Nonlocality sharing for a three-qubit system via multilateral sequential measurements

Changliang Ren,[1,*] Xiaowei Liu,[1] Wenlin Hou,[1] Tianfeng Feng,[2] and Xiaoqi Zhou[2,†]

[1]*Key Laboratory of Low-Dimensional Quantum Structures and Quantum Control of Ministry of Education, Key Laboratory for Matter Microstructure and Function of Hunan Province, Department of Physics and Synergetic Innovation Center for Quantum Effects and Applications, Hunan Normal University, Changsha 410081, China*

[2]*State Key Laboratory of Optoelectronic Materials and Technologies and School of Physics, Sun Yat-sen University, Guangzhou, People's Republic of China*

(Received 6 December 2021; accepted 12 May 2022; published 25 May 2022)

Nonlocality sharing for a three-qubit system via multilateral sequential measurements was deeply discussed. Nonlocality sharing based on the multiple violations of Mermin-Ardehali-Belinskii-Klyshko (MABK) inequality in the trilateral sequential measurements scenario can be observed, where all of eight MABK inequalities can be violated simultaneously. Nevertheless, the genuine nonlocality sharing based on the multiple violation of Svetlichny inequality can be only observed in the unilateral sequential measurements scenario. Compared with two-qubit cases, the nonlocality sharing in a three-qubit system shows more fruitful characteristics.

DOI: 10.1103/PhysRevA.105.052221

I. INTRODUCTION

In 1935, Einstein, Podolsky, and Rosen (EPR) first discussed the distinctive nonclassical properties of quantum physics in their seminal paper [1], which indicated some conflict between quantum mechanics and local realism. Then, Bell proposed a way to exhibit this conflict between classical correlations and quantum correlations, which is referred to as the Bell inequality [2]. Subsequently, Bell-type inequalities have been studied extensively from various perspectives [3–10] and experimentally verified in many different quantum systems [11–19]. These kinds of research are not only crucial to deeply understanding quantum theory, but also play an important role in quantum information protocols, such as quantum key distribution [20], randomness generation [21–25], and entanglement certification [26]. For a background on Bell inequalities, readers could refer to Ref. [27] and references therein.

Inspired by Bell's work, Clauser, Horne, Shimony, and Holt (CHSH) derived a modified inequality [3], which provides a faithful way of experimentally testing the nonlocality property in two-qubit composite systems. However, most discussions of nonlocality based on CHSH inequality focus on one pair of entangled qubits distributed to only two separated observers. Recently, a surprising result—that nonlocality can actually be shared among more than two observers using weak measurements—has been reported by Silva *et al.* [28]. In Silva's scenario, a pair of maximally entangled qubits is distributed to three observers Alice, Bob_1, and Bob_2, in which Alice accesses one qubit and the two Bobs access the other qubit. Alice performs a strong measurement on her own qubit, while Bob_1 performs a weak measurement on his qubit and passes it to Bob_2. Finally Bob_2 carries out a strong measurement. The measurement results reveal that it is possible to observe a simultaneous violation of CHSH inequalities between Alice-Bob_1 and Alice-Bob_2. To date, a series of fruitful related theoretical works [29–45] have been published by tracking this path and several experimental demonstrations have also been performed [46–48]. In particular, this shows that nonlocality sharing can be observed in a wide range even if Bob_1's measurements are close to strong measurements [36,48], which is impossible in the original protocol [28]. Nevertheless, almost all discussions are limited to unilateral sequential cases, i.e., one entangled pair is distributed to one Alice and multiple Bobs. Recently, Zhu *et.al* explored the nonlocality sharing in the bilateral sequential measurements case in which one entangled pair is distributed to multiple Alices and Bobs [43]. But Bell-type nonlocality sharing between Alice_1-Bob_1 and Alice_2-Bob_2 is impossible in such a scenario [43].

In this work, we explored the nonlocality sharing for a three-qubit system via multilateral sequential measurements, where two different Bell-type inequalities, MABK inequality and Svetlichny inequality, were considered. In contrast to two-qubit cases, a complete nonlocality sharing with all of eight MABK inequalities' simultaneous violations can be observed in the trilateral sequential measurements scenario. The genuine nonlocality sharing based on the multiple violation of Svetlichny inequality was also discussed. It is shown that genuine nonlocality sharing can be observed only in the unilateral sequential measurements scenario. These results not only shed new light on the interplay between nonlocality and quantum measurements, especially the emergence of nonlocality sharing via weak measurements, but can also be applied in unbounded randomness certification [49], quantum coherence [33], and quantum steering [34].

The paper is organized as follows. In Sec. II, we described the scenario of a three-qubit system via multilateral sequential

*Corresponding author: renchangliang@hunnu.edu.cn
†Corresponding author: zhouxq8@mail.sysu.edu.cn

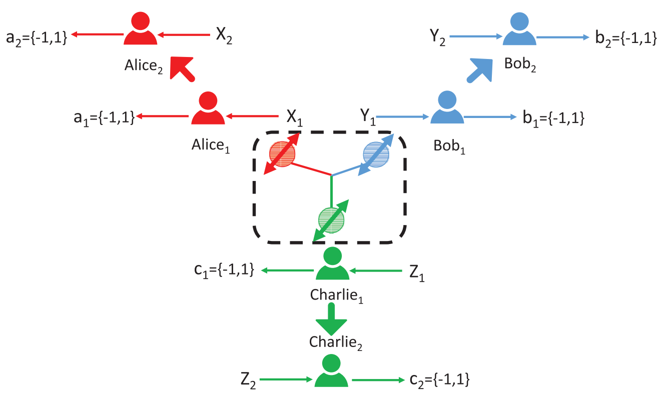

FIG. 1. Scenario of the trilateral sequential case: a three-qubit entangled state is distributed to three sides, and each side has two observers, in which the first observers on each side perform weak measurements and the second observers perform strong measurements. Two Alices occupy one-third of the state. Bobs and Charlies have access to the other different. $\hat{X}_i, \hat{Y}_j, \hat{Z}_k \in \{1, -1\}$ and $a_i, b_j, c_k \in \{1, -1\}$ represent measurement directions and measurement outcomes respectively, where $\{i, j, k\} \in \{1, 2\}$.

measurements. Nonlocality sharing based on the multiple violation of MABK inequality and genuine nonlocality sharing based on the multiple violation of Svetlichny inequality were analyzed in Secs. III and IV, respectively. In Sec. V, we conclude the paper.

II. THE MULTILATERAL SEQUENTIAL MEASUREMENTS SCENARIO

The scenario illustrated in Fig. 1 is considered, where a three-qubit entangled state is distributed to three remote sides, and each side has two observers, which can be named as $\{\text{Alice}_1, \text{Alice}_2\}$, $\{\text{Bob}_1, \text{Bob}_2\}$, and $\{\text{Charlie}_1, \text{Charlie}_2\}$ respectively. Those observers on the same side will measure their shared qubit sequentially. The communication between the observers is forbidden, and the measurement choices of these observers are independent. Each observer randomly chooses one of two observables to measure, which can be defined as $\hat{X}_i = A_{i,l}$ for Alice_i, $\hat{Y}_j = B_{j,m}$ for Bob_j, and $\hat{Z}_k = C_{k,n}$ for Charlie_k, where $A_{i,l}$ is the lth measurement chosen by the observer Alice_i, and it is similar for $B_{j,m}$ and $C_{k,n}$, $\{i, j, k\} \in \{1, 2\}$ where $\{l, m, n\} \in \{1, 2\}$. The binary outcomes of observers' dichotomic measurements are given by a_i, b_j, c_k with $\{a_i, b_j, c_k\} \in \{-1, 1\}$. Such a scenario is characterized by the joint conditional probability of the outcomes $P(a_1, a_2, b_1, b_2, c_1, c_2|\hat{X}_1, \hat{X}_2, \hat{Y}_1, \hat{Y}_2, \hat{Z}_1, \hat{Z}_2)$.

In the scenario, the first observer on each side performs weak measurements, while the second observer of each side carries out strong measurements. In the whole measurement process, we can always obtain the quantum state after measurement according to the selection of measurement and its outcome. Without loss of generality, an arbitrary observable $\hat{O} \in \{\hat{X}_i, \hat{Y}_j, \hat{Z}_k\}$ can be defined as $\hat{O} = \vec{\xi} \cdot \vec{\sigma}$, where $\vec{\sigma}$ is a vector consists of three Pauli matrices $\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$. The weak measurement of $\hat{O}$ can be introduced as follows [28]. As we know, for an arbitrary single-qubit state $|\Phi\rangle$, can be represented by the superposition of the eigenvectors of the measured observable $\hat{O}$, which is $|\Psi\rangle = a|O^+\rangle + b|O^-\rangle$, where $|O^\pm\rangle$ is the eigenvector of $\hat{O}$. In the standard von Neumann measurement scheme, the quantum state interacts with a pointer, which serves as a measurement apparatus, and the outcome of the observable is obtained from the position shift of the pointer. Hence, after an interaction with a pointer, the state $|\Phi\rangle$ becomes $a|O^+\rangle \otimes |\phi(q-1)\rangle + b|O^-|\rangle \otimes |\phi(q+1)\rangle$, where $|\phi(q)\rangle$ is the initial state of the pointer. A weak measurement is defined when the pointer spread is very large, since the pointer position supplies few information and the system has little disturbance. The quality factor F and precision factor G characterize the weak measurement, where $F = \int_{-\infty}^{+\infty} \langle\phi(q+1)|\phi(q-1)\rangle dq$ and $G = \int_{-1}^{1} \phi^2(q) dq$ respectively [28]. F represents the undisturbed extent to the initial state after the measurement and G is the precision factor which quantifies the information gain from the measurement. Hence, the weak measurement of $\hat{O}$ with the outcome ± 1 can be regarded as a positive operator valued measurement (POVM) measurement, where the operator can be given as $E_\pm = G|O^\pm\rangle\langle O^\pm| + (1-G)I/2$ [34]. When $F = 0$ and $G = 1$, it corresponds to a strong measurement, and the operators $E_\pm$ change to the projective measurements. As introduced in Ref. [28], there exists a trade-off between measurement disturbance F and information gain G. For example, the optimal weak measurement requires that $F^2 + G^2 = 1$, where "optimal" means that the most information can be extracted with the same disturbance. We consider that weak measurements in this scenario have the optimal pointer distribution.

We assume that the density matrix of a three-qubit state is ρ. Alice_1 first performs a weak measurement $\hat{X}_1$ on her received qubit with the quality factor F_1 and precision factor G_1. When the measurement outcome is a_1, according to the discussion in Ref. [28], the state changes to

$$\rho_{\hat{X}_1}^{a_1} = \frac{F_1}{2}\rho + \frac{1 + a_1 G_1 - F_1}{2}\left[U_{\hat{X}_1}^{-1}\rho\left(U_{\hat{X}_1}^{-1}\right)^\dagger\right] + \frac{1 - a_1 G_1 - F_1}{2}\left[U_{\hat{X}_1}^{+1}\rho\left(U_{\hat{X}_1}^{+1}\right)^\dagger\right], \tag{1}$$

where $U_{\hat{X}_i}^{a_i} = \Pi_{\hat{X}_i}^{a_i} \otimes I \otimes I$ and $\Pi_{\hat{X}_i}^{a_i} = \frac{I + a_i \hat{X}_i}{2}$. Subsequently, if Alice_2 performs a strong measurement $\hat{X}_2$ with the outcome a_2, the three-qubit state will change to

$$\rho_{\hat{X}_2}^{a_2} = U_{\hat{X}_2}^{a_2} \rho_{\hat{X}_1}^{a_1} U_{\hat{X}_2}^{a_2 \dagger}. \tag{2}$$

Later, Bob_1 performs a weak measurement $\hat{Y}_j$ on his received qubit with the quality factor F_2 and precision factor G_2 of the measurement, with $F_2^2 + G_2^2 = 1$. When the measurement outcome is b_1, the three-qubit state becomes

$$\rho_{\hat{Y}_1}^{b_1} = \frac{F_2}{2}\rho_{\hat{X}_2}^{a_2} + \frac{1 + b_1 G_2 - F_2}{2}\left[U_{\hat{Y}_1}^{-1}\rho_{\hat{X}_2}^{a_2}\left(U_{\hat{Y}_1}^{-1}\right)^\dagger\right] + \frac{1 - b_1 G_2 - F_2}{2}\left[U_{\hat{Y}_1}^{+1}\rho_{\hat{X}_2}^{a_2}\left(U_{\hat{Y}_1}^{+1}\right)^\dagger\right], \tag{3}$$

where $\hat{Y}_j = B_{j,m} = \vec{\xi}_{j,m} \cdot \vec{\sigma}$, $U_{\hat{Y}_j}^{b_j} = I \otimes \Pi_{\hat{Y}_j}^{b_j} \otimes I$. Similarly, if Bob_2 performs a strong measurement $\hat{Y}_2$ with the outcome b_2, the three-qubit state becomes $\rho_{\hat{Y}_2}^{b_2} = U_{\hat{Y}_2}^{b_2} \rho_{\hat{Y}_1}^{b_1} U_{\hat{Y}_2}^{b_2 \dagger}$. Subsequently, Charlie_1 performs a weak measurement $\hat{Z}_k$ on his received qubit with the quality factor F_3 and precision factor G_3 of the measurement, where the optimal weak measurement requires that $F_3^2 + G_3^2 = 1$. When the measurement outcome is c_1, then the three-qubit state changes to

$$\rho_{\hat{Z}_1}^{c_1} = \frac{F_3}{2}\rho_{\hat{Y}_2}^{b_2} + \frac{1 + c_1 G_3 - F_3}{2}\left[U_{\hat{Z}_1}^{-1}\rho_{\hat{Y}_2}^{b_2}\left(U_{\hat{Z}_1}^{-1}\right)^{\dagger}\right] + \frac{1 - c_1 G_3 - F_3}{2}\left[U_{\hat{Z}_1}^{+1}\rho_{\hat{Y}_2}^{b_2}\left(U_{\hat{Z}_1}^{+1}\right)^{\dagger}\right], \tag{4}$$

where $\hat{Z}_k = B_{k,n} = \vec{\xi}_{k,n} \cdot \vec{\sigma}$ and $U_{\hat{Z}_k}^{c_k} = I \otimes I \otimes \Pi_{\hat{Z}_k}^{c_k}$. Finally, when Charlie$_2$ performs a strong measurement $\hat{Z}_2$ with the outcome c_2, the three-qubit state will turn to $\rho_{\hat{Z}_2}^{c_2} = U_{\hat{Z}_2}^{c_2}\rho_{\hat{Z}_1}^{c_1}U_{\hat{Z}_2}^{c_2\dagger}$. So, a cyclic measurement process of this scenario has been completely described. From the unnormalized post-measurement state $\rho_{\hat{Z}_2}^{c_2}$, the joint probability could be obtained, $P(a_1, a_2, b_1, b_2, c_1, c_2|\hat{X}_1, \hat{X}_2, \hat{Y}_1, \hat{Y}_2, \hat{Z}_1, \hat{Z}_2) = \mathrm{Tr}[\rho_{\hat{Z}_2}^{c_2}]$.

In order to investigate nonlocality sharing in such a scenario, we are more concerned about the joint conditional probabilities of the measurement of any three trilateral observers. It is assumed that the observable choosing is unbiased for each observer, which requires that each measurement setting of every observer should be chosen with equal probability. The joint conditional probability $P(a_i, b_j, c_k|\hat{X}_i, \hat{Y}_j, \hat{Z}_k)$ is obtained via marginalizing the corresponding variables,

$$P(a_i, b_j, c_k|\hat{X}_i, \hat{Y}_j, \hat{Z}_k) = \sum_{a_{i'}, b_{j'}, c_{k'}} P(a_i, b_j, c_k, a_{i'}, b_{j'}, c_{k'}|\hat{X}_i, \hat{Y}_j, \hat{Z}_k, \hat{X}_{i'}, \hat{Y}_{j'}, \hat{Z}_{k'}). \tag{5}$$

Based on the joint conditional probability distribution, the expected value $E(\hat{X}_i, \hat{Y}_j, \hat{Z}_k)$ can be given as

$$E(\hat{X}_i, \hat{Y}_j, \hat{Z}_k) = \sum_{a_i, b_j, c_k} a_i b_j c_k \, P(a_i, b_j, c_k|\hat{X}_i, \hat{Y}_j, \hat{Z}_k). \tag{6}$$

III. NONLOCALITY SHARING BASED ON THE MULTIPLE VIOLATION OF MABK INEQUALITY

The quantum nonlocality can be witnessed via violations of corresponding inequalities. To explore the phenomenon of nonlocality sharing for a three-qubit system via multilateral sequential measurements, we first consider the typical N-qubit Bell-type inequality, Mermin-Ardehali-Belinskii-Klyshko (MABK) inequality, which can be described as

$$|-E(A_{\mathrm{i},1}, B_{\mathrm{j},1}, C_{\mathrm{k},1}) + E(A_{\mathrm{i},2}, B_{\mathrm{j},1}, C_{\mathrm{k},2}) + E(A_{\mathrm{i},2}, B_{\mathrm{j},2}, C_{\mathrm{k},1}) + E(A_{\mathrm{i},1}, B_{\mathrm{j},2}, C_{\mathrm{k},2})| \leqslant 2. \tag{7}$$

Obviously, by choosing different observers on each side, we can discuss the violations of eight MABK inequalities. For clarity of discussions, we denote MABK quantity as B_ω, which is the value on the left side of Eq. (7), for the combination of different observers, where B_1 for $(i = j = k = 1)$, B_2 for $(i = k = 1, j = 2)$, B_3 for $(i = j = 1, k = 2)$, B_4 for $(j = k = 1, i = 2)$, B_5 for $(j = k = 2, i = 1)$, B_6 for $(i = k = 2, j = 1)$, B_7 for $(i = j = 2, k = 1)$, and B_8 for $(i = j = k = 2)$. Each expected value in every MABK inequality can be obtained from Eq. (6). Thus, it is possible to check whether there exists a multiple violation via calculation results.

As is known, nonlocality sharing between Alice$_1$-Bob$_1$ and Alice$_2$-Bob$_2$ is impossible in a two-qubit system [43]. We first explore whether nonlocality sharing between Alice$_1$-Bob$_1$-Charlie$_1$ and Alice$_2$-Bob$_2$-Charlie$_2$ exists in a three-qubit system or not.

Without loss of generality, we assume that the observers on the three different sides share a three-qubit GHZ state, which is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle). \tag{8}$$

Each observer has two measurement directions to choose, and the directions of the dichotomic measurements are denoted as $(\{\theta_{11}, \phi_{11}\}, \{\theta_{12}, \phi_{12}\})$ for Alice$_1$, $(\{\theta_{13}, \phi_{13}\}, \{\theta_{14}, \phi_{14}\})$ for Alice$_2$, $(\{\theta_{21}, \phi_{21}\}, \{\theta_{22}, \phi_{22}\})$ for Bob$_1$, $(\{\theta_{23}, \phi_{23}\}, \{\theta_{24}, \phi_{24}\})$ for Bob$_2$, $(\{\theta_{31}, \phi_{31}\}, \{\theta_{32}, \phi_{32}\})$ for Charlie$_1$, and $(\{\theta_{33}, \phi_{33}\}, \{\theta_{34}, \phi_{34}\})$ for Charlie$_2$.

To explore nonlocality sharing between Alice$_1$-Bob$_1$-Charlie$_1$ and Alice$_2$-Bob$_2$-Charlie$_2$, it is necessary to determine whether the MABK quantities, B_1 and B_8, can surpass the classical bound simultaneously or not. To simplify calculations, we require that the measurement directions of every observer always be in the X-Y plane, where $\theta_{11} = \theta_{12} = \theta_{13} = \theta_{14} = \theta_{21} = \theta_{22} = \theta_{23} = \theta_{24} = \theta_{31} = \theta_{32} = \theta_{33} = \theta_{34} = \frac{\pi}{2}$. Unfortunately, it is still too complex to obtain the analytical solution which can show the maximal nonlocality sharing between Alice$_1$-Bob$_1$-Charlie$_1$ and Alice$_2$-Bob$_2$-Charlie$_2$. We have calculated the numerical optimal solution where the optimal double violation of these MABK inequalities is small. Hence, we preferred to exhibit nonlocality sharing with a set of suboptimal analytical solutions. After all, errors are inevitable in numerical solutions. Interestingly, when we chose such simple measurement settings, $\phi_{11} = \phi_{21} = \phi_{31} = 0$, $\phi_{12} = \phi_{22} = \phi_{32} = -\phi_{14} = -\phi_{24} = \frac{\pi}{2}$, $\phi_{13} = -\phi_{23} = -\phi_{33} = \pi$, $\phi_{34} = \frac{3\pi}{2}$, the MABK quantities, B_1 and B_8, turn to

$$B_1 = 4G_1G_2G_3, \tag{9}$$

$$B_8 = \tfrac{1}{2}(1 + F_1)(1 + F_2)(1 + F_3). \tag{10}$$

For simplicity, when $G_1 = G_2 = G_3 = G$, B_1 and B_8 change to $B_1 = 4G^3$ and $B_8 = \frac{1}{2}(1 + \sqrt{1 - G^2})^3$, which can exceed 2 simultaneously in the narrow range of $G \in (\sqrt{2(2^{\frac{2}{3}} - 2^{\frac{1}{3}})}, 2^{-\frac{1}{3}})$ [approximately $G \in (0.793, 0.809)$]. As illustrated in Fig. 2, when $G = 0.8$, $B_1 = B_8 = 2.048$, which is the maximal simultaneous violation for B_1 and B_8. Unlike a two-qubit case, it shows that nonlocality sharing between Alice$_1$-Bob$_1$-Charlie$_1$ and Alice$_2$-Bob$_2$-Charlie$_2$ in a three-qubit system can be observed, where two MABK inequalities can be violated simultaneously. Of course, the magnitude of such a double violation is very small, and it will vanish when the fidelity of the shared state is less than 97.65%.

Second, we can explore whether nonlocality sharing still exists or not for other different combinations of observers. Certainly, it can be analyzed by discussing the simultaneous violation for these MABK quantities, B_2 to B_7. When the same measurement settings mentioned above are used, the MABK

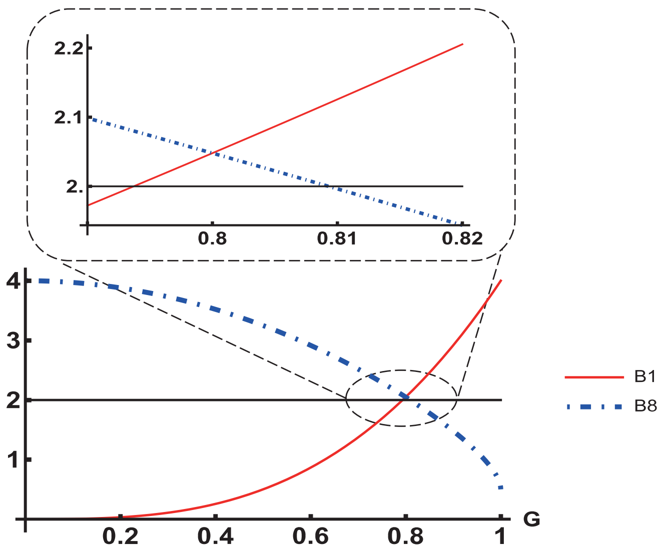

FIG. 2. Plot of MABK quantity B_1 for Alice$_1$-Bob$_1$-Charlie$_1$ and B_8 for Alice$_2$-Bob$_2$-Charlie$_2$ when the state is GHZ state and $G_1 = G_2 = G_3 = G$. The red solid line describes B_1 and the blue dot-dashed line describes B_8. They both exceed the bound of 2 in a narrow range. The picture above is the magnification of the violation part.

quantities, B_2 to B_7, can be written as

$$\begin{aligned} B_2 &= 2(1 + F_2)G_1G_3, \\ B_3 &= 2(1 + F_3)G_1G_2, \\ B_4 &= 2(1 + F_1)G_2G_3, \\ B_5 &= (1 + F_2)(1 + F_3)G_1, \\ B_6 &= (1 + F_1)(1 + F_3)G_2, \\ B_7 &= (1 + F_1)(1 + F_2)G_3. \end{aligned} \tag{11}$$

Similarly, when these MABK quantities in Eq. (11) can exceed 2 simultaneously, the nonlocality sharing phenomenon can be observed. For simplicity, when $G_1 = G_2 = G_3 = G$, the MABK quantities B_2 to B_4 will change to the same value $2G^2(1 + \sqrt{1 - G^2})$, while the MABK quantities B_5 to B_7 will change to another value $G(1 + \sqrt{1 - G^2})^2$. As illustrated in Fig. 3, it is easily to find the MABK quantities B_2 to B_4

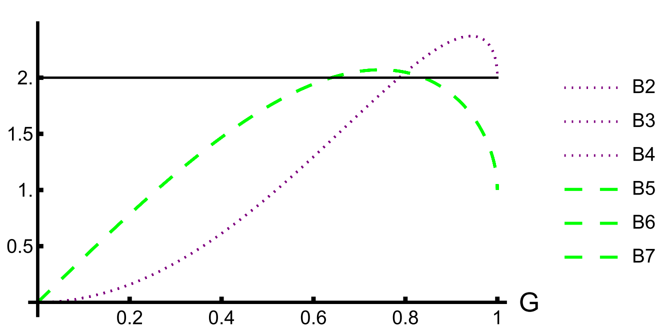

FIG. 3. Plot of MABK quantities B_2 to B_7 for Alice-Bob-Charlie when the state is GHZ state and $G_1 = G_2 = G_3 = G$. Under such conditions, B_2=B_3=B_4 (purple dotted line) and B_5=B_6=B_7 (green dashed line). They can exceed the classical bound simultaneously in a narrow range.

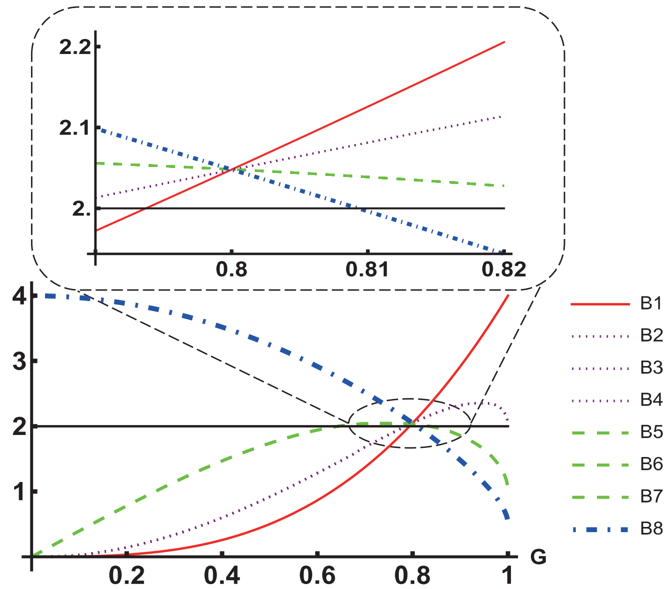

FIG. 4. Plot of MABK quantities B_1 to B_8 for Alice-Bob-Charlie when the state is GHZ state and $G_1 = G_2 = G_3 = G$. Under such conditions, B_2=B_3=B_4 and B_5=B_6=B_7. The red solid line describes B_1, the purple dotted line describes B_2–B_4, the green dashed line describes B_5–B_7, the blue dot-dashed line describes B_8, and they all exceed the bound of 2 in a narrow range. The picture above is the magnification of the violation part.

will exceed 2 simultaneously in the range of $G \in (\sqrt{\frac{\sqrt{5}-1}{2}}, 1)$. When $G = \frac{2\sqrt{2}}{3}$, the MABK quantities B_2 to B_4 achieve the maximal value 2.37 by choosing these measurement settings. The MABK quantities B_5 to B_7 will exceed 2 simultaneously in the range of $G \in (0.638, 0.839)$, and the MABK quantities B_5 to B_7 achieve the maximal value 2.07 by choosing these measurement settings when $G = \frac{\sqrt{5}}{3}$. Hence, the MABK quantities B_2 to B_7 will exceed 2 simultaneously in the range of $G \in (\sqrt{\frac{\sqrt{5}-1}{2}}, 0.839)$. When $G = 0.8$, $B_2 = B_3 = B_4 = B_5 = B_6 = B_7 = 2.048$, which is the maximal simultaneous violation for B_2 to B_7.

Third, when all the eight MABK quantities can exceed 2 simultaneously, the system will exhibit complete nonlocality sharing in such a three-qubit system via multilateral sequential measurements. Actually, the eight MABK inequalities, from B_1 to B_8, can be simultaneously violated. We can easily show such nonlocality sharing by simple measurement settings which are mentioned above, even though they are suboptimal measurement settings. As illustrated in Fig. 4, when $G_1 = G_2 = G_3 = G$, we show all the eight MABK quantities will exceed 2 simultaneously in the range of $G \in (\sqrt{2(2^{\frac{2}{3}} - 2^{\frac{1}{3}})}, 2^{-\frac{1}{3}})$. when $G = 0.8$, $B_1 = B_2 = B_3 = B_4 = B_5 = B_6 = B_7 = B_8 = 2.048$, which is the maximal simultaneous violation for all the eight MABK quantities. Compared with two-qubit cases, the nonlocality sharing in a three-qubit system shows more fruitful characteristics. Nevertheless, it is difficult to observe in current experimental conditions since the value of this simultaneous violation is relatively small.

Obviously, the unilateral ($G_i = G_j = 1$) or bilateral ($G_i = 1$) sequential measurements scenario is a special case of

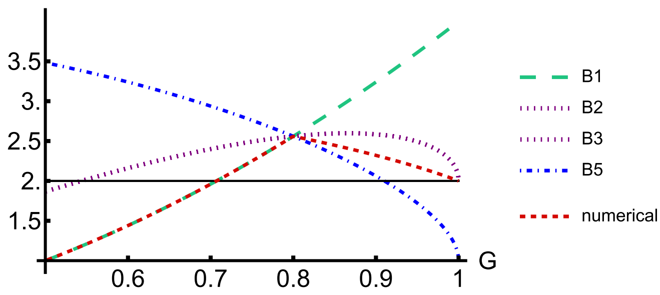

FIG. 5. Plot of numerical solutions (red dashed line) and analytical solutions of MABK quantities B_1 (green wide dashed line), B_2/B_3 (purple dotted line), and B_5 (blue dot-dashed line), when $G_1 = 1$, $G_2 = G_3 = G$. The red dashed line is the optimal solution, which means the simultaneous maximum value that the four quantities (numerical) can reach.

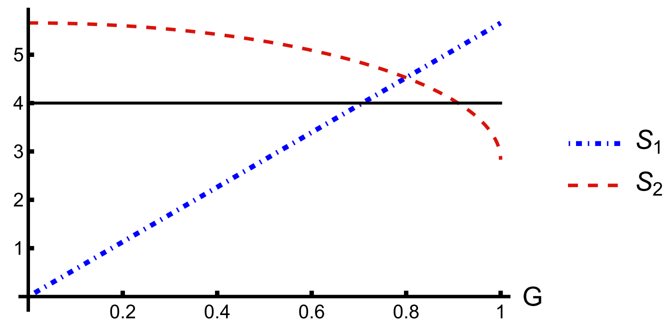

FIG. 6. Plot of Svetlichny quantities S_1 (Alice$_1$-Bob$_1$-Charlie$_1$) and S_2 (Alice$_1$-Bob$_1$-Charlie$_2$), when $G_1 = G_2 = 1$, $G_3 = G$. The blue dot-dashed line describes S_1 and the red dashed line describes S_2. Both of them exceed the classical bound 4 in a narrow range.

the above trilateral scenario. Without loss of generality, we take $G_1 = 1$ and $G_2 = G_3 = G$ as an example, where Alice$_1$ performs strong measurements. In this scenario, nonlocality sharing between four combinations of different observers is worth discussing, Alice$_1$-Bob$_{1,2}$-Charlie$_{1,2}$, where the MABK quantities correspond to B_1, B_2, B_3, B_5. The optimal nonlocality sharing can be obtained by numerical calculations. As illustrated in Fig. 5, the red dashed line is the optimal numerical value. It is shown that those MABK quantities can exceed 2 in the range of $(\frac{1}{\sqrt{2}}, 1)$ and the maximal violation is 2.56 when $G = 0.8$. Likewise, we could also attain the analytical solution under the previous measurement settings. According to Eq. (11), the four quantities change to $B_1 = 4G^2$, $B_2 = 2(1 + F)G$, $B_3 = 2(1 + F)G$, and $B_5 = (1 + F)^2$. When $G = 0.8$, the simultaneous maximal violation of B_1, B_2, B_3, B_5 is 2.56, which is the same value as the optimal numerical solution. However, the range of the nonlocality sharing is narrower than the numerical results.

Obviously, this nonlocality sharing phenomenon could be observed once the fidelity of the shared state is greater than 78%, which is possible to be achieved under the current experimental technologies. Hence, our results are helpful for realizing some device-independent certification schemes, such as self-testing of multipartite GHZ states [50–52].

IV. GENUINE MULTIPARTITE NONLOCALITY SHARING IN THE SEQUENTIAL MEASUREMENTS CASE

Besides using MABK inequality, the multipartite nonlocality sharing based on the multiple violation of other Bell-type inequality can also be investigated in the sequential measurements scenario, such as Svetlichny inequality [53]. As is well known, Svetlichny inequality can be used to detect genuine tripartite nonlocality [27]. Therefore, the multiple violation of Svetlichny inequalities can show genuine tripartite nonlocality sharing, where Svetlichny inequality can be described as [53]

$$\begin{aligned}
&|E(A_{i,1}, B_{j,1}, C_{k,1}) + E(A_{i,1}, B_{j,1}, C_{k,2}) + E(A_{i,1}, B_{j,2}, C_{k,1})\\
&\quad + E(A_{i,2}, B_{j,1}, C_{k,1}) - E(A_{i,2}, B_{j,2}, C_{k,2})\\
&\quad - E(A_{i,2}, B_{j,2}, C_{k,1}) - E(A_{i,2}, B_{j,1}, C_{k,2})\\
&\quad - E(A_{i,1}, B_{j,2}, C_{k,2})| \leqslant 4.
\end{aligned}$$

Similar to the above discussion, we numerically investigated the multiple violation of Svetlichny inequalities in the trilateral, bilateral ($G_1 = 1$), and unilateral ($G_1 = G_2 = 1$) sequential measurement scenarios. It is shown that the multiple violation in the trilateral and bilateral ($G_1 = 1$) cases can never be achieved, which means the genuine nonlocality sharing phenomenon does not exist. However, the genuine nonlocality sharing can be observed in the unilateral case, since the numerical results indicated that the double violation of Svetlichny inequalities between Alice$_1$-Bob$_1$-Charlie$_1$ and Alice$_1$-Bob$_1$-Charlie$_2$ could be observed at the same time. The maximal violation [54] of Svetlichny inequality can be achieved when the measurement settings are fixed in X-Y plane. We can give an analytic solution by choosing the appropriate measurement setting, $\phi_{11} = \phi_{14} = \phi_{23} = \phi_{24} = 0$, $\phi_{12} = \phi_{31} = -\phi_{33} = \frac{\pi}{2}$, $\phi_{21} = \phi_{13} = \frac{\pi}{4}$, $\phi_{32} = -\pi$, and $\phi_{22} = \frac{3\pi}{4}$. The Svetlichny quantities of Alice$_1$-Bob$_1$-Charlie$_1$ and Alice$_1$-Bob$_1$-Charlie$_2$, S_1 and S_2, turn to

$$S_1 = 4\sqrt{2}G, \quad S_2 = 2\sqrt{2}(1 + F), \tag{12}$$

where G and F are the quality factor and precision factor of Charlie$_1$'s weak measurements.

Obviously, the double violation can be observed in the range of $G \in (\frac{1}{\sqrt{2}}, \sqrt{2[-1 + \sqrt{2})}]$, as illustrated in Fig. 6. When $G = 0.8$, the maximal simultaneous violation can be obtained, where $S_1 = S_2 = 4.525$. Compared with the nonlocality sharing based on the multiple violation of MABK inequalities, the genuine nonlocality sharing based on the multiple violation of Svetlichny inequalities is more difficult, which can be only observed in the unilateral sequential measurements scenario.

V. CONCLUSION

The phenomenon of nonlocality sharing for a three-qubit system via multilateral sequential measurements has been deeply discussed. We analyzed the multiple violation of two different Bell-type inequalities, MABK inequality and Svetlichny inequality.

We started with nonlocality sharing based on MABK inequality in the trilateral sequential measurements scenario. In order to compare with a two-qubit case [43], we first explored nonlocality sharing in a three-qubit system between Alice$_1$-Bob$_1$-Charlie$_1$ and Alice$_2$-Bob$_2$-Charlie$_2$. Interestingly, the corresponding MABK inequalities, B_1 and B_8, can exceed 2 simultaneously in the narrow range of $G \in [\sqrt{2(2^{\frac{2}{3}} - 2^{\frac{1}{3}})}, 2^{-\frac{1}{3}}]$. Hence, nonlocality sharing in a three-qubit system between Alice$_1$-Bob$_1$-Charlie$_1$ and Alice$_2$-Bob$_2$-Charlie$_2$ can be observed, while it is impossible in a two-qubit case. Second, we also investigated nonlocality sharing for the other different observers combinations. It is shown that the MABK quantities B_2 to B_7 will exceed 2 simultaneously in the range of $G \in (\sqrt{\frac{\sqrt{5}-1}{2}}, 1)$. Third, all the eight possible MABK inequalities in this scenario were fully explored. Actually, the eight MABK inequalities, from B_1 to B_8, can be violated simultaneously. When $G_1 = G_2 = G_3 = G$, we show all the eight MABK quantities will exceed 2 simultaneously in the range of $G \in [\sqrt{2(2^{\frac{2}{3}} - 2^{\frac{1}{3}})}, 2^{-\frac{1}{3}}]$. when $G = 0.8$, $B_1 = B_2 = B_3 = B_4 = B_5 = B_6 = B_7 = B_8 = 2.048$, which is the maximal simultaneous violation for all the eight MABK quantities.

Furthermore, we investigated nonlocality sharing in the degenerated trilateral sequential measurements scenario, i.e., unilateral or bilateral. When $G_1 = 1$ and $G_2 = G_3 = G$, the optimal numerical solution indicates that the simultaneous maximal value is 2.56 ($G = 0.8$) and the violation range is $G \in (\frac{1}{\sqrt{2}}, 1)$ for Alice$_1$-Bob$_{1,2}$-Charlie$_{1,2}$. The analytic solution reaches the maximal simultaneous violation 2.56, while it has a narrower violation range. In current technologies, it is possible to observe this nonlocality sharing phenomenon, which is beneficial to triggering the investigation of some device-independent certification schemes, such as self-testing of multipartite GHZ states [50–52].

Finally, we studied the genuine nonlocality sharing based on Svetlichny inequality in the trilateral sequential measurements scenario. We numerically demonstrated that the multiple violation of Svetlichny inequalities in the trilateral and bilateral cases does not exist, which means it is impossible to observe the genuine nonlocality sharing in these cases. However, the genuine nonlocality sharing could be observed in the unilateral case. We show that the double violation can be achieved in the range of $G \in [\frac{1}{\sqrt{2}}, \sqrt{2(-1 + \sqrt{2})}]$. When $G = 0.8$, the maximal simultaneous violation reaches $S_1 = S_2 = 4.525$. Obviously, the nonlocality sharing based on Svetlichny inequality is more difficult to observe than MABK inequality.

These results indicate that the nonlocality sharing in a three-qubit system contains more fruitful characteristics. A similar discussion could also be generalized to other inequalities or higher dimensional systems.

ACKNOWLEDGMENTS

C.R. was supported by the National Natural Science Foundation of China (Grant No. 12075245), the Natural Science Foundation of Hunan Province (Grant No. 2021JJ10033), National Key Research and Development Program (Grant No. 2017YFA0305200), and Xiaoxiang Scholars Programme of Hunan Normal University. X.Z. was supported by the National Key Research and Development Program (Grants No. 2017YFA0305200 and No. 2016YFA0301700), the Key Research and Development Program of Guangdong Province of China (Grant No. 2018B030329001 and No. 2018B030325001), the National Natural Science Foundation of China (Grant No. 61974168), the Natural Science Foundation of Guangdong Province of China (Grant No. 2016A030312012), and the National Young 1000 Talents Plan.

[1] A. Einstein, B. Podolsky, and N. Rosen, Phys. Rev. **47**, 777 (1935).
[2] J. S. Bell, Phys. Phys. Fiz. **1**, 195 (1964).
[3] J. Clauser, M. Horne, A. Shimony, and R. Holt, Phys. Rev. Lett. **23**, 880 (1969).
[4] M. Żukowski and Č. Brukner, Phys. Rev. Lett. **88**, 210401 (2002).
[5] N. D. Mermin, Phys. Rev. Lett. **65**, 1838 (1990).
[6] A. V. Belinskĭ and D. N. Klyshko, Phys. Usp. **36**, 653 (1993).
[7] M. Ardehali, Phys. Rev. A **46**, 5375 (1992).
[8] D. Collins, N. Gisin, N. Linden, S. Massar, and S. Popescu, Phys. Rev. Lett. **88**, 040404 (2002).
[9] Č. Brukner, M. Żukowski, and A. Zeilinger, Phys. Rev. Lett. **89**, 197901 (2002).
[10] S. M. Lee, M. Kim, H. Kim, H. S. Moon, and S. W. Kim, Quantum Sci. Technol. **3**, 045006 (2018).
[11] A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. **49**, 1804 (1982).
[12] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. **81**, 5039 (1998).
[13] M. A. Rowe, D. Kielpinski, V. Meyer, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland, Nature (London) **409**, 791 (2001).
[14] J. Hofmann, M. Krug, N. Ortegel, L. Gérard, M. Weber, W. Rosenfeld, and H. Weinfurter, Science **337**, 72 (2012).
[15] M. Giustina, A. Mech, S. Ramelow, B. Wittmann, J. Kofler, J. Beyer, A.Lita, B. Calkins, T. Gerrits, S. W. Nam, R. Ursin, and A. Zeilinger, Nature (London) **497**, 227 (2013).
[16] B. G. Christensen, K. T. McCusker, J. B. Altepeter, B. Calkins, T. Gerrits, A. E. Lita, A. Miller, L. K. Shalm, Y. Zhang, S. W. Nam, N. Brunner, C. C. W. Lim, N. Gisin, and P. G. Kwiat, Phys. Rev. Lett. **111**, 130406 (2013).
[17] B. Hensen *et al.*, Nature (London) **526**, 682 (2015).
[18] M. Giustina, M. A. M. Versteegh, S. Wengerowsky, J. Handsteiner, A. Hochrainer, K. Phelan, F. Steinlechner, J. Kofler, J. A. Larsson, C. Abellan, W. Amaya, V. Pruneri, M. W. Mitchell, J. Beyer, T. Gerrits, A. E. Lita, L. K. Shalm, S. W. Nam, T. Scheidl, R. Ursin, B. Wittmann, and A. Zeilinger, Phys. Rev. Lett. **115**, 250401 (2015).
[19] L. K. Shalm, E. Meyer-Scott, B. G. Christensen, P. Bierhorst, M. A. Wayne, M. J. Stevens, T. Gerrits, S. Glancy, D. R. Hamel, M. S. Allman, K. J. Coakley, S. D. Dyer, C. Hodge, A. E. Lita, V. B. Verma, C. Lambrocco, E. Tortorici, A. L. Migdall, Y. Zhang, D. R. Kumor, W. H. Farr, F. Marsili, M. D. Shaw, J. A. Stern, C. Abellan, W. Amaya, V. Pruneri, T. Jennewein,

M. W. Mitchell, P. G. Kwiat, J. C. Bienfang, R. P. Mirin, E. Knill, and S. W. Nam, Phys. Rev. Lett. **115**, 250402 (2015).
[20] A. Acín, Phys. Rev. Lett. **98**, 230501 (2007).
[21] R. Colbeck, Quantum and relativistic protocols for secure multiparty computation, Ph.D. thesis, University of Cambridge, Cambridge, UK, 2007; arXiv:0911.3814.
[22] S. Pironio, A. Acín, S. Massar, A. Boyer de la Giroday, D. N. Matsukevich, P. Maunz, S. Olmschenk, D. Hayes, L. Luo, T. A. Manning, and C. Monroe, Nature (London) **464**, 1021 (2010).
[23] S. Pirandola, U. L. Andersen, L. Banchi, M. Berta, D. Bunandar, R. Colbeck, D. Englund, T. Gehring, C. Lupo, C. Ottaviani *et al.*, Adv. Opt. Photon. **12**, 1012 (2020).
[24] Y. Liu, Q. Zhao, M. H. Li, J. Y. Guan, Y. B. Zhang, B. Bai, W. J. Zhang, W. Z. Liu, C. Wu, X. Yuan, H. Li, W. J. Munro, Z. Wang, L. X. You, J. Zhang, X. F. Ma, J. Y. Fan, Q. Zhang, and J. W. Pan, Nature (London) **562**, 548 (2018).
[25] P. Bierhorst, E. Knill, S. Glancy, Y. Zhang, A. Mink, S. Jordan, A. Rommal, Y-K. Liu, B. Christensen, S. W. Nam, M. J. Stevens, and L. K. Shalm, Nature (London) **556**, 223 (2018).
[26] J. Bowles, I. Supic, D. Cavalcanti, and A. Acin, Phys. Rev. Lett. **121**, 180503 (2018).
[27] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani, and S. Wehner, Rev. Mod. Phys. **86**, 419 (2014).
[28] R. Silva, N. Gisin, Y. Guryanova, and S. Popescu, Phys. Rev. Lett. **114**, 250401 (2015).
[29] D. Das, A. Ghosal, S. Sasmal, S. Mal, and A. S. Majumdar, Phys. Rev. A **99**, 022305 (2019).
[30] S. Mal, A. S. Majumdar, and D. Home, Mathematics **4**, 48 (2016).
[31] S. Sasmal, D. Das, S. Mal, and A. S. Majumdar, Phys. Rev. A **98**, 012305 (2018).
[32] A. Bera, S. Mal, A. Sen De, and U. Sen, Phys. Rev. A **98**, 062304 (2018).
[33] S. Datta and A. S. Majumdar, Phys. Rev. A **98**, 042311 (2018); Erratum **99**, 019902 (2019).
[34] H. Akshata Shenoy, S. Designolle, F. Hirsch, R. Silva, N. Gisin, and N. Brunner, Phys. Rev. A **99**, 022317 (2019).
[35] A. Kumari and A. K. Pan, Phys. Rev. A **100**, 062130 (2019).
[36] C. L. Ren, T. Feng, D. Yao, H. Shi, J. Chen, and X. Zhou, Phys. Rev. A **100**, 052121 (2019).
[37] S. Saha, D. Das, S. Sasmal, D. Sarkar, K. Mukherjee, A. Roy, and S. S. Bhattacharya, Quantum Inf. Processing **18**, 42 (2019).
[38] K. Mohan, A. Tavakoli, and N. Brunner, New J. Phys. **21**, 083034 (2019).
[39] S. Roy, A. Bera, S. Mal, A. Sen De, and U. Sen, Phys. Lett. A **392**, 127143 (2021).
[40] C. Srivastava, S. Mal, A. Sen De, and U. Sen, Phys. Rev. A **103**, 032408 (2021).
[41] S. Kanjilal, C. Jebarathinam, T. Paterek, and D. Home, arXiv:1912.09900.
[42] D. Yao and C. L. Ren, Phys. Rev. A **103**, 052207 (2021).
[43] J. Zhu, M.-J. Hu, G.-C. Guo, C.-F. Li, and Y.-S. Zhang, Phys. Rev. A **105**, 032211 (2022).
[44] S. Cheng, L. Liu, T. J. Baker, and M. J. W. Hall, Phys. Rev. A **104**, L060201 (2021).
[45] W. L. Hou, X. W. Liu, and C. L. Ren, Phys. Rev. A **105**, 042436 (2022).
[46] M. J. Hu, Z. Y. Zhou, X. M. Hu, C. F. Li, G. C. Guo, and Y. S. Zhang, npj Quantum. Inf. **4**, 63 (2018).
[47] M. Schiavon, L. Calderaro, M. Pittaluga, G. Vallone, and P. Villoresi, Quantum Sci. Technol. **2**, 015010 (2017).
[48] T. Feng, C. Ren, Y. Tian, M. Luo, H. Shi, J. Chen, and X. Zhou, Phys. Rev. A **102**, 032220 (2020).
[49] F. J. Curchod, M. Johansson, R. Augusiak, M. J. Hoban, P. Wittek, and A. Acin, Phys. Rev. A **95**, 020102(R) (2017).
[50] S. Sarkar and R. Augusiak, Phys. Rev. A **105**, 032416 (2022).
[51] R. Augusiak, A. Salavrakos, J. Tura, and A. Acín, New J. Phys. **21**, 113001 (2019)
[52] S. Sarkar, D. Saha, J. Kaniewski, and R. Augusiak, npj Quantum Inf **7**, 151 (2021).
[53] G. Svetlichny, Phys. Rev. D **35**, 3066 (1987).
[54] P. Mitchell, S. Popescu, and D. Roberts, Phys. Rev. A **70**, 060101(R) (2004).