Comment on "Possibility of small electron states"

Iwo Bialynicki-Birula*
Center for Theoretical Physics, Polish Academy of Sciences, Aleja Lotników 32/46, 02-668 Warsaw, Poland

Zofia Bialynicka-Birula
Institute of Physics, Polish Academy of Sciences, Aleja Lotników 32/46, 02-668 Warsaw, Poland

(Received 19 December 2020; accepted 18 February 2022; published 2 March 2022)

It is shown that the claim that the velocity of the electron never exceeds the speed of light is invalid. The velocity of the energy flow, as defined by the author, becomes even infinite at some points. We also show that the proof of the nonexistence of the lower limit on the size of the electron wave function can be obtained from simple dimensional arguments.

DOI: 10.1103/PhysRevA.105.036201

In a recent paper [1] the author studied the problem of the size of the relativistic electron wave functions for the Dirac equation. We show by an explicit calculation that the notions of the energy density and of the velocity of the electron wave packet used by the author raise serious concerns.

In the first part of this Comment we study in some detail the properties of the Gaussian solutions of the Dirac equation evolving from the initial condition studied in Ref. [1]. The author refers to the energy distribution given by Eq. (18) as the *energy density* but this term, in our opinion, is inappropriate. The so-called density of the energy $\rho^{\mathcal{E}}$ discussed in Ref. [1] *does not satisfy* the requirement of positivity. Therefore, one should avoid the use of the term density or, at least, replace it with the term *quasidensity*. In order to underscore this point, we calculate explicitly the quasidensity of the energy for the Gaussian solution of the Dirac equation. The calculation of the Dirac bispinor wave function is simplified if one uses the prescription described in Ref. [2] which connects the solutions of the Dirac equation for the bispinor with the derivatives of the solutions of the Klein-Gordon equation. This will enable us to perform the integration over the angles and obtain the solutions in terms of spherical waves instead of plane waves.

The solution of the Klein-Gordon with positive frequency for the Gaussian wave packet is given by the following integral ($\hbar = 1, c = 1$),

$$\phi(x, y, z, t) = \int_0^\infty dp\, j_0(pr)e^{-iE_pt}e^{-l^2p^2}, \tag{1}$$

where $r = \sqrt{x^2 + y^2 + z^2}$, $E_p = \sqrt{m^2 + p^2}$, and $j_0(pr)$ is the spherical Bessel function. The corresponding solution of the Dirac equation with spin along the z axis is obtained by the differentiation with respect to space-time coordinates,

$$\psi(x, y, z, t) = \begin{bmatrix} (i\partial_t + m)\phi \\ 0 \\ -i\partial_z\phi \\ -i(\partial_x + i\partial_y)\phi \end{bmatrix}. \tag{2}$$

We use in this formula the standard Dirac representation of γ matrices. The derivatives of ϕ with respect to the space-time variables can be evaluated under the integral sign and we arrive at the following representation of the Dirac bispinor ψ in the form of a one-dimensional integral,

$$\begin{aligned}\psi(x, y, z, t) &= \int_0^\infty dp\, u(x, y, z)e^{-iE_pt}e^{-l^2p^2} \\ &= \int_0^\infty dp \begin{bmatrix} (E_p + m)j_0(pr) \\ 0 \\ ipz\, j_1(pr)/r \\ ip(x + iy)j_1(pr)/r \end{bmatrix} e^{-iE_pt}e^{-l^2p^2}.\end{aligned} \tag{3}$$

The bispinor $\psi(x, y, z, t)$ is a solution of the Dirac equation because the bispinor $u(x, y, z)$ obeys the equation

$$[-i(\alpha_x\partial_x + \alpha_y\partial_y + \alpha_z\partial_z) + m\beta - E_p]u(x, y, z) = 0. \tag{4}$$

The formula for $\rho^{\mathcal{E}}$ given in Ref. [1] by Eq. (18),

$$\rho^{\mathcal{E}} = \frac{i}{2}(\psi^\dagger\partial_t\psi - \partial_t\psi^\dagger\psi), \tag{5}$$

coincides with the expression for the time component of the energy-momentum tensor for the Dirac wave function [3]. It contains ψ and also the time derivative of ψ. The expression for the time derivative is obtained by placing the factor E_p inside the integral in (3).

In our calculation the expression for $\rho^{\mathcal{E}}$ contains the products of two integrals over p. These integrals cannot be evaluated analytically but the numerical integration of

*birula@cft.edu.pl

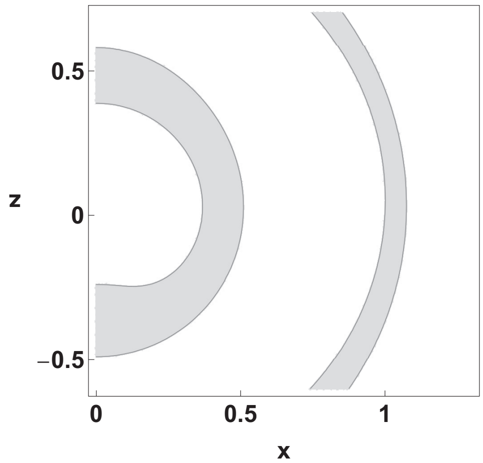

FIG. 1. The quasienergy distribution for the Gaussian wave packet with the following set of parameters: $l = 0.2$, $t = 2.5$. The variables l, x, z, and t are measured in natural units $\hbar/mc$ and $\hbar/mc^2$, respectively.

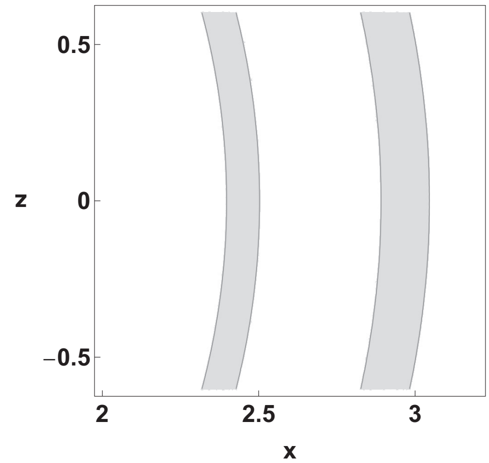

FIG. 2. The quasienergy distribution for the Gaussian wave packet with the additional prefactor $(p^2 - q^2m^2)$ in the integrand. The parameters are $l = 0.1$, $t = 2.5$, $q = 7$.

one-dimensional integrals does not present any problems. The resulting expression for $\rho^{\mathcal{E}}$ has the cylindrical symmetry around the z axis. Therefore, to visualize this function it is sufficient to show the cut by the y plane. The result (for $t = 2.5$) is presented in Fig. 1. In this figure we marked the contour lines corresponding to $\rho^{\mathcal{E}}(\boldsymbol{r}, t) = 0$ in the half of the $y = 0$ plane. The gray stripe between the zero contour lines corresponds to *negative values* of $\rho^{\mathcal{E}}$. In the full three-dimensional plot the gray stripe becomes a thickened ellipsoidal shell obtained by rotation. Of course, the total energy, i.e., the integral over all space, is positive, because it is the quantum-mechanical expectation value of the energy operator and the solutions of the Dirac equation studied here are built solely from positive energy waves. This fact also explains why the regions with negative values are so scarce.

The vanishing of $\rho^{\mathcal{E}}$ contradicts the claim by the author that the speed of light is not exceeded. Since $\rho^{\mathcal{E}}$ and $\vec{G}$ are defined by Eqs. (18) and (19), Eq. (17) is *de facto* the definition of the velocity $\vec{v}$,

$$\vec{v} = \frac{\vec{G}}{\rho^{\mathcal{E}}}. \tag{6}$$

We checked that $\vec{G}$ does not vanish on the surfaces where $\rho^{\mathcal{E}} = 0$. Thus, it follows from (6) that the velocity dramatically exceeds the speed of light; it is infinite.

The appearance of regions with negative values of $\rho^{\mathcal{E}}$ is not a property of the Gaussian solution but it is a common feature. For example, by placing an additional factor $(p^2 - q^2m^2)$ under the integral (3) we obtain not one but two regions of negative values shown in Fig. 2.

The presence of the regions where $\rho^{\mathcal{E}}$ becomes negative can be proven without using a specific model. Let us consider first a monochromatic solution of the Dirac equation with the energy E. In this case we have a positive function $\rho^{\mathcal{E}} = E|\psi|^2$. The situation changes when we have a sum of two monochromatic solutions $\psi_1 + \psi_2$ with the energies E_1 and E_2. The expression for $\rho^{\mathcal{E}}$ now is

$$\rho^{\mathcal{E}} = \tfrac{1}{2}(E_1|\psi_1|^2 + E_1\psi_2^\dagger\psi_1 + E_2\psi_1^\dagger\psi_2 + E_2|\psi_2|^2 + \text{c.c.}). \tag{7}$$

Let us rescale the first bispinor as follows: $\psi_1 = \chi m/E_1$, which gives

$$\rho^{\mathcal{E}} = \frac{m|\chi|^2}{E_1} + E_2|\psi_2|^2 + \mathrm{Re}\left(m^2\psi_2^\dagger\chi + \frac{mE_2\chi^\dagger\psi_2}{E_1}\right). \tag{8}$$

In the limit, when $E_1 \to \infty$ or $m/E_1 \to 0$, we obtain

$$\rho^{\mathcal{E}} \to (m\psi_2^\dagger\chi + E_2|\psi_2|^2 + \text{c.c.})/2. \tag{9}$$

Since the strength and the sign of the bispinor χ is at our disposal, we may easily make this expression negative in a selected region.

The author invokes also the restriction on the size of the electron wave packet imposed by the value $e\hbar/2mc$ of the magnetic moment. The author's argument is based on an invalid assumption that the magnetic moment of the electron [Eq. (6)] is the same as its part due to the orbital motion [Eq. (7)]. The total magnetic moment has two parts: the orbital part and the "intrinsic" part [cf., for example, Eqs. (14)–(16) in Ref. [4]]. Indeed, the orbital part shrinks to zero with the decreasing size of the electron, but the intrinsic part does not depend on the size of the electron wave packet. It always stays finite.

The positive values of the charge density and the negative values of the quasidensity of the energy in some regions characterize spin-1/2 particles. For charged spin-0 and spin-1 particles the situation is reversed. The energy density is always positive but the charge distribution takes on positive and negative values [5,6]. For the electromagnetic field in free space the counterpart of $\vec{v}_q$ is $\vec{v}_{\text{em}}$, the ratio of the Poynting vector to the energy density,

$$\vec{v}_{\text{em}} = \frac{\vec{E}\times\vec{H}}{\frac{1}{2}(\epsilon\vec{E}^2 + \mu\vec{H}^2)}. \tag{10}$$

It is clearly seen that, as $\vec{v}_q$, the velocity of the electromagnetic wave $\vec{v}_{\text{em}}$ cannot exceed the speed of light.

Finally, we would like to point out that the analysis of the "small electron states" with the use of a specific example, such as the Gaussian wave packets that were considered in Ref. [1], can be carried out without reference to any particular model. The conclusion can be reached using a simple general argument based entirely on dimensional analysis.

The general proof that there is no finite lower limit on the mean-square radius $\langle \boldsymbol{r}^2 \rangle$ starts from the observation that every normalizable solution of the Dirac equation must have some scale parameter l that controls the size of the wave packet in space. This parameter in the Gaussian wave packet studied in Re. [1] is equal to $\lambdabar_C/n$, where $\lambdabar_C = \hbar/mc$ is the Compton wavelength. Therefore, the mean-square radius can be expressed in the form

$$\langle \boldsymbol{r}^2 \rangle = l^2 g\left(\frac{\lambdabar_C^2}{l^2}\right), \tag{11}$$

where g is a function of the dimensionless parameter. The function g depends on the chosen solution of the Dirac equation. We cannot say yet that $\langle \boldsymbol{r}^2 \rangle$ goes to zero when l goes to zero because, in principle, the function g may behave as $\lambdabar_C^2/l^2$ when $l \to 0$. This possibility, however, is excluded by the observation that $l \to 0$ in the argument of g is equivalent to $\lambdabar_C \to \infty$, i.e., the particle becomes massless. For a massless particle there is no intrinsic scale parameter and the mean-square radius for every solution must be simply proportional to l^2, i.e., $\langle \boldsymbol{r}^2 \rangle = \text{const}\, l^2$. This argument is quite general. It is valid for all solutions of the free Dirac equation.

[1] C. T. Sebens, Possibility of small electron states, Phys. Rev. A **102**, 052225 (2020).

[2] I. Bialynicki-Birula and Z. Bialynicka-Birula, Relativistic Electron Wave Packets Carrying Angular Momentum, Phys. Rev. Lett. **118**, 114801 (2017).

[3] W. Pauli, *General Principles of Quantum Mechanics* (Springer, Berlin, 1980), p. 166.

[4] K. Gottfried, *Quantum Mechanics* (Benjamin, New York, 1966), p. 313.

[5] I. Bialynicki-Birula and Z. Bialynicka-Birula, Heisenberg uncertainty relations for relativistic electrons, New J. Phys. **21**, 073036 (2019).

[6] I. Bialynicki-Birula and A. Prystupiuk, Heisenberg uncertainty relations for relativistic bosons, Phys. Rev. A **103**, 052211 (2021).