Generalized Bell-like inequality and maximum violation for multiparticle entangled Schrödinger cat states of spin *s*

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This paper proposes a generalized Bell-like inequality (GBI) for multiparticle entangled Schrödinger cat states of arbitrary spin s. Based on quantum probability statistics, the GBI and violation are formulated in a unified manner with the help of a state density operator, which can be separated into local and nonlocal parts. The local part gives rise to the inequality, while the nonlocal part is responsible for the violation. The GBI is not violated at all by the quantum average, except the spin-1/2 entangled states. If the measuring outcomes are restricted in the subspace of the spin coherent state (SCS), namely, only the maximum spin values $\pm s$, the GBI is still meaningful for the incomplete measurement. With the help of SCS quantum probability statistics, it is proved that the violation of GBI can occur only for half-integer spins and not integer spins. Moreover, the maximum violation bound depends on the number parity of the entangled particles, which is 1/2 for the odd particle numbers and 1 for the even numbers.

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I. INTRODUCTION

Nonlocality [1–3] is regarded as the most peculiar characteristic of quantum mechanics since it does not coexist with the relativistic causality in our intuition of space and time. The quantum entangled state was originally introduced by Einstein-Podolsky-Rosen as a critical example for the nonlocality [4] showing apparently contradictory results with the locality and reality criterion in classical theory. The quantum entanglement now has become a key concept of quantum information and computation [5–9]. Quantum correlations between entangled systems are fundamentally different from classical correlations [10,11], especially when these systems are spatially separated. It was Bell who proposed, for the first time, an inequality known as the Bell inequality (BI) to test this difference [12]. The BI is actually a constraint on the correlations compatible with local hidden-variable (or local realistic) theories. Great attention has been paid to theoretical and experimental studies of the inequality [13–17]. Much experimental evidence [18–24] confirms the violation of BI, which provides an overwhelming superiority for the nonlocality in quantum mechanics [25,26].

Stimulated by the original work of Bell, various extensions have been proposed, such as Clauser-Horne-Shimony-Holt (CHSH) [27] inequality and Wigner inequality (WI) [28], in which only the particle-number probability of the positive spin is measured [28,29]. A loophole-free experiment was reported recently to verify the violation of CHSH inequality with the electronic spin of the nitrogen-vacancy defect in a diamond chip [30,31]. The violation was also confirmed experimentally with two-photon entangled states of mutually perpendicular polarizations [32]. By closing the two main loopholes (the "locality loophole" and "detection loophole") at the same time, some teams have independently confirmed that we must definitely renounce local realism [30,33–35].

Theoretical analysis for the violation of BI was presented in the beginning of the 1990s [36,37]. It is certainly of importance that the BI and its violation can be formulated together in the framework of quantum probability statistics in order to have a better understanding of the entanglement nature. In particular, the maximum violation is of significance in the entangled Schrödinger cat states, which play a crucial role in the test of macroscopic quantum effects. Moreover, the measurement of maximum violation may be developed to a device of independent entanglement witnesses. In terms of spin-coherent-state (SCS) quantum probability statistics, the BI, WI, and their maximum violation bounds are studied for arbitrary two-spin entangled states with antiparallel and parallel spin polarizations [38–42] . The density operator of the entangled state is separated into the "local" (classical probability state) and "nonlocal" (quantum interference between two components of entangled state) parts. The local part gives rise to the BI or WI, while the nonlocal part is responsible for the violation.

The Bell correlation for a two-particle entangled state of arbitrary spin s, called the Schrödinger cat state, is also investigated with the SCS quantum probability statistics. The BI is not violated by the entangled Schrödinger cat states at all, except the spin-1/2 case [42]. If, on the other hand, the measuring outcomes are confined in the subspace of SCS, a universal BI is formulated in terms of the local part of the

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density operator. The maximum violation bound is found for the entangled cat states with both antiparallel and parallel polarizations [38,39]. In particular, a spin-parity effect [42] is observed in which the universal BI can be violated only by the entangled cat states of the half-integer and not the integer spins. The violation of universal BI is seen to be a direct result of a nontrivial Berry phase between the SCSs [38–42] of the south- and north-pole gauges for the half-integer spin, while the geometric phase is trivial for the integer spins. This observation [42] provides an example to relate the violation of BI with the geometric phase.

In the present paper, we study the multiparticle entangled cat states, which are just the well-known Greenberger, Horne, and Zeilinger (GHZ) [43–45] state in the spin-$1/2$ case [46,47]. The quantum entanglement was originally generalized to the GHZ state for four spin-$1/2$ particles, while later it was simplified to three spin-$1/2$ particles [48] and was verified experimentally [49,50]. The violations of inequalities for n spin-$1/2$ particles have been studied extensively [51–61]. Bell's inequality has been proposed for n spin-s particles by n distant observers [62].

The main goal of the present study is to extend the two-particle universal BI to a multiparticle entangled cat state of arbitrary spin s. The entanglement in many-particle systems has been thoroughly investigated with significant progress achieved [61,63–69]. The many-particle nonlocal correlation plays an important role in phase transitions and criticality in condensed matter [70]. It might also enhance our understanding of entanglement application in quantum information theory.

Although the nonlocal correlation for a two-particle entangled cat state was investigated, a suitable inequality for an arbitrary many-particle state was not found in our previous paper [42]. In the present paper, a generalized Bell-like inequality (GBI) is discovered to characterize the nonlocality of a multiparticle entangled cat state with arbitrary spin s. The GBI and the maximum violation bound are formulated in a unified formalism with SCS quantum probability statistics.

In Sec. II, we present a brief review of the SCS quantum probability statistics with the BI as an example. The GBI and its maximum violation bound are formulated for n-particle entangled states of spin-$1/2$ in Sec. III. It is demonstrated in Sec. IV that the GBI and the maximum violation exist for the n-particle entangled cat states if the measuring outcomes are restricted in the subspace of SCS. Moreover we observe interesting spin and particle-number parity effects in the violation of GBI. The conclusion and discussion are given in Sec. V, where a possible experiment is proposed to test the spin-parity effect.

II. SCS QUANTUM PROBABILITY STATISTICS AND BI

The original BI is derived based on classical statistics with a hidden-variable assumption. In previous publications [38–42], the Bell-type inequalities and their violation were formulated by means of the SCS quantum probability statistics in a unified manner. The density operator of an entangled state for a bipartite system can be separated to the local (or classical) and nonlocal (or quantum coherent) parts. The former part gives rise to the local realist bound of measuring-outcome correlation, namely, the BIs, while the latter part leads to the violation of the inequalities.

A. Spin-1/2 measuring outcome correlation and violation of BI

We begin with an arbitrary two-spin entangled state of antiparallel polarization in the bases $\hat{\sigma}_z|\pm\rangle = \pm|\pm\rangle$,

$$|\psi\rangle = c_1|+,-\rangle + c_2|-,+\rangle, \tag{1}$$

where the normalized coefficients can be generally parameterized as $c_1 = e^{i\eta}\sin\xi$, $c_2 = e^{-i\eta}\cos\xi$. We assume that two spins are separated to a spacelike distance when the entangled state is prepared. The density operator $\hat{\rho}$ of the entangled state can be divided into two parts,

$$\hat{\rho} = \hat{\rho}_{lc} + \hat{\rho}_{\rm nlc}. \tag{2}$$

The local part,

$$\hat{\rho}_{lc} = \sin^2\xi|+,-\rangle\langle+,-| + \cos^2\xi|-,+\rangle\langle-,+|,$$

which is the classical two-particle probability-density operator, describes the individual spin of the bipartite system separated remotely, while what we called the nonlocal part,

$$\hat{\rho}_{\rm nlc} = \sin\xi\cos\xi(e^{2i\eta}|+,-\rangle\langle-,+| + e^{-2i\eta}|-,+\rangle\langle+,-|),$$

is the quantum coherence density operator between two remote spins.

The measurements of two spins are performed independently along two arbitrary directions, say, $\mathbf{a}$ and $\mathbf{b}$. The measuring outcomes fall into the eigenvalues of projection spin operators $\hat{\sigma}\cdot\mathbf{a}$ and $\hat{\sigma}\cdot\mathbf{b}$, i.e.,

$$\hat{\sigma}\cdot\mathbf{a}|\pm\mathbf{a}\rangle = \pm|\pm\mathbf{a}\rangle, \quad \hat{\sigma}\cdot\mathbf{b}|\pm\mathbf{b}\rangle = \pm|\pm\mathbf{b}\rangle,$$

according to the quantum measurement theory. Solving the eigenvalue equation for each direction denoted by $\mathbf{r} = \mathbf{a}, \mathbf{b}$, we have two orthogonal eigenstates given by

$$\begin{aligned}|+\mathbf{r}\rangle &= \cos\frac{\theta_r}{2}|+\rangle + \sin\frac{\theta_r}{2}e^{i\phi_r}|-\rangle,\\ |-\mathbf{r}\rangle &= \sin\frac{\theta_r}{2}|+\rangle - \cos\frac{\theta_r}{2}e^{i\phi_r}|-\rangle.\end{aligned} \tag{3}$$

In the above solutions, the general unit vector $\mathbf{r} = (\sin\theta_r\cos\phi_r, \sin\theta_r\sin\phi_r, \cos\theta_r)$ is parameterized by the polar and azimuthal angles θ_r, ϕ_r in the coordinate frame with the z axis along the direction of the initial spin polarization. The two orthogonal states $|\pm\mathbf{r}\rangle$ are known as spin-coherent states of the north- and south-pole gauges [42]. The eigenstate product of operators $\hat{\sigma}\cdot\mathbf{a}$ and $\hat{\sigma}\cdot\mathbf{b}$ forms an outcome-independent vector basis for measuring two spins, respectively, along the a, b directions. We label the four basis vectors as

$$\begin{aligned}|1\rangle &= |+\mathbf{a}, +\mathbf{b}\rangle, \quad |2\rangle = |+\mathbf{a}, -\mathbf{b}\rangle,\\ |3\rangle &= |-\mathbf{a}, +\mathbf{b}\rangle, \quad |4\rangle = |-\mathbf{a}, -\mathbf{b}\rangle,\end{aligned} \tag{4}$$

for the sake of simplicity. The measurement correlation operator is denoted by

$$\hat{\Omega}(a,b) = (\hat{\sigma}\cdot\mathbf{a})\otimes(\hat{\sigma}\cdot\mathbf{b}). \tag{5}$$

The correlation probability is obtained as

$$P(a,b) = {\rm Tr}[\hat{\Omega}(a,b)\hat{\rho}], \tag{6}$$

which can also be separated into local and nonlocal parts,

$$P(a, b) = P_{lc}(a, b) + P_{\text{nlc}}(a, b),$$

with

$$P_{lc}(a, b) = \text{Tr}[\hat{\Omega}(a, b)\hat{\rho}_{lc}]$$

and

$$P_{\text{nlc}}(a, b) = \text{Tr}[\hat{\Omega}(a, b)\hat{\rho}_{\text{nlc}}].$$

In terms of the outcome-independent basis vectors given by Eq. (4), we derive the local correlation

$$P_{lc}(a, b) = \rho_{11}^{lc} - \rho_{22}^{lc} - \rho_{33}^{lc} + \rho_{44}^{lc} = -\cos\theta_a \cos\theta_b,$$

which is independent of the state parameters ξ, η, and is valid for the arbitrary normalized entangled states given by Eq. (1). And

$$\rho_{ii} = \langle i|\hat{\rho}|i\rangle = \rho_{ii}^{lc} + \rho_{ii}^{nlc}$$

($i = 1, 2, 3, 4$) denotes matrix elements of the density operator. The BI is recovered with the local correlation. The BI is [12]

$$1 + P_{lc}(b, c) \geqslant |P_{lc}(a, b) - P_{lc}(a, c)| \tag{7}$$

for the antiparallel entangled states given by Eq. (1).

The nonlocal part found as

$$P_{\text{nlc}}(a, b) = \sin(2\xi)\sin\theta_a \sin\theta_b \cos(\phi_a - \phi_b + 2\eta),$$

however, depends on the specific states. The violation of BI is seen to be a direct result of the nonlocal correlation. In particular, when the initial entangled state is the two-spin singlet,

$$|\psi_s\rangle = \frac{1}{\sqrt{2}}(|+, -\rangle - |-, +\rangle),$$

with the state parameters $\xi = (3\pi/4)\text{mod}2\pi$ and $\eta = 0\text{mod}2\pi$, the total correlation $P(a, b)$ becomes a scaler product of the two unit vectors,

$$P(a, b) = -\mathbf{a} \cdot \mathbf{b},$$

from which the BI is violated. For example, we let vector $\mathbf{b}$ be perpendicular to $\mathbf{c}$, then $P(b, c) = -\mathbf{b} \cdot \mathbf{c} = 0$ and the greater side of BI is $1 + P(b, c) = 1$. On the other hand, if the vector $\mathbf{a}$ is parallel to $\mathbf{b} - \mathbf{c}$, the less side $|P(a, b) - P(a, c)| = |\mathbf{a} \cdot (\mathbf{b} - \mathbf{c})| = \sqrt{2}$ becomes greater than $1 + P(b, c)$.

The BI and violation have been extended to the spin-1/2 entangled state of parallel polarization [39].

B. Nonviolation of BI for spin-1 entangled states

The BI is not violated by the quantum statistics average of the measuring-outcome correlation for the spin-1 entangled state,

$$|\psi\rangle = c_1| + 1, -1\rangle + c_2| - 1, +1\rangle.$$

The local part of the density operator is

$$\hat{\rho}_{lc} = \sin^2\xi| + 1, -1\rangle\langle +1, -1| + \cos^2\xi| - 1, +1\rangle\langle -1, +1|,$$

and the nonlocal part is

$$\hat{\rho}_{\text{nlc}} = \sin\xi\cos\xi(e^{2i\eta}| + 1, -1\rangle\langle -1, +1| + e^{-2i\eta}| - 1, +1\rangle\langle +1, -1|).$$

The eigenstates of the spin projection operator obtained from the three equations, $\hat{s} \cdot \mathbf{r}| \pm \mathbf{r}\rangle = \pm 1| \pm \mathbf{r}\rangle$ and $\hat{s} \cdot \mathbf{r}|\mathbf{r}_0\rangle = 0$ (in the unit convention $\hbar = 1$), are found as

$$|+\mathbf{r}\rangle = \cos^2\frac{\theta_r}{2}|+1\rangle + \frac{1}{\sqrt{2}}\sin\theta_r \exp(i\phi_r)|0\rangle + \sin^2\frac{\theta_r}{2}\exp(i2\phi_r)|-1\rangle,$$

$$|-\mathbf{r}\rangle = \sin^2\frac{\theta_r}{2}|+1\rangle - \frac{1}{\sqrt{2}}\sin\theta_r \exp(i\phi_r)|0\rangle + \cos^2\frac{\theta_r}{2}\exp(i2\phi_r)|-1\rangle,$$

$$|\mathbf{r}_0\rangle = -\frac{1}{\sqrt{2}}\sin\theta_r|+1\rangle + \cos\theta_r \exp(i\phi_r)|0\rangle + \frac{1}{\sqrt{2}}\sin\theta_r \exp(i2\phi_r)|-1\rangle.$$

The two-spin measuring-outcome correlation, respectively, along the a, b directions is evaluated by the quantum probability statistics over the nine eigenstates $\hat{\Omega}(a, b)|a_m, b_{m'}\rangle = mm'|a_m, b_{m'}\rangle$, where

$$\hat{\Omega}(a, b) = (\hat{s} \cdot \mathbf{a}) \otimes (\hat{s} \cdot \mathbf{b}),$$

with $m, m' = 1, 0, -1$, respectively. The local part of the correlation is

$$P_{lc}(a, b) = \text{Tr}[\hat{\Omega}(a, b)\hat{\rho}_{lc}] = \sum_{m,m'}\langle a_m, b_{m'}|\hat{\Omega}(a, b)\hat{\rho}_{lc}|a_m, b_{m'}\rangle = -\cos\theta_a \cos\theta_b,$$

while the nonlocal part of the correlation vanishes in the quantum probability average,

$$P_{\text{nlc}}(a, b) = \sum_{m,m'}\langle a_m, b_{m'}|\hat{\Omega}(a, b)\hat{\rho}_{\text{nlc}}|a_m, b_{m'}\rangle = 0.$$

Thus the quantum correlation is equal to the classical one,

$$P(a, b) = P_{lc}(a, b).$$

It is also proven that the nonlocal correlation vanishes [42], $P_s^{nlc}(a, b) = 0$, for Schrödinger cat states of spin-s ($s > 1/2$) with both antiparallel and parallel spin polarizations [41,42,71],

$$|\psi\rangle = c_1| + s, \mp s\rangle + c_2| - s, \pm s\rangle. \tag{8}$$

The reason is that the transition elements $\langle \pm s|\hat{S} \cdot \mathbf{n}|\mp s\rangle = 0$ ($\mathbf{n} = \mathbf{a}, \mathbf{b}$) induced by the spin projection operator vanish in the quantum probability average. The BI is not violated at all.

C. Universal BI and spin-parity effect

When the measurement, however, is restricted in the subspace of SCS, namely, only the maximum spin values $\pm s$ are measured [42], a universal BI,

$$p_{lc}(a,b)p_{lc}(b,c) \leqslant |p_{lc}(a,c)|, \tag{9}$$

is proposed [42] for the incomplete measurements. The universal BI is suitable for complete and partial measurements. It is also valid for entangled states with both antiparallel and parallel spin polarizations, given by Eq. (8).

The SCSs for projection spin operator $\hat{s}\cdot\mathbf{r}$ in the direction of unit vector $\mathbf{r}$ ($\mathbf{r}=\mathbf{a},\mathbf{b}$) can be derived from the eigenstate equations

$$\hat{s}\cdot\mathbf{r}|\pm\mathbf{r}\rangle = \pm s|\pm\mathbf{r}\rangle.$$

The explicit forms of SCSs in the Dicke-state representation are given by [71–73]

$$|+\mathbf{r}\rangle = \sum_{m=-s}^{s}\binom{2s}{s+m}^{\frac{1}{2}} K_r^{s+m}\Gamma_r^{s-m}\exp\left[i(s-m)\phi_r\right]|m\rangle,$$
$$|-\mathbf{r}\rangle = \sum_{m=-s}^{s}\binom{2s}{s+m}^{\frac{1}{2}} K_r^{s-m}\Gamma_r^{s+m}\exp\left[i(s-m)(\phi_r+\pi)\right]|m\rangle, \tag{10}$$

in which

$$K_r^{s\pm m} = \left(\cos\frac{\theta_r}{2}\right)^{s\pm m}$$

and

$$\Gamma_r^{s\pm m} = \left(\sin\frac{\theta_r}{2}\right)^{s\pm m}.$$

The two orthogonal states $|\pm\mathbf{r}\rangle$ are known as SCSs of the north- and south-pole gauges, in which a phase factor $\exp[i(s-m)\pi]$ difference between the two gauges plays a key role in the spin-parity effect. The eigenstates of the projection spin operators $\hat{s}\cdot\mathbf{a}$ and $\hat{s}\cdot\mathbf{b}$ form measuring-outcome-independent basis vectors, if the measurements are restricted in the maximum spin values, $\pm s$. The nonlocal part of measuring the outcome correlation evaluated by the trace over the subspace of SCSs, $P_{\mathrm{nlc}}(a,b)=\mathrm{Tr}[\hat{\Omega}(a,b)\hat{\rho}_{\mathrm{nlc}}]$, is found as

$$p_{\mathrm{nlc}}(a,b) = \rho_{11}^{nlc} - \rho_{22}^{nlc} - \rho_{33}^{nlc} + \rho_{44}^{nlc},$$

in which four eigenvectors labeled by $|i\rangle$ ($i=1,2,3,4$) are the same as in Eq. (4), where

$$\rho_{ii}^{nlc} = \langle i|\hat{\rho}_{\mathrm{nlc}}|i\rangle$$

denotes the matrix elements of the density operator.

The density matrix elements of the nonlocal part are obtained as

$$\rho_{11}^{nlc} = \rho_{44}^{nlc}$$
$$= \sin(2\xi)K_a^{2s}\Gamma_a^{2s}K_b^{2s}\Gamma_b^{2s}\cos\left[2s(\phi_a\mp\phi_b)+2\eta\right], \tag{11}$$

respectively, for antiparallel and parallel spin polarizations, and

$$\rho_{22}^{nlc} = \rho_{33}^{nlc} = (-1)^{2s}\rho_{11}^{nlc}. \tag{12}$$

It may be worthwhile to remark that the two-spin density matrix elements of the same spin polarizations (ρ_{11}^{nlc}, ρ_{44}^{nlc}) differ from that of opposite polarizations (ρ_{22}^{nlc}, ρ_{33}^{nlc}) by a phase factor,

$$(-1)^{2s} = \exp(i2s\pi), \tag{13}$$

which resulted from the geometric phase or Berry phase between SCSs of the north- and south-pole gauges. The nonlocal part of the correlation is simply

$$p_{\mathrm{nlc}}(a,b) = 2[1-(-1)^{2s}]\rho_{11}^{lc}, \tag{14}$$

which vanishes for integer spin s, but does not vanish for the half-integer spin. The Berry phase is trivial for the integer spins and the nonlocal parts of measuring-outcome correlation cancel each other, while the nontrivial Berry phase for half-integer spins leads to the constructive interference of nonlocal correlations. Thus the universal BI given by Eq. (9) can be violated only by half-integer spins and not by the integer spins. A maximum violation bound of universal BI is found for half-integer spin-s states [42]. In the following, we extend the two-particle universal BI to GBI for n particles.

III. MAXIMUM VIOLATION OF GBI FOR *N*-PARTICLE ENTANGLED STATE OF SPIN-1/2 AND PARTICLE-NUMBER PARITY EFFECT

In the present paper, we consider the n-particle entangled cat state of spin s,

$$|\psi\rangle = c_1|+s\rangle^{\otimes n} + c_2|-s\rangle^{\otimes n}, \tag{15}$$

in which $c_1 = e^{i\eta}\sin\xi$, $c_2 = e^{-i\eta}\cos\xi$ characterize the normalized coefficient with two arbitrary real parameters ξ,η. The state density operator can be separated into the local and nonlocal parts,

$$\hat{\rho} = \hat{\rho}_{lc} + \hat{\rho}_{\mathrm{nlc}},$$

with

$$\hat{\rho}_{lc} = \sin^2\xi|+s\rangle^{\otimes n}\langle+s|^{\otimes n} + \cos^2\xi|-s\rangle^{\otimes n}\langle-s|^{\otimes n},$$

$$\hat{\rho}_{\mathrm{nlc}} = \sin\xi\cos\xi\left(e^{i2\eta}|+s\rangle^{\otimes n}\langle-s|^{\otimes n} + e^{-i2\eta}|-s\rangle^{\otimes n}\langle+s|^{\otimes n}\right).$$

The normalized outcome correlation from n observers can also be separated as the local and nonlocal parts,

$$p(a_1,a_2,\ldots,a_n) = \frac{1}{s^n}\mathrm{Tr}[\hat{\rho}\hat{\Omega}(a_1,a_2,\ldots,a_n)]$$
$$= p_{lc}(a_1,a_2,\ldots,a_n) + p_{\mathrm{nlc}}(a_1,a_2,\ldots,a_n), \tag{16}$$

measured, respectively, along the $a_1,a_2,\ldots,a_n$ directions. The measuring correlation operator is

$$\hat{\Omega}(a_1,a_2,\ldots,a_n) = (\hat{s}\cdot\mathbf{a}_1)\otimes(\hat{s}\cdot\mathbf{a}_2)\otimes\cdots\otimes(\hat{s}\cdot\mathbf{a}_n).$$

The universal BI for the two-particle entangled state [42] is then extended directly to a GBI for an n-particle entangled cat state of spin s given by Eq. (15),

$$p_{lc}(a_1,a_2,\ldots,a_n)p_{lc}(a_2,a_3,\ldots,a_{n+1})p_{lc}(a_3,a_4,\ldots,a_{n+2})$$
$$\times\cdots\times p_{lc}(a_n,a_{n+1},\ldots,a_m)$$
$$\leqslant |p_{lc}(a_1,a_3,\ldots,a_m)|, \tag{17}$$

with total measuring directions $m = 2n - 1$. The validity of GBI given by Eq. (17) is obvious according to the hidden-variable classical statistics [42] following Bell [12] since any two-particle normalized measuring outcomes (denoted by $A = \pm 1$ and $B = \pm 1$) have the relation $|A(a)| = |B(a)|$ along the same direction a and $A^2(a) = B^2(a) = 1$. The derivation of GBI given by Eq. (17) is presented in the Appendix.

A. Three-particle entangled state of spin-1/2

As a simple example, we consider the three-particle entangled state of spin-1/2,

$$|\psi\rangle = e^{i\eta}\sin\xi|+,+,+\rangle + e^{-i\eta}\cos\xi|-,-,-\rangle,$$

where $|+\rangle$ and $|-\rangle$ are the spin-up and -down states along the z axis. The local part of density operator $\hat{\rho} = |\psi\rangle\langle\psi|$,

$$\hat{\rho}_{lc} = \sin^2\xi|+,+,+\rangle\langle+,+,+| + \cos^2\xi|-,-,-\rangle\langle-,-,-|,$$

results in the GBI, while the nonlocal part,

$$\hat{\rho}_{\text{nlc}} = \sin\xi\cos\xi\begin{pmatrix} e^{i2\eta}|+,+,+\rangle\langle-,-,-| \\ +e^{-i2\eta}|-,-,-\rangle\langle+,+,+| \end{pmatrix},$$

which describes the quantum interference [38–42] between the two components of the entangled state, leads to the violation of the GBI.

1. Verification of GBI with the local part of correlation

Let us suppose that three spins are measured independently by three observers along arbitrary directions $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$. The measuring outcomes of each spin fall into the eigenvalues of projection spin operator $\hat{s}\cdot\mathbf{r}$ ($\mathbf{r} = \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$), i.e.,

$$\hat{s}\cdot\mathbf{r}|\pm\mathbf{r}\rangle = \pm\tfrac{1}{2}|\pm\mathbf{r}\rangle \tag{18}$$

(in the unit convention $\hbar = 1$), for the case $s = 1/2$. The unit vector of arbitrary direction $\mathbf{r} = (\sin\theta_r\cos\phi_r, \sin\theta_r\sin\phi_r, \cos\theta_r)$ is parameterized with the polar and azimuthal angles θ_r, ϕ_r in spherical coordinates. Two eigenstates of Eq. (18) are given in Eq. (3), which are known as the spin-coherent states of the north- and south-pole gauges [38–41]. The independent measuring-outcome basis vectors are labeled as

$$\begin{aligned} &|1\rangle = |+\mathbf{a}_1, +\mathbf{a}_2, +\mathbf{a}_3\rangle, |2\rangle = |+\mathbf{a}_1, -\mathbf{a}_2, -\mathbf{a}_3\rangle,\\ &|3\rangle = |-\mathbf{a}_1, +\mathbf{a}_2, -\mathbf{a}_3\rangle, |4\rangle = |-\mathbf{a}_1, -\mathbf{a}_2, +\mathbf{a}_3\rangle,\\ &|5\rangle = |+\mathbf{a}_1, +\mathbf{a}_2, -\mathbf{a}_3\rangle, |6\rangle = |+\mathbf{a}_1, -\mathbf{a}_2, +\mathbf{a}_3\rangle,\\ &|7\rangle = |-\mathbf{a}_1, +\mathbf{a}_2, +\mathbf{a}_3\rangle, |8\rangle = |-\mathbf{a}_1, -\mathbf{a}_2, -\mathbf{a}_3\rangle, \end{aligned} \tag{19}$$

for the sake of simplicity. The basis vectors are the eigenstates of the correlation operator such that

$$\hat{\Omega}(a_1, a_2, a_3)|i\rangle = \pm\left(\frac{1}{2}\right)^3|i\rangle,$$

respectively, for $i = 1, 2, 3, 4$ and 5,6,7,8. Thus we have the (normalized) total correlation probability evaluated from the matrix elements of the density operator only,

$$p(a_1, a_2, a_3) = \sum_{i=1}^{4}\rho_{ii} - \sum_{i=5}^{8}\rho_{ii}.$$

From the local matrix elements of the density operator,

$$\rho_{11}^{lc} = \sin^2\xi\prod_{i=1}^{3}K_{a_i}^2 + \cos^2\xi\prod_{i=1}^{3}\Gamma_{a_i}^2,$$

$$\rho_{22}^{lc} = \sin^2\xi K_{a_1}^2\Gamma_{a_2}^2\Gamma_{a_3}^2 + \cos^2\xi\Gamma_{a_1}^2K_{a_2}^2K_{a_3}^2,$$

$$\rho_{33}^{lc} = \sin^2\xi\Gamma_{a_1}^2K_{a_2}^2\Gamma_{a_3}^2 + \cos^2\xi K_{a_1}^2\Gamma_{a_2}^2K_{a_3}^2,$$

$$\rho_{44}^{lc} = \sin^2\xi\Gamma_{a_1}^2\Gamma_{a_2}^2K_{a_3}^2 + \cos^2\xi K_{a_1}^2K_{a_2}^2\Gamma_{a_3}^2,$$

$$\rho_{55}^{lc} = \sin^2\xi K_{a_1}^2K_{a_2}^2\Gamma_{a_3}^2 + \cos^2\xi\Gamma_{a_1}^2\Gamma_{a_2}^2K_{a_3}^2,$$

$$\rho_{66}^{lc} = \sin^2\xi K_{a_1}^2\Gamma_{a_2}^2K_{a_3}^2 + \cos^2\xi\Gamma_{a_1}^2K_{a_2}^2\Gamma_{a_3}^2,$$

$$\rho_{77}^{lc} = \sin^2\xi\Gamma_{a_1}^2K_{a_2}^2K_{a_3}^2 + \cos^2\xi K_{a_1}^2\Gamma_{a_2}^2\Gamma_{a_3}^2,$$

$$\rho_{88}^{lc} = \sin^2\xi\prod_{i=1}^{3}\Gamma_{a_i}^2 + \cos^2\xi\prod_{i=1}^{3}K_{a_i}^2, \tag{20}$$

in which $K_r = \cos\theta_r/2$, $\Gamma_r = \sin\theta_r/2$ for $r = a_1, a_2, a_3$, the local part of the correlation is found as

$$p_{lc}(a_1, a_2, a_3) = -\cos(2\xi)\prod_{i=1}^{3}\cos\theta_{a_i}. \tag{21}$$

The GBI of Eq. (17) for $n = 3$ can be verified with the local part of the correlation probability given by Eq. (21) such that

$$\begin{aligned} &p_{lc}(a_1, a_2, a_3)p_{lc}(a_2, a_3, a_4)p_{lc}(a_3, a_4, a_5)\\ &= \cos^2(2\xi)\left(\prod_{i=2}^{4}\cos^2\theta_{a_i}\right)p_{lc}(a_1, a_3, a_5)\\ &\leqslant |p_{lc}(a_1, a_3, a_5)|. \end{aligned}$$

2. Maximum violation of GBI

The nonlocal parts of the density operator are evaluated as

$$\rho_{11}^{nlc} = \frac{1}{2^3}\sin(2\xi)\left(\prod_{i=1}^{3}\sin\theta_{a_i}\right)\cos\left(\sum_{i=1}^{3}\phi_{a_i} + 2\eta\right),$$

with

$$\rho_{ii}^{nlc} = \rho_{11}^{nlc},$$

for $i = 2, 3, 4$, and

$$\rho_{jj}^{nlc} = -\rho_{11}^{nlc},$$

for $j = 5, 6, 7, 8$. The nonlocal part of the correlation is

$$p_{\text{nlc}}(a_1, a_2, a_3) = \sin(2\xi)\left(\prod_{i=1}^{3}\sin\theta_{a_i}\right)\cos\left(\sum_{i=1}^{3}\phi_{a_i} + 2\eta\right). \tag{22}$$

The entire (normalized) quantum correlation probability becomes

$$p(a_1, a_2, a_3) = -\cos(2\xi)\left(\prod_{i=1}^{3}\cos\theta_{a_i}\right) + \sin(2\xi)\left(\prod_{i=1}^{3}\sin\theta_{a_i}\right)\cos\left(\sum_{i=1}^{3}\phi_{a_i} + 2\eta\right). \tag{23}$$

In order to find the maximum violation of the GBI, we define a quantum correlation-probability difference,

$$p_{GB} = p(a_1, a_2, a_3)p(a_2, a_3, a_4)p(a_3, a_4, a_5) - |p(a_1, a_3, a_5)|. \tag{24}$$

The GBI becomes

$$p_{GB}^{lc} \leqslant 0. \tag{25}$$

Thus any positive value of p_{GB} indicates the violation of GBI. The maximum violation appears with the polar angles $\theta_{a_1} = \theta_{a_2} = \theta_{a_3} = \pi/2$, where the local part of the correlation vanishes. Thus the correlation probability is simplified as

$$p(a_1, a_2, a_3) = \sin(2\xi)\cos\left(\sum_{i=1}^{3}\phi_{a_i} + 2\eta\right).$$

The probability difference given by Eq. (24) reads

$$p_{GB} = \sin^3(2\xi)\cos\left(\sum_{i=1}^{3}\phi_{a_i} + 2\eta\right) \times \cos\left(\sum_{i=2}^{4}\phi_{a_i} + 2\eta\right)\cos\left(\sum_{i=3}^{5}\phi_{a_i} + 2\eta\right) - \left|\sin(2\xi)\cos\left(\sum_{i=0}^{2}\phi_{a_{2i+1}} + 2\eta\right)\right|,$$

which becomes

$$p_{GB} = -\sin\left(\sum_{i=1}^{3}\phi_{a_i}\right)\sin\left(\sum_{i=2}^{4}\phi_{a_i}\right)\sin\left(\sum_{i=3}^{5}\phi_{a_i}\right) - \left|\sin\left(\sum_{i=0}^{2}\phi_{a_{2i+1}}\right)\right|,$$

for the state parameters $\xi = \eta = \pi/4 \mathrm{mod} 2\pi$. With the azimuthal angles of five measuring directions $\phi_{a_{2i+1}} = 0$ ($i = 0, 1, 2$), $\phi_{a_2} = \phi_{a_4} = 3\pi/4$, we have the maximum violation

$$p_{GB}^{\max} = \tfrac{1}{2}. \tag{26}$$

B. Four-particle entangled state of spin-1/2

For the four-particle entangled state, there are 16 independent basis vectors for the arbitrary measuring directions denoted by $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, and $\mathbf{a}_4$,

$$\begin{aligned}
&|1\rangle = |+\mathbf{a}_1, +\mathbf{a}_2, +\mathbf{a}_3, +\mathbf{a}_4\rangle, |2\rangle = |+\mathbf{a}_1, +\mathbf{a}_2, -\mathbf{a}_3, -\mathbf{a}_4\rangle,\\
&|3\rangle = |+\mathbf{a}_1, -\mathbf{a}_2, +\mathbf{a}_3, -\mathbf{a}_4\rangle, |4\rangle = |+\mathbf{a}_1, -\mathbf{a}_2, -\mathbf{a}_3, +\mathbf{a}_4\rangle,\\
&|5\rangle = |-\mathbf{a}_1, +\mathbf{a}_2, +\mathbf{a}_3, -\mathbf{a}_4\rangle, |6\rangle = |-\mathbf{a}_1, +\mathbf{a}_2, -\mathbf{a}_3, +\mathbf{a}_4\rangle,\\
&|7\rangle = |-\mathbf{a}_1, -\mathbf{a}_2, +\mathbf{a}_3, +\mathbf{a}_4\rangle, |8\rangle = |-\mathbf{a}_1, -\mathbf{a}_2, -\mathbf{a}_3, -\mathbf{a}_4\rangle,\\
&|9\rangle = |+\mathbf{a}_1, +\mathbf{a}_2, +\mathbf{a}_3, -\mathbf{a}_4\rangle, |10\rangle = |+\mathbf{a}_1, +\mathbf{a}_2, -\mathbf{a}_3, +\mathbf{a}_4\rangle,\\
&|11\rangle = |+\mathbf{a}_1, -\mathbf{a}_2, +\mathbf{a}_3, +\mathbf{a}_4\rangle, |12\rangle = |+\mathbf{a}_1, -\mathbf{a}_2, -\mathbf{a}_3, -\mathbf{a}_4\rangle,\\
&|13\rangle = |-\mathbf{a}_1, +\mathbf{a}_2, +\mathbf{a}_3, +\mathbf{a}_4\rangle, |14\rangle = |-\mathbf{a}_1, +\mathbf{a}_2, -\mathbf{a}_3, -\mathbf{a}_4\rangle,\\
&|15\rangle = |-\mathbf{a}_1, -\mathbf{a}_2, +\mathbf{a}_3, -\mathbf{a}_4\rangle, |16\rangle = |-\mathbf{a}_1, -\mathbf{a}_2, -\mathbf{a}_3, +\mathbf{a}_4\rangle.
\end{aligned} \tag{27}$$

They are the eigenstates of the spin correlation operator,

$$\hat{\Omega}(a_1, a_2, a_3, a_4) = (\hat{s}\cdot\mathbf{a}_1)\otimes(\hat{s}\cdot\mathbf{a}_2)\otimes(\hat{s}\cdot\mathbf{a}_3)\otimes(\hat{s}\cdot\mathbf{a}_4),$$

with the eigenvalues $\pm(1/2)^4$ for the states labeled, respectively, from 1–8 and 9–16. The average of the measuring-outcome correlation from four observers becomes the algebraic sum of the density operator. The local part of the correlation,

$$p_{lc}(a_1, a_2, a_3, a_4) = \sum_{i=1}^{8}\rho_{ii}^{lc} - \sum_{i=9}^{16}\rho_{ii}^{lc} = \prod_{i=1}^{4}\cos\theta_{a_i}, \tag{28}$$

gives rise to the four-particle GBI such that

$$\begin{aligned}
&p_{lc}(a_1, a_2, a_3, a_4)p_{lc}(a_2, a_3, a_4, a_5)\\
&\quad\times p_{lc}(a_3, a_4, a_5, a_6)p_{lc}(a_4, a_5, a_6, a_7)\\
&= \left(\prod_{i=1}^{4}\cos^2\theta_{a_{2i-1}}\right)\left(\prod_{i=2}^{6}\cos^2\theta_{a_i}\right)\cos^2\theta_{a_4}\\
&\leqslant |p_{lc}(a_1, a_3, a_5, a_7)|.
\end{aligned}$$

It may be worthwhile to remark that the four-particle local part of the correlation given by Eq. (28), which is independent of the state parameters ξ, η, has a positive sign compared with the three-particle case given by Eq. (21).

Maximum violation

The nonlocal elements of the density operator are

$$\rho_{ii}^{nlc} = \pm\frac{1}{2^4}\sin(2\xi)\left(\prod_{i=1}^{4}\sin\theta_{a_i}\right)\cos\left(\sum_{i=1}^{4}\phi_{a_i} + 2\eta\right),$$

respectively, for $i = 1$–8 and 9–16.

The nonlocal part of the correlation is

$$p_{\mathrm{nlc}} = \sin(2\xi)\left(\prod_{i=1}^{4}\sin\theta_{a_i}\right)\cos\left(\sum_{i=1}^{4}\phi_{a_i} + 2\eta\right).$$

The four-particle quantum correlation-probability difference is defined by

$$p_{GB} = p(a_1, a_2, a_3, a_4)p(a_2, a_3, a_4, a_5)p(a_3, a_4, a_5, a_6) \times p(a_4, a_5, a_6, a_7) - |p(a_1, a_3, a_5, a_7)|,$$

any positive value of which indicates the violation of GBI. The maximum violation appears when $\theta_{a_1} = \theta_{a_2} = \theta_{a_3} = \theta_{a_4} = \pi/2$, where we have

$$p(a_1, a_2, a_3, a_4) = \sin(2\xi)\cos\left(\sum_{i=1}^{4}\phi_{a_i} + 2\eta\right).$$

Then the quantum correlation-probability difference becomes

$$p_{GB}=\sin\left(\sum_{i=1}^{4}\phi_{a_i}\right)\sin\left(\sum_{i=2}^{5}\phi_{a_i}\right)\sin\left(\sum_{i=3}^{6}\phi_{a_i}\right)\times\sin\left(\sum_{i=4}^{7}\phi_{a_i}\right)-\left|\sin\left(\sum_{i=0}^{3}\phi_{a_{2i+1}}\right)\right|,$$

with parameters $\xi=\eta=\pi/4$. For the azimuthal angles $\phi_{a_{2i+1}}=0$, with $i=0,1,2,3$, we have

$$p_{GB}=\sin^2(\phi_{a_2}+\phi_{a_4})\sin^2(\phi_{a_4}+\phi_{a_6}).$$

The maximum violation is

$$p_{GB}^{\max}=1, \tag{29}$$

with the azimuthal angles $\phi_{a_{2i}}=\pi/4$ $(i=1,2,3)$ or $\phi_{a_4}=0$, $\phi_{a_2}=\phi_{a_6}=\pi/2$.

C. N-particle entangled state

For the n-particle entangled state,

$$|\psi\rangle=c_1|+\rangle^{\otimes n}+c_2|-\rangle^{\otimes n},$$

the GBI is satisfied by the local realistic mode with correlation

$$p_{lc}(a_1,a_2,\dots,a_n)=\sum_{i=1}^{2^{n-1}}\rho_{ii}^{lc}-\sum_{i=2^{n-1}+1}^{2^n}\rho_{ii}^{lc},$$

which gives rise to

$$p_{lc}(a_1,a_2,\dots,a_n)=-\cos(2\xi)\prod_{i=1}^{n}\cos\theta_{a_i}, \tag{30}$$

for n being an odd number, and

$$p_{lc}(a_1,a_2,\dots,a_n)=\prod_{i=1}^{n}\cos\theta_{a_i}, \tag{31}$$

for even n.

1. Odd n

When n is odd, the local part of the normalized correlation probability given by Eq. (30) satisfies GBI that

$$\begin{aligned}&p_{lc}(a_1,a_2,\dots,a_n)p_{lc}(a_2,a_3,\dots,a_{n+1})\\&\quad\times p_{lc}(a_3,a_4,\dots,a_{n+2})\cdots p_{lc}(a_n,a_{n+1},\dots,a_{2n-1})\\&=\cos^{n-1}(2\xi)\prod_{i=2}^{n+1}\cos^2\theta_{a_i}\prod_{i=4}^{n+3}\cos^2\theta_{a_i}\prod_{i=6}^{n+5}\cos^2\theta_{a_i}\\&\quad\times\cdots\times\prod_{i=n-1}^{2n-2}\cos^2\theta_{a_i}p_{lc}(a_1,a_3,\dots,a_{2n-1})\\&\leqslant|p_{lc}(a_1,a_3,\dots,a_{2n-1})|.\end{aligned} \tag{32}$$

With the nonlocal part of the correlation,

$$\begin{aligned}&p_{\text{nlc}}(a_1,a_2,\dots,a_n)\\&=\sin(2\xi)\left(\prod_{i=1}^{n}\sin\theta_{a_i}\right)\cos\left(\sum_{i=1}^{n}\phi_{a_i}+2\eta\right),\end{aligned} \tag{33}$$

the total correlation becomes

$$\begin{aligned}&p(a_1,a_2,\dots,a_n)\\&=-\cos(2\xi)\prod_{i=1}^{n}\cos\theta_{a_i}\\&\quad+\sin(2\xi)\left(\prod_{i=1}^{n}\sin\theta_{a_i}\right)\cos\left(\sum_{i=1}^{n}\phi_{a_i}+2\eta\right).\end{aligned}$$

For the polar angles $\theta_{a_i}=\pi/2$ and state parameters $\xi=\pi/4\text{mod}2\pi$ and $\eta=\pi/4\text{mod}2\pi$, it is simplified as

$$p(a_1,a_2,\dots,a_n)=-\sin\left(\sum_{i=1}^{n}\phi_{a_i}\right).$$

Then the quantum correlation-probability difference becomes

$$\begin{aligned}p_{GB}&=p(a_1,a_2,\dots,a_n)p(a_2,a_3,\dots,a_{n+1})\\&\quad\times p(a_3,a_4,\dots,a_{n+2})\cdots\times p(a_n,a_{n+1},\dots,a_{2n-1})\\&\quad-|p(a_1,a_3,\dots,a_{2n-1})|\\&=-\sin\left(\sum_{i=1}^{n}\phi_{a_i}\right)\sin\left(\sum_{i=2}^{n+1}\phi_{a_i}\right)\times\cdots\\&\quad\times\sin\left(\sum_{i=n}^{2n-1}\phi_{a_i}\right)-\left|\sin\left(\sum_{i=1}^{n}\phi_{a_{2i-1}}\right)\right|.\end{aligned}$$

When $\phi_{a_{2i-1}}=0$ $(i=1,2,\dots,n)$, we have

$$\begin{aligned}p_{GB}&=-\sin\left(\sum_{i=1}^{(n-1)/2}\phi_{a_{2i}}\right)\sin\left(\sum_{i=1}^{(n+1)/2}\phi_{a_{2i}}\right)\\&\quad\times\sin\left(\sum_{i=2<n-1}^{(n+1)/2}\phi_{a_{2i}}\right)\sin\left(\sum_{i=2<n-1}^{(n+3)/2}\phi_{a_{2i}}\right)\\&\quad\times\sin\left(\sum_{i=3<n-1}^{(n+3)/2}\phi_{a_{2i}}\right)\sin\left(\sum_{i=3<n-1}^{(n+5)/2}\phi_{a_{2i}}\right)\\&\quad\times\cdots\times\sin\left(\sum_{i=(n+1)/2}^{n-1}\phi_{a_{2i}}\right),\end{aligned} \tag{34}$$

which possesses a maximum value with the azimuthal angles given by $\phi_{a_{2i}}=0$, except $i=(n\pm1)/2$, and $\phi_{a_{n-1}}=\phi_{a_{n+1}}=3\pi/4$. Along these measuring conditions, the correlation-probability difference given by Eq. (34) approaches the maximum bound,

$$\begin{aligned}p_{GB}^{\max}&=-\sin\phi_{a_{n-1}}\sin\phi_{a_{n+1}}\sin^{n-2}(\phi_{a_{n-1}}+\phi_{a_{n+1}})\\&=\tfrac{1}{2}.\end{aligned} \tag{35}$$

2. Even n

For even n, the local part of the n-particle correlation probability satisfies the GBI that

$$\begin{aligned}&p_{lc}(a_1,a_2,\dots,a_n)p_{lc}(a_2,a_3,\dots,a_{n+1})\\&\times p_{lc}(a_3,a_4,\dots,a_{n+2})\cdots p_{lc}(a_n,a_{n+1},\dots,a_{2n-1})\end{aligned}$$

$$= \prod_{i=1}^{n}\cos\theta_{a_i}\prod_{i=2}^{n+1}\cos\theta_{a_i}\prod_{i=3}^{n+2}\cos\theta_{a_i}\cdots\prod_{i=n}^{2n-1}\cos\theta_{a_i}$$
$$= \prod_{i=2}^{n}\cos^2\theta_{a_i}\prod_{i=4}^{n+2}\cos^2\theta_{a_i}\prod_{i=6}^{n+4}\cos^2\theta_{a_i}\cdots\prod_{i=n}^{2n-2}\cos^2\theta_{a_i}$$
$$\times\, p_{lc}(a_1, a_3, \dots, a_{2n-1})$$
$$\leqslant |p_{lc}(a_1, a_3, \dots, a_{2n-1})|. \qquad (36)$$

The nonlocal part of the correlation $p_{\mathrm{nlc}}(a_1, a_2, \dots, a_n)$ is the same as that of odd n given in Eq. (33).

The normalized total correlation,

$$p(a_1, a_2, \dots, a_n) = \prod_{i=1}^{n}\cos\theta_{a_i} + \sin(2\xi) \times \left(\prod_{i=1}^{n}\sin\theta_{a_i}\right)\cos\left(\sum_{i=1}^{n}\phi_{a_i} + 2\eta\right),$$

is simplified as

$$p(a_1, a_2, \dots, a_n) = -\sin\left(\sum_{i=1}^{n}\phi_{a_i}\right),$$

with the state parameters $\xi = \eta = \pi/4 \mathrm{mod} 2\pi$, and polar angles $\theta_{a_i} = \pi/2$ of the measuring directions. Then the correlation difference becomes

$$p_{GB} = \sin\left(\sum_{i=1}^{n}\phi_{a_i}\right)\sin\left(\sum_{i=2}^{n+1}\phi_{a_i}\right)\sin\left(\sum_{i=3}^{n+2}\phi_{a_i}\right) \times \cdots \times \sin\left(\sum_{i=n}^{2n-1}\phi_{a_i}\right) - \left|\sin\left(\sum_{i=1}^{n}\phi_{a_{2i-1}}\right)\right|.$$

The maximum value of p_{GB} appears when $\phi_{a_i} = 0$, for $i = 1, 3, 5, \dots, 2n-1$, and we have

$$p_{GB} = \sin^2\left(\sum_{i=1}^{n/2}\phi_{a_{2i}}\right)\sin^2\left(\sum_{i=2}^{n/2+1}\phi_{a_{2i}}\right) \times \sin^2\left(\sum_{i=3}^{n/2+2}\phi_{a_{2i}}\right)\cdots\sin^2\left(\sum_{i=n/2}^{n-1}\phi_{a_{2i}}\right).$$

The maximum bound of the violation is

$$p_{GB}^{\max} = 1, \qquad (37)$$

under the condition of $\phi_{a_{2i}} = \pi/n$ with $i = 1, 2, 3, \dots, n-1$. For the n-particle entangled state of spin-1/2, the GBI is always satisfied by the local correlation. The nonlocal part of the correlation gives rise to the violation of GBI. The maximal violation bound is $p_{GB}^{\max} = 1/2$ for the odd n and 1 for the even n. The maximum violation appears when the $2n-1$ measuring directions are perpendicular to the spin polarization (z axis); the maximum bound depends on the sin function of $n/2$ azimuthal angles $\phi_{a_{2i}}$ for the even n. We always have the possibility to choose the equal value of the angles $\phi_{a_{2i}} = \pi/n$ to approach the maximum bound $p_{GB}^{\max} = 1$, which is in agreement with the previous observation that the violation is larger [54] for even n .

IV. SPIN-PARITY EFFECT IN THE VIOLATION OF GBI FOR *N*-PARTICLE ENTANGLED SCHRÖDINGER CAT STATE OF SPIN *S*

For entangled Schrödinger cat states of spin s, the GBI is always satisfied since the nonlocal correlation vanishes by quantum average. We now consider the measuring outcomes restricted in the subspace of SCS instead, namely, only the maximum spin values $\pm s$ are measured along arbitrary directions.

A. Three-particle case

The local and nonlocal parts of density operator $\hat{\rho}$ are, respectively, written as

$$\hat{\rho}_{lc} = \sin^2\xi|+s, +s, +s\rangle\langle +s, +s, +s| + \cos^2\xi|-s, -s, -s\rangle\langle -s, -s, -s|,$$
$$\hat{\rho}_{\mathrm{nlc}} = \sin\xi\cos\xi\begin{pmatrix} e^{i2\eta}|+s, +s, +s\rangle\langle -s, -s, -s| \\ +e^{-i2\eta}|-s, -s, -s\rangle\langle +s, +s, +s|\end{pmatrix},$$

for the three-particle entangled cat state,

$$|\psi\rangle_{GHZ} = c_1|+s, +s, +s\rangle + c_2|-s, -s, -s\rangle.$$

The SCSs of the projection spin operator in direction $\mathbf{r} = \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ found from the eigenequations $\hat{s}\cdot\mathbf{r}|\pm\mathbf{r}\rangle = \pm s|\pm\mathbf{r}\rangle$ are given by Eq. (10). With the independent measuring-outcome basis vectors labeled in Eq. (19), the local part of the measuring-outcome correlation is

$$P_{lc}(a_1, a_2, a_3) = s^3\left(\sum_{i=1}^{4}\rho_{ii}^{lc} - \sum_{i=5}^{8}\rho_{ii}^{lc}\right),$$

in which the elements of the local density operator are given by the same formulas as Eq. (20), however, with the power index "2" replaced by "4s"; for example, the first one is

$$\rho_{11}^{lc} = \sin^2\xi\prod_{i=1}^{3}K_{a_i}^{4s} + \cos^2\xi\prod_{i=1}^{3}\Gamma_{a_i}^{4s}.$$

The normalized local correlation is found as

$$p_{lc}(a_1, a_2, a_3) = \frac{P_{lc}}{s^3} = -\cos(2\xi)\left(K_{a_1}^{4s} - \Gamma_{a_1}^{4s}\right)\left(K_{a_2}^{4s} - \Gamma_{a_2}^{4s}\right)\left(K_{a_3}^{4s} - \Gamma_{a_3}^{4s}\right),$$

which gives rise to the GBI such that

$$p_{lc}(a_1, a_2, a_3)p_{lc}(a_2, a_3, a_4)p_{lc}(a_3, a_4, a_5) \leqslant -\cos(2\xi)\left(K_{a_1}^{4s} - \Gamma_{a_1}^{4s}\right)\left(K_{a_3}^{4s} - \Gamma_{a_3}^{4s}\right)\left(K_{a_5}^{4s} - \Gamma_{a_5}^{4s}\right) \leqslant |p_{lc}(a_1, a_3, a_5)|.$$

The nonlocal elements of the density operator are seen to be

$$\rho_{11}^{nlc} = \sin(2\xi)K_{a_1}^{2s}\Gamma_{a_1}^{2s}K_{a_2}^{2s}\Gamma_{a_2}^{2s}K_{a_3}^{2s}\Gamma_{a_3}^{2s} \times \cos[2s(\phi_{a_1} + \phi_{a_2} + \phi_{a_3}) + 2\eta] = \rho_{ii}^{nlc},$$

for $i = 2$–4, and

$$\rho_{jj}^{nlc} = (-1)^{2s}\rho_{11}^{nlc}, \qquad (38)$$

with $j = 5$–8. It may be worthwhile to notice that the density matrix elements of the nonlocal part differ by the same phase factor,

$$(-1)^{2s} = \exp(i2s\pi),$$

as Eq. (13), which resulted from the geometric phase or Berry phase between SCSs of the north- and south-pole gauges. For integer spin s, the nonlocal correlation vanishes,

$$p_{\text{nlc}}(a_1, a_2, a_3) = \sum_{i=1}^{4} \rho_{ii}^{nlc} - \sum_{i=5}^{8} \rho_{ii}^{nlc} = 0, \tag{39}$$

which leads to nonviolation of GBI. For half-integer spin s, the nonlocal correlation becomes

$$p_{\text{nlc}}(a_1, a_2, a_3) = 2^{-3(2s-1)} \sin(2\xi) \left(\prod_{i=1}^{3} \sin^{2s} \theta_{a_i} \right) \times \cos\left[2s \left(\sum_{i=1}^{3} \phi_{a_i} \right) + 2\eta \right].$$

The whole quantum correlation probability $p(a_1, a_2, a_3)$ can approach the maximum violation bound with polar angle $\theta_r = \pi/2$,

$$p(a_1, a_2, a_3) = 2^{-3(2s-1)} \sin(2\xi) \cos\left[2s \left(\sum_{i=1}^{3} \phi_{a_i} \right) + 2\eta \right]. \tag{40}$$

The correlation probability for the measurement in the SCS subspace decreases with the increase of spin s since the dimension of the whole Hilbert space is $(2s+1)^3$, while the number of measuring outcome states is only 8. The correlation probability vanishes when $s \to \infty$, in agreement with the known observations [74–77]. We may consider the relative or scaled correlation probability

$$p_{rl}(a_1, a_2, a_3) = \frac{p(a_1, a_2, a_3)}{N},$$

where the normalization constant,

$$N = \sum_{i=1}^{8} |\langle i|\psi\rangle|^2 = \sum_{i=1}^{8} \rho_{ii} = 2^{-3(2s-1)},$$

is the total probability of entangled state $|\psi\rangle$ in the eight measuring basis vectors of SCS given by Eq. (19).

The relative or scaled correlation probability is

$$p_{rl}(a_1, a_2, a_3) = \sin(2\xi) \cos\left[2s \left(\sum_{i=1}^{3} \phi_{a_i} \right) + 2\eta \right]. \tag{41}$$

In the following, the scaled correlation probabilities of Eq. (41) are adopted without the subscript "rl" for the sake of simplicity.

The quantity of the correlation difference is found as

$$p_{GB} = -\sin(2s\phi_{a_2}) \sin[2s(\phi_{a_2} + \phi_{a_4})] \sin(2s\phi_{a_4}),$$

with parameters $\xi = \eta = \pi/4$, and azimuthal angles of measuring directions $\phi_{a_1} = \phi_{a_3} = \phi_{a_5} = 0$. The maximum bound of the violation is

$$p_{GB}^{\max} = \tfrac{1}{2}, \tag{42}$$

when $\phi_{a_2} = \phi_{a_4} = 3\pi/(8s)$, consistent with the case of spin 1/2.

B. Four-particle case

For the four-particle entangled Schrödinger cat state,

$$|\psi\rangle_{\text{GHZ}} = c_1| + s, +s, +s, +s\rangle + c_2| - s, -s, -s, -s\rangle,$$

we have 16 independent basis vectors for measuring along four arbitrary directions $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4$. Following the same procedure, we have the local correlation probability,

$$p_{lc}(a_1, a_2, a_3, a_4) = \prod_{i=1}^{4} \left(K_{a_i}^{4s} - \Gamma_{a_i}^{4s} \right),$$

which gives rise to the GBI that

$$\begin{aligned} &p_{lc}(a_1, a_2, a_3, a_4) p_{lc}(a_2, a_3, a_4, a_5) \\ &\quad \times p_{lc}(a_3, a_4, a_5, a_6) p_{lc}(a_4, a_5, a_6, a_7) \\ &\leqslant \prod_{i=0}^{3} \left(K_{a_{2i+1}}^{4s} - \Gamma_{a_{2i+1}}^{4s} \right) \\ &\leqslant |p_{lc}(a_1, a_3, a_5, a_7)|. \end{aligned}$$

The nonlocal correlation vanishes for integer spin s. For half-integer s, it is

$$p_{\text{nlc}}(a_1, a_2, a_3, a_4) = 2^{-4(2s-1)} \sin(2\xi) \left(\prod_{i=1}^{4} \sin^{2s} \theta_{a_i} \right) \times \cos\left[2s \left(\sum_{i=1}^{4} \phi_{a_i} \right) + 2\eta \right].$$

The total quantum correlation probability is

$$p(a_1, a_2, a_3, a_4) = 2^{-4(2s-1)} \sin(2\xi) \times \cos\left[2s \left(\sum_{i=1}^{4} \phi_{a_i} \right) + 2\eta \right],$$

under the condition of polar angles equal to $\theta_i = \pi/2$. We again consider the relative or scaled correlation probability $p_{rl}(a_1, a_2, a_3, a_4) = p(a_1, a_2, a_3, a_4)/N$ ($N = 2^{-4(2s-1)}$), which becomes

$$p_{rl}(a_1, a_2, a_3, a_4) = \sin(2\xi) \cos\left[2s \left(\sum_{i=1}^{4} \phi_{a_i} \right) + 2\eta \right].$$

Then the quantum correlation-probability difference is

$$\begin{aligned} p_{GB} &= p(a_1, a_2, a_3, a_4) p(a_2, a_3, a_4, a_5) p(a_3, a_4, a_5, a_6) \\ &\quad \times p(a_4, a_5, a_6, a_7) - |p_{lc}(a_1, a_3, a_5, a_7)| \\ &= \sin^2[2s(\phi_{a_2} + \phi_{a_4})] \sin^2[2s(\phi_{a_4} + \phi_{a_6})], \end{aligned}$$

with parameters $\xi = \eta = \pi/4$, and azimuthal angles $\phi_{a_1} = \phi_{a_3} = \phi_{a_5} = \phi_{a_7} = 0$. The maximum violation bound is obviously

$$p_{GB}^{\max} = 1,$$

when $\phi_{a_2} = \phi_{a_4} = \phi_{a_6} = \pi/(8s)$.

C. N-particle case

For the n-particle state,

$$|\psi\rangle = e^{i\eta}\sin\xi|+s\rangle^{\otimes n} + e^{-i\eta}\cos\xi|-s\rangle^{\otimes n},$$

the local part of the n direction measuring-outcome correlation,

$$p_{lc}(a_1, a_2, \dots, a_n) = \sum_{i=1}^{2^{n-1}} \rho_{ii}^{lc} - \sum_{i=2^{n-1}+1}^{2^n} \rho_{ii}^{lc},$$

is found, respectively, as

$$p_{lc}(a_1, a_2, \dots, a_n) = -\cos(2\xi)\prod_{i=1}^{n}\left(K_{a_i}^{4s} - \Gamma_{a_i}^{4s}\right), \quad (43)$$

for odd n, and

$$p_{lc}(a_1, a_2, \dots, a_n) = \prod_{i=1}^{n}\left(K_{a_i}^{4s} - \Gamma_{a_i}^{4s}\right), \quad (44)$$

for even n. The expressions of Eqs. (43) and (44) have the same forms compared with the n-particle state of the spin-1/2 local correlation given by Eqs. (30) and (31), where $\cos\theta_{a_i}$ is replaced by $(K_{a_i}^{4s} - \Gamma_{a_i}^{4s})$. The GBI is satisfied by the n-particle local correlations,

$$\begin{aligned}
&p_{lc}(a_1, a_2, \dots, a_n)p_{lc}(a_2, a_3, \dots, a_{n+1}) \\
&\quad\cdots p_{lc}(a_n, a_{n+1}, \dots, a_{2n-1}) \\
&= \cos^{n-1}(2\xi)\prod_{i=2}^{n+1}\left(K_{a_i}^{4s} - \Gamma_{a_i}^{4s}\right)^2\prod_{i=4}^{n+3}\left(K_{a_i}^{4s} - \Gamma_{a_i}^{4s}\right)^2 \\
&\quad\times\prod_{i=6}^{n+5}\left(K_{a_i}^{4s} - \Gamma_{a_i}^{4s}\right)^2\cdots\prod_{i=n-1}^{2n-2}\left(K_{a_i}^{4s} - \Gamma_{a_i}^{4s}\right)^2 \\
&\quad\times p_{lc}(a_1, a_3, \dots, a_{2n-1}) \\
&\leqslant |p_{lc}(a_1, a_3, \dots, a_{2n-1})|,
\end{aligned}$$

with odd n, and

$$\begin{aligned}
&p_{lc}(a_1, a_2, \dots, a_n)p_{lc}(a_2, a_3, \dots, a_{n+1}) \\
&\quad\cdots p_{lc}(a_n, a_{n+1}, \dots, a_{2n-1}) \\
&= \prod_{i=2}^{n}\left(K_{a_i}^{4s} - \Gamma_{a_i}^{4s}\right)^2\prod_{i=4}^{n+2}\left(K_{a_i}^{4s} - \Gamma_{a_i}^{4s}\right)^2\prod_{i=6}^{n+4}\left(K_{a_i}^{4s} - \Gamma_{a_i}^{4s}\right)^2 \\
&\quad\times\cdots\times\prod_{i=n}^{2n-2}\left(K_{a_i}^{4s} - \Gamma_{a_i}^{4s}\right)^2 p_{lc}(a_1, a_3, \dots, a_{2n-1}) \\
&\leqslant |p_{lc}(a_1, a_3, \dots, a_{2n-1})|,
\end{aligned}$$

with even n. In the following, we find the maximum violation bound including the nonlocal correlations. Since the nonlocal correlation of integer spin s vanishes, the violation of GBI appears only for the half-integer spin s.

Maximum violation bound for odd n

Including the nonlocal part of the correlation,

$$\begin{aligned}
&p_{\text{nlc}}(a_1, a_2, \dots, a_n) \\
&\quad= 2^{-n(2s-1)}\sin(2\xi) \\
&\qquad\times\left(\prod_{i=1}^{n}\sin^{2s}\theta_{a_i}\right)\cos\left[2s\left(\sum_{i=1}^{n}\phi_{a_i}\right) + 2\eta\right],
\end{aligned}$$

the quantum correlation is

$$\begin{aligned}
p(a_1, a_2, \dots, a_n) &= -\cos(2\xi)\prod_{i=1}^{n}\left(K_{a_i}^{4s} - \Gamma_{a_i}^{4s}\right) \\
&\quad + 2^{-n(2s-1)}\sin(2\xi)\left(\prod_{i=1}^{n}\sin^{2s}\theta_{a_i}\right) \\
&\quad\times\cos\left[2s\left(\sum_{i=1}^{n}\phi_{a_i}\right) + 2\eta\right].
\end{aligned}$$

With polar angle $\theta_r = \pi/2$, where the maximum violation appears, we have the simplified quantum correlation,

$$\begin{aligned}
p(a_1, a_2, \dots, a_n) &= 2^{-n(2s-1)}\sin(2\xi) \\
&\quad\times\cos\left[2s\left(\sum_{i=1}^{n}\phi_{a_i}\right) + 2\eta\right].
\end{aligned}$$

The decreasing number factor with s can be scaled out by $p_{rl}(a_1, a_2, \dots, a_n) = p(a_1, a_2, \dots, a_n)/N$, with $N = \sum_{i=1}^{2^n}|\langle i|\psi\rangle|^2 = \sum_{i=1}^{2^n}\rho_{ii} = 2^{-n(2s-1)}$. The relative or scaled correlation probability becomes

$$p_{rl}(a_1, a_2, \dots, a_n) = \sin(2\xi)\cos\left[2s\left(\sum_{i=1}^{n}\phi_{a_i}\right) + 2\eta\right]. \quad (45)$$

In the following, the scaled correlation probabilities of Eq. (45) are adopted without the subscript "*rl*" for the sake of simplicity.

With state parameters $\xi = \pi/4 \text{mod} 2\pi$ and $\eta = \pi/4 \text{mod} 2\pi$, Eq. (45) becomes

$$p(a_1, a_2, \dots, a_n) = -\sin\left(2s\sum_{i=1}^{n}\phi_{a_i}\right).$$

The quantum correlation-probability difference for the spin-s case is seen to be

$$\begin{aligned}
p_{GB} &= -\sin\left(2s\sum_{i=1}^{n}\phi_{a_i}\right)\sin\left(2s\sum_{i=2}^{n+1}\phi_{a_i}\right) \\
&\quad\times\cdots\times\sin\left(2s\sum_{i=n}^{2n-1}\phi_{a_i}\right) - \left|\sin\left(2s\sum_{i=1,3,5}^{2n-1}\phi_{a_i}\right)\right|,
\end{aligned}$$

which reduces to

$$\begin{aligned}
p_{GB} &= -\sin\left(2s\sum_{i=1}^{(n-1)/2}\phi_{a_{2i}}\right)\sin\left(2s\sum_{i=1}^{(n+1)/2}\phi_{a_{2i}}\right) \\
&\quad\times\sin\left(2s\sum_{i=2}^{(n+1)/2}\phi_{a_{2i}}\right)\sin\left(2s\sum_{i=2}^{(n+3)/2}\phi_{a_{2i}}\right)
\end{aligned}$$

$$\times \sin\left(2s\sum_{i=3}^{(n+3)/2}\phi_{a_{2i}}\right)\sin\left(2s\sum_{i=3}^{(n+5)/2}\phi_{a_{2i}}\right)$$
$$\times\cdots\times\sin\left(2s\sum_{i=(n+1)/2}^{n-1}\phi_{a_{2i}}\right),$$

with vanishing angles $\phi_{a_i}=0\,(i=1,3,5,\ldots,2n-1)$ of measuring directions. We, furthermore, let all other angles be zero that $\phi_{a_{2i}}=0$, except the two angles $\phi_{a_{n-1}}=\phi_{a_{n+1}}=3\pi/(8s)$; then the maximum violation bound is approached,

$$\begin{aligned}p_{GB}^{\max}&=-\sin(2s\phi_{a_{n-1}})\sin(2s\phi_{a_{n+1}})\\&\quad\times\sin^{n-2}[2s(\phi_{a_{n-1}}+\phi_{a_{n+1}})]\\&=\tfrac{1}{2}.\end{aligned}\tag{46}$$

1. Even n

For even n, the quantum correlation including the nonlocal part is

$$\begin{aligned}p(a_1,a_2,\ldots,a_n)&=\prod_{i=1}^{n}\left(K_{a_i}^{4s}-\Gamma_{a_i}^{4s}\right)+2^{-n(2s-1)}\\&\quad\times\sin(2\xi)\left(\prod_{i=1}^{n}\sin^{2s}\theta_{a_i}\right)\\&\quad\times\cos\left[2s\left(\sum_{i=1}^{n}\phi_{a_i}\right)+2\eta\right],\end{aligned}$$

which reduces (after rescale) to

$$p(a_1,a_2,\ldots,a_n)=\sin(2\xi)\cos\left[2s\left(\sum_{i=1}^{n}\phi_{a_i}\right)+2\eta\right],\tag{47}$$

with the polar angle $\theta_r=\pi/2$. For the state parameters $\xi=\eta=\pi/4\mathrm{mod}2\pi$, Eq. (47) becomes

$$p(a_1,a_2,\ldots,a_n)=-\sin\left(2s\sum_{i=1}^{n}\phi_{a_i}\right),$$

with which the quantum correlation-probability difference is

$$\begin{aligned}p_{GB}&=\sin\left(2s\sum_{i=1}^{n}\phi_{a_i}\right)\sin\left(2s\sum_{i=2}^{n+1}\phi_{a_i}\right)\sin\left(2s\sum_{i=3}^{n+2}\phi_{a_i}\right)\\&\quad\times\cdots\times\sin\left(2s\sum_{i=n}^{2n-1}\phi_{a_i}\right)-\left|\sin\left(2s\sum_{i=1,3,5}^{2n-1}\phi_{a_i}\right)\right|.\end{aligned}$$

Choosing azimuthal angles $\phi_{a_i}=0\,(i=1,3,5,\ldots,2n-1)$ of the measuring directions, we have

$$\begin{aligned}p_{GB}&=\sin^2\left(2s\sum_{i=1}^{n/2}\phi_{a_{2i}}\right)\sin^2\left(2s\sum_{i=2}^{n/2+1}\phi_{a_{2i}}\right)\\&\quad\times\sin^2\left(2s\sum_{i=3}^{n/2+2}\phi_{a_{2i}}\right)\cdots\sin^2\left(2s\sum_{i=n/2}^{n-1}\phi_{a_{2i}}\right),\end{aligned}$$

which becomes

$$p_{GB}=\sin^n(ns\phi)$$

under the condition that all azimuthal angles are of equal value, $\phi_{a_{2i}}=\phi$. The maximum violation bound is

$$p_{GB}^{\max}=1,\tag{48}$$

when $\phi=\pi/(2ns)$.

The spin-parity effect in the violation of GBI exists for the n-particle entangled Schrödinger cat state of spin s if the measurements are restricted only in the subspace of SCSs. Moreover, a particle-number parity effect is also demonstrated in which the maximal violation bound is $p_{GB}^{\max}=1/2$ for odd number and $p_{GB}^{\max}=1$ for even number.

We have seen that maximum violation appears when the two components of the entangled states have equal probability $|c_1|^2=|c_2|^2=1/2$, and the measuring directions are all perpendicular to the initial spin polarization, i.e., $\theta_{a_i}=\pi/2$. Then, the maximum violation depends on the state phase angles ξ, η and the azimuthal angles ϕ_{a_i} only. The state phase angles are fixed as $\xi=\eta=\pi/4$ in the above evaluation. However, this is not the only choice. It may be more convenient in experiment that $\xi=\pi/4$, $\eta=0$, and the state coefficients are real, $c_1=c_2=1/\sqrt{2}$. The particle-number parity effect is invariant in this case. The azimuthal angles of the measuring directions for even n become $\phi_{a_i}=\pi/(4ns)$ with $i=1,3,5,\ldots,2n-1$, and $\phi_{a_i}=3\pi/(4ns)$ with $i=2,4,6,\ldots,2(n-1)$, while for odd n the angles should be chosen as $\phi_{a_n}=\pi/(4s)$, $\phi_{a_{n-1}}=\phi_{a_{n+1}}=3\pi/(8s)$, and the rest of the angles are all zero.

V. CONCLUSION AND DISCUSSION

We propose in this paper a GBI [$p_{GB}^{lc}\leqslant 0$, Eq. (25)] for the n-particle entangled Schrödinger cat state of spin s. It needs n observers and total $2n-1$ measuring directions following the original BI with $n=2$ and $s=1/2$. The GBI and its violation can be formulated in a unified way by means of the SCS quantum probability statistics. The density operator of the entangled states is divided into a local part and nonlocal part, which describes the quantum interference of the coherent superposition of entangled multiparticle states. The local part leads to the GBI, while the nonlocal part is responsible for the violation in quantum average.

For the n-particle entangled state of spin 1/2, the maximum violation bound depends on the particle number that $p_{GB}^{\max}=1/2, 1$, respectively, for odd and even n, consistent with the known observation of a larger violation [54] for even n. The GBI is never violated by the n -particle entangled Schrödinger cat state with higher spin s under the quantum average. When the measuring outcomes are restricted to the subspace of SCSs, namely, only the maximum spin values $\pm s$ are taken into account, the GBI is violated only by half integer and not integer spin s. This spin-parity effect is seen to be a direct result of the Berry phase between the SCSs of the north- and south-pole gauges. The maximum violation bound of GBI also depends on the particle number, which is the same as the spin-1/2 case. The particle-number parity effect may have some applications in quantum information associated with many-particle entanglement.

Our generic predictions could be tested experimentally with the orbital angular momentum entangled photons [20], macroscopic quantum entanglement of electronic spins of nitrogen-vacancy defect in a diamond chip [30], and superconducting qubits [79] as well. The violation of the BIs or not is related to the quantum probabilities, depending basically on the quantum coherence features. The measurement of maximum violation, which is caused entirely by the nonlocal coherence, indeed should be useful to develop device-independent entanglement witnesses. The two-particle entangled state of spin 1/2 is easily prepared with the usual polarization entanglement of photons. And, for the spin-s entangled Schrödinger cat state, one needs to use the orbital angular momentum entangled states [20,78], for example, $|\psi\rangle = (|l_1 = +1, l_2 = \mp 1\rangle + |l_1 = -1, l_2 = \pm 1\rangle)/\sqrt{2}$. The angular momentum values ± 1 can be measured along three arbitrary directions $\mathbf{a}_1$, $\mathbf{a}_2$, and $\mathbf{a}_3$ independently by two detectors. The violation of BIs should not appear [38,39] at all for this state and thus the spin-parity effect is justified in this example.

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APPENDIX

We present the derivation of GBI for an n-particle spin-s entangled Schrödinger cat state in terms of hidden-variable classical statistics following Bell [12]. For the entangled Schrödinger cat state of $n = 2$,

$$|\psi\rangle = c_1|+s, +s\rangle + c_2|-s, -s\rangle, \tag{A1}$$

with $|c_1|^2 + |c_2|^2 = 1$, the normalized measuring-outcome values of two observers are denoted by

$$A_1(a_1) = \pm 1,$$
$$A_2(a_2) = \pm 1,$$

respectively, along the measuring directions a_1 and a_2. The measuring-outcome correlation according to Bell [12] is evaluated by the classical statistics,

$$p_{lc}(a_1, a_2) = \int \rho(\lambda) A_1(a_1, \lambda) A_2(a_2, \lambda) d\lambda$$
$$\equiv \langle A_1(a_1) A_2(a_2)\rangle,$$

in which $\rho(\lambda)$ is the probability density distribution of hidden variable λ. The product of two correlations is

$$\begin{aligned} &p_{lc}(a_1, a_2) p_{lc}(a_2, a_3) \\ &= \iint \rho(\lambda)\rho(\lambda') A_1(a_1, \lambda) A_2(a_2, \lambda) A_1(a_2, \lambda') A_2(a_3, \lambda') d\lambda d\lambda' \\ &\leqslant \left| \iint \rho(\lambda)\rho(\lambda') A_1(a_1, \lambda) A_2(a_3, \lambda') d\lambda d\lambda' \right| \\ &= |\langle A_1(a_1)\rangle\langle A_2(a_3)\rangle|, \end{aligned} \tag{A2}$$

since $A_2(a_2) = A_1(a_2)$ for the parallel spin polarization of the entangled state given by Eq. (A1) and $A_2^2(a_2) = 1$. We define the classical probability mean deviation by

$$\Delta A_1 \equiv A_1(a_1) - \langle A_1(a_1)\rangle,$$
$$\Delta A_2 \equiv A_2(a_3) - \langle A_2(a_3)\rangle,$$

with $\langle A_1(a_1)\rangle = \int \rho(\lambda) A_1(a_1, \lambda) d\lambda$, and $\langle A_2(a_3)\rangle = \int \rho(\lambda) A_2(a_3, \lambda) d\lambda$ being the average values of the measuring outcomes. The average of the deviation product is evaluated as

$$\begin{aligned} \langle \Delta A_1 \Delta A_2\rangle &= \langle [A_1(a_1) - \langle A_1(a_1)\rangle][A_2(a_3) - \langle A_2(a_3)\rangle]\rangle \\ &= \langle A_1(a_1) A_2(a_3)\rangle - \langle A_1(a_1)\rangle\langle A_2(a_3)\rangle, \end{aligned}$$

from which the right-hand side of Eq. (A2) equals

$$|\langle A_1(a_1)\rangle\langle A_2(a_3)\rangle| = |\langle A_1(a_1) A_2(a_3)\rangle - \langle \Delta A_1 \Delta A_2\rangle|.$$

Because $\langle A_1(a_1) A_2(a_3)\rangle$, $\langle \Delta A_1 \Delta A_2\rangle$ have the same sign and

$$|\langle A_1(a_1) A_2(a_3)\rangle| \geqslant |\langle \Delta A_1 \Delta A_2\rangle|,$$

we have the inequality

$$|\langle A_1(a_1)\rangle\langle A_2(a_3)\rangle| \leqslant |\langle A_1(a_1) A_2(a_3)\rangle|. \tag{A3}$$

Then, Eq. (A2) becomes

$$\begin{aligned} p_{lc}(a_1, a_2) p_{lc}(a_2, a_3) &\leqslant |\langle A_1(a_1)\rangle\langle A_2(a_3)\rangle| \\ &\leqslant |\langle A_1(a_1) A_2(a_3)\rangle| \\ &= |p_{lc}(a_1, a_3)|. \end{aligned} \tag{A4}$$

Also, the validity of the GBI given by Eq. (A4) for $n = 2$ can be easily verified in terms of our quantum probability average with the local part of the density operator,

$$p_{lc}(a_1, a_2) = \frac{1}{s^2} \mathrm{Tr}\big[\hat{\rho}_{lc} \hat{\Omega}(a_1, a_2)\big].$$

The explicit forms of the local correlation probabilities for $s = 1/2$ are given by $p_{lc}(a_1, a_2) = \cos\theta_{a_1} \cos\theta_{a_2}$, $p_{lc}(a_2, a_3) = \cos\theta_{a_2} \cos\theta_{a_3}$, and $p_{lc}(a_1, a_3) = \cos\theta_{a_1} \cos\theta_{a_3}$. We then have

$$\begin{aligned} p_{lc}(a_1, a_2) p_{lc}(a_2, a_3) &= \cos\theta_{a_1} \cos^2\theta_{a_2} \cos\theta_{a_3} \\ &\leqslant |\cos\theta_{a_1} \cos\theta_{a_3}| \\ &= |p_{lc}(a_1, a_3)|, \end{aligned} \tag{A5}$$

which is valid in general for three arbitrary directions a_1, a_2, a_3 measured by two observers.

For $n = 3$,

$$|\psi\rangle = c_1|+s, +s, +s\rangle + c_2|-s, -s, -s\rangle,$$

the measuring-outcome correlations for three observers along the directions $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ are

$$p_{lc}(a_1, a_2, a_3) = \int \rho(\lambda) A_1(a_1, \lambda) A_2(a_2, \lambda) A_3(a_3, \lambda) d\lambda$$
$$\equiv \langle A_1(a_1) A_2(a_2) A_3(a_3)\rangle.$$

The product of three correlations is

$$
\begin{aligned}
&p_{lc}(a_1,a_2,a_3)p_{lc}(a_2,a_3,a_4)p_{lc}(a_3,a_4,a_5)\\
&=\iiint \rho(\lambda)\rho(\lambda')\rho(\lambda'')\begin{bmatrix} A_1(a_1,\lambda)A_2(a_2,\lambda)\\ A_3(a_3,\lambda)A_1(a_2,\lambda')\\ A_2(a_3,\lambda')A_3(a_4,\lambda')\\ A_1(a_3,\lambda'')A_2(a_4,\lambda'')\\ A_3(a_5,\lambda'')\end{bmatrix} d\lambda d\lambda' d\lambda''\\
&\leqslant \left|\begin{matrix}\iiint \rho(\lambda)\rho(\lambda')\rho(\lambda'')A_1(a_1,\lambda)A_2(a_3,\lambda')\\ \times A_3(a_5,\lambda'')d\lambda d\lambda' d\lambda''\end{matrix}\right|\\
&=|\langle A_1(a_1)\rangle\langle A_2(a_3)\rangle\langle A_3(a_5)\rangle|,
\end{aligned}
$$

since $A_2(a_2)=A_1(a_2)$, $A_3(a_3)=A_1(a_3)$, $A_3(a_4)=A_2(a_4)$, and $A_i^2(a_i)=1$. From the inequality given by Eq. (A3), it is easy to have

$$
\begin{aligned}
p_{lc}(a_1,a_2,a_3)&p_{lc}(a_2,a_3,a_4)p_{lc}(a_3,a_4,a_5)\\
&\leqslant |\langle A_1(a_1)\rangle\langle A_2(a_3)\rangle\langle A_3(a_5)\rangle|\\
&\leqslant |\langle A_1(a_1)A_2(a_3)\rangle\langle A_3(a_5)\rangle|\\
&\leqslant |\langle A_1(a_1)A_2(a_3)A_3(a_5)\rangle|\\
&=|p_{lc}(a_1,a_3,a_5)|.
\end{aligned}
\tag{A6}
$$

For arbitrary n,

$$|\psi\rangle = c_1|+s\rangle^{\otimes n}+c_2|-s\rangle^{\otimes n},$$

we need total $2n-1$ independent measuring directions labeled, respectively, by a_1, a_2, a_3,..., a_{2n-1} for n observers. The product of n correlations leads directly to the GBI that

$$
\begin{aligned}
&p_{lc}(a_1,a_2,\dots,a_n)p_{lc}(a_2,a_3,\dots,a_{n+1})p_{lc}(a_3,a_4,\dots,a_{n+2})\\
&\quad\times\cdots p_{lc}(a_n,a_{n+1},\dots,a_{2n-1})\\
&=\int\cdots\int\left(\prod_{i=1}^{n}\rho(\lambda_i)d\lambda_i\right)\begin{bmatrix} A_1(a_1,\lambda_1)A_2(a_2,\lambda_1)\\ A_3(a_3,\lambda_1)\dots A_n(a_n,\lambda_1)\\ \times A_1(a_2,\lambda_2)A_2(a_3,\lambda_2)\\ A_3(a_4,\lambda_2)\dots A_n(a_{n+1},\lambda_2)\\ \times\cdots\\ \times A_1(a_n,\lambda_n)A_2(a_{n+1},\lambda_n)\\ A_3(a_{n+2},\lambda_n)\dots A_n(a_{2n-1},\lambda_n)\end{bmatrix}\\
&\leqslant\left|\int\cdots\int\left(\prod_{i=1}^{n}\rho(\lambda_i)d\lambda_i\right)\right.\\
&\quad\left.\times[A_1(a_1,\lambda_1)A_2(a_3,\lambda_2)\dots A_n(a_{2n-1},\lambda_n)]\right|\\
&=|\langle A_1(a_1)\rangle\langle A_2(a_3)\rangle\dots\langle A_n(a_{2n-1})\rangle|,
\end{aligned}
$$

with $A_i(a_i)=A_j(a_i)$, $A_i^2(a_i)=1$. According to Eq. (A3), the above equation becomes

$$
\begin{aligned}
&p_{lc}(a_1,a_2,\dots,a_n)p_{lc}(a_2,a_3,\dots,a_{n+1})p_{lc}(a_3,a_4,\dots,a_{n+2})\\
&\quad\times\cdots p_{lc}(a_n,a_{n+1},\dots,a_{2n-1})\\
&\quad\leqslant|\langle A_1(a_1)\rangle\langle A_2(a_3)\rangle\cdots\langle A_n(a_{2n-1})\rangle|\\
&\quad\leqslant|\langle A_1(a_1)A_2(a_3)\cdots A_n(a_{2n-1})\rangle|\\
&\quad=|p_{lc}(a_1,a_3,\dots,a_{2n-1})|.
\end{aligned}
\tag{A7}
$$

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