimensional GHZ state $(\frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle)_{ABC} = (\frac{1}{3}[(|00\rangle + |11\rangle + |22\rangle)|0\rangle + (|00\rangle + e^{i\frac{2\pi}{3}}|11\rangle + e^{i\frac{4\pi}{3}}|22\rangle)|1\rangle + (|00\rangle + e^{i\frac{4\pi}{3}}|11\rangle + e^{i\frac{2\pi}{3}}|22\rangle)|2\rangle]_{ABC}$, Alice and Charlie can control transmit 00, 01, or 10 through relative-phase information of the quantum channel to Bob. Without the controller's assistance, when Alice transmits 00 to Bob, Bob has a probability of 1/3 to obtain 2 bits of correct information, and the probability of 2/3 can only obtain 1 bit of correct information; when Alice transmits 01 or 10 to Bob, there is a 1/3 probability that Bob can obtain 2 bits of correct information, a 1/3 probability that Bob only obtains 1 bit of correct information, and a 1/3 probability that Bob cannot obtain correct information at all. Hence, the average error rate is 4/9. By using bit information of this quantum channel, Alice can control transmit 00, 01, or 10 to Bob without giving a determined coding basis, independently. On the premise of not changing the channel capacity, Alice can encode the classical information 00 in three different ways: $\frac{1}{\sqrt{3}}(|10\rangle + |01\rangle + |22\rangle)_{ABC}$, $\frac{1}{\sqrt{3}}(|20\rangle + |11\rangle + |02\rangle)_{ABC}$, or $\frac{1}{\sqrt{3}}(|00\rangle + |21\rangle + |12\rangle)_{ABC}$ with local operations. With the aid of three-dimensional qudit-flip operation $\sigma_x' = \sum_{i=0}^{2}|i+1\rangle\langle i|$ (mod 3) or $\sigma_x'^2$, 01 or 10 can be encoded on qudit A in the above different coding bases by Alice. Hence, without Alice's assistance, the average error rate is also 4/9 for Bob. When d is equal to other values, we can use the same calculation method to deduce the corresponding controlled error rate by analog.

If using a 2^N-dimensional GHZ state to transmit d-dimensional classical information ($2^N > d > 2^{N-1}$), the control power P_c can be improved to 0.5 by exploiting the decentralized CDC scheme. For example, as described in Sec. II, by using the decentralized CDC scheme and decreasing the dimension of bit information from 4 to 3, the control power is 2 bits, and the corresponding controlled error rate is 0.5. To clearly illustrate the different performances of CPs, we numerically calculated P_q and P_c of optimal-state channels as a function of the dimension parameter d ($16 \geqslant d \geqslant 2$). The results are shown in Fig. 3. One can see that there is an improvement for P_{q_2} by reducing the dimension of the teleported information. However in the CDC scheme, even if the classical capacity is intentionally reduced, the controlled error rate can only be increased to 0.5 for controlled transmission of classical information, and it will not be greater than when the channel capacity is not consumed. Therefore, it does not make sense to consume capacity in the optimal-state channel.

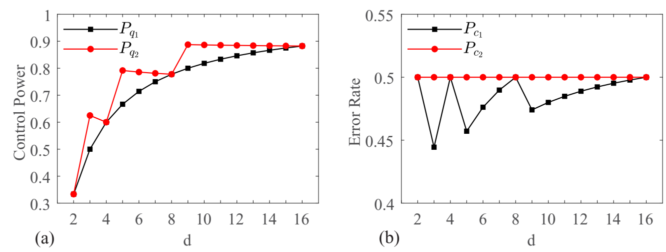

FIG. 3. Control power vs dimensional parameter d ($16 \geqslant d \geqslant 2$) in optimal-state channels. (a) The CP for the controlled teleportation scheme (CP here is defined according to fidelity and dimensionless). P_{q_1} (black line with square) is the CP for controlled teleporting a d-dimensional qudit by using the standard d-dimensional GHZ state. P_{q_2} (red line with circle) is the CP for controlled teleporting a d-dimensional qudit by using the standard 2^N-dimensional GHZ state ($2^N > d > 2^{N-1}$). (b) The controlled error rate for the controlled CDC scheme. P_{c_1} (black line with square) is the CP for controlled transmitting $2\log_2 d$ bits of classical information by using the standard d-dimensional GHZ state. P_{c_2} (red line with circle) is the CP for controlled transmitting $2\log_2 d$ bits of classical information by using the standard 2^N-dimensional GHZ state ($2^N > d > 2^{N-1}$).

V. CONCLUSION AND SUMMARY

To summarize, we have proposed a decentralized CDC scheme to improve the control power of the quantum channel by decoding the classical information in a suitable lower-dimensional basis. By calculating the control power of the decentralized CDC scheme in a four-dimensional extended GHZ-class state channel, we found that there is no tradeoff between the control power and the classical capacity of the four-dimensional standard GHZ state quantum channel. Only when the channel is noisy can the decentralized CDC scheme activate the tradeoff between the control power and the classical capacity. This noise-induced characteristic is very different from that of transmitting quantum information.

For the antinoise strategy proposed in this paper, we calculate it under different noise channel conditions and the results are different. For three-qubit GHZ-like state channels $|\psi_{\text{GHZ-like}}\rangle_{ABC} = \lambda_1|000\rangle + \lambda_2|011\rangle + \lambda_3|101\rangle + \lambda_4|110\rangle$, where $\lambda_i \geqslant 0$ and $\sum_i \lambda_i^2 = 1$, $\{i = 1, 2, 3, 4\}$, the two schemes have no different effect on the control power of the channel. For the channel Eq. (9) mentioned in this paper, if it is simplified to the form $|\psi'_{e-\text{GHZ}}\rangle_{ABC} = \lambda_1|000\rangle + \lambda_2|110\rangle + \lambda_3|111\rangle$, where $\lambda_i \geqslant 0$ and $\sum_i \lambda_i^2 = 1$, $\{i = 1, 2, 3\}$, we can get the conclusion that the control power is improved for the decentralized scheme, but the improvement effect is not as good as the channel Eq. (9). From the above conclusion, we can see that the classical control authority gained by Alice, as the compensation for quantum control power, is a new component of control power. When the quantum control power determined by the control qubit C is reduced by the noise, the classical control power performed by Alice can well compensate the loss of quantum control power, and the antinoise strategy proposed in this paper will work.

On the other hand, in the so-called degree-mismatch problem [54], the information carrier and the channel have different dimensions, and the control power of an asymmetrical scheme will not follow the rules of low-dimensional channels. Using a quantum channel to transmit classical information is a very effective secure transmission method for the future, which has a very wide range of applications. The application scenario of controlled quantum communication is the secure access of ordinary users to quantum networks. An ordinary user is not only the sender of information, but also the controller of information. She/he has the authority to control the effective release of information. Our scheme is to study the antinoise effect of the control power in this situation.. Our scheme also has good application prospects in quantum secure text transmission scenarios such as controlled quantum secure direct communication [27,28,55,56]. If Bob can perform a complete 3×4 dimensional BSA measurement, it is not necessary to DUOs. He can directly use the 3×4 dimensional BSA measurement to decode directly, and the calculated results of the control power are the same as those with DUOs. A similar asymmetric 3×4 dimensional BSA has been constructed in our previous work [57]. With the help of similar physical systems, one can carry out the BSA required in this paper.

Our scheme may provide new ideas for improving control power by transforming the redundant dimension of quantum channels into the control authority of the sender to maximize the utilization of channel resources. Our scheme can also provide more personalized choices for users in the networks. Finally, our framework of nonlocal controlled classical information transmission in high dimensions provides a feasible method and directions of exploration for the future construction of quantum networks. Furthermore, it may provide new ideas and methods to develop the resource theory of quantum channels.

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