Error-disturbance uncertainty relations in Faraday measurements

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We examine error-disturbance relations in the quantum measurement of spin systems using an atom-light interface scheme. We model a single spin-1/2 system that interacts with a polarized light meter via a Faraday interaction. We formulate the error and disturbance of the model and examine the uncertainty relations. We find that, for the coherent light meter in pure polarization, both the error and disturbance behave like the cyclic oscillations due to the Faraday rotation in both the light and spin polarizations. We also examine a class of polarization-squeezed light meter, where we apply the phase-space approximation and characterize the role of squeezing. We derive the error-disturbance relations for these cases and find that the Heisenberg-Arthurs-Kelly uncertainty is violated while the tight Branciard-Ozawa uncertainty always holds. We note that, in the limit of weak interaction strength, the error and disturbance come to obey the unbiasedness condition and hence the Heisenberg-Arthurs-Kelly relation holds. The work contributes to our understanding of the quantum measurement of spin systems under the atom-light interface framework and may hold potential applications in quantum metrology, quantum state estimation, and control.

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I. INTRODUCTION

Quantum measurements play a crucial role in characterizing physical systems, which elucidate hidden quantum properties to the classical world [1]. Moreover, many measurements come with more than just one observable that does not commute, and thus have an enormous impact from the fundamental verification such as the Bell nonlocality and entanglement [2–4], quantum steering [5], quantum metrology [6,7] to quantum information technologies including quantum key distribution [8], quantum dense coding [9–12], quantum cryptography [9,13], and nonlocal quantum measurement [14–16].

An important intrinsic property of quantum measurements is the uncertainty relation in which it is unfeasible to measure incompatible observables with arbitrary precision. This is the fundamental restriction in the attainable precision of quantum measurements. In the early stage of quantum mechanics, Heisenberg [17] first formulated such an uncertainty relation between the position measurement and the disturbance of the momentum that satisfies $\epsilon_q \eta_p \approx \hbar/2$, where $\epsilon_\circ$ and $\eta_\star$ represent the *root-mean-square* error and *root-mean-square* disturbance, respectively. The study then was paraphrased under the form of the standard deviations by Kennard [18], Weyl [19], and later Robertson [20], for a general pair of operators $\boldsymbol{A}$ and $\boldsymbol{B}$, which reads $\sigma_{\boldsymbol{A}}\sigma_{\boldsymbol{B}} \geqslant \mathcal{C}_{\boldsymbol{A},\boldsymbol{B}}$, where $\mathcal{C}_{\boldsymbol{A},\boldsymbol{B}} = |\langle\psi|[\boldsymbol{A},\boldsymbol{B}]|\psi\rangle|/2$ and $\sigma_{\boldsymbol{\Lambda}} = \sqrt{\langle\boldsymbol{\Lambda}^2\rangle - \langle\boldsymbol{\Lambda}\rangle^2}$ represents the standard deviation of $\boldsymbol{\Lambda}$, with $\langle\boldsymbol{\Lambda}\rangle = \langle\psi|\boldsymbol{\Lambda}|\psi\rangle$ as the expectation value for a pure quantum state $|\psi\rangle$, and $\boldsymbol{\Lambda} \equiv \boldsymbol{A}$ or $\boldsymbol{B}$. However, this mathematical relation in the form of standard derivation has no direct connection to the limitation on measurements and thus could not cover Heisenberg's interpretation uncertainty. Preferably, Arthurs and Kelly [21] provided an error-disturbance relation, which was then generalized to [22,23]

$$\epsilon_{\boldsymbol{A}}\eta_{\boldsymbol{B}} \geqslant \mathcal{C}_{\boldsymbol{A},\boldsymbol{B}}, \tag{1}$$

which states that if the measurement of an observable $\boldsymbol{A}$ with an error $\epsilon_{\boldsymbol{A}}$ then it also disturbs an observable $\boldsymbol{B}$ with a disturbance $\eta_{\boldsymbol{B}}$, satisfying such a relation. So far, it is known that this relation is not universally valid (see, for example, Ref. [24].) Hereafter, we call Eq. (1) the Heisenberg-Arthurs-Kelly uncertainty.

Ozawa theoretically derived a universal error-disturbance relation [25,26] through an indirect measurement following the von Neumann paradigms [27]. The measurement consists of an interaction between a quantum system and a meter. A measurement of $\boldsymbol{A}$ in the system was done indirectly via a measurement of $\boldsymbol{M}$ in the meter. At the same time, this process reflects back to the system, and thus it disturbs the subsequent measurement of observable $\boldsymbol{B}$ in the system. According to Ozawa, the error-disturbance relation for any input state $|\psi\rangle$ is expressed by [25,26]

$$\epsilon_{\boldsymbol{A}}\eta_{\boldsymbol{B}} + \epsilon_{\boldsymbol{A}}\sigma_{\boldsymbol{B}} + \eta_{\boldsymbol{B}}\sigma_{\boldsymbol{A}} \geqslant \mathcal{C}_{\boldsymbol{A},\boldsymbol{B}}. \tag{2}$$

This relation was experimentally confirmed recently by using a state-preparation method [28–32], weak probe method [33–36], continuous-variable entangled states [37,38], and others [39,40].

Subsequently, Branciard [41,42] and Ozawa [43] considered a rigorous relation that reads

$$\epsilon_{\boldsymbol{A}}^2\sigma_{\boldsymbol{B}}^2 + \sigma_{\boldsymbol{A}}^2\eta_{\boldsymbol{B}}^2 + 2\epsilon_{\boldsymbol{A}}\eta_{\boldsymbol{B}}\sqrt{\sigma_{\boldsymbol{A}}^2\sigma_{\boldsymbol{B}}^2 - \mathcal{C}_{\boldsymbol{A},\boldsymbol{B}}^2} \geqslant \mathcal{C}_{\boldsymbol{A},\boldsymbol{B}}^2, \tag{3}$$

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which claimed to be tighter than relation (2) and was experimentally verified [31,36–38]. Hereafter, we call Eq. (3) the Branciard-Ozawa uncertainty. Recently, numerous alternative approaches were used to revisit the uncertainty relation theoretically and experimentally [39,44–67].

Recently, the Faraday measurements of spin based on an atom-light interface framework were actively studied [68–78]. This has contributed to our understating of quantum measurement and has various applications in the quantum metrology of atomic ensemble [79], quantum information processing [80], strongly correlated systems [81], and many-body systems [82]. The Faraday effect causes the rotation of polarized light via the interaction with the spin system and thus allows indirect measurement of the spin system through the polarized light meter. Such a measurement contains fundamental limits in the sensitivity caused by the quantum nature of light. Likewise, the back-action of the polarized light meter perturbs the spin state, which causes a disturbance on the subsequent measurements of the spin system.

Recently, the uncertainty relation in the Faraday measurementwas studied by examining the relation between preparation (prediction) and postselection (retrodiction) [75], where the authors considered the Kennard-Weyl-Robertson relation for the approximate canonical position and momentum of the spin of atoms. However, the obtained relation cannot be considered as the error-disturbance relation. Also, the approximate canonical observables used there are only applicable in the case of weak interaction and unbiased measurements. Thus, a more precise and appropriate analysis of the error-disturbance uncertainty relation in the Faraday measurement is necessary.

In this paper, we formulate an atom-light interface scheme in the Faraday measurement and evaluate the error, disturbance, and their uncertainty relations. We consider an atom as a single spin-$1/2$ particle interacting with a polarized light meter. We first consider a classical coherent polarized light as the light meter. Without approximation, we derive the error and disturbance and their trade-off relation as functions of the interaction strength. Next, we investigate the case of polarization-squeezed light using the canonical phase-space approximation for the light meter, where the squeezing parameter is regarded as one of the parameters that defines the measurement strength. We also examine the case of weak interaction strength, where the error and disturbance satisfy the joint unbiasedness, i.e., the condition in which the Heisenberg-Arthurs-Kelly uncertainty holds. We further formulate the error-disturbance relations in these cases and provide that the Heisenberg-Arthurs-Kelly uncertainty [21] can be violated while the tight Branciard-Ozawa uncertainty for the qubit system [41] always holds. Our analysis would contribute to the understanding of the effect of error and disturbance as well as their uncertainty relations in the quantum measurement under the atom-light interface framework.

This paper is organized as follows. We introduce the concept of the atom-light interface in Sec. II. In Sec. III, we derive the error and disturbance under the atom-light interface framework for classical coherent light meter and polarization-squeezed light meter. The error-disturbance relations are provided in Sec. IV. We give a brief summary and outlook in Sec. V.

II. MEASUREMENT PROCESS

We consider a measurement model in which a spin-$1/2$ system interacts with a polarized light meter based on the Faraday interaction under the standard von Neumann paradigm [27]. The spin system is a single-particle characterized by Pauli matrices $\boldsymbol{\sigma}_i$, with $i = x, y, z$, while the polarized light meter is given by the Stokes operators $\boldsymbol{S}_i$ [83]. For light propagating along the z direction, we explicitly have

$$\boldsymbol{S}_0 = \boldsymbol{a}_H^\dagger \boldsymbol{a}_H + \boldsymbol{a}_V^\dagger \boldsymbol{a}_V = \boldsymbol{n}_H + \boldsymbol{n}_V, \tag{4}$$

$$\boldsymbol{S}_x = \boldsymbol{a}_H^\dagger \boldsymbol{a}_H - \boldsymbol{a}_V^\dagger \boldsymbol{a}_V = \boldsymbol{n}_H - \boldsymbol{n}_V, \tag{5}$$

$$\boldsymbol{S}_y = \boldsymbol{a}_H^\dagger \boldsymbol{a}_V + \boldsymbol{a}_H \boldsymbol{a}_V^\dagger, \tag{6}$$

$$\boldsymbol{S}_z = -i(\boldsymbol{a}_H^\dagger \boldsymbol{a}_V - \boldsymbol{a}_H \boldsymbol{a}_V^\dagger), \tag{7}$$

where H and V stand for the light modes of horizontal and vertical linear polarizations, respectively, $\boldsymbol{a}_{H,V}$ ($\boldsymbol{a}_{H,V}^\dagger$) are the annihilation (creation) operators in the corresponding polarization modes, and $\boldsymbol{n} = \boldsymbol{a}^\dagger \boldsymbol{a}$ the photon number operator. The Stokes operators obey the angular momentum commutation relation $[\boldsymbol{S}_x, \boldsymbol{S}_y] = 2i\boldsymbol{S}_z$, and cyclic permutations.

The unitary evolution of the Faraday interaction is given by

$$\boldsymbol{U}_T = e^{-ig\boldsymbol{A}\otimes\boldsymbol{S}_z}, \tag{8}$$

where $g = \int_0^T g(t)\,dt$ is the interaction strength over the time interval T. Here $\boldsymbol{A}$ is the observable being measured in the system. Under such an atom-light interface, the polarization state of the light meter rotates through the Faraday effect by an amount proportional to $\boldsymbol{A}$, and thus allows the indirect measurement of $\boldsymbol{A}$. Likewise, under the back-action effect, the system state is rotated around the z axis by an amount proportional to $\boldsymbol{S}_z$, and thus disturbs the system.

Assume that the spin system is prepared in state $|\psi\rangle$ and the light meter state is $|\xi\rangle$. They are initially uncorrelated, so that $|\Psi\rangle = |\psi\rangle \otimes |\xi\rangle$. The unitary operator $\boldsymbol{U}_T$ in Eq. (8) describes the time evolution of the joint system meter during the interaction time. After the interaction, the joint state is given by $|\Psi'\rangle = \boldsymbol{U}_T|\Psi\rangle$, and the measuring expectation value of an observable $\boldsymbol{M}$ in the meter will be

$$\begin{aligned}\langle(\boldsymbol{I}\otimes\boldsymbol{M})\rangle &= \langle\Psi'|(\boldsymbol{I}\otimes\boldsymbol{M})|\Psi'\rangle \\ &= \langle\Psi|\boldsymbol{U}_T^\dagger(\boldsymbol{I}\otimes\boldsymbol{M})\boldsymbol{U}_T|\Psi\rangle.\end{aligned} \tag{9}$$

Let us choose $\boldsymbol{M} = \boldsymbol{S}_y$, and in the Heisenberg picture, we consider $(\boldsymbol{I}\otimes\boldsymbol{S}_y)_T = \boldsymbol{U}_T^\dagger(\boldsymbol{I}\otimes\boldsymbol{S}_y)_0\boldsymbol{U}_T$ is the time-dependent operator after the interaction. Particularly, for $\boldsymbol{A}^2 = \boldsymbol{I}$, as in the Pauli operators, using the Baker-Campbell-Hausdorff (BCH) formula [84], we obtain (see Appendix A)

$$(\boldsymbol{I}\otimes\boldsymbol{S}_y)_T = (\boldsymbol{I}\otimes\boldsymbol{S}_y)_0\cos(2g) + (\boldsymbol{A}\otimes\boldsymbol{S}_x)_0\sin(2g). \tag{10}$$

The subscripts T and 0 stand for the time dependent on the times T and 0, respectively. Equation (10) means that the Stokes operators rotate about the z axis with the angle $2g$, i.e., the Faraday rotation. The rotation direction is determined by the sign of $\boldsymbol{A}$; note that the eigenvalues of $\boldsymbol{A}$ are ± 1 since $\boldsymbol{A}^2 = \boldsymbol{I}$. Then, we measure the expectation value of the meter $\langle(\boldsymbol{I}\otimes\boldsymbol{S}_y)_T\rangle$ that provides the information of an indirect measurement performed on the system. In our model, the expectation value gives

$$\langle(\boldsymbol{I}\otimes\boldsymbol{S}_y)_T\rangle = \langle\boldsymbol{S}_y\rangle_\xi \cos(2g) + \langle\boldsymbol{A}\rangle_\psi \langle\boldsymbol{S}_x\rangle_\xi \sin(2g). \quad (11)$$

Here and hereafter, the bra-ket symbol $\langle\cdots\rangle$ means $\langle\Psi|\cdots|\Psi\rangle = \langle\psi|\langle\xi|\cdots|\psi\rangle|\xi\rangle$ whereas $\langle\cdots\rangle_\psi$ stands for $\langle\psi|\cdots|\psi\rangle$ and $\langle\cdots\rangle_\xi$ for $\langle\xi|\cdots|\xi\rangle$. We omit the subscript 0 in the right-hand side (R.H.S.) without confusion. Here, the mean value of the meter's observable will shift from the initial value by an amount proportional to the mean value of the system's observable $\langle A\rangle_\psi$. Without loss of generality, we can choose the initial mean of the meter is zero, i.e., $\langle\boldsymbol{S}_y\rangle_\xi = 0$. We thus can indirectly measure the value of the system operator $\boldsymbol{A}$ via a calibrated meter operator $\boldsymbol{M}_T = (\boldsymbol{I}\otimes\boldsymbol{S}_y)_T/\langle\boldsymbol{S}_x\rangle_\xi \sin(2g)$. The calibration is designed so that $\boldsymbol{M}_T$ is *unbiased*, i.e., $\langle\boldsymbol{M}_T - (\boldsymbol{A}\otimes\boldsymbol{I})_0\rangle = 0$ irrespective of $|\psi\rangle$, given that $\langle\boldsymbol{S}_y\rangle_\xi = 0$. The calibration factor $1/\langle\boldsymbol{S}_x\rangle_\xi \sin(2g)$ can be determined independently in practical experiments.

In this scenario, to measure $\boldsymbol{A}$ of the system before the interaction, we measure $\boldsymbol{S}_y$ of the meter after the interaction. If these two observables are perfectly correlated in any given system state $|\psi\rangle$, the measurement is said to be accurate [33,85]. However, in general, they would not be perfectly correlated and thus become inaccurate because of the possible noise and error in the measurement process. Moreover, when another observable $\boldsymbol{B}$ in the system is measured after the measurement of $\boldsymbol{A}$, it would be disturbed by the back-action effect caused by the prior interaction in the $\boldsymbol{A}$ measurement. In the following, we will consider the error and disturbance in our measurement model.

III. ERROR AND DISTURBANCE

A. Exact solution for classical coherent light meter

In the following, we will consider the measurements of $\boldsymbol{A} = \boldsymbol{\sigma}_z$ and $\boldsymbol{B} = \boldsymbol{\sigma}_x$ in a single spin system. In the joint space, we denote

$$\boldsymbol{A}_0 = (\boldsymbol{\sigma}_z\otimes\boldsymbol{I})_0, \text{ and } \boldsymbol{B}_0 = (\boldsymbol{\sigma}_x\otimes\boldsymbol{I})_0, \quad (12)$$

for the operators at the time 0. We denote the measurement operators at time T as

$$\boldsymbol{M}_T = \frac{(\boldsymbol{I}\otimes\boldsymbol{S}_y)_T}{\langle\boldsymbol{S}_x\rangle_\xi\ \sin(2g)}, \text{ and } \boldsymbol{B}_T = (\boldsymbol{\sigma}_x\otimes\boldsymbol{I})_T. \quad (13)$$

We get [see Eqs. (B1) and (B8) in Appendix B]

$$\boldsymbol{M}_T = \frac{(\boldsymbol{I}\otimes\boldsymbol{S}_y)_0\cot(2g)}{\langle\boldsymbol{S}_x\rangle_\xi} + \frac{(\boldsymbol{\sigma}_z\otimes\boldsymbol{S}_x)_0}{\langle\boldsymbol{S}_x\rangle_\xi}, \quad (14)$$

$$\boldsymbol{B}_T = [\boldsymbol{\sigma}_x\otimes\cos(2g\boldsymbol{S}_z)]_0 - [\boldsymbol{\sigma}_y\otimes\sin(2g\boldsymbol{S}_z)]_0. \quad (15)$$

The error can be evaluated by the error operator $\boldsymbol{N}_{\sigma_z}$, and the disturbance is defined through the disturbance operator $\boldsymbol{D}_{\sigma_x}$ as follows:

$$\boldsymbol{N}_{\sigma_z} = \boldsymbol{M}_T - \boldsymbol{A}_0, \text{ and } \boldsymbol{D}_{\sigma_x} = \boldsymbol{B}_T - \boldsymbol{B}_0. \quad (16)$$

Then, the square error and the square disturbance are given by [25,26,86],

$$\epsilon^2_{\sigma_z} = \langle\boldsymbol{N}^2_{\sigma_z}\rangle, \text{ and } \eta^2_{\sigma_x} = \langle\boldsymbol{D}^2_{\sigma_x}\rangle. \quad (17)$$

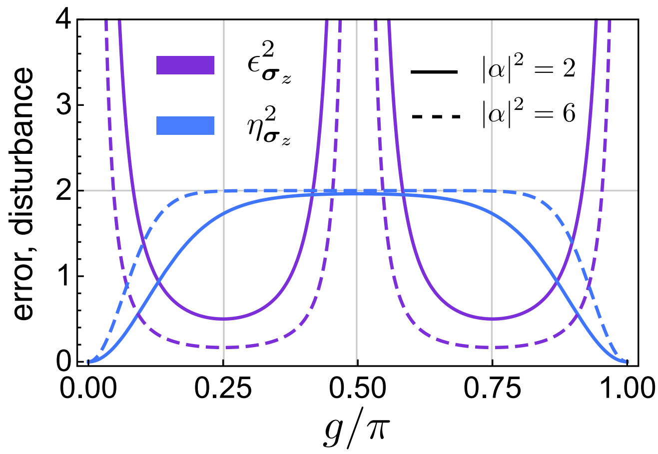

FIG. 1. The plot of the square error and square disturbance as functions of interaction strength g for some values of amplitude $|\alpha|^2$ as shown in the figure. The square error is large for small g and reaches the minimum when $g = \pi/4$ and increases again for g increases to $\pi/2$. The procedure is repeated when continuously increasing g. Similarly, the square disturbance increases along with g and reaches the maximum of 2, then it reduces to 0 as g increases to π.

In the following, we choose the initial system state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, an eigenstate of $\boldsymbol{\sigma}_y$ that maximize the R.H.S. of the error-disturbance relations, Eqs. (1), (2), and (3). We also choose the light-meter state to be a coherent state in the horizontal linear polarization

$$|\xi\rangle \equiv |\alpha\rangle_H|0\rangle_V = \exp(\alpha\boldsymbol{a}^\dagger_H - \alpha^*\boldsymbol{a}_H)|0\rangle_H|0\rangle_V, \quad (18)$$

where $|\alpha\rangle$ is the coherent state with the coherent amplitude α and $|0\rangle$ is the vacuum state of light. For this meter state, $\langle\boldsymbol{S}_x\rangle_\xi = |\alpha|^2$ and $\langle\boldsymbol{S}_y\rangle_\xi = \langle\boldsymbol{S}_z\rangle_\xi = 0$. We readily find $\langle\boldsymbol{N}_{\sigma_z}\rangle_\xi = 0$ and thus $\langle\boldsymbol{N}_{\sigma_z}\rangle = 0$ irrespective of $|\psi\rangle$, i.e., $\boldsymbol{M}_T$ is unbiased as mentioned earlier. For the disturbance, however, $\langle\boldsymbol{D}_{\sigma_x}\rangle_\xi \neq 0$ in general, since $\langle\cos(2g\boldsymbol{S}_z)\rangle_\xi \neq \boldsymbol{I}$ even though $\langle\boldsymbol{S}_z\rangle_\xi = 0$ and $\langle\sin(2g\boldsymbol{S}_z)\rangle_\xi = 0$. The nonzero means that the disturbance comes from the noise (fluctuation) in $\boldsymbol{S}_z$, which randomly rotates the spin system about the z axis and effectively reduces the x component of the spin. This behavior is the imprint of the back-action effect on the spin system caused by the light meter, which disturbs (rotates) the spin system on its Bloch sphere.

Then the square error and disturbance read (see the detailed calculation in Appendix B)

$$\epsilon^2_{\sigma_z} = \frac{1}{|\alpha|^2\sin^2(2g)}, \quad (19)$$

$$\eta^2_{\sigma_x} = 2(1 - e^{-2|\alpha|^2\sin^2 g}). \quad (20)$$

Note that, for the polarized coherent state of light, the root-mean-square noise in $\boldsymbol{S}_y$ (and also in $\boldsymbol{S}_x$ and $\boldsymbol{S}_z$) is $|\alpha|$. This noise is imprinted in $\boldsymbol{M}_T$ as $|\alpha|/\langle\boldsymbol{S}_x\rangle_\xi \sin(2g) = 1/|\alpha|\sin(2g)$, and thus results in the square error $\epsilon^2_{\sigma_z}$ in Eq. (19). Also, as mentioned above, the noise in $\boldsymbol{S}_z$ contributes to the disturbance in $\boldsymbol{\sigma}_x$ with a bias and thus results in the square disturbance $\eta^2_{\sigma_x}$ given in Eq. (20).

In Fig. 1, we show the square error $\epsilon^2_{\sigma_z}$ and square disturbance $\eta^2_{\sigma_x}$ as functions of the interaction strength g for several coherent amplitudes $|\alpha|^2$. When $g = n\pi/2$ where n is an integer number, $\epsilon^2_{\sigma_z}$ diverges because no shift of $\boldsymbol{S}_y$ is expected in the meter. With increasing g, due to the rotation of the light polarization that causes a certain amount of shift of $\boldsymbol{S}_y$ in the meter depending on $\boldsymbol{\sigma}_z$ in the system, the square error gradually decreases as $\epsilon^2_{\sigma_z} \approx g^{-2}$. When $g = \pi/4 + n\pi/2$, $\epsilon^2_{\sigma_z}$ reaches its minimum value $1/|\alpha|^2$, i.e., the minimum square error that can be achieved by the coherent light meter. Likewise, the square disturbance $\eta^2_{\sigma_x}$ exhibits periodic behavior as a function of g. When $g = n\pi$, the square disturbance $\eta^2_{\sigma_x}$ vanishes because the spin system is rotated by integer multiples of 2π for any integer values of $\boldsymbol{S}_z$ and thus returns to its original state. This phenomenon can be regarded as a kind of *quantum revival*, which essentially reflects the discrete nature of the observable, i.e., $\boldsymbol{S}_z$. When $g = \pi/2 + n\pi$, $\eta^2_{\sigma_x}$ reaches its maximum $2(1 - e^{-2|\alpha|^2}) \sim 2$ for large $|\alpha|$. In this case, the spin system is rotated about the z axis by 0 or π at almost even probabilities depending on the even or odd numbers of $\boldsymbol{S}_z$, so that the square disturbance becomes approximately $(2^2 + 0^2)/2 = 2$. This analysis provides us a complete and accurate insight on the quantum measurement of spin systems via the Faraday interaction.

B. Phase-space approximation (canonical approximation) for polarization-squeezed light meter

To further investigate the error and disturbance in various light meter states, we apply the phase-space approximation (PSA) for the light system. We introduce two canonical operators as $\boldsymbol{q} \equiv \boldsymbol{S}_y/\sqrt{|\langle \boldsymbol{S}_x\rangle|}$ and $\boldsymbol{p} \equiv \boldsymbol{S}_z/\sqrt{|\langle \boldsymbol{S}_x\rangle|}$ for a finite $\langle \boldsymbol{S}_x\rangle$ [68,75]. These operators approximately obey the canonical communicator relation $[\boldsymbol{q}, \boldsymbol{p}] = 2i\frac{\boldsymbol{S}_x}{|\langle \boldsymbol{S}_x\rangle|} \simeq 2i$. This approximation is valid when $\boldsymbol{S}_x$ can be regarded as a classical positive constant that does not change during the measurement process. Practically, under the PSA the evolution of $\boldsymbol{q}$ in the BCH formula [Eq. (A2) in Appendix A] is approximated up to its first order (first and second terms).

Here, we discuss the error and disturbance using the impact of a class of the polarization-squeezed state in the light meter space, which is given by

$$|\xi\rangle = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \int e^{-\frac{q^2}{4\sigma^2}} |q\rangle\, dq, \tag{21}$$

where σ represents the squeezing parameter. For $\sigma = 1$, it is a coherent state, the cases $\sigma < 1$ and $\sigma > 1$ correspond to an amplitude-squeezed state and a phase-squeezed state, respectively [87] (see Appendix C for details). Here, q and $|q\rangle$ are the eigenvalue and eigenstate of the position operator $\boldsymbol{q}$, such that $\boldsymbol{q}|q\rangle = q|q\rangle$. We illustrate such a polarization-squeezed state in a Poincaré sphere in Fig. 2.

The interaction evolution (8) is recast as

$$\boldsymbol{U}_T = e^{-ig|\alpha|\boldsymbol{\sigma}_z \otimes \boldsymbol{p}}, \tag{22}$$

where we set $\sqrt{|\langle \boldsymbol{S}_x\rangle|} = |\alpha|$. Under the PSA as mentioned above and using the BCH formula, we have

$$(\boldsymbol{I} \otimes \boldsymbol{q})_T = (\boldsymbol{I} \otimes \boldsymbol{q})_0 + 2g|\alpha|(\boldsymbol{\sigma}_z \otimes \boldsymbol{I})_0. \tag{23}$$

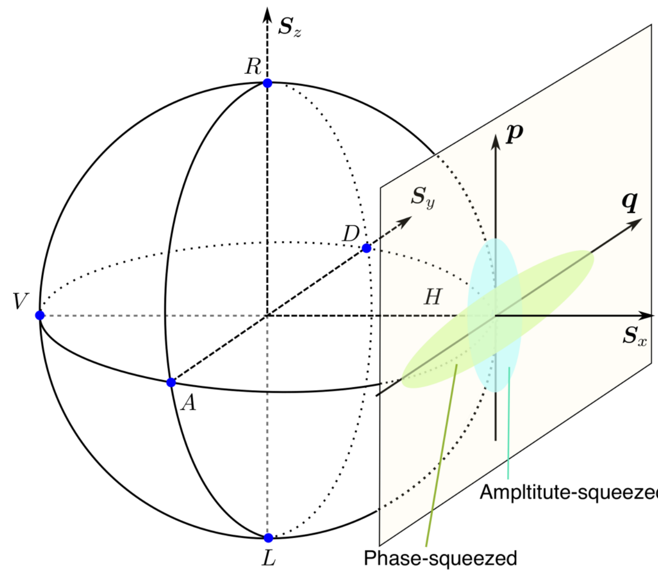

FIG. 2. Illustration of the class of polarization-squeezed light in the Poincaré sphere.

Using the calibrated meter operator $(\boldsymbol{I} \otimes \boldsymbol{q})_T/2g|\alpha|$, we obtain the corresponding information of $(\boldsymbol{\sigma}_z \otimes \boldsymbol{I})_0$ in the system. Thus, the error operator is given by

$$N_{\sigma_z} = (\boldsymbol{I} \otimes \boldsymbol{q})_T/2g|\alpha| - (\boldsymbol{\sigma}_z \otimes \boldsymbol{I})_0 = (\boldsymbol{I} \otimes \boldsymbol{q})_0/2g|\alpha|. \tag{24}$$

As a result, the square error is appropriate to the variance of the meter, i.e., $\langle \boldsymbol{q}^2\rangle_\xi/4g^2|\alpha|^2$ when $\langle \boldsymbol{q}\rangle_\xi = 0$. Similarly, the square disturbance operator is given by $2(1 - \langle \cos(2g|\alpha|\boldsymbol{p})\rangle_\xi)$ (see Appendix C). Straightforward calculating gives

$$\epsilon^2_{\sigma_z} = \frac{1}{4\chi^2}, \quad \text{and } \eta^2_{\sigma_x} = 2(1 - e^{-2\chi^2}), \tag{25}$$

where $\chi = g|\alpha|/\sigma$ represents the measurement strength.

In Fig. 3, we show the square error (solid curve) and disturbance (short-dashed curve) for PSA as functions of χ. When $g|\alpha|$ is fixed (not required to be small), hence, the squeezing parameter σ plays the role of the measurement strength: for large σ (or small χ) the measurement is weak, likewise, for small σ (or large χ) the measurement is strong. It is natural that for weak measurement, the square error is large and gradually reduces when increasing the measurement strength (solid curve). Inversely, the square disturbance is small for weak measurement and gradually increases when increasing the measurement strength and reaches the maximum of 2 (short-dashed curve). These results can be explained by the "squeezed" of the meter state: larger σ means a broader Gaussian shape in the class of the polarization-squeezed state, which is equivalent to "weak measurement," while small σ refers to the narrow Gaussian shape, which results in a strong measurement. Thus, using the class of the polarization-squeezed state will be more convenient in some particular cases, such as using squeezed states instead of the large photon-number coherence states.

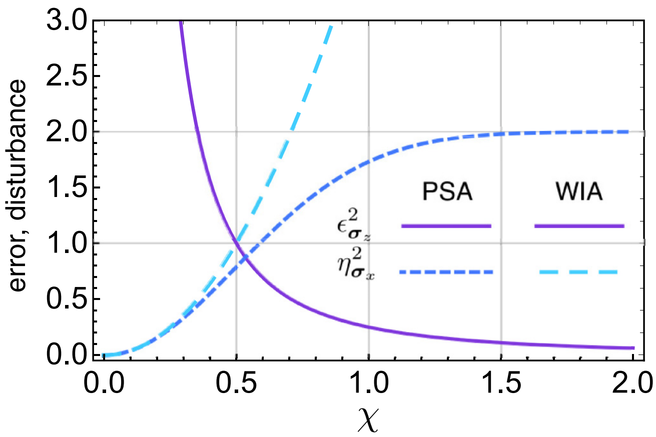

FIG. 3. Phase-space approximation (PSA): The plot of square error (solid curve) and square disturbance (short-dashed curve) in the class of squeezed coherent state in the meter as functions of the measurement strength $\chi = g|\alpha|/\sigma$. For a fixed of $g|\alpha|$, then σ plays the role of the measurement strength: for large σ the measurement is weak, for small σ the measurement is strong. Correspondingly, small (large) χ implies weak (strong) measurement. Weak interaction approximation (WIA): The plot of square error (solid curve) and square disturbance (long-dashed curve) in the WIA as functions of the measurement strength χ.

C. Weak interaction approximation

In many realistic models, the Faraday interaction is used in the far off-resonant light to avoid completely absorbing the classical field [68]. For adaptability with such models, a weak interaction in the atom-light can be made (see, for example, Ref. [70]).

In this subsection, to investigate the impact of the error and disturbance in such a far off-resonant region, we consider a weak interaction approximation (WIA), i.e., $\chi \ll 1$. In this approximation, we have

$$\epsilon^2_{\sigma_z} \approx \frac{1}{4\chi^2}, \text{ and } \eta^2_{\sigma_x} \approx 4\chi^2, \tag{26}$$

where $\chi = g|\alpha|$ (see Appendix D). Noting that, in the coherent-state case, $\sigma = 1$. We show Fig. 3 for the square error (solid curve) and square disturbance (long-dashed curve), denoted by "WIA." While the square error is the same as in the PSA case, the square disturbance gradually increases from zero when increasing χ. This square disturbance is different from that of the PSA case because the WIA is applied to both the spin system and the light meter (we can neglect the higher-order terms of g in both the spin system and the light meter). It thus provides us with the impact of weak Faraday interaction on the error and disturbance in spin measurements. We also confirm that the joint unbiasedness condition is satisfied, i.e., $\langle \boldsymbol{N}_{\sigma_z}\rangle = \langle \boldsymbol{D}_{\sigma_x}\rangle = 0$ irrespective of the initial system state $|\psi\rangle$ (see Appendix D), which is sufficient for holding the Heisenberg-Arthurs-Kelly uncertainty [25], i.e., $\epsilon^2_{\sigma_z}\eta^2_{\sigma_x} = 1$.

IV. ERROR-DISTURBANCE RELATIONS

This section examines the error-disturbance relations for the measurement of a single spin system with two cases of the meter state: the exact solution of the classical

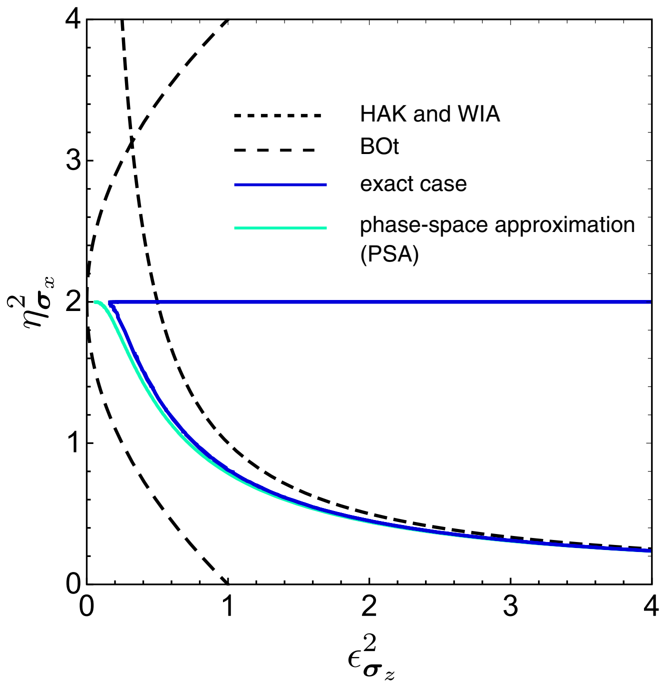

FIG. 4. The error-disturbance tradeoffs. The short-dashed curve is the Heisenberg-Arthurs-Kelly bound given in Eq. (27) and denoted by HAK. The left region is the forbidden region where the HAK relation is violated. Similarly, the long-dashed curve is the tight Branciard-Ozawa bound given in Eq. (29) and denoted by BOt. The solid curves show the error-disturbance tradeoff obtained from the atom-light interface in this work for two cases of exact solution and PSA. For the WIA it follows the HAK bound.

coherent light [Eq. (18)] and PSA of the polarization-squeezed light [Eq. (21)]. We consider the Heisenberg-Arthurs-Kelly relation [the left-hand side (L.H.S.) of Eq. (1)] and the Brainciard-Ozawa relation [the L.H.S. of Eq. (3)], which are denoted as HAK and BO, respectively. With our choice of the spin system, we have $\sigma_{\sigma_z} = 1, \sigma_{\sigma_x} = 1$, and $\mathcal{C}_{\sigma_z,\sigma_x} = 1$. We straightforwardly rewrite these relations as

$$\epsilon^2_{\sigma_z}\eta^2_{\sigma_x} \geqslant 1 \qquad \text{for HAK relation,} \tag{27}$$

$$\epsilon^2_{\sigma_z} + \eta^2_{\sigma_x} \geqslant 1 \quad \text{for BO relation,} \tag{28}$$

We also consider a tighter Brainciard-Ozawa relation (BOt), where the condition of $\boldsymbol{B}^2 = \boldsymbol{I}$ is satisfied, here $\boldsymbol{B} = \boldsymbol{\sigma}_x$. Following the authors of Ref. [41], we replace η_{σ_x} by $\eta_{\sigma_x}\sqrt{1 - \frac{\eta^2_{\sigma_x}}{4}}$ in Eq. (28) and recast it as

$$\epsilon^2_{\sigma_z} + \eta^2_{\sigma_x}\left(1 - \frac{\eta^2_{\sigma_x}}{4}\right) \geqslant 1. \tag{29}$$

We examine an error-disturbance tradeoff, which shows the dependence of the disturbance on the error and vice versa. The result is shown in Fig. 4. It can be seen that the error-disturbance tradeoff in the exact case behaves the same as the dependence on the Faraday and spin rotations: for small g, the error is large and the disturbance is small, then increasing g results in reducing of the error and increasing of the disturbance. After the error reaches the minimum, the disturbance grows towards 2, while the error gradually increases and results in a straight line in the tradeoff. Here, we show the result for $|\alpha|^2 = 6$. For large $|\alpha|^2$, the tradeoff asymptotically reaches that of the PSA. Moreover, the tradeoff can reach the BOt relation in the PSA, while the HAK relation is violated. Concretely, it can be seen that for large square error (small χ), the error-disturbance tradeoff reaches the HAK bound, while for small square error (large χ), the error-disturbance tradeoff reaches the maximum of 2, the BOt bound.

Finally, let us briefly discuss the comparison between our work and the work by Bao *et al.* [75]. Our work discusses the error-disturbance relations in a single measurement, while the work of the authors of Ref. [75] considers the uncertainty relation between the standard deviation of observables, i.e., the Kennard-Weyl-Robertson (KWR) relation, in the presence of prediction (preparation) and retrodiction (postselection) measurements. Because the two works are conceptually different they cannot be directly compared with each other. Nevertheless, it is interesting to compare the origins of the violation of the standard HAK or KWR relation. The violation of the HAK relation in our work originates from the fact that the unbiasedness condition is broken except for the case of WIA. In fact, the disturbance (back-action) onto the spin system is biased, as was pointed out earlier. In the case of Ref. [75], the postselection is the origin of the violation of the standard KWR relation, which becomes irrelevant in the presence of conditional postprocesses such as postselection. It may be regarded, in some sense, as the relation between the accuracies of state preparation (prediction) and the estimation (retrodiction), where the standard uncertainly relations are not applicable. It is also suggested that the error-disturbance relations could also be violated in such a situation [75].

V. CONCLUSION

We discussed the error, disturbance, and their uncertainly relations in Faraday measurements. For a single spin interacting with coherent polarization of the light meter, we derived the exact behaviors of error and disturbance without approximation. Under the Faraday rotation of the coherent light polarization and its back-action to the spin system, the error and disturbance behave as cyclic oscillations. In these cases of the polarization-squeezed light meter, to which we apply the canonical phase-space approximation, the squeezing parameter acts as a factor that modifies the measurement strength. In this approximation, the square error monotonically decreases to 0 while the square disturbance monotonically increases and approaches to 2 with increasing measurement strength. In the cases above, the Heisenberg-Arthurs-Kelly uncertainty is violated while the tight Branciard-Ozawa uncertainty always holds. It is worth mentioning that, under the weak interaction approximation, the Heisenberg-Arthurs-Kelly uncertainty holds because the error and disturbance both satisfy the unbiasedness.

Our analysis contributes to the deeper understanding of error, disturbance, and uncertainty relations in quantum measurements under the atom-light interface and provides an insight into quantum metrology [88,89], quantum sensing [90], and quantum state estimation [91]. This analytical work is also a testbed for further experimental studies.

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APPENDIX A: HEISENBERG EQUATION OF MOTIONS

1. Heisenberg equation for $\boldsymbol{S}_y$

In the Heisenberg picture, the meter's operator $\boldsymbol{S}_y$ evolves with time according to

$$(\boldsymbol{I}\otimes\boldsymbol{S}_y)_T = \boldsymbol{U}_T^\dagger(\boldsymbol{I}\otimes\boldsymbol{S}_y)_0\boldsymbol{U}_T, \tag{A1}$$

where $\boldsymbol{U}_T = e^{-ig\boldsymbol{A}\otimes\boldsymbol{S}_z}$.

Using the Baker-Campbell-Hausdorff formula [84]

$$e^{\boldsymbol{E}}\boldsymbol{F}e^{-\boldsymbol{E}} = \boldsymbol{F} + [\boldsymbol{E},\boldsymbol{F}] + \frac{1}{2!}[\boldsymbol{E},[\boldsymbol{E},\boldsymbol{F}]] + \cdots, \tag{A2}$$

for $\boldsymbol{E} = ig\boldsymbol{A}\otimes\boldsymbol{S}_z$ and $\boldsymbol{F} = \boldsymbol{I}\otimes\boldsymbol{S}_y$, we have

$$\begin{aligned}
\boldsymbol{F} &= \boldsymbol{I}\otimes\boldsymbol{S}_y,\\
[\boldsymbol{E},\boldsymbol{F}] &= ig[\boldsymbol{A}\otimes\boldsymbol{S}_z,\boldsymbol{I}\otimes\boldsymbol{S}_y]\\
&= ig\boldsymbol{A}\otimes[\boldsymbol{S}_z,\boldsymbol{S}_y]\\
&= 2g\boldsymbol{A}\otimes\boldsymbol{S}_x,\\
\frac{1}{2!}[\boldsymbol{E},[\boldsymbol{E},\boldsymbol{F}]] &= -\frac{(2g)^2}{2!}\boldsymbol{A}^2\otimes\boldsymbol{S}_y,\\
\frac{1}{3!}[\boldsymbol{E},[\boldsymbol{E},[\boldsymbol{E},\boldsymbol{F}]]] &= -\frac{(2g)^3}{3!}\boldsymbol{A}^3\otimes\boldsymbol{S}_x,\\
&\cdots.
\end{aligned}$$

Then Eq. (A2) is recast as

$$e^{\boldsymbol{E}}\boldsymbol{F}e^{-\boldsymbol{E}} = [\cos(2g\boldsymbol{A})\otimes\boldsymbol{S}_y]_0 + [\sin(2g\boldsymbol{A})\otimes\boldsymbol{S}_x]_0. \tag{A3}$$

Submitting Eq. (A3) into the R.H.S. of Eq. (A1), we have

$$(\boldsymbol{I}\otimes\boldsymbol{S}_y)_T = [\cos(2g\boldsymbol{A})\otimes\boldsymbol{S}_y]_0 + [\sin(2g\boldsymbol{A})\otimes\boldsymbol{S}_x]_0. \tag{A4}$$

In the case of $\boldsymbol{A}^2 = \boldsymbol{I}$, as in the Pauli operators, we obtain

$$(\boldsymbol{I}\otimes\boldsymbol{S}_y)_T = (\boldsymbol{I}\otimes\boldsymbol{S}_y)_0\cos(2g) + (\boldsymbol{A}\otimes\boldsymbol{S}_x)_0\sin(2g), \tag{A5}$$

which is given in Eq. (10) in the main text.

2. Heisenberg equation for $\boldsymbol{\sigma}_x$

Next, we consider the particular case where $\boldsymbol{A} = \boldsymbol{\sigma}_z$ and $\boldsymbol{B} = \boldsymbol{\sigma}_x$, and calculate the Heisenberg equation of motion for $\boldsymbol{\sigma}_x$ in the spin system. We consider

$$(\boldsymbol{\sigma}_x\otimes\boldsymbol{I})_T = \boldsymbol{U}_T^\dagger(\boldsymbol{\sigma}_x\otimes\boldsymbol{I})_0\boldsymbol{U}_T, \tag{A6}$$

where $\boldsymbol{U}_T = e^{-ig\boldsymbol{\sigma}_z\otimes\boldsymbol{S}_z}$. Using the BCH formula for $\boldsymbol{E} = ig\boldsymbol{\sigma}_z\otimes\boldsymbol{S}_z$ and $\boldsymbol{F} = \boldsymbol{\sigma}_x\otimes\boldsymbol{I}$, we have

$$\begin{aligned}
\boldsymbol{F} &= \boldsymbol{\sigma}_x\otimes\boldsymbol{I},\\
[\boldsymbol{E},\boldsymbol{F}] &= ig[\boldsymbol{\sigma}_z\otimes\boldsymbol{S}_z,\boldsymbol{\sigma}_x\otimes\boldsymbol{I}]\\
&= ig[\boldsymbol{\sigma}_z,\boldsymbol{\sigma}_x]\otimes\boldsymbol{S}_z\\
&= -2g\boldsymbol{\sigma}_y\otimes\boldsymbol{S}_z,\\
\frac{1}{2!}[\boldsymbol{E},[\boldsymbol{E},\boldsymbol{F}]] &= -\frac{(2g)^2}{2!}\boldsymbol{\sigma}_x\otimes\boldsymbol{S}_z^2,\\
&\cdots.
\end{aligned}$$

Then Eq. (A6) gives

$$(\boldsymbol{\sigma}_x \otimes \boldsymbol{I})_T = [\boldsymbol{\sigma}_x \otimes \cos(2g\boldsymbol{S}_z)]_0 - [\boldsymbol{\sigma}_y \otimes \sin(2g\boldsymbol{S}_z)]_0. \quad (A7)$$

APPENDIX B: ERROR AND DISTURBANCE

In this section, we provide a detailed calculation *root-mean-square* (rms) error ϵ_{σ_z} and rms disturbance η_{σ_x}.

1. Error

We first consider the $\boldsymbol{M}_T$ operator as follows:

$$\boldsymbol{M}_T = \frac{1}{\langle \boldsymbol{S}_x\rangle_\xi \sin(2g)}(\boldsymbol{I}\otimes \boldsymbol{S}_y)_T = \frac{(\boldsymbol{I}\otimes \boldsymbol{S}_y)_0 \cot(2g)}{\langle \boldsymbol{S}_x\rangle_\xi} + \frac{(\boldsymbol{\sigma}_z\otimes \boldsymbol{S}_x)_0}{\langle \boldsymbol{S}_x\rangle_\xi}, \quad (B1)$$

where we used $\boldsymbol{A} = \boldsymbol{\sigma}_z$ in Eq. (A5). Then, the noise operator is given by

$$\boldsymbol{N}_{\sigma_z} = \frac{(\boldsymbol{I}\otimes \boldsymbol{S}_y)_0 \cot(2g)}{\langle \boldsymbol{S}_x\rangle_\xi} + \frac{(\boldsymbol{\sigma}_z\otimes \boldsymbol{S}_x)_0}{\langle \boldsymbol{S}_x\rangle_\xi} - (\boldsymbol{\sigma}_z\otimes \boldsymbol{I})_0. \quad (B2)$$

Then, we obtain

$$\boldsymbol{N}^2_{\sigma_z} = \Bigg[\underbrace{\frac{(\boldsymbol{I}\otimes \boldsymbol{S}_y)_0 \cot(2g)}{\langle \boldsymbol{S}_x\rangle_\xi}}_{\boldsymbol{Y}} + \underbrace{\frac{(\boldsymbol{\sigma}_z\otimes \boldsymbol{S}_x)_0}{\langle \boldsymbol{S}_x\rangle_\xi}}_{\boldsymbol{X}} - \underbrace{(\boldsymbol{\sigma}_z\otimes \boldsymbol{I})_0}_{\boldsymbol{Z}}\Bigg]^2. \quad (B3)$$

Now, we calculate the average $\langle \boldsymbol{N}^2_{\sigma_z}\rangle$ over the initial joint state $|\psi\rangle \otimes |\xi\rangle$. We have

$$\langle \boldsymbol{Y}^2\rangle = \frac{\langle \boldsymbol{S}_y^2\rangle_\xi \cot^2(2g)}{\langle \boldsymbol{S}_x\rangle_\xi^2}; \quad \langle \boldsymbol{X}^2\rangle = \frac{\langle \boldsymbol{S}_x^2\rangle_\xi}{\langle \boldsymbol{S}_x\rangle_\xi^2}; \langle \boldsymbol{Z}^2\rangle = 1,$$
$$\langle \boldsymbol{YX}\rangle = \langle \boldsymbol{XY}\rangle = 0, \langle \boldsymbol{YZ}\rangle = \langle \boldsymbol{ZY}\rangle = 0,$$
$$\langle \boldsymbol{XZ}\rangle = \langle \boldsymbol{ZX}\rangle = 1.$$

Explicitly, we express the meter-coherent state $|\xi\rangle$ into two modes as $|\xi\rangle = |\alpha_H, 0_V\rangle$. We have

$$\langle \boldsymbol{S}_x\rangle_\xi = \langle \alpha_H, 0_V|(\boldsymbol{a}_H^\dagger \boldsymbol{a}_H - \boldsymbol{a}_V^\dagger \boldsymbol{a}_V)|\alpha_H, 0_V\rangle = |\alpha|^2, \quad (B4)$$

$$\begin{aligned}\langle \boldsymbol{S}_y^2\rangle_\xi &= \langle \alpha_H, 0_V|(\boldsymbol{a}_H^\dagger \boldsymbol{a}_V + \boldsymbol{a}_H \boldsymbol{a}_V^\dagger)^2|\alpha_H, 0_V\rangle \\ &= \langle \alpha_H, 0_V|(\boldsymbol{a}_H^\dagger \boldsymbol{a}_V \boldsymbol{a}_H^\dagger \boldsymbol{a}_V + \boldsymbol{a}_H^\dagger \boldsymbol{a}_V \boldsymbol{a}_H \boldsymbol{a}_V^\dagger \\ &\quad + \boldsymbol{a}_H \boldsymbol{a}_V^\dagger \boldsymbol{a}_H^\dagger \boldsymbol{a}_V + \boldsymbol{a}_H \boldsymbol{a}_V^\dagger \boldsymbol{a}_H \boldsymbol{a}_V^\dagger)|\alpha_H, 0_V\rangle \\ &= |\alpha|^2, \end{aligned} \quad (B5)$$

and

$$\begin{aligned}\langle \boldsymbol{S}_x^2\rangle_\xi &= \langle \alpha_H, 0_V|(\boldsymbol{a}_x^\dagger \boldsymbol{a}_x - \boldsymbol{a}_y^\dagger \boldsymbol{a}_y)^2|\alpha_H, 0_V\rangle \\ &= |\alpha|^2 + |\alpha|^4. \end{aligned} \quad (B6)$$

Then, we obtain the square error:

$$\begin{aligned}\epsilon^2_{\sigma_z} = \langle \boldsymbol{N}^2_{\sigma_z}\rangle &= \frac{1}{|\alpha|^2}[\cot^2(2g) + 1] \\ &= \frac{1}{|\alpha|^2 \sin^2(2g)}. \end{aligned} \quad (B7)$$

2. Disturbance

Next, we calculate the rms disturbance, starting from the $\boldsymbol{B}_T$ operator in Eq. (A7),

$$\begin{aligned}\boldsymbol{B}_T &\equiv (\boldsymbol{\sigma}_x \otimes \boldsymbol{I})_T \\ &= [\boldsymbol{\sigma}_x \otimes \cos(2g\boldsymbol{S}_z)]_0 - [\boldsymbol{\sigma}_y \otimes \sin(2g\boldsymbol{S}_z)]_0. \end{aligned} \quad (B8)$$

The disturbance operator reads

$$\boldsymbol{D}_{\sigma_x} = (\boldsymbol{\sigma}_x \otimes \boldsymbol{I})_T - (\boldsymbol{\sigma}_x \otimes \boldsymbol{I})_0. \quad (B9)$$

Substituting Eq. (B8) into Eq. (B9) and taking the square of both sides, we have

$$\boldsymbol{D}^2_{\sigma_x} = \Big[\underbrace{(\boldsymbol{\sigma}_x \otimes [\cos(2g\boldsymbol{S}_z) - \boldsymbol{I}])_0}_{\boldsymbol{X}} - \underbrace{[\boldsymbol{\sigma}_y \otimes \sin(2g\boldsymbol{S}_z)]_0}_{\boldsymbol{Y}}\Big]^2. \quad (B10)$$

Now, we calculate the average $\langle \boldsymbol{D}^2_{\sigma_x}\rangle$ over the initial joint state $|\psi\rangle \otimes |\xi\rangle$. We have

$$\begin{aligned}\langle \boldsymbol{X}^2\rangle &= \langle [\cos(2g\boldsymbol{S}_z) - \boldsymbol{I}]^2\rangle_\xi, \\ \langle \boldsymbol{Y}^2\rangle &= \langle \sin^2(2g\boldsymbol{S}_z)\rangle_\xi, \\ \langle \boldsymbol{XY}\rangle &= \langle \boldsymbol{YX}\rangle = 0.\end{aligned}$$

Finally, we have

$$\langle \boldsymbol{D}^2_{\sigma_x}\rangle = 2[1 - \langle \cos(2g\boldsymbol{S}_z)\rangle_\xi]. \quad (B11)$$

For $|\xi\rangle = |\alpha_H, 0_V\rangle$ and using the operator ordering relation [92–94], such as $e^{\kappa \boldsymbol{a}^\dagger \boldsymbol{a}} =: e^{(e^\kappa - 1)\boldsymbol{a}^\dagger \boldsymbol{a}}$:, we have

$$\langle \cos(2g\boldsymbol{S}_z)\rangle_\xi = e^{-2|\alpha|^2 \sin^2 g}. \quad (B12)$$

Finally, we obtain the square disturbance:

$$\eta^2_{\sigma_x} = \langle \boldsymbol{D}^2_{\sigma_x}\rangle = 2(1 - e^{-2|\alpha|^2 \sin^2 g}), \quad (B13)$$

as shown in Eq. (20) in the main text.

APPENDIX C: PHASE-SPACE APPROXIMATION AND POLARIZATION-SQUEEZED LIGHT METER

In this Appendix, we examine a class of the polarization-squeezed coherent meter states using the phase-space approximation. Let us consider a class of the squeezed coherent state of the two polarization modes as

$$|\xi\rangle = |\alpha, z\rangle = \mathcal{D}(\alpha)\mathcal{S}(z)|0\rangle_H|0\rangle_V, \quad (C1)$$

where $\mathcal{D}(\alpha) = \exp[\alpha \boldsymbol{a}_H^\dagger - \alpha^* \boldsymbol{a}_H]$ is the displacement operator with $\alpha = |\alpha|e^{i\phi}$ in the polar form and the two-mode squeezing operator is chosen to be $\mathcal{S}(z) = \exp[z^* \boldsymbol{a}_L \boldsymbol{a}_R - z \boldsymbol{a}_L^\dagger \boldsymbol{a}_R^\dagger]$ with $z = re^{i\vartheta}$, and $\boldsymbol{a}_{L(R)} = (\boldsymbol{a}_H \pm i\boldsymbol{a}_V)/\sqrt{2}$. This squeezing operator is equivalent to $\mathcal{S}(z) = \mathcal{S}_H(z)\mathcal{S}_V(z)$ where $\mathcal{S}_H(z) = \exp[\frac{1}{2}z^* \boldsymbol{a}_H^2 - \frac{1}{2}z(\boldsymbol{a}_H^\dagger)^2]$ and $\mathcal{S}_V(z) = \exp[\frac{1}{2}z^* \boldsymbol{a}_V^2 - \frac{1}{2}z(\boldsymbol{a}_V^\dagger)^2]$.

It is known that when $r > 0$, $\phi - \vartheta/2 = 0$ results in the amplitude-squeezed coherent state, while the phase-squeezed coherent state will happen when $\phi - \vartheta/2 = \pm\pi/2$ [95]. In the following, we choose $\phi - \vartheta/2 = 0$.

The mean photon numbers in the two modes are

$$\begin{aligned}\langle n_H\rangle &= \langle \boldsymbol{a}_H^\dagger \boldsymbol{a}_H\rangle \\ &= {}_H\langle 0|\mathcal{S}_H^\dagger(z)\mathcal{D}^\dagger(\alpha)\boldsymbol{a}_H^\dagger \boldsymbol{a}_H \mathcal{D}(\alpha)\mathcal{S}_H(z)|0\rangle_H \\ &= |\alpha|^2 + \sinh^2 r,\end{aligned} \tag{C2}$$

$$\langle n_V\rangle = \langle \boldsymbol{a}_V^\dagger \boldsymbol{a}_V\rangle = \sinh^2 r, \tag{C3}$$

and the variances are

$$\begin{aligned}\sigma_{n_H}^2 &= |\alpha|^2 e^{-2r} + 2\cosh^2 r \sinh^2 r \\ &= |\alpha|^2 e^{-2r} + \tfrac{1}{2}\sinh^2 2r,\end{aligned} \tag{C4}$$

$$\sigma_{n_V}^2 = \tfrac{1}{2}\sinh^2 2r. \tag{C5}$$

The expectation values of the Stokes operators give

$$\langle \boldsymbol{S}_0\rangle = |\alpha|^2 + 2\sinh^2 r, \tag{C6}$$

$$\langle \boldsymbol{S}_x\rangle = |\alpha|^2, \text{ and } \langle \boldsymbol{S}_y\rangle = \langle \boldsymbol{S}_z\rangle = 0. \tag{C7}$$

The variances give

$$\sigma_{S_0}^2 = \sigma_{S_x}^2 = \sigma_{S_y}^2 = |\alpha|^2 e^{-2r} + \sinh^2 2r \tag{C8}$$

$$\sigma_{S_z}^2 = |\alpha|^2 e^{2r}. \tag{C9}$$

For $|\alpha| \gg e^{3r}$, we can ignore the term $\sinh^2 2r$ and read

$$\sigma_{S_0}^2 = \sigma_{S_x}^2 = \sigma_{S_y}^2 = |\alpha|^2 e^{-2r}, \tag{C10}$$

$$\sigma_{S_z}^2 = |\alpha|^2 e^{2r}. \tag{C11}$$

Thus, for $r > 0$, we observe squeezing in $\sigma_{S_0}^2$, $\sigma_{S_x}^2$, and $\sigma_{S_y}^2$, while antisqueezing appears in $\sigma_{S_z}^2$.

We introduce the canonical operators

$$\boldsymbol{q} = \frac{\boldsymbol{S}_y}{\sqrt{|\langle \boldsymbol{S}_x\rangle|}}, \text{ and } \boldsymbol{p} = \frac{\boldsymbol{S}_z}{\sqrt{|\langle \boldsymbol{S}_x\rangle|}}, \tag{C12}$$

which is proportional to the polarized Stokes operators $\boldsymbol{S}_y$ and $\boldsymbol{S}_z$, respectively. The commutation relation $[\boldsymbol{q},\boldsymbol{p}] = 2i\boldsymbol{S}_x/|\langle \boldsymbol{S}_x\rangle| \simeq 2i$ indicates that $\boldsymbol{q}$ and $\boldsymbol{p}$ can be regarded as a pair of canonical operators when $\boldsymbol{S}_x$ can be approximated by a classical positive constant such as $|\langle \boldsymbol{S}_x\rangle|$ so that $[\boldsymbol{S}_x,\boldsymbol{S}_y] \simeq 0$ and $[\boldsymbol{S}_x,\boldsymbol{S}_z] \simeq 0$. It means that the third and higher terms in Eq. (A2) can be ignored as in the case where $g \ll 1$, and that the Stokes operators, which essentially hold the discrete nature of the photon number, are replaced by the canonical continuous variable operators. Again, this approximation is valid only when $g \ll 1$ so that the change in $\boldsymbol{S}_x/|\langle \boldsymbol{S}_x\rangle|$, which is in the order of g^2, is sufficiently small.

Then, the variances in $\boldsymbol{q}$ and $\boldsymbol{p}$ read

$$\sigma_{\boldsymbol{q}}^2 = e^{-2r}; \quad \sigma_{\boldsymbol{p}}^2 = e^{2r}. \tag{C13}$$

We can define the wave function for the polarization-squeezed coherent state as

$$\psi(q) = \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} e^{-\frac{q^2}{4\sigma^2}}, \tag{C14}$$

where $\sigma = \sqrt{\sigma_{\boldsymbol{q}}^2} = e^{-r}$ represents a squeezing parameter:

$$\begin{cases}\sigma < 1 \to r > 0: & \text{amplitude-squeezed,}\\ \sigma = 1 \to r = 0: & \text{no squeezed,}\\ \sigma > 1 \to r < 0: & \text{phase-squeezed,}\end{cases}$$

Then, the meter light state $|\xi\rangle$ can be defined as

$$\begin{aligned}|\xi\rangle &= \int \psi(q)|q\rangle dq \\ &= \left(\frac{1}{2\pi\sigma^2}\right)^{1/4} \int e^{-\frac{q^2}{4\sigma^2}}|q\rangle dq,\end{aligned} \tag{C15}$$

which we name as the polarization-squeezed coherent state.

1. Error

The interaction evolution is defined by

$$\boldsymbol{U}_T = e^{-ig|\alpha|\boldsymbol{\sigma}_z\otimes\boldsymbol{p}}. \tag{C16}$$

Using the BCH formula, we have

$$\begin{aligned}(\boldsymbol{I}\otimes\boldsymbol{q})_T &= e^{ig|\alpha|\boldsymbol{\sigma}_z\otimes\boldsymbol{p}}\,(\boldsymbol{I}\otimes\boldsymbol{q})_0\,e^{-ig|\alpha|\boldsymbol{\sigma}_z\otimes\boldsymbol{p}} \\ &= (\boldsymbol{I}\otimes\boldsymbol{q})_0 + 2g|\alpha|(\boldsymbol{\sigma}_z\otimes\boldsymbol{I})_0.\end{aligned} \tag{C17}$$

Therefore, if we measure the calibrated meter operator $(\boldsymbol{I}\otimes\boldsymbol{q})_T/2g|\alpha|$, we will obtain the corresponding information of $(\boldsymbol{\sigma}_z\otimes\boldsymbol{I})_0$ in the system.

We first calculate the error $\epsilon_{\boldsymbol{\sigma}_z}^2 = \langle N_{\boldsymbol{\sigma}_z}^2\rangle_\xi = \langle \boldsymbol{q}^2\rangle_\xi/4g^2|\alpha|^2$. Particularly, for $|\xi\rangle = (\frac{1}{2\pi\sigma^2})^{1/4}\int e^{-\frac{q^2}{4\sigma^2}}|q\rangle\, dq$, we have

$$\begin{aligned}\langle \boldsymbol{q}^2\rangle_\xi &= \left(\frac{1}{2\pi\sigma^2}\right)^{1/4}\int e^{-\frac{q^2}{4\sigma^2}}\langle q|dq \cdot \int q_1^2|q_1\rangle\langle q_1|dq_1 \\ &\quad\times\left(\frac{1}{2\pi\sigma^2}\right)^{1/4}\int e^{-\frac{q_2^2}{4\sigma^2}}|q_2\rangle dq_2 \\ &= \left(\frac{1}{2\pi\sigma^2}\right)^{1/2}\int q^2 e^{-\frac{q^2}{2\sigma^2}}dq \\ &= \left(\frac{1}{2\pi\sigma^2}\right)^{1/2}\frac{1}{2}\sqrt{8\pi\sigma^6} \\ &= \sigma^2.\end{aligned} \tag{C18}$$

Then, we obtain the square error

$$\epsilon_{\boldsymbol{\sigma}_z}^2 = \frac{\langle \boldsymbol{q}^2\rangle_\xi}{4g^2|\alpha|^2} = \frac{1}{4\chi^2}, \tag{C19}$$

where in the last equality we have set $\chi = g|\alpha|/\sigma$.

2. Disturbance

In a similar manner as in Appendix B, we obtain

$$\left\langle \boldsymbol{D}_{\boldsymbol{\sigma}_x}^2\right\rangle = 2[1 - \langle \cos(2g|\alpha|\boldsymbol{p})\rangle_\xi]. \tag{C20}$$

Using the Fourier transformation, we recast the meter state $|\xi\rangle$ in the momentum representation as $|\xi\rangle = (\frac{2\sigma^2}{\pi})^{1/4}\int e^{\frac{-\sigma^2p^2}{4}}|p\rangle\, dp$. Then, we obtain

$$\eta_{\boldsymbol{\sigma}_x}^2 = \left\langle \boldsymbol{D}_{\boldsymbol{\sigma}_x}^2\right\rangle = 2(1 - e^{-2\chi^2}), \tag{C21}$$

as shown in Eq. (25) in the main text.

APPENDIX D: WEAK INTERACTION APPROXIMATION

Under the WIA, we assume $g \ll 1$ and $g|\alpha| \ll 1$. From Eqs. (B7) and (B13), we obtain

$$\epsilon_{\sigma_z}^2 \approx \frac{1}{4g^2|\alpha|^2}, \quad \text{and} \quad \eta_{\sigma_x}^2 \approx 4g^2|\alpha|^2. \tag{D1}$$

From Eqs. (C19) and (C21), we obtain for $\chi \ll 1$,

$$\epsilon_{\sigma_z}^2 \approx \frac{1}{4\chi^2}, \quad \text{and} \quad \eta_{\sigma_x}^2 \approx 4\chi^2, \tag{D2}$$

which are equivalent to Eq. (D1) when $\sigma = 1$. Note that, in Eq. (C19), we already assumed that $g \ll 1$ in the phase-space approximation. In Eqs. (D1) and (D2), we observe $\epsilon_{\sigma_z}^2 \eta_{\sigma_x}^2 = 1$ and thus the Heisenberg-Arthurs-Kelly uncertainty is valid with minimal uncertainty.

We also show that, under the WIA, both the noise and disturbance are unbiased. As described in the main text, the noise operator (B2) is already unbiased, i.e., $\langle \boldsymbol{N}_{\sigma_z} \rangle_\xi = 0$ and thus $\langle \boldsymbol{N}_{\sigma_z} \rangle = 0$ irrespective of $|\psi\rangle$, provided that $\langle \boldsymbol{S}_y \rangle_\xi = 0$. Obviously, it is also true in the case of WIA. For the disturbance operator, from Eqs. (B8) and (B9), we get

$$\boldsymbol{D}_{\sigma_x} = (\boldsymbol{\sigma}_x \otimes [\cos(2g\boldsymbol{S}_z) - \boldsymbol{I}])_0 - [\boldsymbol{\sigma}_y \otimes \sin(2g\boldsymbol{S}_z)]_0 \tag{D3}$$

and

$$\begin{aligned} \langle \boldsymbol{D}_{\sigma_x} \rangle_\xi &= \boldsymbol{\sigma}_x[\langle \cos(2g\boldsymbol{S}_z) \rangle_\xi - 1] - \boldsymbol{\sigma}_y \langle \sin(2g\boldsymbol{S}_z) \rangle_\xi \\ &= \boldsymbol{\sigma}_x(e^{-2|\alpha|^2 \sin^2 g} - 1). \end{aligned} \tag{D4}$$

Here, we use Eq. (B12) and $\langle \sin(2g\boldsymbol{S}_z) \rangle_\xi = 0$. Thus, $\langle \boldsymbol{D}_{\sigma_x} \rangle_\xi \approx 0$ when $g \ll 1$ and $g|\alpha| \ll 1$. Consequently, for our initial meter state under the WIA, both the noise and disturbance operators are unbiased, i.e., $\langle \boldsymbol{N}_{\sigma_z} \rangle = \langle \boldsymbol{D}_{\sigma_x} \rangle = 0$ irrespective of the initial system state $|\psi\rangle$. This *joint-unbiassedness* condition is sufficient for holding the Heisenberg-Arthurs-Kelly uncertainty [25].

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