Enhanced optimal quantum communication by a generalized phase-shift-keying coherent signal

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It is well known that the maximal success probability of the binary quantum communication can be improved by using a sub-Poissonian nonstandard coherent state as an information carrier. In the present article, we consider the quantum communication with N-ary phase-shift-keying (N-PSK) signal for an arbitrary positive integer $N > 1$. By using nonstandard coherent state, we analytically provide the maximal success probability of the quantum communication with N-PSK. Unlike the binary case, there is a case that the guessing probability of N-PSK quantum communication can be improved by some non-sub-Poissonian, nonstandard coherent state.

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I. INTRODUCTION

In optical communication, a sender encodes a message in an optical signal and sends it to a receiver who detects the signal to decode the message [1]. Thus, the success probability of the optical communication is determined by the physical and statistical properties of the optical signal together with the structure of the receiver's measurement device. In classical optical communication, the receiver can use an on-off detector to decode a sender's message encoded in an on-off keying signal [2,3], and a homodyne detector for the binary phase-shift-keying signal [4]. However, the maximal success probability for decoding encoded messages by using *conventional measurements* such as the on-off and the homodyne detectors cannot exceed the standard quantum limit.

One of the goals in quantum communication is to design a novel measurement so that the maximal success probability to decode messages can surpass the standard quantum limit [5]. According to the quantum theory, the optical signal is described as a density operator on a Hilbert space and a measurement is described as a positive-operator-valued measure (POVM); therefore, the quantum communication is described as a quantum state discrimination protocol [6,7].

Minimum error discrimination [8,9] is one representative state discrimination strategy used in various quantum communication protocols. When a one-bit message is encoded by binary coherent states, minimum error discrimination between the binary coherent states can be implemented via the Dolinar receiver [10]. However, when several bits are encoded and sequentially sent, the photon number detector used for the discrimination may not efficiently react along the received states [5]. For this reason, N-ary coherent states such as the N-amplitude-shift-keying (N-ASK) signal [2] and the N-phase-shift-keying (N-PSK) signal [4] have been considered to send $\log_2 N$-bit messages.

According to a recent work [11], the maximal success probability (or guessing probability) of discriminating a message encoded in the 2-PSK signal composed of *nonstandard coherent states (NS-CS)* with a novel quantum measurement can be improved by the sub-Poissonianity of the NS-CS. Moreover, the experimental method for implementing the quantum measurement reaching for the guessing probability has recently been proposed [12]. Since the negative Mandel parameter to quantify the sub-Poissonianity is considered as a resource in a nonclassical light [13], this result implies that the sub-Poissonianity can be a resource for improving the performance of the quantum communication.

In the present article, we consider the quantum communication with the N-PSK signal for an arbitrary positive integer $N > 1$. By using the nonstandard coherent state, we analytically provide the maximal success probability of the quantum communication with N-PSK. Unlike the binary case, there is a case that the guessing probability of N-PSK quantum communication can be improved by some non-sub-Poissonian NS-CS [14].

For $N > 2$, the N-PSK signal enables us to transmit a $\log_2 N$-bit message per a signal pulse, which is a better information exchange rate than binary-PSK. Moreover, it is also known that the N-PSK signal can provide an improved information exchange rate between the sender and receiver even though the receiver's measurement is slow [5]. However, the maximal success probability of discriminating a message encoded in the N-PSK signal generally decreases as N is getting large. Thus our results about the possible enhancement of the maximal success probability in N-PSK quantum communication by NS-CS is important and even necessary to design efficient quantum communication schemes.

The present article is organized as follows. In Sec. II, we briefly review the problem of minimum error discrimination among N symmetric pure states. In Sec. III, we provide the analytical guessing probability of the N-PSK signal composed of NS-CS. In Sec. IV, we investigate the guessing probability of the N-PSK signal composed of optical spin coherent states (OS-CS), Perelomov coherent states (P-CS), Barut-Girardello coherent states (BG-CS), and modified Susskind-Glogower coherent states (mSG-CS), and discuss the relation between the sub-Poissonianity of the nonclassical light and the

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performance of the N-PSK quantum communication. Finally, in Sec. V, we propose the conclusion of the present article.

II. PRELIMINARIES: MINIMUM ERROR DISCRIMINATION AMONG SYMMETRIC PURE STATES

In quantum communication, Alice (sender) prepares her message $x \in \{1, \dots, N\}$ with a prior probability $q_x \in \{q_1, \dots, q_N\}$, encodes the message in a quantum state $\rho_x \in \{\rho_1, \dots, \rho_N\}$, and sends the quantum state to Bob (receiver). Bob performs a quantum measurement described as a POVM $\{M_1, \dots, M_N\}$ to discriminate the encoded message. In the POVM, M_x is a POVM element with respect to a result x.

For a given ensemble $\mathcal{E} = \{q_x, \rho_x\}_{x=1}^N$ of Alice and a POVM $\mathcal{M} = \{M_x\}_{x=1}^N$ of Bob, the success probability of the quantum communication between Alice and Bob is described by the success probability of the state discrimination,

$$P_s(\mathcal{E}, \mathcal{M}) = \sum_{x=1}^{N} q_x \mathrm{tr}\{\rho_x M_x\}. \tag{1}$$

One way to optimize the efficiency of quantum communication is to consider a POVM that maximizes the success probability in Eq. (1). In this case, the maximization of the success probability in Eq. (1) is equivalent to the minimization of the error probability,

$$P_e(\mathcal{E}, \mathcal{M}) = 1 - P_s(\mathcal{E}, \mathcal{M}) = \sum_{x=1}^{N} \sum_{y \neq x} q_x \mathrm{tr}\{\rho_x M_y\}. \tag{2}$$

Minimum error discrimination is to minimize the error probability in Eq. (2) over all possible POVMs $\mathcal{M}$ of Bob.

For a given ensemble $\mathcal{E}$, it is known that the following inequality is a necessary and sufficient condition for POVM $\mathcal{M}$ minimizing the error probability [2,15],

$$\sum_{z=1}^{N} q_z \rho_z M_z - q_x \rho_x \geqslant 0, \quad \forall x \in \{1, \dots, N\}. \tag{3}$$

Moreover, it is known that the following equality is a useful necessary condition to characterize the structure of the POVM,

$$M_x(q_x\rho_x - q_y\rho_y)M_y = 0, \quad \forall x, y \in \{1, \dots, N\}. \tag{4}$$

If all quantum state ρ_x are pure (that is, $\rho_x = |\psi_x\rangle\langle\psi_x|$) and linearly independent, the optimal POVM is given by a rank-1 projective measurement [2]. In other words, $M_x = |\pi_x\rangle\langle\pi_x|$ for every $x \in \{1, \dots, N\}$.

Now, we focus on the minimum error discrimination among a specific class of pure states, called *symmetric pure states*.

Definition 1. [16] For a positive integer N, the distinct pure states $|\psi_1\rangle, \dots, |\psi_N\rangle$ are called *symmetric*, if there exists a unitary operator V such that

$$|\psi_x\rangle = V^{x-1}|\psi_1\rangle, \tag{5}$$

for $x = 1, 2, \dots, N$, and

$$V^N = \mathbb{I}, \tag{6}$$

where $\mathbb{I}$ is an identity operator on a subspace spanned by $\{|\psi_1\rangle, \dots, |\psi_N\rangle\}$.

The Gram matrix composed of the symmetric pure states in Definition 1 is

$$G = (\langle\psi_x|\psi_y\rangle)_{x,y=1}^N. \tag{7}$$

From a straightforward calculation, the eigenvalues of the Gram matrix in Eq. (7) are in forms of

$$\lambda_p = \sum_{k=1}^{N} \langle\psi_j|\psi_k\rangle e^{-\frac{2\pi i(p-1)(j-k)}{N}}, \quad p = 1, 2, \dots, N, \tag{8}$$

for any choice of $j \in \{1, 2, \dots, N\}$. We note that the set $\{\lambda_p\}_{p=1}^N$ is invariant under the choice of j due to the symmetry of the pure state $\{|\psi_1\rangle, \dots, |\psi_N\rangle\}$. The following proposition provides the maximal success probability of the minimum error discrimination among the symmetric pure states in Definition 1.

Proposition 1. [17,18] Let $\mathcal{E}_{\rm sym}$ be an equiprobable ensemble of symmetric pure states $|\psi_1\rangle, \dots, |\psi_N\rangle$. Then, the maximal success probability is given as

$$P_{\rm guess}(\mathcal{E}_{\rm sym}) = \frac{1}{N^2}\left(\sum_{p=1}^{N} \sqrt{\lambda_p}\right)^2, \tag{9}$$

where λ_p are the eigenvalues of the Gram matrix composed of $\{|\psi_1\rangle, \dots, |\psi_N\rangle\}$ in Eq. (7).

Equation (9) is also called *guessing probability*, and $1 - P_{\rm guess}(\mathcal{E}_{\rm sym})$ is called *minimum error probability*.

III. OPTIMAL COMMUNICATION WITH PHASE-SHIFT-KEYING (PSK) SIGNAL

In quantum optical communication, the phase-shift-keying (PSK) signal is expressed as equiprobable symmetric pure states [1]. In this section, we derive the maximal success probability of the quantum communication with the PSK signal composed of generalized coherent states. First, we provide a definition of the generalized coherent state which encapsulates the standard coherent state (S-CS) and nonstandard coherent state (NS-CS) as special cases.

Definition 2. [11] If a pure state takes the form

$$|\alpha, \vec{h}\rangle = \sum_{n=0}^{\infty} \alpha^n h_n(|\alpha|^2)|n\rangle, \tag{10}$$

where R is a positive real number, α is a complex number such that $|\alpha| < R$, $\{|n\rangle | n \in \mathbb{Z}^+ \cup \{0\}\}$ is the Fock basis, and $\vec{h}$ is a tuple of real-valued functions $h_n : [0, R^2] \to \mathbb{R}$ satisfying

$$\sum_{n=0}^{\infty} u^n\{h_n(u)\}^2 = 1, \tag{11}$$

$$\sum_{n=0}^{\infty} nu^n\{h_n(u)\}^2 \tag{12}$$

is a strictly increasing function of u,

$$\int_0^{R^2} du\, w(u) u^n \{h_n(u)\}^2 = 1 \tag{13}$$

for a real-valued function w: $[0, R^2] \to \mathbb{R}^+$, then the pure state is called *generalized coherent state*. If every real-valued function h_n in Eq. (10) takes the form

$$h_n(u) = \frac{1}{\sqrt{n!}} e^{-\frac{1}{2}u}, \quad \forall n \in \mathbb{Z}^+ \cup \{0\}, \tag{14}$$

then Eq. (10) is called *the standard coherent state* (S-CS) [19]. Otherwise, Eq. (10) is called *the nonstandard coherent state* (NS-CS).

Remark 1. The mean photon number of S-CS is given by

$$\langle n\rangle = |\alpha|^2. \tag{15}$$

Several examples of NS-CS have been introduced such as the optical spin coherent state (OS-CS) [14], the Perelomov coherent state (P-CS) [14], the Barut-Girardello coherent state (BG-CS) [20], and the modified Susskind-Glogower coherent state (mSG-CS) [21].

Example 1. For a given non-negative integer $\widetilde{n}$, if h_n takes the form

$$h_n(u) = \sqrt{\frac{\widetilde{n}!}{n!(\widetilde{n}-n)!}}(1+u)^{-\frac{\widetilde{n}}{2}}, \tag{16}$$

for $0 \leqslant n \leqslant \widetilde{n}$ and $h_n(u) = 0$ for $n > \widetilde{n}$, then the generalized coherent state in Eq. (10) is called *the optical spin coherent state* (OS-CS). The mean photon number of OS-CS is given by [11]

$$\langle n\rangle = \widetilde{n}\frac{|\alpha|^2}{1+|\alpha|^2}. \tag{17}$$

Example 2. For all non-negative integer n and a real number ς with $\varsigma \geqslant 1/2$, if h_n takes the form

$$h_n(u) = \frac{1}{\sqrt{\mathcal{N}(u)}}\sqrt{\frac{\Gamma(2\varsigma)}{n!\Gamma(2\varsigma+n)}}, \tag{18}$$

then the generalized coherent state in Eq.(10) is called the *Barut-Girardello coherent state* (BG-CS). Here, Γ is the Gamma function of the first kind and $\mathcal{N}(u)$ is a normalization factor,

$$\mathcal{N}(u) = \Gamma(2\varsigma)u^{1/2-u}I_{2\varsigma-1}(2\sqrt{u}), \tag{19}$$

where I_ν is the modified Bessel function of the first kind. The mean photon number of BG-CS is given by [11]

$$\langle n\rangle = |\alpha|\frac{I_{2\varsigma}(2|\alpha|)}{I_{2\varsigma-1}(2|\alpha|)}. \tag{20}$$

Example 3. For all non-negative integer n, if h_n takes the form

$$h_n(u) = \sqrt{\frac{n+1}{\bar{\mathcal{N}}(u)}}\frac{1}{u^{\frac{n+1}{2}}}J_{n+1}(2\sqrt{u}), \tag{21}$$

then the generalized coherent state in Eq. (10) is called *the modified Susskind-Glogower coherent state* (mSG-CS). Here, J_n is the Bessel function of the first kind and $\bar{\mathcal{N}}(u)$ is a normalization factor,

$$\bar{\mathcal{N}}(u) = \frac{1}{u}[2u\{J_0(2\sqrt{u})\}^2 - \sqrt{u}J_0(2\sqrt{u})J_1(2\sqrt{u}) + 2u\{J_1(2\sqrt{u})\}^2]. \tag{22}$$

The mean photon number of mSG-CS is given by [11]

$$\langle n\rangle = \frac{1}{\bar{\mathcal{N}}(|\alpha|^2)} - 1. \tag{23}$$

Example 4. For all non-negative integer n, ς and an integer or half-integer with $\varsigma \geqslant 1/2$, if h_n takes the form

$$h_n(u) = \sqrt{\frac{(2\varsigma-1+n)!}{n!(2\varsigma-1)!}}(1-u)^{\varsigma}, \tag{24}$$

then the generalized coherent state in Eq. (10) is called *the Perelomov coherent state* (P-CS). The mean photon number of P-CS is given by [11]

$$\langle n\rangle = 2\varsigma\frac{|\alpha|^2}{1-|\alpha|^2}. \tag{25}$$

We mainly focus on which NS-CS provided in the examples can give the advantage to the N-ary PSK quantum communication. For this reason, we define the N-ary generalized PSK (N-GPSK) signal as follows.

Definition 3. If an equiprobable ensemble $\mathcal{E}_{\rm gcs}$ consists of generalized coherent states,

$$\{|\alpha_x, \vec{h}\rangle | x \in \{1, 2, \cdots, N\}\}, \tag{26}$$

where $N \in \mathbb{Z}^+$ and $\alpha_x \in \mathbb{C}$ such that

$$\alpha_x = \alpha e^{\frac{2\pi i}{N}x}, \tag{27}$$

with a non-negative integer α, then the ensemble $\mathcal{E}_{\rm gcs}$ is called the N-ary generalized PSK (N-GPSK) signal.

Moreover, the N-GPSK signal is called the N-ary standard PSK (N-SPSK) signal [1] if every coherent state in Eq. (26) is S-CS, and the N-PSK signal is called the N-ary nonstandard PSK (N-NSPSK) signal if every coherent state in Eq. (26) is NS-CS. The following theorem shows that the generalized coherent states in Definition 3 are symmetric.

Theorem 1. For given distinct generalized coherent states $|\alpha_1, \vec{h}\rangle, \ldots, |\alpha_N, \vec{h}\rangle$, there exists a unitary operator U such that

$$|\alpha_x, \vec{h}\rangle = U^{x-1}|\alpha_1, \vec{h}\rangle, \quad \forall x \in \{1, 2, \ldots, N\}, \tag{28}$$

for $x = 1, 2, \ldots, N$, and

$$U^N = \mathbb{I}, \tag{29}$$

where $\mathbb{I}$ is an identity operator on a subspace spanned by $\{|\alpha_1, \vec{h}\rangle, \ldots, |\alpha_N, \vec{h}\rangle\}$.

Proof. Consider a unitary operator,

$$U = e^{\frac{2\pi i}{N}a^\dagger a}, \tag{30}$$

where a and $a^\dagger$ are the annihilation and creation operators satisfying

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad \forall n \in \mathbb{Z}^+, \tag{31}$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle, \quad \forall n \in \mathbb{Z}^+ \cup \{0\}, \tag{32}$$

respectively. It is straightforward to show that the unitary operator U in Eq. (30) satisfies Eq. (29).

We also note that

$$U|n\rangle = e^{\frac{2\pi i}{N}n}|n\rangle, \tag{33}$$

for any non-negative integer n, therefore we have that

$$U|\alpha_x,\vec{h}\rangle=\sum_{n=0}^{\infty}\alpha_x^n h_n(|\alpha_x|^2)e^{\frac{2\pi i}{N}a^\dagger a}|n\rangle$$
$$=\sum_{n=0}^{\infty}\alpha_x^n h_n(|\alpha_x|^2)e^{\frac{2\pi i}{N}n}|n\rangle$$
$$=\sum_{n=0}^{\infty}\left(\alpha_x e^{\frac{2\pi i}{N}}\right)^n h_n(|\alpha_x|^2)|n\rangle, \quad (34)$$

for every $x\in\{1,2,\dots,N-1\}$. Moreover, Eq. (27) leads us to

$$\alpha_x e^{\frac{2\pi i}{N}}=\alpha_{x+1}, \quad (35)$$

for $x\in\{1,2,\dots,N-1\}$ and

$$|\alpha_x|=\alpha, \quad (36)$$

for $x\in\{1,2,\dots,N\}$. From Eqs. (34), (35), and (36), we have

$$U|\alpha_x,\vec{h}\rangle=\sum_{n=0}^{\infty}(\alpha_{x+1})^n h_n(|\alpha_{x+1}|^2)|n\rangle=|\alpha_{x+1},\vec{h}\rangle. \quad (37)$$

Equation (28) can be shown by an inductive use of Eq. (37), which completes the proof. ■

Theorem 1 means that the guessing probability of quantum communication with the N-GPSK signal is given by Eq. (9) in Proposition 1, which is encapsulated in the following theorem.

Theorem 2. The guessing probability of the N-GPSK signal is given by

$$P_{\text{guess}}(\mathcal{E}_{\text{gcs}})=\frac{1}{N^2}\left(\sum_{p=1}^{N}\sqrt{\lambda_p^{(G)}}\right)^2, \quad (38)$$

where $\lambda_p^{(G)}$ takes the form of

$$\lambda_p^{(G)}=\sum_{k=0}^{N-1}\left[\sum_{n=0}^{\infty}\alpha^{2n}\cos\left\{\frac{2\pi}{N}k(n+p-1)\right\}\{h_n(\alpha^2)\}^2\right], \quad (39)$$

for every $p\in\{1,2,\dots,N\}$.

Proof. For every $j,k\in\{1,2,\dots,N\}$, the inner product $\langle\alpha_j,\vec{h}|\alpha_k,\vec{h}\rangle$ is

$$\langle\alpha_j,\vec{h}|\alpha_k,\vec{h}\rangle=\sum_{n=0}^{\infty}\left\{\alpha^2 e^{i\frac{2\pi}{N}(k-j)}\right\}^n\{h_n(\alpha^2)\}^2. \quad (40)$$

From Eq. (40) together with Eq. (8), $\lambda_p^{(G)}$ is also obtained by

$$\lambda_p^{(G)}=\sum_{k=1}^{N}\left[\sum_{n=0}^{\infty}\left\{\alpha^2 e^{i\frac{2\pi}{N}(k-j)}\right\}^n\{h_n(\alpha^2)\}^2\right]e^{-\frac{2\pi i(p-1)(j-k)}{N}}$$
$$=\sum_{k=1}^{N}\left[\sum_{n=0}^{\infty}\alpha^{2n}e^{i\frac{2\pi}{N}(k-j)(n+p-1)}\{h_n(\alpha^2)\}^2\right]. \quad (41)$$

As mentioned before, the set $\{\lambda_p^{(G)}\}_{p=1}^N$ is invariant under the choice of $j\in\{1,2,\dots,N\}$. By choosing $j=1$ and substituting k to $k-1$, $\lambda_p^{(G)}$ can be rewritten by

$$\lambda_p^{(G)}=\sum_{k=0}^{N-1}\left[\sum_{n=0}^{\infty}\alpha^{2n}e^{i\frac{2\pi}{N}k(n+p-1)}\{h_n(\alpha^2)\}^2\right]. \quad (42)$$

Since the Gram matrix is Hermitian, $\lambda_p^{(G)}$ is a real number. Thus, by using the relation,

$$\lambda_p^{(G)}=\frac{\lambda_p^{(G)}+\lambda_p^{(G)*}}{2}, \quad (43)$$

together with Eq. (42), we have Eq. (39). Due to Theorem 1, every generalized coherent state in the N-GPSK signal is symmetric. Thus, Proposition 1 and Eq. (39) lead us to the guessing probability in Eq. (38). ■

IV. SUB-POISSONIANITY OF NS-CS AND THE GUESSING PROBABILITY

For $N=3$, 4, and 8, we provide illustrative results of the guessing probability of the N-NSPSK signal of Eq. (38) in the cases of OS-CS, P-CS, BG-CS, and mSG-CS. We also compare these results with the case of the N-SPSK signal.

A. Optical spin coherent states (OS-CS)

The minimum error probabilities of the N-SPSK signal and N-NSPSK signal composed of OS-CS are illustrated in Fig. 1, where Figs. 1(a), 1(b), and 1(c) show the case of $N=3$, 4, and 8, respectively. In these figures, thick solid purple, dotted red, dashed-dotted blue, and dashed green lines show the case of the N-NSPSK signal with $\widetilde{n}=3$, $\widetilde{n}=5$, $\widetilde{n}=7$, and $\widetilde{n}=11$, respectively. Solid black lines in the figures show the case of the N-SPSK signal.

In Fig. 1(a), the minimum error probabilities of the 3-NSPSK signal composed of OS-CS is smaller than that of the 3-SPSK signal when the mean photon number is large ($\langle n\rangle>0.45$, $\langle n\rangle>0.42$, $\langle n\rangle>0.38$, and $\langle n\rangle>0.37$ in the cases of $\widetilde{n}=3$, $\widetilde{n}=5$, $\widetilde{n}=7$, and $\widetilde{n}=11$, respectively). In other words, *3-PSK quantum communication can be enhanced by a nonstandard coherent state using OS-CS.* However, in Fig. 1(b), each minimum error probability of the 4-NSPSK signal is larger than that of the 4-SPSK signal for the arbitrary mean photon number. This aspect repeatedly happens in Fig. 1(c) where the 8-NSPSK signal is considered. These results imply that *4-PSK and 8-PSK quantum communication cannot be enhanced by OS-CS.*

B. Barut-Girardello coherent states (BG-CS)

The minimum error probabilities of the N-SPSK signal and the N-NSPSK signal composed of BG-CS are illustrated in Fig. 2, where Figs. 2(a), 2(b), and 2(c) show the case of $N=3$, 4, and 8, respectively. In these figures, dashed blue and dotted red lines show the case of the N-NSPSK signal with $\varsigma=0.5$ and $\varsigma=1.5$, respectively. Solid black lines show the case of N-SPSK.

In Fig. 2(a), each minimum error probabilty of 3-NSPSK signal with $\varsigma=1.5$ is smaller than that of the 3-SPSK signal when the mean photon number is larger than 0.48. Meanwhile,

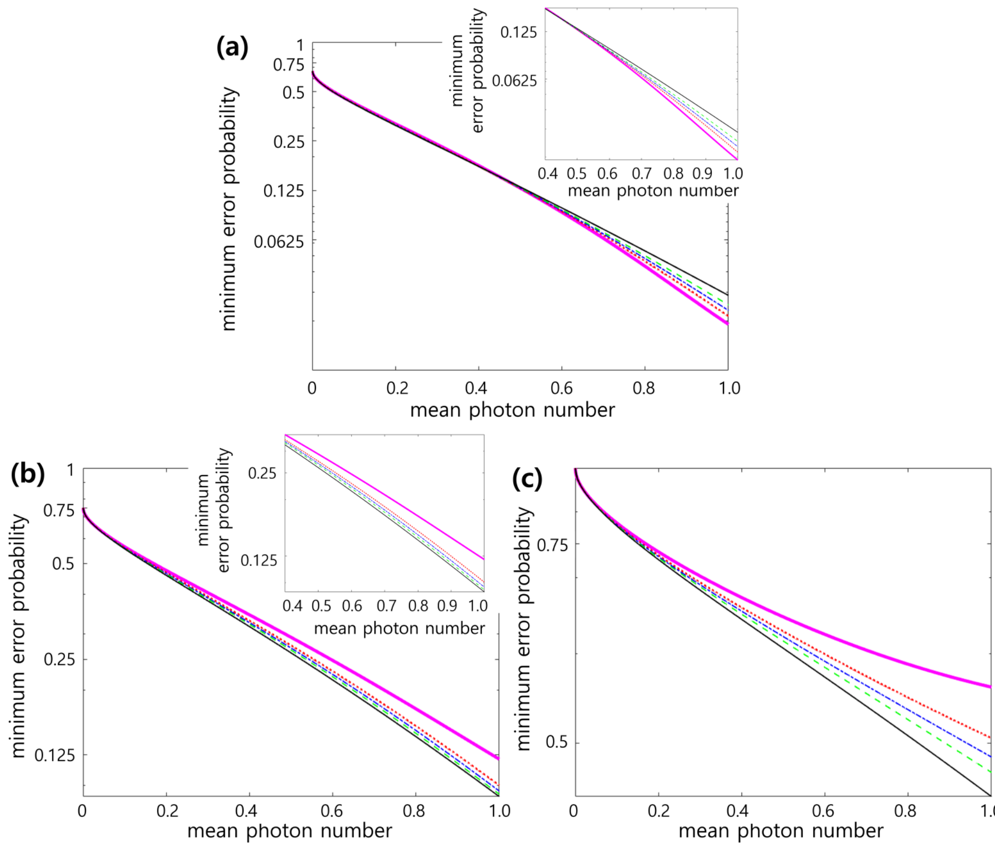

FIG. 1. The minimum error probabilities of the N-SPSK and N-NSPSK signals composed of OS-CS, where (a), (b), and (c) show the case of $N = 3$, 4, and 8, respectively. In these figures, thick solid purple, dotted red, dashed-dotted blue, and dashed green lines show the case of the N-NSPSK signal with $\widetilde{n} = 3$, $\widetilde{n} = 5$, $\widetilde{n} = 7$, and $\widetilde{n} = 11$, respectively. Thin solid black lines in the figures show the case of the N-SPSK signal.

each minimum error probability of the 3-NSPSK signal with $\varsigma = 0.5$ is larger than that of the 3-SPSK signal for the arbitrary mean photon number. *Thus, enhancing 3-PSK quantum communication by the nonstandard coherent state using BG-CS depends on the parameter* ς. However, in Fig. 2(b), each minimum error probability of the 4-NSPSK signal is larger than that of the 4-SPSK signal for the arbitrary mean photon number. This aspect repeatedly happens in Fig. 2(c) where the 8-NSPSK signal is considered. These results imply that *4-PSK and 8-PSK quantum communication cannot be enhanced by BG-CS*.

C. Modified Susskind-Glogower coherent states (mSG-CS)

The minimum error probabilities of the N-SPSK signal and the N-NSPSK signal composed of mSG-CS are illustrated in Fig. 3, where Figs. 3(a), 3(b), and 3(c) show the case of $N = 3$, 4, and 8, respectively. In these figures, dotted red lines show the case of the N-NSPSK signal and solid black lines show the case of N-SPSK.

In Fig. 3, each minimum error probability of the N-NSPSK signal is larger than that of the N-SPSK signal for any $N = 3$, 4, and 8 and any mean photon number. This result implies that *3-PSK, 4-PSK, and 8-PSK quantum communication cannot*

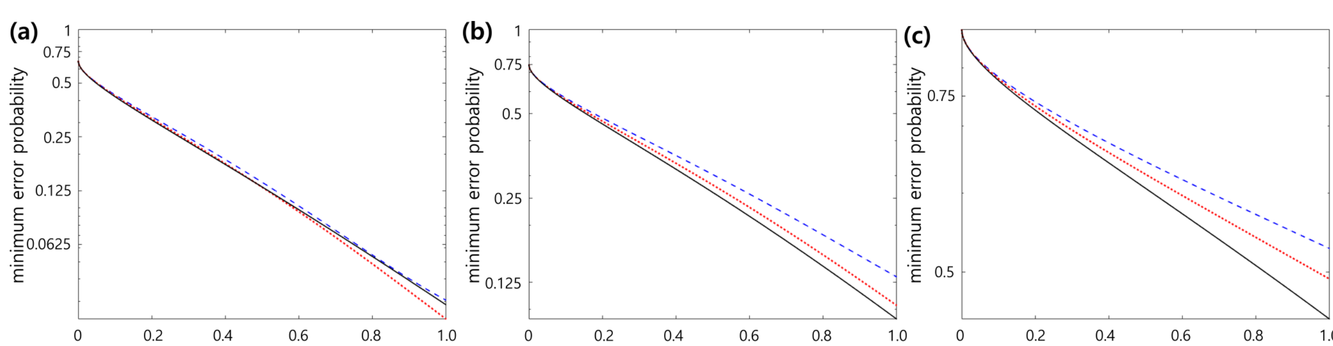

FIG. 2. The minimum error probabilities of the N-SPSK and N-NSPSK signals composed of BG-CS, where (a), (b), and (c) shows the case of $N = 3$, 4, and 8, respectively. In these figures, dashed blue and dotted red lines show the case of the N-NSPSK signal with $\varsigma = 0.5$ and $\varsigma = 1.5$, respectively. Solid black lines show the case of N-SPSK.

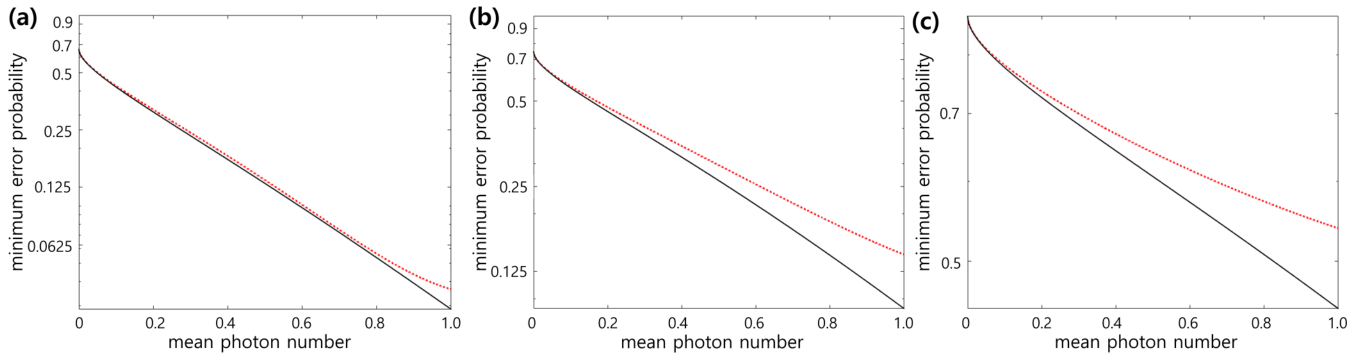

FIG. 3. The minimum error probabilities of the N-SPSK signal and the N-NSPSK signal composed of mSG-CS, where (a), (b), and (c) show the case of $N = 3$, 4, and 8, respectively. In these figures, dotted red lines show the case of the N-NSPSK signal and solid black lines show the case of N-SPSK.

be enhanced by mSG-CS. We also compare this result with the previous work about the on-off keying signal [11]; it is known that the minimum error probability of the on-off keying signal composed of mSG-CS has a singular point where the logarithm of the minimum error probability diverges to $-\infty$. This implies that the minimum error probability can achieve to zero. Unlike the result in the previous work [11], the minimum error probabilities of the 3-, 4-, and 8-NSPSK signals in Fig. 3 do not have such singular points.

D. Perelomov coherent states (P-CS)

The minimum error probabilities of the N-SPSK signal and the N-NSPSK signal composed of P-CS are illustrated in Fig. 4, where Figs. 4(a), 4(b), and 4(c) shows the case of $N = 3$, 4, and 8, respectively. In these figures, dashed blue and dotted red lines show the case of P-CS with $\varsigma = 0.5$ and $\varsigma = 1.5$, respectively. Solid black lines show the case of S-CS.

In Fig. 4(a), each minimum error probability of the 3-NSPSK signal composed of P-CS with $\varsigma = 0.5$ and 1.5 is larger than that of the 3-SPSK signal for the arbitrary mean photon number. In other words, *3-PSK quantum communication cannot be enhanced by the nonstandard coherent state using P-CS*. However, in Fig. 4(b), each minimum error probability of the 4-NSPSK signal composed of P-CS is smaller than that of the 4-SPSK signal when mean photon number is small ($\langle n\rangle < 0.585$ and $\langle n\rangle < 0.786$ in case of $\varsigma = 0.5$ and $\varsigma = 1.5$, respectively). This implies that *4-PSK quantum communication can be enhanced by P-CS*. In Fig. 4(c), each minimum error probability of the 8-NSPSK signal composed of P-CS is smaller than that of the 8-SPSK signal for the arbitrary mean photon number. This means that 4-PSK and 8-PSK quantum communication can be improved by P-CS which is not sub-Poissonian. We discuss the details in the next section.

E. Mandel parameter and N-NSPSK quantum communication

It is known that sub-Poissonianity of nonclassical light is one of the important statistical properties for improving guessing probability of binary quantum optical communication [11]. For this reason, we consider the following *Mandel parameter*,

$$Q_M^{(\mathrm{NS})} = \frac{(\Delta n)^2}{\langle n\rangle} - 1, \tag{44}$$

where $\langle n\rangle$ is mean photon number and Δn is standard deviation of the number of photons. It is known that if $Q_M^{(NS)} > 0(< 0)$, then the generalized coherent state is *super-Poissonian* (*sub-Poissonian*) [22,23]. If $Q_M^{(NS)} = 0$ (for example, S-CS), then the generalized coherent state is *Poissonian*. Here, we consider the relation between the performance of the N-PSK quantum communication and the Mandel parameter.

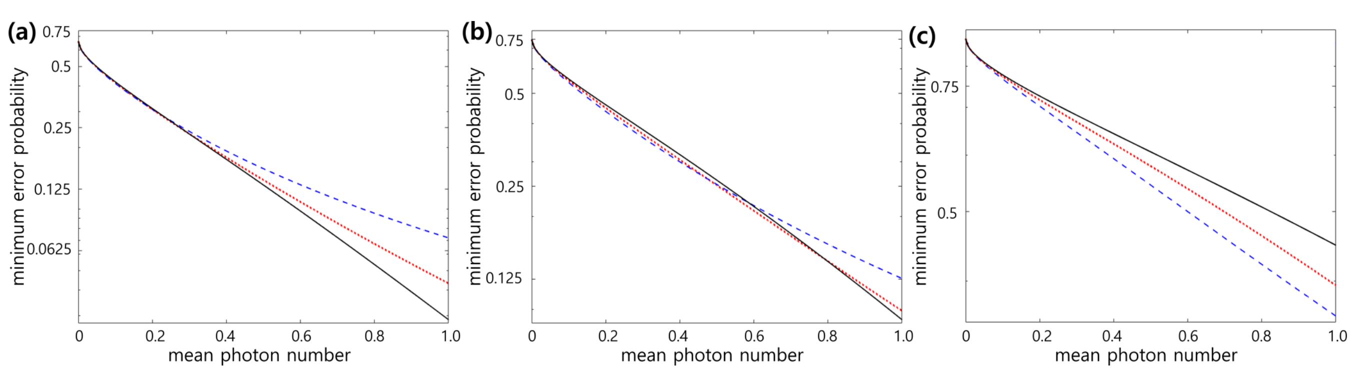

FIG. 4. The minimum error probabilities of the N-SPSK signal and the N-NSPSK signal composed of P-CS, where (a), (b), and (c) show the case of $N = 3$, 4, and 8, respectively. In these figures, dashed blue and dotted red lines show the case of P-CS with $\varsigma = 0.5$ and $\varsigma = 1.5$, respectively. Solid black lines show the case of N-SPSK.

(1) In the case of OS-CS, the Mandel parameter is analytically driven as [11]

$$Q_M^{(\mathrm{OS})} = -\frac{\langle n\rangle}{\widetilde{n}}, \tag{45}$$

which means that OS-CS is always sub-Poissonian. According to Fig. 1, we note that sub-Poissonianity of OS-CS does not always guarantee the enhancement of the N-PSK quantum communication.

(2) In the case of BG-CS, the Mandel parameter is analytically driven in terms of the modified Bessel function of the first kind as [11]

$$Q_M^{(\mathrm{BG})} = \alpha\left[\frac{I_{2\varsigma+1}(2\alpha)}{I_{2\varsigma}(2\alpha)} - \frac{I_{2\varsigma}(2\alpha)}{I_{2\varsigma-1}(2\alpha)}\right]. \tag{46}$$

Since the inequality $\{I_{\nu+1}(x)\}^2 \geqslant I_\nu(x)I_{\nu+2}(x)$ holds for every $x \geqslant 0$, $Q_M^{(\mathrm{BG})}$ is negative semidefinite. Therefore, BG-CS is always sub-Poissonian or Poissonian. Moreover, $Q_M^{(\mathrm{BG})}$ is known to be strictly negative for the nonzero mean photon number [11]. Nevertheless, Fig. 2 shows that sub-Poissonianity of BG-CS does not always guarantee the enhancement of the N-PSK quantum communication.

(3) Since the analytic form of the Mandel parameter of the mSG-CS Mandel parameter is too complex [11], we do not introduce the analytic form here. According to the result of [11], the Mandel parameter of mSG-CS is negative when the mean photon number is not too large. Nevertheless, Fig. 3 shows that the sub-Poissonianity of the mSG-CS cannot provide any advantage on the N-PSK quantum communication.

(4) In case of P-CS, the Mandel parameter is analytically driven as [11]

$$Q_M^{(P)} = \frac{\langle n\rangle}{2\varsigma}, \tag{47}$$

which means that P-CS is super-Poissonian. However, Fig. 4 shows that P-CS can enhance the N-PSK quantum communication for $N = 3$, 4, or 8.

V. CONCLUSION

In the present article, we have considered the quantum communication with the N-ary phase-shift-keying (N-PSK) signal for an arbitrary positive integer $N > 1$. By using NS-CS, we have analytically provided the guessing probability of the quantum communication with N-PSK. Unlike the binary case [11,12], we note that the guessing probability of N-PSK quantum communication can be improved by P-CS, which is not sub-Poissonian NS-CS: The guessing probability can be improved by considering the sub-Poissonian NS-CS for $N = 3$; meanwhile the P-CS can improve the guessing probability for $N = 4$ and $N = 8$.

Using the N-PSK signal with $N > 2$, we can achieve a better transmission rate per a signal pulse than that of binary-BPSK even if the receiver's measurement is slow [5]. On the other hand, the maximal success probability of discriminating a message encoded in the N-PSK signal generally decreases as N is getting large. Thus our results about the possible enhancement of the maximal success probability in N-PSK quantum communication by NS-CS is important and even necessary to design efficient quantum communication schemes.

In the present article, we have only the considered PSK signal with equal prior probabilities, which is composed of symmetric pure states. However, it is interesting and even important to consider a nonequiprobable or asymmetric ensemble of NS-CS for several reasons: First, it is practically difficult to implement the PSK signal having perfect symmetry or equal prior probabilities. Moreover, in discriminating three nonequiprobable and asymmetric pure states, there is the possibility that sub-Poissonianity of nonclassical light can enhance the guessing probability. We note that it is also interesting to consider unambiguous discrimination [24–29] of NS-CS since this strategy can provide us with better confidence than the minimum error discrimination.

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