Compatibility of quantum instruments

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Incompatibility of quantum devices is a useful resource in various quantum information theoretical tasks, and it is at the heart of some fundamental features of quantum theory. While the incompatibility of measurements and quantum channels is well studied, the incompatibility of quantum instruments has not been explored in much detail. In this work, we revise a notion of instrument compatibility introduced in the literature that we call traditional compatibility. Then, we introduce the notion of parallel compatibility and show that these two notions are inequivalent. We then argue that the notion of traditional compatibility is conceptually incomplete and prove that, while parallel compatibility captures measurement and channel compatibility, traditional compatibility does not capture channel compatibility. Hence, we propose parallel compatibility as the conceptually complete definition of the compatibility of quantum instruments.

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I. INTRODUCTION

Incompatibility (the lack of compatibility) is one of the features. of quantum theory that sets it apart from classical physics [1]. Intuitively, two quantum devices are compatible if there exists a joint device such that implementing the joint device is equivalent to simultaneously implementing the two original devices. Otherwise, they are called incompatible. While incompatibility may at first sound like a drawback, in fact, the incompatibility of quantum measurements leads to practical advantage in various quantum information processing tasks [2–5]. From the foundational point of view, the incompatibility of quantum channels is intimately linked to the well-known no-cloning theorem [6,7], and the incompatibility of the identity channel and a nontrivial measurement is linked to the uncertainty principle [8].

While the incompatibility of measurements and quantum channels is well studied, much less effort has been designated to studying the incompatibility of quantum *instruments*, a more general class of quantum devices capturing measurement processes in their full detail. In this work, we review a definition of instrument compatibility used in the literature (which we call *traditional* compatibility) and address its conceptual adequacy and its relation to measurement and channel compatibility. We then define a different notion of instrument compatibility (which we call *parallel* compatibility) and argue that this notion is conceptually more in line with the well-established notions of measurement and channel compatibility. We further prove that parallel compatibility captures measurement and channel compatibility in a well-defined manner, while traditional compatibility cannot capture channel compatibility. We therefore propose to adopt the notion of parallel compatibility of instruments instead of traditional compatibility.

The rest of this paper is organized as follows. In Sec. II, we discuss the mathematical background, various quantum devices, and their compatibility. In Sec. III, we discuss the compatibility of instruments. In particular, in Sec. III A, we review the definition of traditional compatibility, introduce the concept of parallel compatibility, and show that these two notions are inequivalent. In Sec. III B, we argue that the traditional definition of compatibility of instruments is conceptually incomplete—unlike parallel compatibility—and show that it cannot capture channel compatibility. In Sec. III C, we prove that parallel compatibility can capture the idea of measurement compatibility, channel compatibility, and measurement-channel compatibility. We conclude in Sec. IV and lay out potential connections of parallel compatibility to certain information-theoretic tasks.

II. PRELIMINARIES

In this section, we discuss the mathematical background, different types of quantum devices, and their compatibility. In general, in quantum theory, to every physical system there is an associated Hilbert space, $\mathcal{H}$, which we assume to be finite dimensional. The set of linear operators on $\mathcal{H}$ is denoted by $\mathcal{L}(\mathcal{H})$ and its subset of positive semidefinite operators is denoted by $\mathcal{L}^+(\mathcal{H})$. A quantum state is described by a positive semidefinite operator, $\rho \in \mathcal{L}^+(\mathcal{H})$, with unit trace, and the set of all states on $\mathcal{H}$ is denoted by $\mathcal{S}(\mathcal{H})$.

A. Measurements

Measurements can be thought of as quantum devices that take a quantum state as an input and produce a classical output

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(the measurement outcome). Mathematically, measurements are described by positive operator-valued measures (POVMs), which in the n-outcome case correspond to a set of n positive semidefinite operators, $A = \{A(x)\}_{x=1}^{n}$, such that $\sum_x A(x) = \mathbb{I}$, where $\mathbb{I}$ is the identity operator. In the following, we denote the outcome set of A by Ω_A.

B. Quantum channels

Quantum channels map quantum states to quantum states; that is, they are devices with a quantum input and a quantum output. Mathematically, quantum channels are described by completely positive trace-preserving (CPTP) maps $\Lambda : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$ [9,10]. A useful characterization of quantum channels is the *Kraus representation*. In fact, any completely positive (CP) map Λ can be written as $\Lambda(\rho) = \sum_i K_i \rho K_i^\dagger$, where the K_i's are linear operators called the Kraus operators. The trace-preserving property corresponds to the relation $\sum_i K_i^\dagger K_i = \mathbb{I}$.

The dual of a map $\Lambda : \mathcal{C}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ [where $\mathcal{C}(\mathcal{H}) \subseteq \mathcal{L}(\mathcal{H})$] is a map $\Lambda^* : \mathcal{L}(\mathcal{K}) \to \mathcal{L}(\mathcal{H})$ such that $\mathrm{tr}[\Lambda(X)Y] = \mathrm{tr}[X\Lambda^*(Y)]$ for all $X \in \mathcal{L}(\mathcal{H})$ and $Y \in \mathcal{L}(\mathcal{K})$. It is easy to show that the dual of a CPTP map is a CP unital map, i.e., a CP map that maps the identity to the identity. The following observation on dual channels will be useful later:

Observation 1. Let $\Gamma : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2)$ be a CP map and $\Lambda : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1)$ be a CP map such that $\Lambda(\rho) = \mathrm{tr}_{\mathcal{K}_2}[\Gamma(\rho)]$ for all $\rho \in \mathcal{S}(\mathcal{H})$. Then $\mathrm{tr}[\Lambda(\rho)] = \mathrm{tr}[\Gamma(\rho)]$, and thus $\mathrm{tr}[\rho\Lambda^*(\mathbb{I}_{\mathcal{K}_1})] = \mathrm{tr}[\rho\Gamma^*(\mathbb{I}_{\mathcal{K}_1\otimes\mathcal{K}_2})]$ for all $\rho \in \mathcal{S}(\mathcal{H})$. Using the fact that the dual map of a CP map is CP and that one can always find a positive semidefinite basis of Hermitian matrices, we conclude that $\Lambda^*(\mathbb{I}_{\mathcal{K}_1}) = \Gamma^*(\mathbb{I}_{\mathcal{K}_1\otimes\mathcal{K}_2})$.

C. Quantum instruments

Quantum instruments simultaneously generalize measurements and quantum channels: they take a quantum state as an input and provide both a classical and a quantum output. One may think of a quantum instrument as a measurement process, by associating the classical output with the measurement outcome and associating the quantum output with the post-measurement state. Mathematically, a quantum instrument $\mathcal{I}$ is defined as a set of CP maps $\{\Phi_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})\}$, such that $\Phi^{\mathcal{I}} \equiv \sum_x \Phi_x$ is a CPTP map [10]. Given a quantum state ρ, the classical output of the above instrument is x and the quantum output is $\Phi_x(\rho)$, both with probability $\mathrm{tr}[\Phi_x(\rho)]$.

Given the measurement $A = \{A(x)\}$, we say that the above instrument is A *compatible* if $\mathrm{tr}[\Phi_x(\rho)] = \mathrm{tr}[\rho A(x)]$ for all $\rho \in \mathcal{S}(\mathcal{H})$ and all $x \in \Omega_A$. Note that for every instrument $\mathcal{I} = \{\Phi_x\}$, there exists a unique measurement A, such that $\mathcal{I}$ is A compatible. Indeed, we have that $\mathrm{tr}[\Phi_x(\rho)] = \mathrm{tr}[\rho\Phi_x^*(\mathbb{I})]$. Thus, defining $A(x) \equiv \Phi_x^*(\mathbb{I})$, we have that $\mathrm{tr}[\Phi_x(\rho)] = \mathrm{tr}[\rho A(x)]$ for all $\rho \in \mathcal{S}(\mathcal{H})$, and this $A(x)$ is unique, positive semidefinite and $\sum_x A(x) = \mathbb{I}$, which follows from the fact that the dual of a CPTP map is a CP unital map. However, in general, there exist multiple different instruments compatible with the same measurement, corresponding to different implementations of the same measurement. For more results regarding quantum instruments, we refer the reader to Refs. [11–25].

D. Three kinds of compatibility in quantum theory

One possible definition of the compatibility of quantum devices is that they can be performed jointly. That is, a pair of devices is compatible if there exists a joint device, such that applying the joint device reproduces *both* of the outcomes of the compatible devices. If two devices are not compatible, we say that they are *incompatible*. Arguably, the most studied notions of compatibility in quantum theory are the following [1].

(i) Measurement compatibility. Two measurements, $A = \{A(x)\}$ and $B = \{B(y)\}$, on $\mathcal{H}$ are compatible if there exists a measurement $\mathcal{G} = \{G(x, y)\}$ on $\mathcal{H}$ with the outcome set $\Omega_{\mathcal{G}} = \Omega_A \times \Omega_B$ such that

$$A(x) = \sum_y G(x, y);\; B(y) = \sum_x G(x, y), \tag{1}$$

for all $x \in \Omega_A$ and $y \in \Omega_B$. Through measuring G, one can simultaneously recover the outputs of both A and B. That is, the distribution $p_G(x, y) \equiv \mathrm{tr}[G(x, y)\rho]$ is a joint distribution of $p_A(x) \equiv \mathrm{tr}[\rho A(x)]$ and $p_B(y) \equiv \mathrm{tr}[\rho B(y)]$ for all $\rho \in \mathcal{S}(\mathcal{H})$.

(ii) Channel compatibility. Two quantum channels, $\Lambda_1 : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K}_1)$ and $\Lambda_2 : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K}_2)$, are compatible if there exists a quantum channel $\Lambda : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K}_1 \otimes \mathcal{K}_2)$ such that $\Lambda_1(\rho) = \mathrm{tr}_{\mathcal{K}_2}[\Lambda(\rho)]$ and $\Lambda_2(\rho) = \mathrm{tr}_{\mathcal{K}_1}[\Lambda(\rho)]$ for all $\rho \in \mathcal{S}(\mathcal{H})$. Through implementing the channel Λ, one can simultaneously recover the outputs of both Λ_1 and Λ_2. That is, $\Lambda(\rho)$ is a joint state of $\Lambda_1(\rho)$ and $\Lambda_2(\rho)$ for all $\rho \in \mathcal{S}(\mathcal{H})$.

(iii) Measurement-channel compatibility. A measurement $A = \{A(x)\}$ on $\mathcal{H}$ and a quantum channel $\Lambda : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$ are compatible if there exists a quantum instrument $\mathcal{I} = \{\Phi_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})\}$ such that $\mathrm{tr}[\Phi_x(\rho)] = \mathrm{tr}[\rho A(x)]$ for all $x \in \Omega_A$ and $\rho \in \mathcal{S}(\mathcal{H})$ and $\sum_x \Phi_x = \Lambda$. Through implementing the quantum instrument $\mathcal{I}$, one can simultaneously recover the outputs of both A and Λ (for the latter, one needs to perform classical postprocessing).

III. COMPATIBILITY OF QUANTUM INSTRUMENTS

A. Definitions and concepts

While the compatibility of instruments has been studied in far less detail than that of measurements or channels, there is an existing definition in the literature, which we refer to as the *traditional* definition.

Definition 1 (Traditional compatibility). Two quantum instruments, $\mathcal{I}_1 = \{\Phi_x^1 : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})\}$ and $\mathcal{I}_2 = \{\Phi_y^2 : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})\}$, are (traditionally) compatible if there exists an instrument $\mathcal{I} = \{\Phi_{xy} : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})\}$ such that $\sum_y \Phi_{xy} = \Phi_x^1$ and $\sum_x \Phi_{xy} = \Phi_y^2$ for all x and y.

This definition appears in Ref. [25], [Definition 3], and in Ref. [15], [Definition 2.5]. The same definition is also given in Ref. [16], [p. 15], under the name "coexistence" (this notion is understood to be different from "joint measurability" in the context of measurement compatibility [26]). Notice that traditional compatibility can only be defined for instruments with the same quantum output space. Intuitively, the joint instrument $\mathcal{I}$ in Definition 1 reproduces both of the classical outputs x and y of $\mathcal{I}_1$ and $\mathcal{I}_2$, and by classical postprocessing, one can recover *either* one of the two quantum outputs, $\Phi_x^1(\rho)$

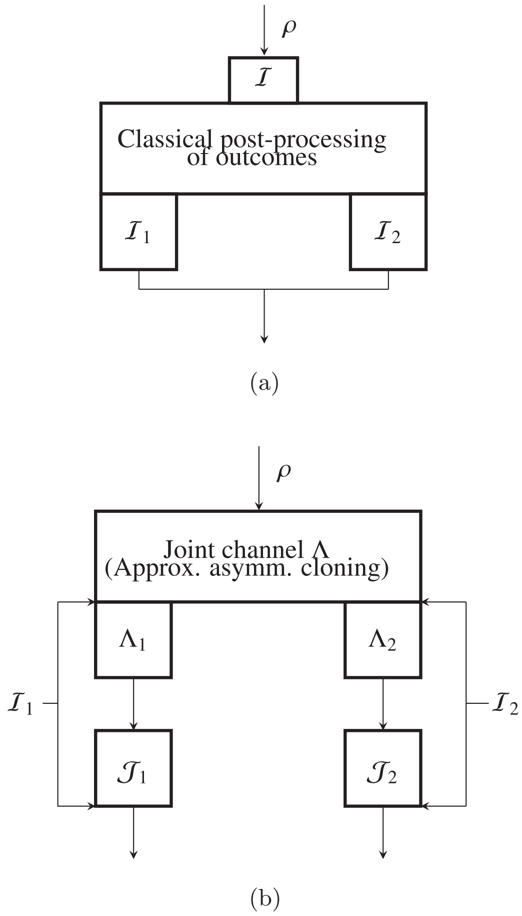

FIG. 1. Schematic representation of joint instruments for (a) traditionally compatible and (b) parallel compatible instruments. (a) Traditional compatibility: Schematic representation of Definition 1. Recovering the quantum output of *either* $\mathcal{I}_1$ or $\mathcal{I}_2$ can be done by first implementing the joint instrument $\mathcal{I}$ on the state ρ and then performing the postprocessing of outcomes, i.e., taking the marginal over either x or y. The downward arrows represent quantum systems. Clearly, in this case there is only one output quantum system. (b) Parallel compatibility: An example of parallel simultaneous implementation of two instruments (according to Definition 3), corresponding to Example 1. The simultaneous implementation of $\mathcal{I}_1$ and $\mathcal{I}_2$ can be done through the following steps: (i) implementing the channel Λ on the state ρ which is the joint channel of the compatible channels Λ_1 and Λ_2 [where $\Lambda_1(\rho)$ and $\Lambda_2(\rho)$ can be considered as the approximate unequal clones (unless $\Lambda_1 = \Lambda_2$) of the state ρ, in general, and therefore, it can be considered as approximate asymmetric cloning], and then (ii) applying the instruments $\mathcal{J}_1$ and $\mathcal{J}_2$ on $\Lambda_1(\rho)$ and $\Lambda_2(\rho)$, respectively, such that $\mathcal{J}_1 \circ \Lambda_1 = \mathcal{I}_1$ and $\mathcal{J}_2 \circ \Lambda_2 = \mathcal{I}_2$. The existence of such a channel Λ and such instruments $\mathcal{J}_1$ and $\mathcal{J}_2$ implies the parallel compatibility of the instruments $\mathcal{I}_1$ and $\mathcal{I}_2$, as explained in Example 1. The downward arrows represent quantum systems. Clearly, in this case there are two output quantum systems.

or $\Phi^2_y(\rho)$. For a schematic representation of the joint instrument of traditionally compatible instruments, see Fig. 1(a).

A concept related to traditional compatibility is that of *weak* compatibility.

Definition 2 (Weak compatibility). Two quantum instruments, $\mathcal{I}_1 = \{\Phi^1_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})\}$ and $\mathcal{I}_2 = \{\Phi^2_y : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})\}$, are weakly compatible if there exists a quantum channel $\Lambda : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K})$ such that $\sum_x \Phi^1_x = \sum_y \Phi^2_y = \Lambda$.

It is known that if a set of instruments is traditionally compatible then it is also weakly compatible, but it is easily seen that the converse is not true, in general [15].

Here, we propose a different definition of instrument compatibility, which we refer to as *parallel* compatibility.

Definition 3 (Parallel compatibility). Two quantum instruments, $\mathcal{I}_1 = \{\Phi^1_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1)\}$ and $\mathcal{I}_2 = \{\Phi^2_y : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_2)\}$, are parallel compatible if there exists an instrument $\mathcal{I} = \{\Phi_{xy} : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2)\}$ such that $\sum_y \mathrm{tr}_{\mathcal{K}_2} \Phi_{xy} = \Phi^1_x$ and $\sum_x \mathrm{tr}_{\mathcal{K}_1} \Phi_{xy} = \Phi^2_y$ for all x and y.

Notice that parallel compatibility can be defined for instruments with arbitrary quantum output spaces (not necessarily the same). Intuitively, the joint instrument $\mathcal{I}$ reproduces both of the classical outputs x and y of $\mathcal{I}_1$ and $\mathcal{I}_2$, and *both* of the quantum outputs $\Phi^1_x(\rho)$ and $\Phi^2_y(\rho)$ on a tensor product Hilbert space. To recover the quantum outputs, one needs to perform classical postprocessing, which can be done independently on the two quantum output spaces. Furthermore, if $\mathcal{I}_1 = \{\Phi^1_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1)\}$ and $\mathcal{I}_2 = \{\Phi^2_y : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_2)\}$ are parallel compatible with the joint instrument $\mathcal{I} = \{\Phi_{xy} : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2)\}$, then the channels $\Phi^1 \equiv \sum_x \Phi^1_x$ and $\Phi^2 \equiv \sum_y \Phi^2_y$ are compatible with the joint channel $\Phi \equiv \sum_{xy} \Phi_{xy}$.

For later convenience, we provide an alternative (but equivalent) definition of parallel compatibility.

Definition 4. Two quantum instruments, $\mathcal{I}_1 = \{\Phi^1_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1)\}$ and $\mathcal{I}_2 = \{\Phi^2_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_2)\}$, are parallel compatible if there exists a quantum instrument $\mathcal{I} = \{\Phi_z : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2)\}$ such that $\Phi^1_x = \sum_z p_1(x|z)\mathrm{tr}_{\mathcal{K}_2}\Phi_z$ and $\Phi^2_y = \sum_z p_2(y|z)\mathrm{tr}_{\mathcal{K}_1}\Phi_z$, where p_1 and p_2 are conditional probability distributions.

Proposition 1. Definition 4 is equivalent to Definition 3.

Proof. The proof is completely analogous to the related proof of equivalent definitions of observable compatibility in Ref. [1], Eqs. (15)–(17). First, it is clear that Definition 3 is a special case of Definition 4, by taking $z = (x', y')$, $p_1[x|(x', y')] = \delta_{xx'}$, and $p_2[y|(x', y')] = \delta_{yy'}$, where δ is the Kronecker delta. Hence, we just need to show that if a joint instrument $\mathcal{I} = \{\Phi_z : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2)\}$ such as the one in Definition 4 exists, then there also exists a joint instrument as in Definition 3. In particular, pick

$$\mathcal{I}' = \{\Phi'_{xy} : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2)\}, \tag{2}$$

with $\Phi'_{xy} = \sum_z p_1(x|z)p_2(y|z)\Phi_z$. One can readily check that this is a valid instrument and that $\Phi^1_x = \sum_y \mathrm{tr}_{\mathcal{K}_2}\Phi'_{xy}$ and $\Phi^2_y = \sum_x \mathrm{tr}_{\mathcal{K}_1}\Phi'_{xy}$.

We further illuminate the concept of parallel compatibility through an example and the accompanying figure, Fig. 1(b).

Example 1 (An example of parallel compatible instruments). Consider two compatible quantum channels, $\Lambda_1 : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}_1)$ and $\Lambda_2 : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}_2)$, with the joint channel $\Lambda : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. Since the implementation of a local channel on a subsystem does not change the quantum state on the other subsystem, for arbitrary channels

$\Gamma_1 : \mathcal{S}(\mathcal{H}_1) \rightarrow \mathcal{S}(\mathcal{K}_1)$ and $\Gamma_2 : \mathcal{S}(\mathcal{H}_2) \rightarrow \mathcal{S}(\mathcal{K}_2)$, we have that $\mathrm{tr}_{\mathcal{K}_2}(\mathbb{I} \otimes \Gamma_2) \circ \Lambda = \Lambda_1$ and $\mathrm{tr}_{\mathcal{K}_1}(\Gamma_1 \otimes \mathbb{I}) \circ \Lambda = \Lambda_2$.

Now consider a pair of arbitrary quantum instruments $\mathcal{J}_1 = \{\Phi'^1_x : \mathcal{S}(\mathcal{H}_1) \rightarrow \mathcal{L}^+(\mathcal{K}_1)\}$ and $\mathcal{J}_2 = \{\Phi'^2_y : \mathcal{S}(\mathcal{H}_2) \rightarrow \mathcal{L}^+(\mathcal{K}_2)\}$ such that $\sum_x \Phi'^1_x = \Gamma_1$ and $\sum_y \Phi'^2_y = \Gamma_2$. Consider the pair of instruments $\mathcal{I}_1 = \mathcal{J}_1 \circ \Lambda_1 = \{\Phi^1_x = \Phi'^1_x \circ \Lambda_1 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_1)\}$ and $\mathcal{I}_2 = \mathcal{J}_2 \circ \Lambda_2 = \{\Phi^2_y = \Phi'^2_y \circ \Lambda_2 : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_2)\}$. We show that the instrument $\mathcal{I} = \{\Phi_{xy} = (\Phi'^1_x \otimes \Phi'^2_y) \circ \Lambda : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2)\}$ is a joint instrument of $\mathcal{I}_1$ and $\mathcal{I}_2$.

Clearly, for all x

$$\begin{aligned}
\Phi^1_x &= \Phi'^1_x \circ \Lambda_1 \\
&= \Phi'^1_x \circ \mathrm{tr}_{\mathcal{K}_2}(\mathbb{I} \otimes \Gamma_2) \circ \Lambda \\
&= \Phi'^1_x \circ \mathrm{tr}_{\mathcal{K}_2}\left(\mathbb{I} \otimes \sum_y \Phi'^2_y\right) \circ \Lambda \\
&= \mathrm{tr}_{\mathcal{K}_2} \sum_y (\Phi'^1_x \otimes \Phi'^2_y) \circ \Lambda \\
&= \sum_y \mathrm{tr}_{\mathcal{K}_2} \Phi_{xy}.
\end{aligned} \tag{3}$$

Similarly, $\Phi^2_y = \sum_x \mathrm{tr}_{\mathcal{K}_1} \Phi_{xy}$ for all x. Hence, $\mathcal{I}_1$ and $\mathcal{I}_2$ are parallel compatible with the joint instrument $\mathcal{I}$.

Next, we show that traditional compatibility and parallel compatibility are conceptually different.

Proposition 2. There exist pairs of quantum instruments which are parallel compatible, but not traditionally compatible.

Proof. In Example 1, Γ_1 and Γ_2 can be arbitrary, and therefore, $\Gamma_1 \circ \Lambda_1$ and $\Gamma_2 \circ \Lambda_2$ are not equal, in general. Therefore, for the case where $\Gamma_1 \circ \Lambda_1 \neq \Gamma_2 \circ \Lambda_2$, $\mathcal{I}_1$ and $\mathcal{I}_2$ are parallel compatible, but not weakly compatible and, therefore, not traditionally compatible.

Proposition 3. There exist pairs of quantum instruments which are traditionally compatible, but not parallel compatible.

Proof. Consider two quantum instruments, $\mathcal{I}^p = \{\Phi^p_1 = p_1\mathfrak{I}, \Phi^p_2 = p_2\mathfrak{I}\}$ and $\mathcal{I}^q = \{\Phi^q_1 = q_1\mathfrak{I}, \Phi^q_2 = q_2\mathfrak{I}\}$, where $\mathfrak{I}$ is the identity channel and $p_i = \sum_j r_{ij}$ and $q_j = \sum_i r_{ij}$ for some $\{r_{ij} \geqslant 0\}_{ij=\{1,2\}}$, with $\sum_{ij} r_{ij} = 1$. Clearly, $\mathcal{I}^p$ and $\mathcal{I}^q$ are traditionally compatible with the joint instrument $\mathcal{I}^r = \{r_{ij}\mathfrak{I}\}_{ij=\{1,2\}}$. However, as discussed earlier, if $\mathcal{I}^p$ and $\mathcal{I}^q$ are parallel compatible, then $\Phi^p = \sum_i \Phi^p_i = \mathfrak{I}$ and $\Phi^q = \sum_j \Phi^q_j = \mathfrak{I}$ are compatible. But since the identity channel $\mathfrak{I}$ is not compatible with itself (due to the no-cloning theorem), Φ^p and Φ^q cannot be compatible, and therefore, $\mathcal{I}^p$ and $\mathcal{I}^q$ cannot be parallel compatible.

B. Drawbacks of traditional compatibility

In the previous section, we have introduced two notions of instrument compatibility and showed that these notions are conceptually different (neither of them implies the other). Here, we argue that the traditional notion has significant drawbacks.

Let us recall from Sec. II that measurements are devices with a quantum input and a classical output, while channels are devices with a quantum input and a quantum output. Furthermore, we say that a pair of such devices is compatible if there exists a joint device that upon taking a quantum input reproduces *both* of the outputs of the original devices. For measurements, this means that the joint measurement produces a classical output that is the joint measurement outcome of the two compatible measurements. For channels, this means that the joint channel produces a quantum output that is the joint state of the outputs of the compatible channels. According to this principle, when one is looking for a definition of compatibility of instruments, one should look for a joint instrument that reproduces *both* the joint classical and the joint quantum output of the compatible instruments.

It is clear from Definition 1 that the traditional notion of instrument compatibility provides a joint instrument with *a single quantum output*. Thus, by design, the traditional definition does not allow for producing a *simultaneous* quantum output of *both* of the compatible quantum instruments. Furthermore, this definition only applies to instruments with the same output Hilbert space. Note that, for traditionally compatible instruments, one can only recover a single quantum output via classical postprocessing. This is not the case for parallel compatibility, where the joint instrument produces a joint state, whose marginals (after classical postprocessing) coincide with the quantum outputs of the compatible instruments. Indeed, after performing the joint instrument, one has access to *both* of the quantum outputs, and one can perform further operations on both of them simultaneously. To illuminate this argument, we recall Proposition 3, which shows that two "identity instruments" (instruments with all their channels being proportional to the identity channel) are traditionally compatible. In any notion of compatibility for which the joint instrument recovers both of the quantum outputs, this would mean that one can recover two copies of the output of an identity channel, which is in clear contradiction with the no-cloning theorem. Thus, we argue that the traditional notion of instrument compatibility does not capture compatibility in the same way as the well-established notions of measurement and channel compatibility do.

As further justification for our argument, the following two propositions show that, while traditional compatibility of instruments captures measurement compatibility, it can never capture channel compatibility.

Proposition 4. Two measurements, A and B, are compatible if and only if there exist an A-compatible instrument $\mathcal{I}_A$ and a B-compatible instrument $\mathcal{I}_B$ such that $\mathcal{I}_A$ and $\mathcal{I}_B$ are traditionally compatible.

Proof. For the "if" part, suppose that there exists an A-compatible instrument $\mathcal{I}_A = \{\Phi^A_x : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K})\}$ and a B-compatible instrument $\mathcal{I}_B = \{\Phi^B_y : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K})\}$ such that $\mathcal{I}_A$ and $\mathcal{I}_B$ are traditionally compatible. Then there exists an instrument $\mathcal{I} = \{\Phi_{xy} : \mathcal{S}(\mathcal{H}) \rightarrow \mathcal{L}^+(\mathcal{K})\}$ such that $\sum_y \Phi_{xy} = \Phi^A_x$ and $\sum_x \Phi_{xy} = \Phi^B_y$ for all x and y. Let us consider the unique measurement $G = \{G(x, y)\}$ such that $\mathcal{I}$ is G compatible. Then, for any $\rho \in \mathcal{S}(\mathcal{H})$, we have that $\mathrm{tr}[\rho G(x, y)] = \mathrm{tr}[\Phi_{xy}(\rho)]$ for all x and y and for all $\rho \in \mathcal{S}(\mathcal{H})$. Therefore, $\sum_y \mathrm{tr}[\rho G(x, y)] = \sum_y \mathrm{tr}[\Phi_{xy}(\rho)] = \mathrm{tr}[\Phi^A_x(\rho)] = \mathrm{tr}[\rho A(x)]$ and $\sum_x \mathrm{tr}[\rho G(x, y)] = \sum_x \mathrm{tr}[\Phi_{xy}(\rho)] = \mathrm{tr}[\Phi^B_y(\rho)] = \mathrm{tr}[\rho B(y)]$ for all x and y and for all $\rho \in \mathcal{S}(\mathcal{H})$. That is, A and B are compatible with the joint measurement $G = \{G(x, y)\}$.

For the "only if" part, suppose that A and B are compatible with the joint measurement $G = \{G(x, y)\}$, and let $\mathcal{I} = \{\Phi_{xy}\}$ be a G-compatible instrument [e.g., choose the Lüders instrument, $\Phi_{xy}(\rho) = \sqrt{G(x, y)}\rho\sqrt{G(x, y)}$]. Then it is easy to check that the instrument $\mathcal{I}^A \equiv \{\Phi^A_x = \sum_y \Phi_{xy}\}$ is an A-compatible instrument and $\mathcal{I}^B \equiv \{\Phi^B_x = \sum_y \Phi_{xy}\}$ is a B-compatible instrument, and by definition they are traditionally compatible.

Proposition 5. There exist compatible channels Φ^A and Φ^B with the same output Hilbert space such that there exist no traditionally compatible instruments $\mathcal{I}_A = \{\Phi^A_x\}$ and $\mathcal{I}_B = \{\Phi^B_y\}$, with $\sum_x \Phi^A_x = \Phi^A$ and $\sum_y \Phi^B_y = \Phi^B$.

Proof. Take two instruments, $\mathcal{I}_A = \{\Phi^A_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})\}$ and $\mathcal{I}_B = \{\Phi^B_y : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K})\}$, such that they are traditionally compatible. Taking $\Phi^A \equiv \sum_x \Phi^A_x$ and $\Phi^B = \sum_y \Phi^B_y$, we recall that traditional compatibility implies $\Phi^A = \Phi^B$. Since there exist compatible channels that are not equal, the proposition follows.

C. Arguments for parallel compatibility

In this section, we argue that parallel compatibility does not have the flaws of traditional compatibility. In the previous section, we already argued for this from the conceptual viewpoint—that is, parallel compatibility allows for the simultaneous recovery of both of the quantum outputs of the compatible instruments. Here, we further justify the adequacy of parallel compatibility by showing that this notion captures the idea of measurement compatibility, channel compatibility, and measurement-channel compatibility. We summarize these findings in the following theorem.

Theorem 1. Parallel compatibility of instruments captures measurement compatibility, channel compatibility, and measurement-channel compatibility.

(i) Two measurements, A and B, are compatible if and only if there exist an A-compatible instrument $\mathcal{I}_A$ and a B-compatible instrument $\mathcal{I}_B$ such that $\mathcal{I}_A$ and $\mathcal{I}_B$ are parallel compatible.

(ii) Two quantum channels, $\Phi^1 : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K}_1)$ and $\Phi^2 : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{K}_2)$, are compatible if and only if there exist two parallel compatible instruments $\mathcal{I}_1 = \{\Phi^1_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1)\}$ and $\mathcal{I}_2 = \{\Phi^2_y : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_2)\}$ such that $\sum_x \Phi^1_x = \Phi^1$ and $\sum_y \Phi^2_y = \Phi^2$.

(iii) If an A-compatible instrument $\mathcal{I}_A = \{\Phi^A_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1)\}$ and a B-compatible instrument $\mathcal{I}_B = \{\Phi^B_y : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_2)\}$, such that $\sum_x \Phi^A_x = \Phi^A$ and $\sum_y \Phi^B_y = \Phi^B$, are parallel compatible, then A and B are both compatible with both Φ^A and Φ^B.

Proof. We start with the "if" part of statement 1. Suppose that there exists an A-compatible quantum instrument $\mathcal{I}_A = \{\Phi^A_x : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1)\}$ and a B-compatible quantum instrument $\mathcal{I}_B = \{\Phi^B_y : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_2)\}$ such that they are parallel compatible. Then there exists an instrument $\mathcal{I} = \{\Phi_{xy} : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2)\}$ such that $\sum_y \mathrm{tr}_{\mathcal{K}_2} \Phi_{xy} = \Phi^A_x$ and $\sum_x \mathrm{tr}_{\mathcal{K}_1} \Phi_{xy} = \Phi^B_y$ for all x and y. Since $\mathcal{I}_A$ is A compatible, we have that $\mathrm{tr}[\rho A(x)] = \mathrm{tr}[\Phi^A_x(\rho)] = \mathrm{tr}[\rho(\Phi^A_x)^*(\mathbb{I}_{\mathcal{K}_1})]$ for all $\rho \in \mathcal{S}(\mathcal{H})$ and all x. This implies

$$\sum_y \Phi^*_{xy}(\mathbb{I}_{\mathcal{K}_1 \otimes \mathcal{K}_2}) = (\Phi^A_x)^*(\mathbb{I}_{\mathcal{K}_1}) = A(x) \quad \forall x, \tag{4}$$

where the first equality is a consequence of Observation 1. Similarly, we have that $\sum_x \Phi^*_{xy}(\mathbb{I}_{\mathcal{K}_1 \otimes \mathcal{K}_2}) = (\Phi^B_y)^*(\mathbb{I}_{\mathcal{K}_2}) = B(y)$ for all y. Therefore, defining the measurement $G = \{G(x, y) = \Phi^*_{xy}(\mathbb{I}_{\mathcal{K}_1 \otimes \mathcal{K}_2})\}$ (which is the unique measurement compatible with $\mathcal{I}$), it is clear that A and B are compatible via the joint measurement G.

Now we move on to the "only if" part. Let $\{A(x)\}$ and $\{B(y)\}$ be compatible measurements on the Hilbert space $\mathcal{H}$. Let $\{G(x, y)\}$ denote a joint measurement for A and B, and consider the Naimark dilation $\{\Pi(x, y)\}$ on the Hilbert space $\mathcal{K} \equiv \mathcal{H} \otimes \mathcal{H}'$. That is, for every state $\rho \in \mathcal{S}(\mathcal{H})$ we have that $\mathrm{tr}[G(x, y)\rho] = \mathrm{tr}[\Pi(x, y)(\rho \otimes |0\rangle\langle 0|)]$ for some fixed state $|0\rangle$ on $\mathcal{H}'$, and $\{\Pi(x, y)\}$ is a projection-valued measure (PVM), i.e., $\Pi^2(x, y) = \Pi(x, y)$ for all x and y. Furthermore, consider a rank-1 "fine-graining," $\tilde{\Pi}(z) = |\phi_z\rangle\langle\phi_z|$, of $\Pi(x, y)$, i.e., a rank-1 projective measurement such that

$$\Pi(x, y) = \sum_{z \in P(x,y)} \tilde{\Pi}(z), \tag{5}$$

where $P(x, y)$ is the subset of all the possible values of z such that $\tilde{\Pi}(z)$ is in the support of $\Pi(x, y)$.

Consider the instrument

$$\begin{aligned}\mathcal{I}_{\tilde{\Pi}} = \big\{&\Phi^{\tilde{\Pi}}_z : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}) \mid \Phi^{\tilde{\Pi}}_z(\rho) \\ &= \tilde{\Pi}(z)(\rho \otimes |0\rangle\langle 0|)\tilde{\Pi}(z)\big\}\end{aligned} \tag{6}$$

defined by the channels mapping ρ to the (un-normalized) postmeasurement state of the dilated and fine-grained measurement $\tilde{\Pi}$. It is clear that $\Phi^{\tilde{\Pi}}_z$ is CP with the single Kraus operator $\tilde{\Pi}(z)(\mathbb{I}_{\mathcal{H}} \otimes |0\rangle_{\mathcal{H}'})$. Further, it is also clear that $\Phi^{\tilde{\Pi}} \equiv \sum_z \Phi^{\tilde{\Pi}}_z$ is CPTP, since

$$\begin{aligned}\mathrm{tr}[\Phi^{\tilde{\Pi}}(\rho)] &= \sum_z \mathrm{tr}[\tilde{\Pi}(z)(\rho \otimes |0\rangle\langle 0|)\tilde{\Pi}(z)] \\ &= \mathrm{tr}\left[\sum_z \tilde{\Pi}(z)(\rho \otimes |0\rangle\langle 0|)\right] \\ &= \mathrm{tr}[\rho \otimes |0\rangle\langle 0|] \\ &= \mathrm{tr}\rho\end{aligned} \tag{7}$$

for all $\rho \in \mathcal{L}(\mathcal{H})$.

Since $\tilde{\Pi}$ is a PVM, its un-normalized postmeasurement states, $\tilde{\rho}_z$, are rank-1 and pairwise orthogonal. Explicitly, they are given by

$$\tilde{\rho}_z = \tilde{\Pi}(z)(\rho \otimes |0\rangle\langle 0|)\tilde{\Pi}(z) = \lambda_z(\rho)|\phi_z\rangle\langle\phi_z|, \tag{8}$$

where $\lambda_z(\rho) = \mathrm{tr}[\tilde{\Pi}(z)(\rho \otimes |0\rangle\langle 0|)]$. Consider then an isometry $V : \mathcal{K} \to \mathcal{K}_1 \otimes \mathcal{K}_2$ (with $\mathcal{K}_1 \cong \mathcal{K}_2 \cong \mathcal{K}$) such that

$$V\,|\phi_z\rangle_{\mathcal{K}} = |\phi_z\rangle_{\mathcal{K}_1} \otimes |\phi_z\rangle_{\mathcal{K}_2} \quad \forall z, \tag{9}$$

which always exists, since one can always clone a set of fixed orthogonal states. Hence,

$$V\tilde{\rho}_z V^\dagger = \lambda_z(\rho)|\phi_z\rangle\langle\phi_z|_{\mathcal{K}_1} \otimes |\phi_z\rangle\langle\phi_z|_{\mathcal{K}_2}. \tag{10}$$

Let us then define the instrument

$$\mathcal{I} = \{\Phi_z : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1 \otimes \mathcal{K}_2)\}, \tag{11}$$

with $\Phi_z(\rho) = V\tilde{\Pi}(z)(\rho \otimes |0\rangle\langle 0|)\tilde{\Pi}(z)V^\dagger = \lambda_z(\rho)|\phi_z\rangle\langle\phi_z|_{\mathcal{K}_1} \otimes |\phi_z\rangle\langle\phi_z|_{\mathcal{K}_2}$. Since the Φ_z are just the composition of $\Phi_z^{\tilde{\Pi}}$ with the isometry V, it is clear that these are also CP maps and that $\Phi \equiv \sum_z \Phi_z$ is a CPTP map, and hence $\mathcal{I}$ is a valid instrument.

Let us now define the instruments

$$\mathcal{I}_A \equiv \{\Phi_x^A : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1)\}, \tag{12}$$

where $\Phi_x^A(\rho) = \sum_y \sum_{z\in P(x,y)} \mathrm{tr}_{\mathcal{K}_2}\Phi_z(\rho) = \sum_y \sum_{z\in P(x,y)} \lambda_z(\rho)|\phi_z\rangle\langle\phi_z|_{\mathcal{K}_1}$, and

$$\mathcal{I}_B \equiv \{\Phi_y^B : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_2)\}, \tag{13}$$

where $\Phi_y^B(\rho) = \sum_x \sum_{z\in P(x,y)} \mathrm{tr}_{\mathcal{K}_1}\Phi_z(\rho) = \sum_x \sum_{z\in P(x,y)} \lambda_z(\rho)|\phi_z\rangle\langle\phi_z|_{\mathcal{K}_2}\}$. It is clear that these are valid instruments, and by definition, $\mathcal{I}_A$ and $\mathcal{I}_B$ are parallel compatible with the joint instrument $\mathcal{I}$. Furthermore, we have that

$$\begin{aligned}
\mathrm{tr}\big[\Phi_x^A(\rho)\big] &= \mathrm{tr}\Bigg[\sum_y \sum_{z\in P(x,y)} \lambda_z(\rho)|\phi_z\rangle\langle\phi_z|_{\mathcal{K}_1}\Bigg] \\
&= \sum_y \sum_{z\in P(x,y)} \lambda_z(\rho) \\
&= \sum_y \sum_{z\in P(x,y)} \mathrm{tr}[\tilde{\Pi}(z)(\rho\otimes|0\rangle\langle 0|)] \\
&= \sum_y \mathrm{tr}[\Pi(x,y)(\rho\otimes|0\rangle\langle 0|)] \\
&= \sum_y \mathrm{tr}[G(x,y)\rho] \\
&= \mathrm{tr}[A(x)\rho],
\end{aligned} \tag{14}$$

that is, $\mathcal{I}_A$ is A compatible, and similarly we have that $\mathcal{I}_B$ is B compatible. This finishes the proof of statement (i) of Theorem 1.

We continue with the proof of statement (ii) of Theorem 1. To show the "only if" part, notice that a quantum channel Λ can be considered as a single-outcome quantum instrument $\mathcal{I}_\Lambda$. Therefore, the compatibility of the two quantum channels Φ^1 and Φ^2 implies the parallel compatibility of the two instruments $\mathcal{I}_1 \equiv \{\Phi^1\}$ and $\mathcal{I}_2 \equiv \{\Phi^2\}$. The "if" part follows straightforwardly from the definition of parallel compatibility, and it is already explained below Definition 3.

Last, we prove statement (iii) of Theorem 1. By definition, A is compatible with Φ^A and B is compatible with Φ^B. Since $\mathcal{I}_A$ and $\mathcal{I}_B$ are parallel compatible, there exists a quantum instrument $\mathcal{I} = \{\Phi_{xy} : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_1\otimes\mathcal{K}_2)\}$ such that $\Phi_x^A = \sum_y \mathrm{tr}_{\mathcal{K}_2}\Phi_{xy}$ and $\Phi_y^B = \sum_x \mathrm{tr}_{\mathcal{K}_1}\Phi_{xy}$. Since $\mathcal{I}_A$ is an A-compatible instrument, we have that, for all $\rho\in\mathcal{S}(\mathcal{H})$ and $x\in\Omega_A$,

$$\begin{aligned}
\mathrm{tr}_{\mathcal{H}}[\rho A(x)] &= \mathrm{tr}_{\mathcal{K}_1}\big[\Phi_x^A(\rho)\big] \\
&= \mathrm{tr}_{\mathcal{K}_1}\Bigg[\sum_y \mathrm{tr}_{\mathcal{K}_2}[\Phi_{xy}(\rho)]\Bigg] \\
&= \sum_y \{\mathrm{tr}_{\mathcal{K}_1}\mathrm{tr}_{\mathcal{K}_2}[\Phi_{xy}(\rho)]\} \\
&= \mathrm{tr}_{\mathcal{K}_2}\Bigg[\sum_y \mathrm{tr}_{\mathcal{K}_1}[\Phi_{xy}(\rho)]\Bigg] \\
&= \mathrm{tr}_{\mathcal{K}_2}\big[\Phi_x'^A(\rho)\big],
\end{aligned} \tag{15}$$

where $\Phi_x'^A \equiv \sum_y \mathrm{tr}_{\mathcal{K}_1}\Phi_{xy}$. Clearly, $\Phi_x'^A : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_2)$ for all $x\in\Omega_A$, and

$$\begin{aligned}
\sum_x \Phi_x'^A &= \sum_x\sum_y \mathrm{tr}_{\mathcal{K}_1}\Phi_{xy} \\
&= \sum_y \Phi_y^B \\
&= \Phi^B,
\end{aligned} \tag{16}$$

and thus, $\mathcal{I}'_A \equiv \{\Phi_x'^A : \mathcal{S}(\mathcal{H}) \to \mathcal{L}^+(\mathcal{K}_2)\}$ is a quantum instrument. Hence, A and Φ^B are compatible through the instrument $\mathcal{I}'_A$. Similarly one can prove that B and Φ^A are compatible as well.

Thus, we have proved that parallel compatibility captures the three kinds of compatibilities between basic quantum devices, that is, measurement compatibility, channel compatibility, and measurement-channel compatibility.

Last, we recall that while every instrument has a unique measurement it is compatible with, the converse is not true: there are multiple instruments that are compatible with the same measurement (corresponding to different implementations of the same measurement). One consequence of this is that, while for every pair of compatible measurements A and B there exists an A-compatible instrument $\mathcal{I}_A$ and a B-compatible instrument $\mathcal{I}_B$ such that they are parallel compatible [statement (i) of Theorem 1], not *every* A-compatible and B-compatible instrument will be parallel compatible. Even for a single measurement, two different instruments that are compatible with it may not be parallel compatible, as the following example shows.

Example 2 (Two parallel incompatible instruments associated with the same measurement). A trivial measurement $J = \{J(x) = p_x\mathbb{I}\}$ is compatible with any quantum channel Λ through the instrument $\mathcal{I}_{J,\Lambda} = \{p_x\Lambda\}$ [1]. Let us consider a channel Γ, which is incompatible with Λ. Clearly, J is also compatible with the quantum channel Γ through the instrument $\mathcal{I}_{J,\Gamma} = \{p_x\Gamma\}$. From Theorem 1, we know that if two instruments are parallel compatible then their corresponding channels are compatible. Then, since Γ and Λ were chosen to be incompatible, the instruments $\mathcal{I}_{J,\Gamma}$ and $\mathcal{I}_{J,\Lambda}$ cannot be parallel compatible.

IV. CONCLUSION

In this paper, we introduced the concept of parallel compatibility of instruments and showed that this concept is different from the traditional definition of instrument compatibility. We argued that the traditional definition of compatibility of instruments is conceptually incomplete, and we provided arguments for the adequacy of parallel compatibility. We showed that the definition of parallel compatibility of quantum instruments can capture the idea of measurement compatibility, channel compatibility, and measurement-channel compatibility.

The notion of parallel compatibility may be relevant to various information theoretic tasks. First, suppose that Charlie wants to simultaneously transfer information to two parties, Alice and Bob, separated by a long distance. The information is transmitted through a quantum state, and Alice is retrieving the information via a measurement A, while Bob is retrieving the information via a measurement B. Then, if A and B are compatible, Fig. 1(b) suggests that the transmission can be done via the joint instrument of the corresponding A- and B-compatible instruments, $\mathcal{I}_A$ and $\mathcal{I}_B$ (which always exists, according to Theorem 1). Second, consider the same scenario in a cryptographic setting, where Charlie and Alice aim to perform a key distribution task and Bob is an eavesdropper. Then, Fig. 1(b) suggests that cloning-type attacks can be modeled through parallel compatibility of quantum instruments. We leave the exploration of the role of parallel compatibility in such information theoretic tasks, as well as the further characterization of parallel compatible instruments, for future work.

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