Degenerate local-dimension-invariant stabilizer codes and an alternative bound for the distance preservation condition

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One hurdle to performing reliable quantum computations is overcoming noise. One possibility is to reduce the number of particles needing to be protected from noise and instead use systems with more states, so-called qudit quantum computers. In this paper we show that codes for these systems can be derived from already known codes, and, in particular, that degenerate stabilizer codes can have their distance also promised upon sufficiently large local dimension as well as an alternative bound on the local dimension required to preserve the distance of local-dimension-invariant codes, which is a result which could prove to be useful for error-corrected qudit quantum computers.

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I. INTRODUCTION

Having protected quantum information is an essential piece of being able to perform quantum computations. There are a variety of methods to help protect quantum information, such as those discussed in Ref. [1]. In this paper we focus on stabilizer codes as they are the quantum analog of classical linear codes. Even with error-correcting codes, having sufficient amounts of protected quantum information to perform useful tasks is still an unresolved challenge. A way to retain a similarly sized computational space whereas reducing the number of particles that need precise controls is to replace the standard choice of qubits with *qudits*, quantum particles with q levels, also known as local-dimension q [2]. Throughout this paper we require q to be a prime so that each nonzero element has a unique multiplicative inverse over $\mathbb{Z}_q$. This restriction can likely be removed, but for simplicity and clarity we only consider this case. Experimental realizations of qudit systems are currently underway [3–6], so having more error-correcting codes will aid in protecting such systems.

Prior work on qudit error-correcting codes have at times had challenging restrictions between the parameters of the code [7–9], and we have already made progress on reducing this barrier in a prior paper [10]. Our prior work showed the ability to make error-correcting codes that preserved their parameters even upon changing the local dimension of the system, provided the local dimension is sufficiently large. Unfortunately the ability to promise the distance of the codes was only shown for nondegenerate codes and with a large local-dimension value required. Beyond this, qudits also have proven connections to foundational aspects of physics [11]. Seeing these potential reasons for using qudits, this paper builds off of our prior work to expand the local-dimension-invariant (LDI) framework to the case of degenerate codes as well as providing a roughly quadratic improvement in the size of the local dimension needed to still promise the distance of the code. With these results the practicality of using this method is improved as well as now providing the option of applying the result to the essential class of degenerate codes, such as quantum versions of low-density parity-check (LDPC) codes.

II. DEFINITIONS FOR QUDIT STABILIZER CODES

In this section we review some key facts about qudit stabilizer codes. For a more complete guide on qudit stabilizer codes, we recommend Ref. [7]. The definitions laid out here will be used throughout this paper. Let q be the local dimension of a system, where q is a prime number. We will denote by $\mathbb{Z}_q$ the set $\{0, 1, \dots q-1\}$. When $q=2$ we refer to each register as a qubit, whereas for any value of q we call each register a qudit. In order to speak more generally and not specify q, we will often times refer to each register as a particle instead. We now begin to define the operations for these registers.

Definition 1. Generalized Paulis for a particle over q orthogonal levels (local-dimension q) are given by

$$X_q|j\rangle = |(j+1) \mod q\rangle, \qquad Z_q|j\rangle = \omega^j|j\rangle, \qquad (1)$$

with $\omega = e^{2\pi i/q}$, where $j \in \mathbb{Z}_q$. These Paulis form a group, denoted $\mathbb{P}_q$.

When $q=2$, these are the standard qubit operators X and Z with $Y = iXZ$. This group structure is preserved over tensor products since each of these Paulis has order q. A generalized Pauli over n registers is a tensor product of n generalized Pauli group members over a single register.

A commuting subgroup of generalized Pauli operators with $n-k$ generators but not including any nontrivial coefficient for the identity operator is equivalent to a stabilizer code. The number of orthogonal eigenvectors, which form bases called codewords for these $n-k$ generators is q^k. In effect, we have constructed *k-logical* particles from the *n-physical* particles. If we are to use these subgroups for error-correction purposes

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then they ought to be able to have some accidental operator occur and still have the codewords be discernible. We will work under the assumption that errors on distinct particles are independent, and we will assume the error model on each qudit is the depolarizing channel. Given this error model we will predominantly be interested in the number of nonidentity terms in any error as the exponent of the error term increases with this.

Definition 2. The weight of an n-qudit Pauli operator is the number of nonidentity operators in it.

Definition 3. A stabilizer code, specified by its $n-k$ generators, is characterized by the following set of parameters:

(1) n: The number of (physical) particles that are used to protect the information.

(2) k: The number of encoded (logical) particles.

(3) d: The distance of the code, given by the lowest weight of an undetectable generalized Pauli error. An undetectable generalized Pauli error is an n-qudit Pauli operator which commutes with all elements of the stabilizer group but is not in the group itself.

These values are specified for a particular code as $[n, k, d]_q$, where q is the local dimension of the qudits.

We pause for a moment here to discuss how degenerate codes differ from nondegenerate codes. Degenerate codes are different in the following equivalent ways. First, they may have multiple errors with the same syndrome value and that map to different physical states but upon recovery still map back to the same logical state. Second, degenerate codes may have generators, aside from the identity operator, which have lower weight than the distance of the code. These two differences make degenerate codes markedly different from their nondegenerate counterpart. Degenerate codes, whereas having these extra nuances, are a crucial class of stabilizer codes as any quantum analog of a LDPC code with high distance will need to be a degenerate code. We will begin our results by focusing on nondegenerate codes, then move to the degenerate case in Theorem 15, however, there are more tools needed before discussing the results.

Working with tensors of operators can be challenging, and so we make use of the following well-known mapping from these to vectors, following the notation from Ref. [10]. This representation is often times called the symplectic representation for the operators, but we use this notation instead to allow for greater flexibility, particularly, in specifying the local dimension of the mapping. This linear algebraic representation will be used for our proofs.

Definition 4 (ϕ representation of a qudit operator). We define the linear surjective map,

$$\phi_q\colon \mathbb{P}_q^n \mapsto \mathbb{Z}_q^{2n}, \tag{2}$$

which carries an n-qudit Pauli in $\mathbb{P}_q^n$ to a $2n$ vector mod q, where we define this mapping by:

$$I^{\otimes i-1} X_q^a Z_q^b I^{\otimes n-i} \mapsto (0^{i-1}\, a\, 0^{n-i} | 0^{i-1}\, b\, 0^{n-i}), \tag{3}$$

which puts the power of the ith X operator in the ith position and the power of the ith Z operator in the $(n+i)$-th position of the output vector. This mapping is defined as a homomorphism with $\phi_q(s_1 \circ s_2) = \phi_q(s_1) \oplus \phi_q(s_2)$, where $\oplus$ is componentwise addition mod q. We denote the first half of the vector as $\phi_{q,x}$ and the second half as $\phi_{q,z}$.

We may invert the map to return to the original n-qudit Pauli operator with the global phase being undetermined. We make note of a special case of the ϕ representation:

Definition 5. Let q be the dimension of the initial system. Then we denote by ϕ_∞ the mapping,

$$\phi_\infty\colon \mathbb{P}_q^n \mapsto \mathbb{Z}^{2n}, \tag{4}$$

where no longer are any operations taken mod some base, but instead, carried over the full set of integers.

The ability to define ϕ_∞ as a homomorphism still (and with the same rule) is a portion of the results of Ref. [10]. ϕ_q is the standard choice for working over q bases, however, our ϕ_∞ allows us to avoid being dependent on the local dimension of our system when working with our code. Formally we will write a code in ϕ_q, perform some operations, then write it in ϕ_∞, then select a new local dimension q' and use $\phi_{q'}$. We shorten this to write it as ϕ_∞, and can later select to write it as $\phi_{q'}$ for some prime q' by taking elementwise mod q'. Whereas the operators in ϕ_∞ all commute, normalization of the codewords for infinitely many levels becomes a potential problem.

The commutator of two operators in this picture is given by the following definition:

Definition 6. Let s_i, s_j be two qudit Pauli operators over q bases, then these commute if and only if,

$$\phi_q(s_i) \odot \phi_q(s_j) = 0 \mod q, \tag{5}$$

where $\odot$ is the symplectic product, defined by

$$\phi_q(s_i) \odot \phi_q(s_j) = \oplus_k [\phi_{q,z}(s_j)_k \cdot \phi_{q,x}(s_i)_k - \phi_{q,x}(s_j)_k \cdot \phi_{q,z}(s_i)_k], \tag{6}$$

where $\cdot$ is the standard integer multiplication mod q and $\oplus$ is addition mod q.

When the commutator of s_i and s_j is not zero, this provides the difference in the number of X operators in s_i that must pass a Z operator in s_j and the number of Z operators in s_i that must pass an X operator in s_j when attempting to switch the order of these two operators.

Before finishing, we make a brief list of some possible operations we can perform on our ϕ representation:

(1) We may perform elementary row operations over $\mathbb{Z}_q$, corresponding to relabeling and composing generators together.

(2) We may swap registers (qudits) in the following ways:

(1) We may swap columns $(i, i+n)$ and $(j, j+n)$ for $1 \leqslant i,\ j \leqslant n$, corresponding to relabeling qudits.

(2) We may swap columns i and $(-1)\cdot(i+n)$, for $1 \leqslant i \leqslant n$, corresponding to conjugating by a Hadamard gate on particle i (or discrete Fourier transforms in the qudit case [12]), thus, swapping X and Z's roles on that qudit.

All of these operations leave the code parameters n, k, and d alone but can be used in proofs.

A. Local-dimension-invariant codes

In this section we recall the results relating to LDI stabilizer codes. These codes answer the question of when we can apply a code from one local-dimension q on a system with a different local-dimension p. Whereas an unusual property, a LDI code would permit the importing of smaller local-dimension codes for larger local-dimension systems. Some codes with particular parameters may not be known, and so this fills in some of these gaps. Additionally, this framework could potentially provide insights into local-dimension-invariant measurements. Few examples of LDI codes, although not by this name, were known, notable the five-particle code [13] and the nine-particle code [14], until the recent work in Ref. [10] which showed that all codes can satisfy the commutation requirements, and, at least, for sufficiently large local dimensions the distance can also be, at least, preserved. We will review next the primary results from that work.

Definition 7. A stabilizer code S is called local-dimension-invariant (LDI) iff,

$$\phi_\infty(s_i)\odot\phi_\infty(s_j)=0,\ \forall s_i,s_j\in S. \tag{7}$$

As an example, consider the two-qubit code generated by $\langle X\otimes X, Z\otimes Z\rangle$. The symplectic product between the two generators is 2, so it makes it a valid qubit code, however, $2 \mod p\neq 0$ unless $p=2$, so it is not a valid qudit code for $p\neq 2$. If we instead transform the code into one generated by $\langle X\otimes X^{-1}, Z\otimes Z\rangle$, then the symplectic product is now 0, and so it can be used as generators for any choice of local dimension and so is an LDI code. The next statement explains that it is always possible to do so [10]:

Theorem 8. All stabilizer codes S can be put into a LDI form. One such method is to put S into canonical form $[I_k\ X_2|Z_1\ Z_2]$ then transform the code into $[I_kX_2|Z_1+LZ_2]$ with $L_{ij}=\phi_\infty(s_i)\odot\phi_\infty(s_j)$ when $i>j$ and 0 otherwise.

Note that this does not say all codes have a *unique* LDI form just that there exists one. The proof used is useful as it gives a prescriptive method for turning a code into a LDI form, however, if one does not put the code into canonical form, the code can still be transformed into a LDI form as this process is equivalent to finding solutions to an integer linear program with an abundance of variables. As the code is put into canonical form in this prescriptive method, we know that the rank of the matrix will be preserved by this operation. All LDI forms ought to also preserve the rank or, equivalently, the number of independent generators.

As of this point we have merely generated a set of commuting operators that are local-dimension independent. This does not provide for any claims on the distance of the code produced through this method aside from promising that the procedure does not change the distance of the code over the initial local-dimension q. For this, we have the following theorem:

Theorem 9. For all primes $p>p^*$ with p^* a cutoff value greater than q, the distance of a LDI form of a nondegenerate stabilizer code $[n,k,d]_q$ applied over p bases $[n,k,d']_p$ has $d'\geqslant d$.

There are two caveats to this result, one of which we resolve here, the other of which we provide an improvement on. Let B be the maximal entry in $\phi_\infty(S)$. First, this result is only for the case of nondegenerate codes. We will resolve this with Theorem 15. Second, the initially proven bound was $p^*=B^{2(d-1)}[2(d-1)]^{(d-1)}$, which grows very rapidly. Whereas it was true that all primes below p^* could have their distances checked computationally, this still left a large number of primes to check in most cases. In this paper we manage to prove an alternative bound that has a nearly quadratic improvement on the dependency on B. In the next section we show this alternative cutoff bound, whereas in the section thereafter the ability to provide a distance promise for degenerate codes is proven and differences between the cases are discussed.

III. ALTERNATIVE CUTOFF BOUND FOR THE DISTANCE PROMISE

Whereas the proof of Theorem 8 from Ref. [10] used $L_{ij}=\phi_\infty(s_i)\odot\phi_\infty(s_j)$ in order to generate a single LDI form, we may generate other LDI forms by altering the added L matrix. We note two of these now: $L^{(+)}$ and $L^{(-)}$.

Definition 10. $L^{(+)}$ ($L^{(-)}$) has $L_{ij}^{(+)}$ ($L_{ij}^{(-)}$) is $\phi_\infty(s_i)\odot\phi_\infty(s_j)$ if the symplectic product is greater than zero (less than zero).

These alternative L matrices each provide a different property. First, using $L^{(+)}$ allows $\phi_\infty(S)$ to have only nonnegative entries. There are certain properties that are only generally true for matrices with non-negative entries, so this can perhaps be of use. Additionally, this could be of use for systems formally with countably infinite local dimension, such as bosonic systems where operators with negative powers are not feasible. Second, $L^{(-)}$ permits a slight reduction in the bound for the maximal entry in $\phi_\infty(S)$ as the following lemma shows:

Lemma 11. The maximal entry in $\phi_\infty(S)$ B can be at most $[1+k(q-1)](q-1)$, and generally $B\leqslant\max_{i,j}|\phi_\infty(s_i)\odot\phi_\infty(s_j)|$.

Upon putting the code into canonical form this follows immediately from the definition of $L^{(-)}$ as each entry will be whatever value was already in that location (values in $\mathbb{Z}_q$) minus the absolute value of the inner product, which will be, at most, an absolute value of the inner product. Whereas this is a small improvement on the value of B since it is the base of an exponential expression this amounts to a larger improvement in the overall cutoff value.

We will now move to proving an alternative bound on the local dimension needed in order to promise the distance is, at least, preserved. The first proof of the cutoff bound for the distance promise for LDI codes used random permutations of the entries in ϕ_∞. Here we utilize the structure of the symplectic product as well as that of the partitions of the code in terms of its X component and Z component to obtain an alternative bound for all nondegenerate codes. Whereas this bound is looser when d increases, for small d and large k this bound will typically be roughly quadraticly smaller. In particular we will show:

Theorem 12. For all primes $p>p^*$ the distance of a LDI representation of a nondegenerate stabilizer code $[n,k,d]_q$ over p bases $[n,k,d']_p$ has $d'\geqslant d$ where we may use as p^* the value,

$$\{B(q-1)(d-1)[1+(d-1)^2(q-1)^{d-1}(d-2)^{(d-2)/2}]\}^{d-1}, \tag{8}$$

with q as the initial local dimension, d as the distance of the initial code, and B as the maximal entry in the ϕ_∞ representation of the code.

To make claims about the distance of the code we begin by breaking down the set of undetectable errors into two sets. These definitions highlight the subtle possibility of the distance reducing upon changing the local dimension.

Definition 13. An unavoidable error is an error that commutes with all stabilizers and produces the $\vec{0}$ syndrome over the integers.

These correspond to undetectable errors that would remain undetectable regardless of the number of bases for the code since they always exactly commute under the symplectic inner product with all stabilizer generators—-and so all members of the stabilizer group. Since these errors are always undetectable we call them unavoidable errors as changing the number of bases would not allow this code to detect this error.

We also define the other possible kind of undetectable error for a given number of bases, which corresponds to the case where some syndromes are multiples of the number of bases:

Definition 14. An artifact error is an error that commutes with all stabilizers but produces, at least, one syndrome that is only zero modulo the base.

These are named artifact errors as their undetectability is an artifact of the number of bases selected and could become detectable if a different number of bases were used with this code. Each undetectable error is either an unavoidable error or an artifact error. We utilize this fact to show our theorem.

Proof. Let us begin with a code with local-dimension q and apply it to a system with local-dimension p. The errors for the original code are the vectors in the kernel of ϕ_q for the code. These errors are either unavoidable errors or are artifact errors. The stabilizers that generate these multiples of q entries in the syndrome are members of the null space of the minor formed using the corresponding stabilizers.

Now, consider the extension of the code to p bases. Building up the qudit Pauli operators by weight j, we consider the minors of the matrix. These minors of size $2j \times 2j$ can have a nontrivial null space in two possible ways:

(1) If the determinant is 0 over the integers then this is either an unavoidable error or an error whose existence did not occur due to the choice of the number of bases.

(2) If the determinant is not 0 over the integers but takes the value of some multiple of p, then it is $0 \mod p$ and so a null space exists.

Thus, we can only introduce artifact errors to decrease the distance. By bounding the determinant by p^*, any choice of $p > p^*$ will ensure that the determinant is a unit in $\mathbb{Z}_p$, and, hence, have a trivial null space since the matrix is invertible.

We next utilize the structure of the symplectic product more heavily in order to reduce the cutoff local dimension. Note that for a pair of Paulis in the ϕ representation, we may write

$$\phi(s_1) \odot \phi(s_2) = \phi(s_1)\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}\phi(s_2)^T \tag{9}$$

$$:= \phi(s_1) g \phi(s_2)^T, \tag{10}$$

and so we may consider the commutation for the generators with some Pauli u as being given by $\bigoplus_{i=1}^{n-k}[\phi(s_i)g]\phi(u)^T$, where $\bigoplus$ is a direct sum symbol here, indicating that a vector of syndrome values is returned. This removes the distinction between the two components and allows the symplectic product to act, such as the normal matrix-vector product. Now, note that for any Pauli weight j operator, we will have up to j nonzero entries in the X component of the ϕ representation and up to j nonzero entries in the Z component. This means that up to j columns in each component will be involved in any commutator.

Next, note that to ensure that an artifact error is not induced it suffices to ensure that there is a nontrivial kernel, induced by the local-dimension choice, which is ensured so long as any $2(d-1)2(d-1)$ minor does not have a determinant which is congruent to the local dimension. This can be promised by requiring the local dimension to be larger than the largest possible determinant for such a matrix. Since there will be, at most, j nonzero entries in each component it suffices to consider j columns from each component and subsets of $2j$ rows of this.

From this reduction, we need only ensure that the local dimension is larger than the largest possible determinant for this $2j2j$ minor. Let us denote this minor by

$$\begin{bmatrix} X_1 & Z_1 \\ X_2 & Z_2 \end{bmatrix}, \tag{11}$$

where each block has dimensions jj. The maximal entries are $q-1$ for X_1 and X_2, whereas for Z_1 and Z_2 it is bounded by B. We now use the block matrix identity,

$$\det\begin{bmatrix} X_1 & Z_1 \\ X_2 & Z_2 \end{bmatrix} = \det(X_1)\det\big(Z_2 - X_2X_1^{-1}Z_1\big). \tag{12}$$

Since all entries in X_1 are integers and the determinant is, by construction, nonzero, the maximal entry in X_1^{-1} will be, at most, that of the largest cofactor of X_1. The largest cofactor $\tilde{C}$ will be, at most, $(q-1)^{d-2}(d-2)^{(d-2)/2}$ as provided by Hadamard's inequality. The largest entry in $Z_2 - X_2X_1^{-1}Z_1$ is then upper bounded by $B[1+(q-1)\tilde{C}(d-1)^2]$. From here, we may apply Hadamard's inequality for determinants again using the given entry bounds using that each block has dimensions up to $(d-1)(d-1)$, which provides $p^* = (q-1)^{d-1}(d-1)^{d-1}\{B[1+(q-1)\tilde{C}(d-1)^2]\}^{d-1}$ or, alternatively, expressed in terms of our fundamental variables as

$$\{B(q-1)(d-1)[1+(d-1)^2(q-1)^{d-1}(d-2)^{(d-2)/2}]\}^{d-1}. \tag{13}$$

In the case of $q=2$ this reduces to $[B(d-1)(1+(d-1)^2(d-2)^{(d-2)/2})]^{d-1}$.

Lastly, when $j = d$, we can either encounter an unavoidable error in which case the distance of the code is d or we could obtain an artifact error also causing the distance to be d. It is possible that neither of these occur at $j = d$ in which case the distance becomes some d' with $d < d' \leqslant d^*$ with d^* being the distance of the code over the integers. ■

Before concluding this section, we provide a brief comparison of this bound to the original one of $B^{2(d-1)}[2(d-1)]^{(d-1)}$. The new bound only depends on B^{d-1} opposed to the original $B^{2(d-1)}$, which as the bound on B depends on k means that for codes or code families with larger k values the new bound can provide a tighter expression. Unfortunately, however, this alternative bound is doubly exponential in the distance of the code d, having a dependency of roughly d^{d^2} opposed to the prior dependency of d^d, so if one is attempting to promise the distance of a code with a larger distance, this new bound is likely to be far less tight. To summarize, this alternative bound is not *per se* better, however, since one may simply use whichever of the bounds is tighter this alternative bound may provide a lower requirement for the local dimension needed in order to ensure that the distance of the code is, at least, preserved.

IV. DEGENERATE CODES

Degenerate codes are a uniquely quantum phenomenon, which suggests that they are a crucial class of quantum error-correcting codes in order to obtain certain properties. For a degenerate quantum error-correcting code we must avoid undetectable errors, but also detectable errors which produce the same syndrome but do not map to the same physical codeword. Any LDPC-like quantum error-correcting code will be degenerate as, equivalently, a quantum error-correcting code is degenerate if there is some stabilizer group member with lower weight than the distance of the code and by construction one would aim to have a long distance for a quantum LDPC code but still $O(1)$ weight for each generator. We show now that a similar distance promise may be made in the degenerate case as was possible in the nondegenerate case and remark on what differences exist between the two classes in the local-dimension-invariant framework.

Theorem 15. For all primes $p > p^*$ the distance of an LDI representation of a degenerate stabilizer code $[n, k, d]_q$ over p bases, $[n, k, d']_p$ has $d' \geqslant d$, where p^* is the same function of n, k, d, and q as before.

Proof. In the case of nondegenerate codes all undetectable errors up to distance d, were in the normalizer of the generators $\mathcal{N}(S)$ as the weight of all members of the stabilizer group have weight, at least, d. For degenerate codes we only need to be concerned about elements in $\mathcal{N}(S)/S$ as now there are some members of the stabilizer group which might have weight below d. The latter set is a subset of the former $[\mathcal{N}(S)/S \subset \mathcal{N}(S)]$, and so the same distance promise is obtained as before. ∎

Note that all Paulis with weight less than d that are in S produce a syndrome that is all zeros over the integers and so may appear to be within the category of unavoidable errors when syndromes are computed. This means that when checking the distance this must carefully be taken into account, otherwise, the members in S may be mistaken for these errors leading to an erroneous distance value.

This means that just like nondegenerate quantum codes, we may also promise the distance of the code in the degenerate case and with the same cutoff bound. Whereas this cutoff value is large, it provides some local-dimension value beyond which the distance will be kept and bounds the set of local-dimension values for which the distance must be manually verified.

This provides information about when the distance of the code must be preserved, however, if we apply a code over q levels to a system with $p < q$ levels, is there some range of values for p whereby we know that the distance must decrease? In the nondegenerate case, we denoted this by p^{**}, which was given by

$$\sqrt{1+\binom{n}{t}^{1/[(n-k)-t]}}, \qquad t=\left\lfloor \frac{d-1}{2}\right\rfloor. \tag{14}$$

Whenever $p < p^{**}$, it must be the case that the distance of the code must decrease. The expression for p^{**} was derived by using the generalized quantum Hamming bound, which holds for all nondegenerate codes, however, for degenerate codes this bound does not always hold. This means that for a general degenerate code we have the following lemma:

Lemma 16. There is no corresponding p^{**} that holds for arbitrary degenerate codes.

Whereas not all degenerate quantum codes obey the generalized quantum Hamming bound, there are certain code families which do [7,15]. For those code families the exact same expression for p^{**} holds as it did for nondegenerate codes.

The nonexistence of a p^{**} expression for arbitrary degenerate codes provides an opportunity. Consider a code whose initial local-dimension q is far larger than 2. In the nondegenerate case this p^{**} provides a local-dimension value below which the distance of the code must decrease, but for degenerate codes the lack of this means that it may be possible to apply the code over a far smaller local dimension, even local-dimension 2, and still preserve all of the parameters and, particularly, the distance. This suggests that it may be possible to import codes into lower local-dimension values than previously expected.

To ground some of the discussions, we provide some examples next.

Example 17. Consider the qubit code with generators $\langle s_1, s_2\rangle = \langle X^{\otimes n}, Z^{\otimes n}\rangle$ with $n \geqslant 4$ being an even number. This code has parameters $[n, n-2, 2]_2$. $|\phi_\infty(s_1) \odot \phi_\infty(s_2)| = n$, so directly applying Lemma 11, $B = n - 1$ is obtained. This provides a bound of $2(n-1)^2$ via the prior bound, whereas with the alternative bound shown here this is $(n - 1)$. The results then say that the distance is preserved for $p > n - 1$.

Let us take the LDI form for the code as $\langle X^{1-n}X^{\otimes(n-1)}, Z^{\otimes n}\rangle$. Observe that all weight one Paulis do not commute with, at least, one generator for the code, whereas $IZZ^{-1}I^{\otimes(n-3)}$ is an unavoidable error, so the distance is always $d = 2$.

Whereas this suggests that the determinant bound we showed is incredibly loose, we can write the qubit code in a different LDI form as $\langle (XX^{-1})^{\otimes(n/2)}, Z^{\otimes n}\rangle$. For this form $B = 1$, which provides $p^* = 2$ using either bound, which means that the distance is always, at least, preserved. This illustrates the impact of careful selection of the LDI form used and suggests that perhaps with a careful choice of LDI form the bounds provided can be tight for a given code.

Example 18. As another example let us consider the Shor code with parameters $[9, 1, 3]_2$, and consider a local-dimension-invariant form for it. The Shor code is a degenerate code as the inner blocks of the code have some repeated syndromes. The code has a maximal symplectic product of 2, meaning that there is an LDI form which has $B = 1$. One such option is the following set of generators:

$$\langle XX^{-1}II^{\otimes 6},\ IXX^{-1}I^{\otimes 6},\ I^{\otimes 3}XX^{-1}II^{\otimes 3}, I^{\otimes 3}IXX^{-1}I^{\otimes 3},\\ I^{\otimes 6}XX^{-1}I,\ I^{\otimes 6}IXX^{-1}, Z^{\otimes 6}I^{\otimes 3},\ I^{\otimes 3}Z^{\otimes 6}\rangle. \tag{15}$$

Using $B = 1$, the bound from Ref. [10] is tighter, which provides $p^* = 16$, meaning that so long as the local dimension is 17 or larger the distance will be, at least, 3. From here, manual checking for local-dimensions 3, 5, 7, 11, and 13 verifies that the distance is always preserved. There already was a nine-register code [14], however, this contextualizes the result within the local-dimension-invariant framework. For completeness, a set of logical operators for this code is given by $\bar{X} = XX^{-1}XX^{-1}XX^{-1}XX^{-1}X$ and $\bar{Z} = Z^{\otimes 9}$. For the logical operators we only require that they do not commute with each other but do commute with the generators for the code—here $\bar{X} \odot \bar{Z} \neq 0$.

Lastly, generally for the logical operators of the local-dimension-invariant representation of the degenerate code the same argument holds as was given in Ref. [10]. With all of these pieces we have an equally complete description of degenerate LDI codes, and their slight differences as existed for nondegenerate LDI codes.

V. DISCUSSION

The LDI representation of stabilizer codes allows these codes to be applied regardless of the local dimension of the underlying system. When introduced only nondegenerate codes could be written in local-dimension-invariant form and have their distance promised to be, at least, preserved, once the system had sufficiently many levels. In this paper we have shown an alternative bound for how many levels are needed for the distance to be promised. Whereas this bound suffers a severe dependency on the distance of the code, it does provide a nearly quadratic improvement on the dependency of the largest entry in the LDI form of the code, given by B. So whereas this bound is less helpful in some cases than the original bound it can be a tighter bound in others. Of particular note is the situation where one does not need to guarantee the same distance as the original code, but just some smaller distance δ or larger. In this case the value for B does not change, however, everywhere that a d appears in the expressions for p^* may be replaced by δ. In these cases the quadratic improvement on the dependency on B shown here can become particularly advantageous.

Beyond this, this paper has shown that the LDI representation's associated distance promise also exists for degenerate quantum codes using the same argument as before but overlooked, and so completes the application of this technique to both families of standard stabilizer codes. Degenerate codes are of particular appeal since they are not restricted by the generalized quantum Hamming bound and can at times protect more logical particles than permitted by nondegenerate codes for a given distance and number of physical particles.

Unfortunately, the utility of this method is somewhat limited as both bounds on the required local dimension are quite large as indicated in Table I, but as seen in the examples this bound can often be significantly reduced through careful construction of the LDI form. In order to improve the practicality of this technique the value for p^* must be significantly decreased. One way to reduce these bounds is to reduce the expression for B, the maximal entry in the LDI representation. To do so, other analysis techniques will be needed beyond simple counting arguments. Since the LDI form for a code is not unique, one possible method may be to solve systems of homogeneous linear diophantine equations, which given the surplus of variables (additions to entries) compared to variables (requirement of commutations to be zero) is likely to yield far smaller bounds on B. A starting point for this might include the following works: Refs. [17,18].

TABLE I. This table compares the bounds on p^*, above which the distance of the code is known to be preserved for a few example codes. The bound on B is used for the value of B. Examples taken from Ref. [16] for the qubit codes and Ref. [7] for qudit cases.

Code parameters	Bound from Ref. [10]	Bound shown here
$[9, 1, 3]_2$	256	400
$[13, 7, 3]_2$	65536	6400
$[21, 13, 3]_2$	614656	19600
$[29, 19, 4]_2$	13824000000	1481544000
$[13, 7, 3]_3$	12960000	4161600
$[27, 22, 3]_3$	1049760000	37454400
$[91, 85, 3]_3$	218889236736	540841536
$[25, 22, 3]_5$	213813760000	31258240000

The results shown here extend the utility of local-dimension-invariant stabilizer codes, and so naturally there are questions as to what other uses this technique will have. Is it possible to apply this technique to show some foundational aspect of quantum measurements? Can this technique in some way be used for other varieties of stabilizerlike codes, such as entanglement-assisted quantum error-correcting codes [19,20]? If this method can be applied in this situation it is possible that it could remove the need for entanglement use in these codes, so long as the local dimension is altered. However, even still, the local dimension required would likely be quite large so the importance of decreasing the bounds for p^* would become that much more.

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