Scaling of the finite-size effect of the α-Rényi entropy in disjointed intervals under dilation

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The α-Rényi entropy in the gapless models has been obtained by the conformal field theory, which is exact in the thermodynamic limit. However, the calculation of its finite-size effect (FSE) is challenging. So far only the FSE in a single interval in the XX model has been understood and the FSE in the other models and in the other conditions is totally unknown. Here we report the FSE of this entropy in disjointed intervals $A = \cup_i A_i$ under a uniform dilation λA in the XY model, showing a universal scaling law as $\Delta^\alpha_{\lambda A} = \Delta^\alpha_A \lambda^{-\eta}\mathcal{B}(A, \lambda)$, where $|\mathcal{B}(A, \lambda)| \leqslant 1$ is a bounded function and $\eta = \min(2, 2/\alpha)$ when $\alpha < 10$. We verify this relation in the phase boundaries of the XY model, in which the different central charges correspond to the physics of free fermion and free boson models. We find that in the disjointed intervals two FSEs, termed as extrinsic FSE and intrinsic FSE, are required to fully account for the FSE of the entropy. Physically, we find that only the edge modes of the correlation matrix localized at the open ends ∂A have contribution to the total entropy and its FSE. Our results provide some incisive insight into the entanglement entropy in the many-body systems.

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I. INTRODUCTION

Entanglement has played a more and more important role in quantum information and many-body physics. A large number of investigations have shown that the ground state of the gapped and gapless phases will have totally different entanglement entropies. For a regime A [see Fig. 1(a)], we can denote the reduced density matrix as ρ_A, then the Shannon entropy can be calculated using $S_A = -\mathrm{Tr}(\rho_A \ln \rho_A)$. In the gapped phase, its entropy satisfies the area law [1–5]:

$$S_A = \tilde{\alpha}\partial A - \tilde{\gamma} \sim L^{d-1}. \quad (1)$$

However, in the gapless phase, it satisfies a different area law with logarithmic correlation as [6,7]

$$S_A \sim L^{d-1} \ln_2 L. \quad (2)$$

In the above two equations, L is the system size and d is its system dimension. When $d = 1$, it yields the logarithm divergence of the entropy with the increasing of system size (see below). Similar features may also be found for their low-lying excited states [8–11]. By generalizing this concept in terms of α-Rényi entropy, one finds that in the one-dimensional gapless phase [12,13]

$$S^\alpha_A = \frac{1}{1-\alpha} \log_2 \mathrm{Tr}\rho^\alpha_A = \frac{c+\bar{c}}{12(1+\alpha)} \log_2 L + s^\alpha_0 + \Delta^\alpha_A, \quad (3)$$

where c and $\bar{c}$ are the holomorphic and antiholomorphic central charges, respectively [14,15]; s_0 is a nonuniversal constant; and Δ^α_A is its finite-size effect (FSE), satisfying

$$\lim_{L\to\infty} \Delta^\alpha_A = 0, \quad (4)$$

by its definition. The expression of α-Rényi entropy has been examined numerically in some of the solvable models [13,16–22], which can be more rigorously obtained by the conformal field theory (CFT) [14,23–25]. Since the gapped and gapless phases have totally different entanglement properties, these features are used to diagnose the phase transitions in some of the many-body models [26–32].

The scaling laws of the FSE in Eq. (4), which in the gapped and gapless phases should exhibit totally different behaviors, are the major concern of this paper. To date, they have been rarely investigated. In the XX model with free fermions [13,19,20,33], they have been calculated using the Jin-Korepin (JK) approach [13,34], yielding an extremely complicated polynomial of length L with exponents $\eta = 2$ and $2n/\alpha$ for $n \in \mathbb{Z}^+$ [34–38] (see discussion in Sec. III). However, the FSEs in the other models and in disjointed intervals are unknown [see Fig. 1(b)], and are also challenging to calculate by the JK approach [39–43]. In Ref. [56], Facchi *et al.* have considered the entanglement of two blocks of the same size L, separated by d, in the critical Ising model, finding that the Shannon entropy ($\alpha = 1$) is given by

$$S_A = \tfrac{1}{6}[2\ln_2(L-a) - 2\ln_2(L+d) + \ln_2(2L+d-a) + \ln_2(d+a) - 2\ln_2(a)], \quad (5)$$

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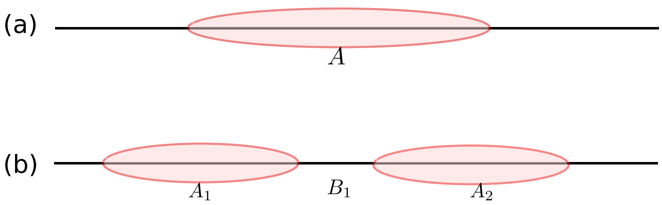

FIG. 1. (a) Entanglement entropy in a single interval A and (b) entanglement entropy in two disjointed intervals A_1 and A_2 separated by an interval B_1. The configuration in (b) can be generalized to disjointed intervals; see results in Figs. 5 and 6. The correlation between different intervals will be calculated by the correlation functions $\mathcal{G}(x)$.

where $a = 0.056\,622\,6$. This formula yields the following finite-size effect in the limit of $d \gg L \gg 1$:

$$S_A = 2S_L + \Delta_A^1, \quad \Delta_A^1 \propto \frac{1}{d^2}. \tag{6}$$

Thus $\eta = 2$, consistent with the theoretical results by the JK approach. This is the only available result for the finite-size effect of entanglement entropy in multiple intervals. A great endeavor has been made trying to explore this FSE in disjointed intervals [44–46], and failed to find some universal scaling behaviors in them.

This paper aims to explore the scaling law of the above FSE in multiple intervals [see Fig. 1(b)] in free fermion ($c = \bar{c} = 1$) and boson ($c = \bar{c} = 1/2$) models, in which the correlators exhibit some kind of scaling laws under uniform dilation, such as $\langle\phi(\lambda x)\phi(\lambda y)\rangle = \lambda^{-\nu}\langle\phi(x)\phi(y)\rangle$. This feature can give rise to a scaling law in Δ_A^α if it is a function of these correlators, which has not yet been unveiled in the previous literature. Let us denote $A = \cup_i A_i$ to be the jointed structure of some disjointed intervals A_i and let λA (with $\lambda \in \mathbb{Z}^+$) denote its uniform dilation. The set of open ends is denoted as $\partial A = \cup_i \partial A_i$. The key result of this paper in the large size limit can be formulated as

$$\Delta_{\lambda A}^\alpha = \Delta_A^\alpha \lambda^{-\eta}\mathcal{B}(A, \lambda), \tag{7}$$

where $|\mathcal{B}(A, \lambda)| \leqslant 1$ is a bounded function and $\eta = \min(2, 2/\alpha)$ when $\alpha < 10$. We find that only the edge modes of the correlation matrix with wave functions localized near the open ends contribute to the Rényi entropy and its FSE. We confirm Eq. (7) in both free fermion and free boson models. Our results may provide insight into the FSE of Rényi entropy in multiple intervals in other many-body systems.

This paper is organized as following. In Sec. II, we present the XY model, in which the properties of the correlation function are discussed in detail. These correlation functions are essential for the scaling laws of the entanglement entropy. We will show that the correlation functions in the gapped and gapless phases are totally different. In Sec. III, we will discuss the major results by JK. In Sec. IV, two different FSEs are defined in disjointed intervals, and their features in two and three intervals are discussed. At the end of this section, the results in the gapped phases, which are trivial, will also be briefly discussed. In Sec. V, we conclude our results. In Sec. A, we show that the entropy defined in this way is well defined.

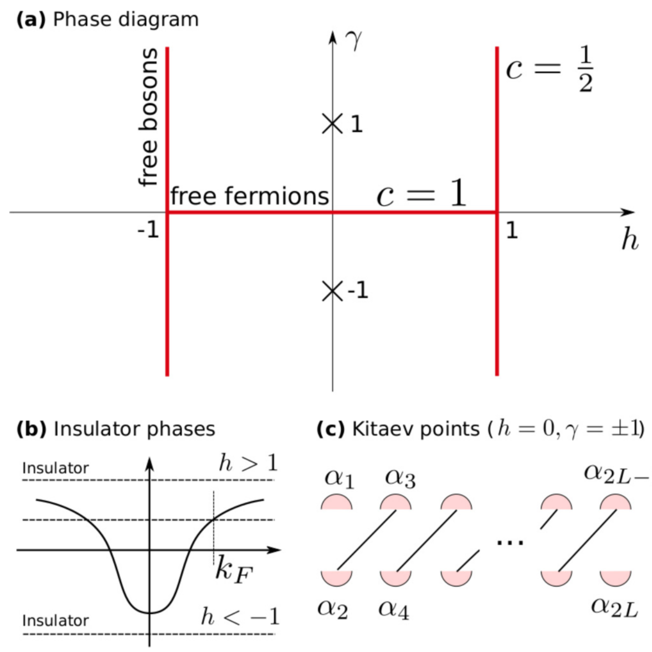

FIG. 2. (a) Phase diagram of the transverse XY model. The thick lines correspond to the gapless phase with $c = \bar{c} = 1$ for free fermions and $c = \bar{c} = 1/2$ for free bosons. (b) Insulator phases with $\gamma = 0$ and $|h| > 1$ and gapless phase with Fermi points $\cos(k_F) = h$ when $|h| < 1$. (c) The special points with $\gamma = \pm 1$ and $h = 0$. In the Majorana fermion representation, this model is decoupled into paired Majorana fermions (represented by the hemicycles), with α_1 and α_{2L} unpaired, giving rise to degenerate zero modes. This special case has been studied by Kitaev [47]. The insulator phases for $|h| > 1$ and $\gamma = 0$ in (b) and the Kitaev points in (c) have zero range correlation with $\mathcal{G}(x) = 0$.

II. XY SPIN CHAIN

We illustrated the above conclusion using the following exact solvable one-dimensional XY spin chain:

$$H = \sum_i \left(\frac{1+\gamma}{2}\right) s_i^x s_{i+1}^x + \left(\frac{1-\gamma}{2}\right) s_i^y s_{i+1}^y + h s_i^z, \tag{8}$$

where s_i^α ($\alpha = x, y, z$) are Pauli matrices and h is the transverse Zeeman field. After a Jordan-Wigner transformation by assigning fermion operators c_i and $c_i^\dagger$ to each site, it is mapped to a free fermion model as

$$H = -\sum_i c_i^\dagger c_{i+1} + \gamma c_i^\dagger c_{i+1}^\dagger + \text{H.c.} + h(1 - 2c_i^\dagger c_i), \tag{9}$$

with excitation gap

$$\epsilon_k = \sqrt{[\cos(k) - h]^2 + \gamma^2 \sin(k)^2}. \tag{10}$$

The phase boundary is determined by $\epsilon_k = 0$, which yields three phase boundaries in Fig. 2. When $\gamma \neq 0$, we have $|h| = 1$; when $\gamma = 0$, we have $|h| \leqslant 1$. The phase transition in this model is characterized by $\mathbb{Z}_2$ symmetry breaking [48]. We choose this model because the two gapless boundaries correspond to free fermions and free bosons, respectively [see Fig. 2(a)], thus this model automatically yields the physics in these two distinct free particles.

The density matrix of $A = \cup_i A_i$ can be calculated exactly using the same approach as that used in a single interval, for the reason that the density matrix of several disjointed intervals can be expressed as [19,45]

$$\rho_A \propto \exp(H_A), \quad H_A = \frac{i\alpha^T W \alpha}{4}, \quad \tanh\frac{W}{2} = \Gamma, \quad (11)$$

based on the Majorana operators $\alpha_{2l-1} = (\prod_{m<l}\sigma_m^z)\sigma_l^x$ and $\alpha_{2l} = (\prod_{m<l}\sigma_m^z)\sigma_l^y$. Here, Γ is a skew matrix with entries given by

$$\Gamma_{i,i+x} = -i(\langle\alpha_i\alpha_{i+x}\rangle - \delta_{x0}) = \begin{pmatrix} 0 & \mathcal{G}(x) \\ -\mathcal{G}(-x) & 0 \end{pmatrix}, \quad (12)$$

where [13,19,20]

$$\mathcal{G}(x) = \int_{-\pi}^{\pi} \frac{\gamma\sin(k)\sin(kx) - e_k\cos(kx)}{2\pi\epsilon_k} dk, \quad (13)$$

with $e_k = h - \cos(k)$. This integral determines all the properties of the Rényi entropy. It has a number of salient features [20]. In the gapless phases, it decays algebraically as

$$\mathcal{G}(x) = \frac{K(x)}{x}, \quad (14)$$

where $K(x)$ is a bounded oscillating function. Specifically, we find the following.

(I) For free fermions with $\gamma = 0$ and $|h| \leqslant 1$, we have

$$K(x) = \frac{\{2\sin[x\arccos(|h|)] - \sin(\pi x)\}}{\pi}. \quad (15)$$

When $h = \pm 1$, the spectrum is gapless with quadratic dispersion as $E_k \propto k^2$ and $\mathcal{G}(x) = 0$, which violate conformal symmetry. In the fully gapped phase with $|h| > 1$, we always have $\mathcal{G}(x) = 0$ due to the presence of a vacuum state or a fully filled state [see Fig. 2(b)], hence Γ is always equal to zero [see Eq. (12)].

(II) For free bosons with $|h| = 1$ and $\gamma \neq 0$, we have

$$K(x) \simeq -2\,\mathrm{sign}(\gamma)[1 - \cos(\pi x)], \quad (16)$$

which is long-range correlated. The above two $K(x)$ are bounded functions, that is, $|K(x)| \leqslant C$ for some positive constant C; $\mathcal{G}(x)$ decays according to $1/x$ and their oscillating behaviors. This long-range correlator is essential for the logarithm relation of the entanglement entropy, as shown in Eqs. (2) and (3).

(III) In the gapped phases, $\mathcal{G}(x)$ is short-range correlated with an exponential decaying behavior. Based on Eq. (13), we can even show at the Kitaev points with $h = 0$ and $\gamma = \pm 1$ [see Fig. 2(c)] that $\mathcal{G}(x)$ is zero range correlated since

$$K(x) = \sin(\pi x) = 0, \quad x \in \mathbb{Z}, \quad (17)$$

which can be understood as due to the fact that in these two points α_{2i} and α_{2i+1} are paired, leaving only α_1 and α_{2L} to be the dangling operators left out from the Hamiltonian. In the gapped phases with finite energy gap, by expanding $\epsilon_k = a + bk^2$, where $a = |h \pm 1|$ and $b = (|h \pm 1| + \gamma^2)/2|h \pm 1|$ for the energy gap at $k = 0$ $(-)$ or π $(+)$, we obtain

$$\mathcal{G}(x) \sim e^{-|x|/\xi}, \quad (18)$$

where the decay length $\xi = |\gamma|/(\sqrt{2}|h \pm 1|)$. This implies the area law [49], as shown in Eq. (1). In Fig. 3, we plot the results of $K(x)$ in the gapped and gapless phases, showing excellent agreement with the above analysis. In the gapped phase, $K(x)$ will always vanish at large $x \in \mathbb{Z}$.

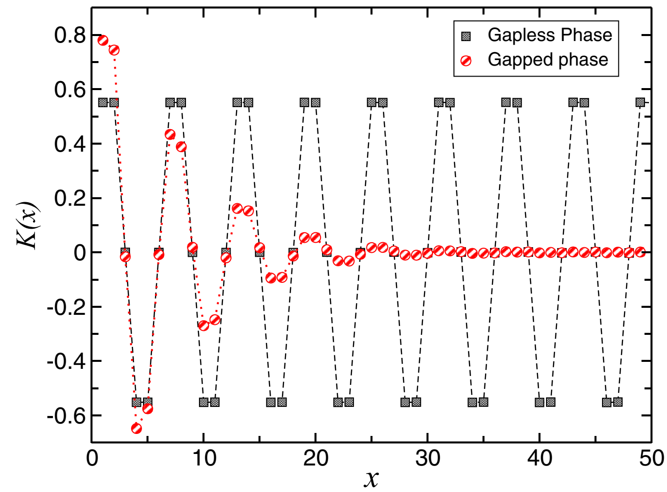

FIG. 3. $K(x)$ in the gapped phase and gapless phase. In the gapped phase, we have chosen $\gamma = 0.4$ and $h = 0.5$, while in the gapless phase we used $\gamma = 0.0$ and $h = 0.5$. Only values at $x \in \mathbb{Z}$ are plotted.

Equation (11) is essential to calculate the density matrix of ρ_A and its eigenvalues numerically. Notice that W and Γ are real skew matrices, and can be solved by an orthogonal matrix, then we have $\rho_A = \prod_l \otimes\rho_l$, with $\rho_l = \mathrm{diag}(\frac{1-\nu_l}{2}, \frac{1+\nu_l}{2})$, with eigenvalues as $\lambda = \prod_{s_l=1,-1}(\frac{1+s_l\nu_l}{2})$. By definition we have [50]

$$S_A^\alpha = \frac{1}{1-\alpha}\sum_l \log_2\left[\left(\frac{1+\nu_l}{2}\right)^\alpha + \left(\frac{1-\nu_l}{2}\right)^\alpha\right], \quad (19)$$

which is reduced to the Shannon entropy when $\alpha \to 1$, with $S_A = -\sum_{l=1}^{L}(\frac{1\pm\nu_l}{2})\log_2(\frac{1\pm\nu_l}{2})$. Furthermore, with the increasing of α, the total entropy decreases monotonically, which finally saturates to $S_A^\infty = -\sum_{\nu_l<1}\log_2(\frac{1+\nu_l}{2})$. The eigenvalues and eigenvectors of the Hermite operator $i\Gamma$ in Fig. 4 show that only the modes localized at the edges have contribution to the Rényi entropy, while the extended modes with $\nu_l \to \pm 1$ will not. This is expected, since for the extended modes the wave functions are extended in the whole interval and their amplitudes at the open ends ∂A are vanishingly small with the increasing of system size [see the wave functions in Figs. 4(b) and 4(d)]. In this way, the coupling between regime A with its complement $\bar{A}$ is negligible in the large L limit. However, for the localized edge modes, the coupling between the sites near the boundary ∂A is strong, and $\nu_l < 1$. This is also the essential origin of the area law quoted above.

III. FSE BY JIN-KOREPIN

The FSE of this entropy is defined as the difference between the exact (numerical) Rényi entropy and the prediction from CFT [35,51–53]. In a single interval with $\gamma = 0$ for free fermions, Δ_A^α was obtained by the JK approach [13]. To the

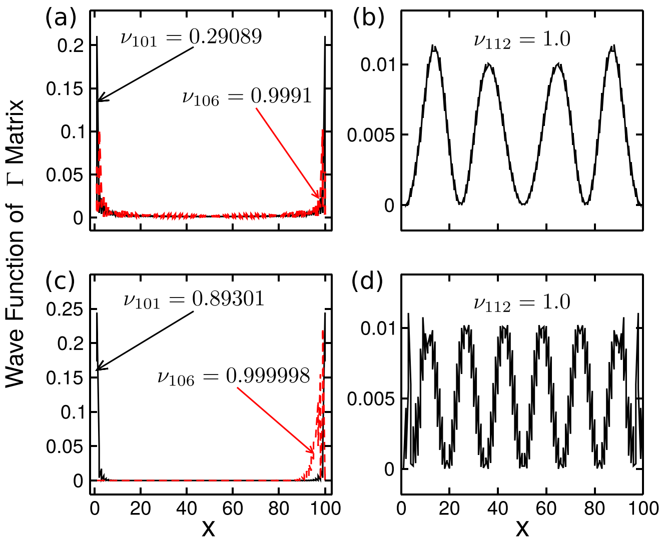

FIG. 4. Wave functions of the correlation matrix $i\Gamma$ in a single interval with $L = 100$. (a, b) Wave function in the gapless phase ($\gamma = 1.36$ and $h = 1.0$) for ν_{101}, ν_{106}, and ν_{112} (using $\nu_{100+l} = -\nu_{100-l}$ from the particle-hole symmetry of $i\Gamma$). (c, d) Results for $\gamma = 1.36$ and $h = 1.5$ in the gapped phase for ν_{101}, ν_{106}, and ν_{112}. Here $i\Gamma$ is a $2L \times 2L$ matrix, thus we have in total $2L = 200$ eigenvalues.

leading term [34,35]

$$\Delta_A^\alpha = \frac{\mathcal{A}_1}{[2L|\sin(k_F)|]^2} + \frac{\mathcal{A}_2}{[2L|\sin(k_F)|]^{\frac{2}{\alpha}}} + \mathcal{O}(L^{-\frac{2n}{\alpha}}), \quad (20)$$

where

$$\mathcal{A}_1 = \frac{[12(3\alpha^2 - 7) + (49 - \alpha^2)\sin^2(k_F)](1+\alpha)}{(285\alpha^3)} \quad (21)$$

and

$$\mathcal{A}_2 = \frac{2Q\cos(2k_F L)}{(1-\alpha)}, \quad (22)$$

with $Q = \Gamma(1/2 + 1/(2\alpha))^2/\Gamma[1/2 - 1/(2\alpha)]^2$, and $k_F = \arccos(|h|)$ is the Fermi momentum. The next leading terms correspond to $n > 1$. Note that the second term oscillates periodically with spatial period $d = \pi/k_F$, reflecting the coupling between the scatterings near the two Fermi momenta $\pm k_F$. This result shows that the first term is irrelevant when $\alpha > 1$ with $\mathcal{B} = \cos(2k_F L)$, and the second term is irrelevant when $\alpha < 1$ with $\mathcal{B} = 1$, while both terms are important near the Shannon entropy with $\alpha \sim 1$. Thus $\eta = \min(2, 2/\alpha)$. Our data in Figs. 5(a) and 5(b) show excellent agreement with this prediction. One should notice that $\mathcal{A}_1$ and $\mathcal{A}_2$ may have similar amplitudes but opposite signs near $\alpha \sim 1$ with a proper choice of k_F, which may yield strong cancellation between them, thus it cannot be fitted well using Eq. (7) at the regime with $\alpha \sim 1$ in Fig. 5(b).

The following section will generalize the results in Eq. (20) to disjointed intervals, showing great similarity between them. In the disjointed intervals, the FSE is much more complicated, and two FSEs—the extrinsic and intrinsic FSEs—should be defined, all of which have similar scaling laws under uniform dilation, including the basic feature of the bounded function $\mathcal{B}(L)$.

IV. TWO FSEs IN DISJOINTED INTERVALS

To characterize the FSE in multiple intervals, we need to define two more different FSEs, in addition to Δ_A^α in a single interval. To this end, we first consider the FSE in two intervals A_1 and A_2 separated by B_1 [see configuration in Fig. 1(b)], which reads as

$$S_{A_1A_2}^\alpha = S_{A_1B_1A_2}^\alpha - S_{A_1B_1}^\alpha - S_{B_1A_2}^\alpha + S_{A_1}^\alpha + S_{B_1}^\alpha + S_{A_2}^\alpha + \Delta_2^\alpha(A_1, B_1, A_2). \quad (23)$$

Obviously, for two intervals, the nonuniversal constant is twice s_0^α. This definition is well defined (see the Appendix). The right-hand side contains the entropies in all possible single intervals, while the left-hand side is the entropy of two disjointed intervals [45,54–56]. Here we have introduced Δ_2^α to account for the difference between the left- and right-hand sides, making it an exact identity. When the sizes of A_1, B_1, and A_2 approach infinity, we expect that

$$\lim_{L_{A_1,A_2,B_1}\to\infty} \Delta_2^\alpha(A_1, B_1, A_2) = 0. \quad (24)$$

Since it is an extrinsic effect to force the above identity, it is termed as the extrinsic FSE of the disjointed intervals. Meanwhile, we can write

$$S_{A_1A_2}^\alpha = S_{A_1A_2}^{\alpha,\mathrm{CFT}} + \Delta_{A_1A_2}^\alpha \quad (25)$$

using the definition in Eq. (3), where $S_{A_1A_2}^{\alpha,\mathrm{CFT}}$ is the entropy from CFT that can be found in Refs. [52,57]. This FSE is an intrinsic effect, not related to the identity above; it is termed as the intrinsic FSE of the disjointed intervals. By definition, we also expect this FSE vanishes when the intervals and their separation approach infinity.

The expression of $S_{A_1A_2}^{\alpha,\mathrm{CFT}}$ may also be inferred from Eq. (23), assuming negligible FSE and S_A^α in a single interval given by Eq. (3). Thus we have an identity between all FSEs as

$$\Delta_{A_1A_2}^\alpha = \Delta_{A_1B_1A_2}^\alpha - \Delta_{A_1B_1}^\alpha - \Delta_{B_1A_2}^\alpha + \Delta_{A_1}^\alpha + \Delta_{B_1}^\alpha + \Delta_{A_2}^\alpha + \Delta_2^\alpha(A_1, B_1, A_2). \quad (26)$$

From the facts that CFT is analytically exact when the system size is large enough and that the correlation between the two intervals decreases with the increasing of separation, we expect all Δ^α in Eq. (26) to approach zero at large separation according to $\Delta^\alpha \propto 1/L^\eta$. This limit can be used to extract the nonuniversal constant of s_0 in Eq. (3). In the gapless phase, all these FSEs are in the same order of magnitude, thus all of them are important; they reflect the FSE of A from different aspects.

This definition can be easily generalized to three (or many) disjointed intervals using the finding from the CFT that

$$S_{A_1A_2A_3}^\alpha = S_{A_1B_1A_2B_2A_3}^\alpha - S_{A_1B_1A_2B_2}^\alpha - S_{B_1A_2B_2A_3}^\alpha + \cdots + S_{B_1}^\alpha + S_{A_2}^\alpha + S_{B_2}^\alpha + S_{A_3}^\alpha + \Delta_3^\alpha, \quad (27)$$

where $\Delta_3^\alpha = \Delta_3^\alpha(A_1, B_1, A_2, B_2, A_3)$ is the extrinsic FSE. This expression can also be obtained from Eq. (23), assuming A_2 to be a union of two disjointed intervals (see Sec. A). This

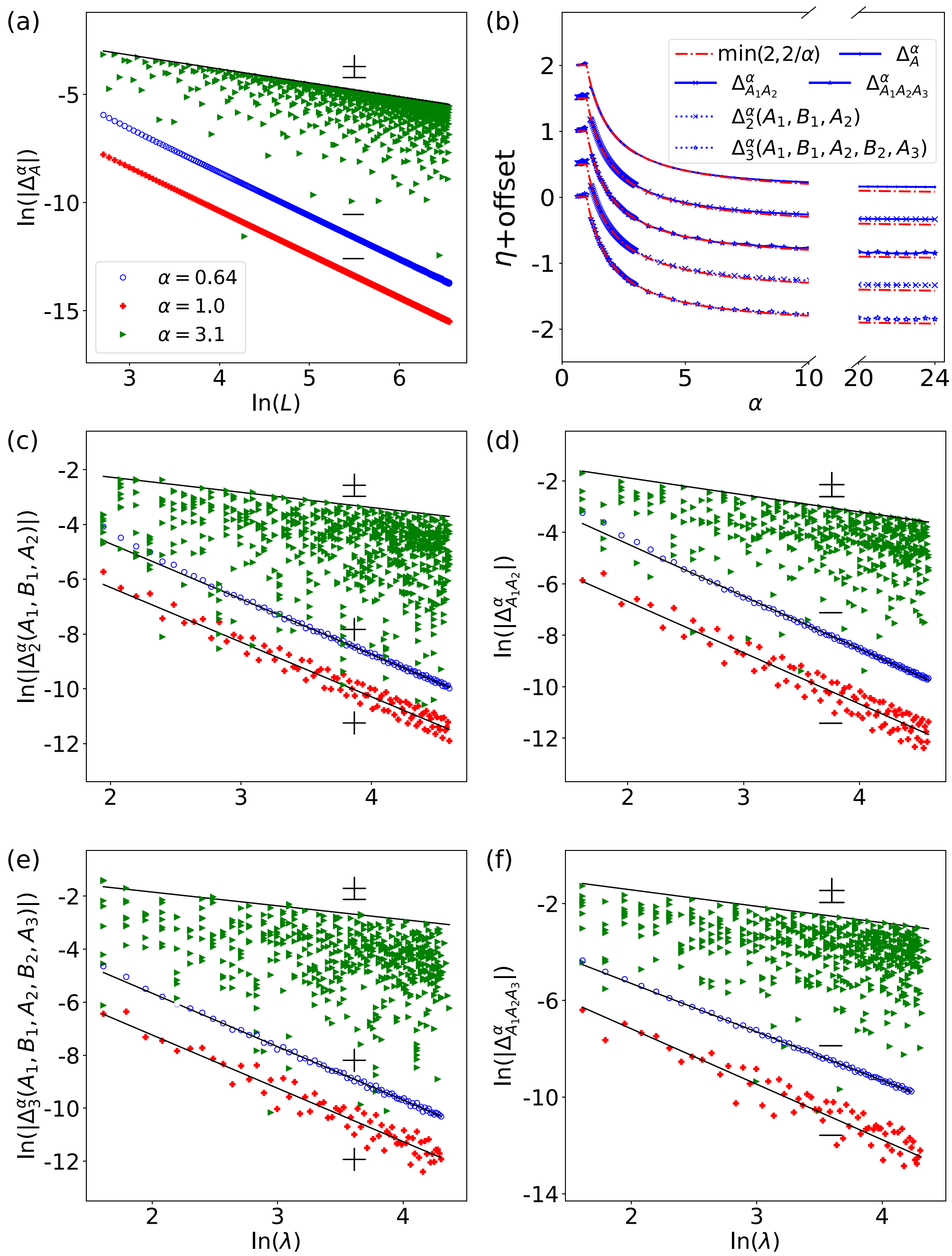

FIG. 5. FSE of Δ_A^α for the free fermion model at $\gamma = 0$ and $|h| = 0.6$. (a) The FSE in a single interval. (c–f) FSE of the entropy for the two different definitions (see text) based on two and three disjointed intervals (c, d) $\lambda A = (A_1, B_1, A_2) = (\lambda, 3\lambda, 2\lambda)$ and (e, f) $\lambda A = (A_1, B_1, A_2, B_2, A_3) = (\lambda, 2\lambda, \lambda, 2\lambda, 4\lambda)$. The plus and minus signs next to each line indicate the sign of this FSE, thus $\pm$ corresponds to the oscillation behavior with period $d = \pi/k_F$ [with $k_F = \arccos(|h|)$] arising from $\mathcal{B}(A, \lambda)$. (b) The fitted values of η for these five cases, which for the sake of convenience are offset by 0.5; otherwise, they will collapse to the same curve given by $\eta = \min(2, 2/\alpha)$ when α is not large enough. The regime for $\alpha \sim 1$ cannot be fitted well using Eq. (7) from the cancellation effect in Eq. (20).

identity has the same structure as Eq. (23). Similarly, we define

$$S_{A_1A_2A_3}^\alpha = S_{A_1A_2A_3}^{\alpha,\mathrm{CFT}} + \Delta_{A_1A_2A_3}^\alpha, \tag{28}$$

where $\Delta_{A_1A_2A_3}^\alpha$ is the intrinsic FSE of the three intervals. We can find an identity between all FSEs exactly the same as Eq. (26). Thus we see that the FSE can be fully characterized by these two FSEs in many intervals. In our numerical simulation, we can extract these two FSEs, and discuss their effect under uniform dilation.

We present the data for these Δ^α in Figs. 5(c)–5(f) under uniform dilation for free fermions with $c = \bar{c} = 1$ [see Fig. 2(a)]. We find a strong oscillation of Δ_A^α for $\alpha > 1$ in Figs. 5(c)–5(f), which is consistent with Eq. (20) with a somewhat modified $\mathcal{A}_1$ and $\mathcal{A}_2$; unfortunately, these values cannot

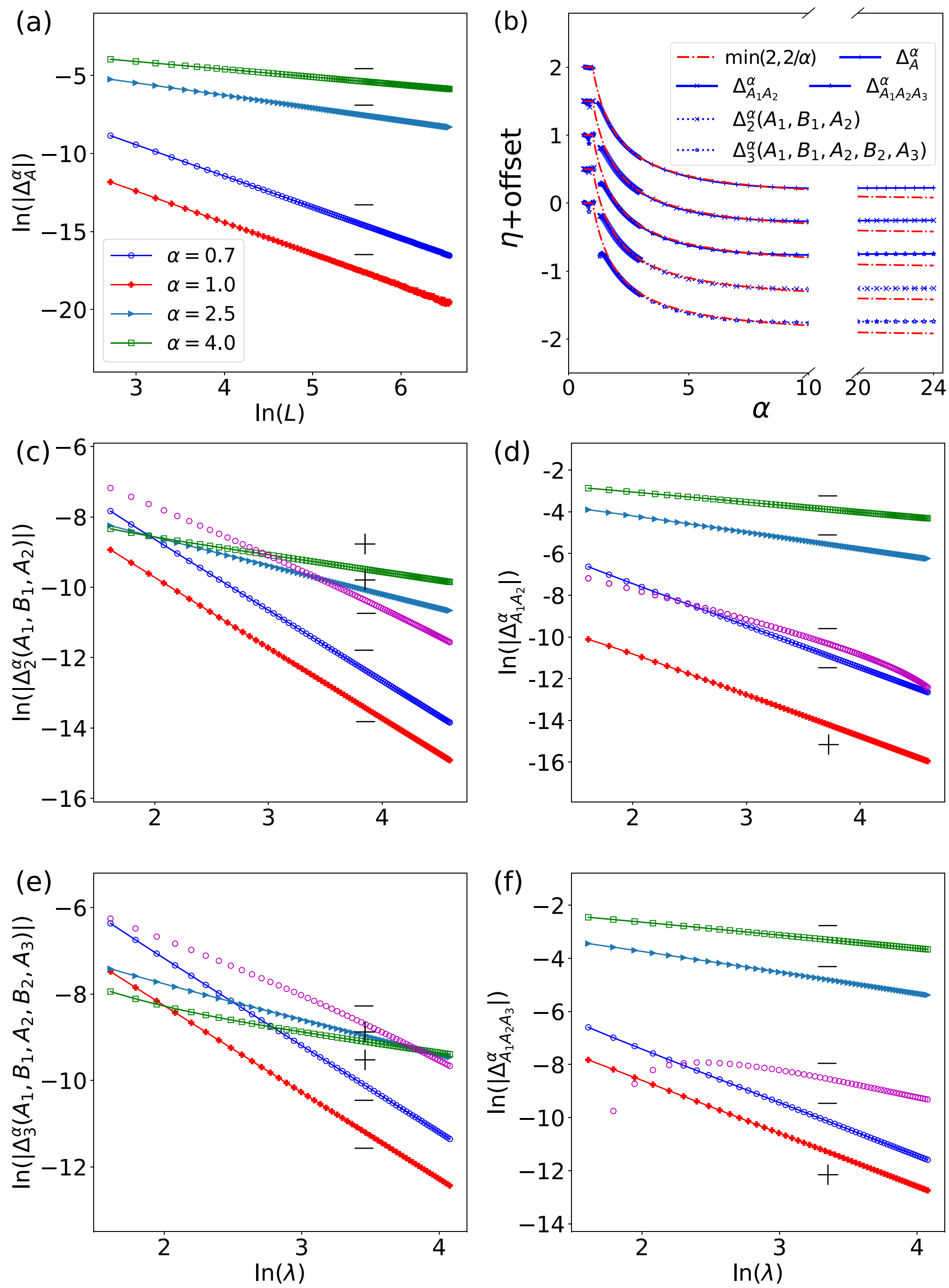

FIG. 6. FSE of Δ_A^α for the free boson model at $\gamma = 1.36$ and $h = 1$. The meaning of each curve is the same as that in Fig. 5; however, since $k_F = \pi$ as shown from the correlator $\mathcal{G}(x)$, oscillation of the FSE is absent, and the scaling law of Eq. (7) can be well reproduced. For comparison, we also show the condition of nonuniform dilation using $(A_1, B_1, A_2) = (28, \lambda, 24)$ and $(A_1, B_1, A_2, B_2, A_3) = (18, \lambda, 12, \lambda, 23)$ with open circles, which violate Eq. (20) obviously.

be determined analytically. We find that when $\alpha < 1$ the $\mathcal{A}_1$ term is always relevant with $\mathcal{B} = 1$, while when $\alpha > 1$ the $\mathcal{A}_2$ term is relevant with $|\mathcal{B}(A, \lambda)|$ being a complicated yet nonanalytical bounded function. These observations yield the major conclusion of Eq. (7). In Fig. 5(b), we present the fitted exponent η as a function of α for all these Δ_A^α, all of which fall to the same expression $\eta = \min(2, 2/\alpha)$ when $\alpha < 10$. When $\alpha \sim 1$, it may not be well fitted using Eq. (7) for the same reason of cancellation in Eq. (20). Moreover, in Fig. 5(b), saturation of entropy happens when $\alpha > 10$, at which the exponent η will deviate from $2/\alpha$ from the unspecified high-order terms $L^{-2n/\alpha}$ $(n > 1)$ in Eq. (20); see Refs. [13,34,35].

These results can also be found for free bosons with $c = \bar{c} = 1/2$ in Fig. 6. However, the correlator $\mathcal{G}(x)$ oscillates with period $k_F = \pi$, thus the oscillation of the FSE from $\mathcal{A}_2$ disappears and $\mathcal{B}(A, \lambda) = 1$ for all disjointed intervals A. As a result, for all the FSEs in the one, two, and three disjointed intervals, all the FSEs decay monotonically with the increasing of dilation ratio λ, following the claim of Eq. (7). We also show that when the dilation is nonuniform this general relation fails. In Fig. 6(b), the summarized η is also the same as that in Fig. 5(b), showing a similar cancellation effect prescribed by Eq. (20) even in multiple intervals, though their analytical expressions are impossible.

Finally, we briefly discuss the FSE in the gapped phases. We find that all the Δ_A^α decay exponentially under dilation in the multiple intervals, with S_A^α satisfies the area law [49]. In these phases, the correlator $\mathcal{G}(x)$ also decays exponentially as a function of x. From Figs. 4(c) and 4(d), we show that only the edge modes of $i\Gamma$ contribute to S_A^α. Due to the short-range correlation from $\mathcal{G}(x)$, the FSEs will quickly disappear following $\Delta_A^\alpha \sim e^{-x/\xi}$, where x is the minimal separation between the open ends in ∂A. This result is trivial, thus it is not presented in this paper. For this reason, Δ_A^α in multiple intervals also exhibit different kinds of scaling laws in the gapped and gapless phases, which can be used for the diagnostication of phase transitions [26–32].

V. CONCLUSION

To conclude, we examine the FSE of α-Rényi entropy in the free fermion and free boson models in the XY model, which exhibit the same scaling law during uniform dilation that $\Delta_{\lambda A}^\alpha = \lambda^{-\eta}\Delta_A^\alpha \mathcal{B}(A,\lambda)$. We find that the regimes $\alpha < 1$ and $\alpha > 1$ are described by different relevant terms, and thus exhibit different scaling behaviors. When α is not large enough, we find $\eta = \min(2, 2/\alpha)$. For the Shannon entropy, we thus have $\eta = 2$ exactly. From the correlation matrix $i\Gamma$, we find that only the edge modes localized at the open ends ∂A contribute to the α-Rényi entropy as well as its FSE. Our results in multiple intervals provide some incisive insight into the entanglement entropy in the many-body system, in which the analytical calculation is scarcely possible. Since this FSE is different in the gapped and gapless phases, the FSE of the disjointed intervals can also be used to characterize this difference and their phase transitions.

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APPENDIX: WELL-DEFINED ENTROPY S_A

The formula of entropy in disjointed intervals [see for example Eq. (23) for two intervals] depends only on the separation between A_1 and A_2, which is independent of the other part of the infinity system. The same conclusion holds for more complicated structures. To understand this, let us assume a ring geometry in Fig. 7, in which A_1 and A_2 are separated by either B_1 or B_1. We assume that their sizes are large enough, then the FSEs are negligible. We have two different methods—using A_1-B_1-A_2 and A_1-B_2-A_2—to account for the entropy of A_1 and A_2, that is,

$$S_{A_1A_2}^\alpha = S_{A_1B_iA_2}^\alpha - S_{A_1B_i}^\alpha - S_{B_iA_2}^\alpha + S_{A_1}^\alpha + S_{A_2}^\alpha + S_{B_i}^\alpha, \quad (\mathrm{A1})$$

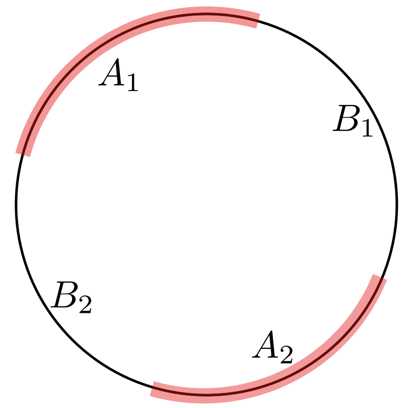

FIG. 7. Entanglement entropy in a ring geometry, in which the total entropy of $A = A_1 \cup A_2$ calculated using A_1-B_1-A_2 and A_1-B_2-A_2 will yield the same result.

for $i = 1$ and 2. Using the fact that $S_A = S_{\bar{A}}$ by definition, we have $S_{A_1B_1A_2}^\alpha = S_{B_2}^\alpha$, $S_{A_1B_2A_2}^\alpha = S_{B_1}^\alpha$, $S_{A_1B_2}^\alpha = S_{B_1A_2}^\alpha$, and $S_{A_1B_1}^\alpha = S_{B_2A_2}^\alpha$, and we can show directly that the above two calculations will yield the same result. This method can be generalized to much more complicated disjointed structures. For this reason, we also expect well-defined FSEs of these entropies.

The above result can also be understood intuitively in the following way. Let us assume that $S_{A_1A_2}^\alpha = x_1 S_{A_1B_iA_2}^\alpha + x_2 S_{A_1B_i}^\alpha + x_3 S_{B_iA_2}^\alpha + x_4 S_{A_1}^\alpha + x_5 S_{A_2}^\alpha + x_6 S_{B_i}^\alpha$, where x_i are undetermined coefficients. This definition should satisfy some basic features.

(1) The above two calculations should yield the same result.

(2) When the separations B_1 and B_2 are much larger than the sizes of A_1 and A_2, we will recover the limit that $S_{A_1A_2}^\alpha = S_{A_1}^\alpha + S_{A_2}^\alpha$.

(3) This expression has well-defined symmetry, that is, $S_{A_1A_2}^\alpha = S_{A_2A_1}^\alpha$.

These three constraints will yield uniquely the above entropy in two disjointed intervals.

Finally, we assume that the entropy of two intervals is correct even when A_2 is a union of two disjointed intervals. Then we assume $A_2 \to A_2 \cup A_3$, where A_2 and A_3 are separated by B_2. In this way, we will find that the right-hand side of Eq. (23) is made by disjointed intervals. For instance, $S_{A_1B_1(A_2A_3)}$ is the total entropy of $A_1 \cup B_1 \cup A_2$ and A_3, which can be calculated, again, using Eq. (23). Collecting all these results will yield Eq. (27). In this way, we can derive the expression of entropy in many disjointed intervals. In the large size limit, the same expression can be found by CFT. This result suggests that

$$S_{\lambda A}^\alpha = S_A^\alpha + \frac{c+\bar{c}}{12(1+\alpha)} k \ln \lambda + \Delta_{\lambda A}^\alpha, \quad (\mathrm{A2})$$

for k disjointed intervals. Obviously, in this case, the nonuniversal constant is given by $k s_0^\alpha$, where s_0^α is the constant in a single interval [see Eq. (3)].

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