Master equation for the quantum Rabi model in the adiabatic regime

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A qubit–harmonic-oscillator open system coupled to thermal baths was considered and the master equation describing the evolution of the open system was deduced when the qubit transition frequency is $\lesssim 0.1$ the oscillator frequency. The master equation is valid in all the qubit-oscillator state space and holds for all values of the qubit-oscillator coupling including the ultrastrong and deep strong coupling regimes. It only requires an oscillator frequency much larger than the relaxation rates. The qubit-oscillator coupling can enhance or decrease both the relaxation rates and the frequency shifts induced by the thermal baths. It was found that weak, sinusoidal qubit driving forces the qubit-oscillator open system to behave like a driven qubit whose evolution is governed by equations similar to those of the Bloch vector in the optical Bloch equations and whose transition frequency decreases with increasing qubit-oscillator coupling strength. Finally, it was shown how one can reach the adiabatic regime by using sinusoidal qubit driving with large driving frequency and the concepts of π and $\pi/2$ pulses were generalized to manipulate transitions between dressed states.

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I. INTRODUCTION

The qubit–harmonic-oscillator open system described by the quantum Rabi model (QRM) subject to dissipation is a fundamental model in open quantum systems used to describe experiments in many areas such as circuit quantum electrodynamics (QED) where the ultrastrong coupling (USC) and deep strong coupling (DSC) regimes can be reached [1–8].

The way in which dissipation is introduced is fundamental to obtain an adequate description of physical phenomena. When the qubit-oscillator coupling strength $|g|$ is small with respect to the oscillator frequency ω_r ($|g| \ll \omega_r$), the qubit-oscillator interaction is quasiresonant ($|\omega_r - \omega_q| < |g|$ with ω_q the qubit transition frequency), and the qubit-oscillator open system is weakly coupled to thermal baths, such as in cavity QED [9,10], a phenomenological master equation [11,12] governing the evolution of the qubit-oscillator system is usually deduced by applying the sometimes called *approximation of independent rates of variation* [13]. In this approximation one first calculates the master equation of the qubit-oscillator system neglecting the interaction between them and then one simply adds the qubit-oscillator interaction which can be simplified using the rotating-wave approximation (RWA). This master equation can lead to false results when one considers the dispersive and RWA regime ($|g| \ll |\omega_r - \omega_q| \ll \omega_r + \omega_q$) where it can predict a large amount of qubit flipping induced by dephasing noise [12] and there can be differences when one considers a thermal bath density of states that does not correspond to white noise [12,14].

When one attempts to apply the approximation of independent rates of variation to situations in the USC and DSC regimes but the open system is still weakly coupled to thermal baths, then one obtains nonphysical results [12]. For example, the phenomenological master equation incorrectly describes Purcell decay (the relaxation of the the qubit by emission of photons outside the cavity) [12], dressed dephasing (how dephasing can produce relaxation) [15,16], and the vacuum Rabi splitting spectrum [12]. In addition, at zero temperature it incorrectly tends to bring the qubit-oscillator system to the state formed by the tensor product of the individual ground states of the qubit and the oscillator. The latter is not the ground state of the qubit-oscillator system in the USC and DSC regimes. In order to obtain a master equation that correctly describes physical phenomena one must use the *dressed states* (energy eigenstates) of the qubit-oscillator system [12,17]. This eliminates the aforementioned nonphysical results and leads to a different interpretation of the physics. In the phenomenological model one of the subsystems (qubit or harmonic oscillator) decays because it is directly coupled to a reservoir and because it is coupled to the other subsystem which is the one that decays due to its interaction with a reservoir. Using the dressed states, it is the whole qubit-oscillator system that decays even though only one of the subsystems is directly coupled to a reservoir. Moreover, the situation also changes when the qubit-oscillator system is strongly coupled to an environment which can be non-Markovian [18,19].

A particularly interesting case occurs when the qubit has quasidegenerate levels ($\omega_q \ll \omega_r$) because the adiabatic approximation [20] gives a very accurate approximation of the dressed states. In this case the deduction of a master equation describing the dynamics of the qubit-oscillator open system is challenging because the frequency difference between some dressed states is very small and this makes the application of the secular approximation difficult. In particular, [21] managed to deduce a Born-Markov secular master equation by restricting both the qubit-oscillator coupling to values $|g/\omega_r| \leqslant 0.2$ and the qubit-oscillator dynamics to a

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subspace which has small dimension when $|g/\omega_r| \sim 0.2$. Also, the master equation presented in [21] cannot consider very small values of the qubit frequency ω_q and does not reduce to the correct master equation when $\omega_q = 0$. In this article we deduce a simple master equation for the qubit-oscillator open system that is valid for arbitrary values of the qubit-oscillator coupling, the whole qubit-oscillator state space, and all values $\omega_q \lesssim 0.1\omega_r$ and that reduces to the correct master equation when $\omega_q = 0$. Moreover, we also determine the effects of sinusoidal qubit driving and show how one can reach the adiabatic and dispersive regimes by using the driving. Finally, it is also shown how one can manipulate transitions between dressed states by generalizing π and $\pi/2$ pulses. The adiabatic regime is also interesting because Schrödinger cat states can be generated in the oscillator when $\omega_q = 0$ [22] and these are useful for quantum error correction [23]. In addition, oscillator driving can be used to probe the energy of the qubit-oscillator system in the adiabatic regime [21].

The article is organized as follows. Section II introduces the system under consideration, while Sec. III presents the adiabatic regime and a method to reach it. Section IV establishes the master equation describing the evolution of the system, compares it with the one presented in [21], and considers the effects of qubit driving. Finally, the conclusions are given in Sec. V.

II. THE MODEL

The complete system consists of a qubit (q) with transition angular frequency $\omega_q > 0$, a harmonic oscillator (r) with angular frequency $\omega_r > 0$, and two independent thermal baths of harmonic oscillators B_1 and B_2. The thermal baths are responsible for introducing dissipation to the qubit-oscillator open system. The Hamiltonian of the complete system is

$$\begin{aligned}\hat{H}_T &= \hat{H} + \hat{H}_{B_1} + \hat{H}_{B_2} \\ &\quad - \hbar\hat{\sigma}_x \sum_k (g_{1k}\hat{a}_{1k}^\dagger + g_{1k}^*\hat{a}_{1k}) \\ &\quad - \hbar(\hat{a}^\dagger + \hat{a}) \sum_k (\kappa_k \hat{a}_{2k}^\dagger + \kappa_k^* \hat{a}_{2k}), \end{aligned} \quad (1)$$

where the qubit-oscillator free Hamiltonian is

$$\hat{H} = \frac{\hbar\omega_q}{2}\hat{\sigma}_z + \hbar\omega_r \hat{a}^\dagger \hat{a} - \hbar\hat{\sigma}_x(g\hat{a}^\dagger + g^*\hat{a}), \quad (2)$$

and the free Hamiltonian of B_j is given by

$$\hat{H}_{B_j} = \sum_k \hbar\omega_{jk}\hat{a}_{jk}^\dagger \hat{a}_{jk}. \quad (3)$$

Here, $\hat{\sigma}_x$ and $\hat{\sigma}_z$ are the Pauli operators and $\hat{a}^\dagger$ and $\hat{a}$ are the oscillator creation and annihilation operators. Also, $\omega_{jk} > 0$, $\hat{a}_{jk}^\dagger$, and $\hat{a}_{jk}$ are the angular frequency and the creation and annihilation operators of the kth-harmonic oscillator of B_j, respectively. Finally, g, g_{1k}, and κ_k are complex numbers with units $1/s$ whose magnitude represents the strength of the couplings between the various subsystems and z^* denotes the complex conjugate of complex number z. We assume that the qubit-oscillator open system is weakly coupled to the thermal baths.

Observe that (2) is the Hamiltonian of the QRM. Since we are interested in the adiabatic regime, in the rest of the article we assume

$$\omega_q \ll \omega_r. \quad (4)$$

In order to deduce the Born-Markov secular master equation describing the evolution of the qubit-oscillator density operator, we first determine the spectrum of $\hat{H}$ in the adiabatic regime. This is done in the next sections.

III. QRM WITH SMALL QUBIT FREQUENCY

In order to calculate the eigenvalues and eigenvectors of $\hat{H}$ it is convenient to express it as

$$\hat{H} = \frac{\hbar\omega_q}{2}\hat{\sigma}_z + \hat{H}_{\text{DL}}, \quad (5)$$

where $\hat{H}_{\text{DL}}$ is the Hamiltonian of the QRM when the qubit has degenerate energy levels:

$$\hat{H}_{\text{DL}} = \hbar\omega_r \hat{a}^\dagger \hat{a} - \hbar\hat{\sigma}_x(g\hat{a}^\dagger + g^*\hat{a}). \quad (6)$$

In the next section we calculate the eigenvectors and eigenvalues of $\hat{H}_{\text{DL}}$, an intermediate step in applying the adiabatic approximation.

A. Eigenvalues and eigenvectors of $\hat{H}_{\text{DL}}$

We present a summary of results presented in Sec. V A of [22] (it uses exactly the same notation as this article except that $\omega_q = \omega_s$ and $\omega_r = \Omega$). We frequently use coherent states $|\gamma\rangle$ (γ a complex number) and the displacement operator $\hat{D}(\gamma)$ of the oscillator:

$$\hat{D}(\gamma) = e^{\gamma\hat{a}^\dagger - \gamma^*\hat{a}}. \quad (7)$$

An orthonormal basis for the qubit-oscillator state space is given by $\beta = \{|\omega_n, \pm\rangle : n = 0, 1, 2, \ldots\}$ with

$$|\omega_n, \pm\rangle = |\pm\rangle_x \otimes \hat{D}\left(\pm\frac{g}{\omega_r}\right)|n\rangle \quad (n = 0, 1, 2, \ldots). \quad (8)$$

Here $|n\rangle (n = 0, 1, 2, \ldots)$ are the harmonic oscillator number states and $|\pm\rangle_x$ are normalized eigenvectors of $\hat{\sigma}_x$ with corresponding eigenvalues ± 1:

$$|\pm\rangle_x = \frac{1}{\sqrt{2}}(|2\rangle \pm |1\rangle), \quad (9)$$

with $|2\rangle$ and $|1\rangle$ the excited and ground states of the qubit, respectively. The kets $|\omega_n, \pm\rangle$ are eigenvectors of $\hat{H}_{\text{DL}}$:

$$\hat{H}_{\text{DL}}|\omega_n, \pm\rangle = \hbar\omega_n|\omega_n, \pm\rangle \quad (n = 0, 1, 2, \ldots), \quad (10)$$

with the corresponding eigenvalues

$$\hbar\omega_n = \hbar\omega_r\left(n - \frac{|g|^2}{\omega_r^2}\right) \quad (n = 0, 1, 2, \ldots). \quad (11)$$

Observe that the spectrum of $\hat{H}_{\text{DL}}$ is equal to that of the harmonic oscillator shifted by the quantity $-\hbar|g|^2/\omega_r$ and that each eigenvalue of $\hat{H}_{\text{DL}}$ is two degenerate.

These results are used in the next section to determine the eigenvalues and eigenvectors of $\hat{H}$ within the adiabatic approximation.

B. Adiabatic approximation

The assumption in (4) allows one to apply the *averaging theorem* [24] so that $\hat{H}$ can be approximated by the *adiabatic limit Hamiltonian* (see the Appendix)

$$\hat{H}_{\mathrm{AL}} = \frac{\hbar\omega_q}{2}\sum_{n=0}^{+\infty} d_{nn}(|\omega_n,-\rangle\langle\omega_n,+|+|\omega_n,+\rangle\langle\omega_n,-|) + \hat{H}_{\mathrm{DL}}, \tag{12}$$

where

$$d_{nn} = e^{-2|g/\omega_r|^2} L_n\left(4\left|\frac{g}{\omega_r}\right|^2\right), \tag{13}$$

with $L_n(x)$ the nth Laguerre polynomial [25]. Using a bound on the Laguerre polynomials presented in [26], it can be shown that $|d_{nn}| \leqslant 1$ for $n = 0, 1, 2, \ldots$.

We emphasize that no assumption is made on the value of the coupling g. The adiabatic approximation only requires the assumption in (4). Also, $\hat{H}_{\mathrm{AL}}$ is going to be a more accurate approximation to $\hat{H}$ for smaller values of $|\omega_q/\omega_r|$ and larger values of $|g/\omega_r|$ (see the Appendix).

We now determine the eigenvalues and eigenvectors of $\hat{H}_{\mathrm{AL}}$. First observe that

$$\hat{H}_{\mathrm{AL}}|\omega_n,\pm\rangle = \hbar\omega_n|\omega_n,\pm\rangle + \frac{\hbar\omega_q}{2}d_{nn}|\omega_n,\mp\rangle. \tag{14}$$

Then, the orthonormal set $\{|\omega_n,+\rangle, |\omega_n,-\rangle\}$ spans an invariant subspace of $\hat{H}_{\mathrm{AL}}$ for each $n = 0, 1, 2, \ldots$ and one can diagonalize $\hat{H}_{\mathrm{AL}}$ in each of these subspaces. One obtains that the eigenvalues of $\hat{H}_{\mathrm{AL}}$ are

$$E_{n\pm} = \hbar\omega_n \pm \frac{\hbar\omega_q}{2}d_{nn} \quad (n = 0, 1, 2, \ldots). \tag{15}$$

Since $|d_{nn}| \leqslant 1$ for $n = 0, 1, 2, \ldots$, observe from (15) that the assumption in (4) guarantees that the addend $\hbar\omega_q d_{nn}/2$ is a small correction to $\hbar\omega_n$ if $\hbar\omega_n \neq 0$.

The corresponding eigenvectors of $\hat{H}_{\mathrm{AL}}$ are

$$|E_{n\pm}\rangle = \frac{1}{\sqrt{2}}(|\omega_n,+\rangle \pm |\omega_n,-\rangle). \tag{16}$$

These form an orthonormal basis for the qubit-oscillator state space. Using (8) one can write

$$|E_{n\pm}\rangle = \tfrac{1}{2}\{|2\rangle\otimes|\Psi_{n\pm}\rangle + |1\rangle\otimes|\Psi_{n\mp}\rangle\} \tag{17}$$

with

$$|\Psi_{n\pm}\rangle = \hat{D}\left(\frac{g}{\omega_r}\right)|n\rangle \pm \hat{D}\left(-\frac{g}{\omega_r}\right)|n\rangle. \tag{18}$$

Notice that the overlap of the states on the right-hand side of (18) is

$$\left|\langle n|\hat{D}^\dagger\left(-\frac{g}{\omega_r}\right)\hat{D}\left(\frac{g}{\omega_r}\right)|n\rangle\right|^2 = d_{nn}^2. \tag{19}$$

Since $|d_{nn}| \to 0$ as $n \to +\infty$, it follows that this overlap decreases to zero as n tends to infinity. Hence, the states composing $|\Psi_{n\pm}\rangle$ tend to be orthogonal as $n \to +\infty$. Also, observe that $|\Psi_{0\pm}\rangle$ are Schrödinger cat states of the oscillator and that the ground state of $\hat{H}_{\mathrm{AL}}$ is $|E_{0-}\rangle$. This state is very different from the tensor product $|1\rangle\otimes|0\rangle$ of the individual ground states of the qubit and the oscillator and contains virtual photons [27].

The eigenvalues and eigenvectors of $\hat{H}_{\mathrm{AL}}$ can be used to approximate the eigenvalues and eigenvectors of $\hat{H}$ and are usually called the eigenvalues and eigenvectors of $\hat{H}$ in the adiabatic approximation. Moreover, the eigenvectors of $\hat{H}$ are called *dressed states*. According to [20], the eigenvalues in (15) with $n = 0, \ldots, 7$ are very accurate approximations to the exact eigenvalues of $\hat{H}$ for all values of $|g/\omega_r|$ when $\omega_q/\omega_r \leqslant \frac{1}{3}$. Also, observe that the eigenvectors of $\hat{H}_{\mathrm{AL}}$ are independent of the value of ω_q, so all Hamiltonians with the same form as $\hat{H}$ but with different value of ω_q have the same eigenvectors in the adiabatic approximation. Improvements on the adiabatic approximation that lead to more accurate expressions for both the eigenvalues and eigenvectors of $\hat{H}_{\mathrm{DL}}$ can be found in [28,29].

In the next section we use the eigenvalues and eigenvectors of $\hat{H}$ in the adiabatic approximation to calculate the associated evolution operator.

C. Evolution operator in the adiabatic approximation

Using the approximate Hamiltonian in (12) and its eigenvalues and eigenvectors in (15) and (16), one obtains a very accurate approximation to the evolution operator associated with $\hat{H}$:

$$\begin{aligned} e^{-\frac{i}{\hbar}\hat{H}t} &\simeq e^{-\frac{i}{\hbar}\hat{H}_{\mathrm{AL}}t} \\ &= \sum_{n=0}^{+\infty} e^{-i\omega_n t}\Bigg\{\cos\left(\frac{d_{nn}}{2}\omega_q t\right)[|\omega_n,+\rangle\langle\omega_n,+| \\ &\quad + |\omega_n,-\rangle\langle\omega_n,-|] \\ &\quad - i\sin\left(\frac{d_{nn}}{2}\omega_q t\right)[|\omega_n,+\rangle\langle\omega_n,-| \\ &\quad + |\omega_n,-\rangle\langle\omega_n,+|]\Bigg\}. \end{aligned} \tag{20}$$

Observe that one recovers the evolution operator $e^{-i\hat{H}_{\mathrm{DL}}t/\hbar}$ associated with $\hat{H}_{\mathrm{DL}}$ for small times $|\omega_q t| \ll 1$: since $|d_{nn}| \leqslant 1$, for small times $|\omega_q t| \ll 1$ one can make an approximation to order zero in $\omega_q t$ in the cosine and sine in (20) to obtain $e^{-i\hat{H}_{\mathrm{DL}}t/\hbar}$.

We now comment on the accuracy of the approximation on the right-hand side of (20). The adiabatic limit Hamiltonian $\hat{H}_{\mathrm{AL}}$ in (12) was obtained by applying the *averaging theorem* of dynamical systems [24] with the perturbation parameter $\omega_q/(2\omega_r)$ (see the Appendix). If $|\psi_E(t)\rangle$ is the exact state of the system obtained by application of $\exp(-i\hat{H}t/\hbar)$, $|\psi(t)\rangle$ is the approximate state of the system obtained by application of $\exp(-i\hat{H}_{\mathrm{AL}}t/\hbar)$, the coupling strength satisfies $|g/\omega_r| \leqslant 2$, and the dynamics of the qubit-oscillator system are restricted to the subspace spanned by the orthonormal set $\beta_N = \{|\omega_n,\pm\rangle : n = 0, 1, \ldots, N = 10^3\}$, then $||\psi_E(t) - \psi(t)|| = O[\omega_q/(2\omega_r)]$ for $0 \leqslant \omega_r t \leqslant O(2\omega_r/\omega_q)$ with $\psi_E(t)$ and $\psi(t)$ the coordinate vectors of $|\psi_E(t)\rangle$ and $|\psi(t)\rangle$ in the basis β_N, $||\cdot||$ the Euclidean norm, and O the Landau symbol [24]. The conditions $|g/\omega_r| \leqslant 2$ and dynamics restricted to the subspace spanned by β_N can be changed if the bound in (A9) holds for larger values of m, n, and $|g/\omega_r|$ (see the Appendix).

In addition, the approximation to $\exp(-i\hat{H}t/\hbar)$ obtained using the method of multiple scales with the fast timescale $\omega_{oo}t$ and the slow timescale $\omega_q t$ with

$$\omega_{oo} = \min\{|\omega_n| : \omega_n \neq 0, n = 0, 1, 2, \dots\} \tag{21}$$

is exactly the same as (20). This also leads to the fact that (20) is an accurate approximation to the exact evolution operator for $0 \leqslant t \leqslant O(1/\omega_q)$ [30].

D. A method to reach the adiabatic regime

In this section and only this section we drop the assumption $\omega_q \ll \omega_r$ in (4) and we present how one can reach the adiabatic regime in the QRM if one drives the qubit with a sufficiently large frequency. The adiabatic regime is attractive because it has been shown that nonclassical states such as Schrödinger cat states appear naturally in the evolution of the qubit-oscillator system if $\omega_q = 0$ [22] and these are useful for quantum error correction [23]. Moreover, the dispersive regime can also be attained. In general, the dispersive regime is defined by the condition that the qubit-oscillator coupling strength $|g|$ is much smaller than the detuning $|\omega_r - \omega_q|$ between the oscillator frequency and the qubit transition frequency:

$$|g| \ll |\omega_r - \omega_q|. \tag{22}$$

Since the adiabatic regime is valid for arbitrary values of the qubit-oscillator coupling g and only requires a small qubit transition frequency with respect to the oscillator frequency, it also encompasses part of the dispersive regime, namely, the parameter region $|g| \ll \omega_r - \omega_q \sim \omega_r$. The importance of the dispersive regime is evidenced by its use in circuit QED: a quantum nondemolition measurement of a qubit [31], measurements of the probability to find photon-number states in a cavity and a plausible photon statistics analyzer [32], efficient detection of two-qubit correlations in two-qubit conditional phase gates [33], and a quantum switch formed by a qubit dispersively coupled to two cavities [34]. All these applications consider the case $|\omega_r - \omega_q| \ll \omega_r + \omega_q$ where the usual RWA can be applied to the qubit-oscillator interaction. Nevertheless, the method presented here could be used to reach the dispersive regime where $\omega_r - \omega_q \sim \omega_r$ and the RWA cannot be applied to the qubit-oscillator interaction. The dispersive regime where only (22) is required was studied in [35].

To present how to reach the adiabatic regime we use the results of Secs. II–IV of [22] (it uses exactly the same notation as this article except that $\omega_q = \omega_s$, $\omega_r = \Omega$, and $\hat{\rho} = \hat{\rho}_{\mathrm{SF}}$).

Consider a sinusoidally driven qubit interacting with a harmonic oscillator such that the Hamiltonian of the qubit-oscillator system is given by

$$\hat{H}_{dq}(t) = \hat{H} - \hbar\Omega_d \cos(\omega_d t)\hat{\sigma}_x, \tag{23}$$

with $\hat{H}$ in (2) and ω_d and Ω_d the driving angular frequency and strength, respectively. Note that (23) is equal to the Hamiltonians $\hat{H}(t)$ and $\hat{H}_0(t)$ presented in Eqs. (10) and (19) of [22] when there are no thermal baths.

The qubit-oscillator density operator $\hat{\rho}(t)$ evolves according to the von Neumann equation [which is Eq. (21) of [22]]

$$i\hbar \frac{d}{dt}\hat{\rho}(t) = [\hat{H}_{dq}(t), \hat{\rho}(t)]. \tag{24}$$

Now change to the interaction picture (IP) defined by the unitary transformation

$$\hat{U}_{IS}(t) = e^{-ib_0 t_2} e^{\frac{i}{2}\theta(t_1)\hat{\sigma}_x} \tag{25}$$

with

$$t_1 = \omega_d t, \quad \theta(t_1) = 2\left(\frac{\Omega_d}{\omega_d}\right)\sin(t_1),$$
$$t_2 = \omega_q t, \quad b_0 = \frac{1}{2}\left[1 - J_0\left(2\frac{\Omega_d}{\omega_d}\right)\right]. \tag{26}$$

Here J_0 is the Bessel function of the first kind of order zero [25] and $\hat{U}_{IS}(t)$ is designed to eliminate the driving term. It is part of the evolution operator of a sinusoidally driven qubit in the large, blue detuned regime [22,36].

For simplicity, operators in the IP have a subindex I. If $\hat{A}(t)$ is a linear operator in the Schrödinger picture (SP), then the corresponding operator in the IP is

$$\hat{A}_I(t) = \hat{U}_{IS}^\dagger(t)\hat{A}(t)\hat{U}_{IS}(t). \tag{27}$$

Assume that the driving frequency ω_d is much larger than the rest of the parameters in $\hat{H}_{dq}(t)$ with the exception of the driving strength Ω_d:

$$\omega_q, \omega_r, |g| \ll \omega_d. \tag{28}$$

Using assumption (28), Ref. [22] showed that one can average the IP von Neumann equation for $\hat{\rho}_I(t)$ in a t interval of length $2\pi/\omega_d$ to obtain an approximate equation that accurately describes the evolution of $\hat{\rho}_I(t)$ [see Eq. (31) of [22]]:

$$\frac{d}{dt}\hat{\rho}_I(t) = -\frac{i}{\hbar}[\hat{H}_I^{\mathrm{avg}}, \hat{\rho}_I(t)], \tag{29}$$

with the effective Hamiltonian [see Eq. (32) of [22] without the thermal baths]

$$\hat{H}_I^{\mathrm{avg}} = \frac{\hbar\omega_{qo}}{2}\hat{\sigma}_z + \hbar\omega_r \hat{a}^\dagger\hat{a} - \hbar\hat{\sigma}_x(g\hat{a}^\dagger + g^*\hat{a}), \tag{30}$$

and the effective qubit frequency [see Eq. (33) of [22] and eliminate the shift δ_3 due to the interaction with a thermal bath]

$$\omega_{qo} = \omega_q J_0\left(2\frac{\Omega_d}{\omega_d}\right). \tag{31}$$

Observe that (30) is identical to the Hamiltonian $\hat{H}$ of the QRM in (2), except that the qubit frequency is ω_{qo} in (31). Since $-0.403 \leqslant J_0(2\Omega_d/\omega_d) \leqslant 1$, one has $|\omega_{qo}| \leqslant \omega_q$ and one can adjust the driving parameters Ω_d/ω_d so that $|\omega_{qo}| \ll \omega_r$ and one reaches the adiabatic regime. For example, if $\Omega_d/\omega_d = 1.2025$, then $J_0(2\Omega_d/\omega_d) = 0$ and, consequently, $\omega_{qo} = 0$. Also notice that the ground and excited states of the qubit are interchanged if $\omega_{qo} < 0$. This adjusting of the driving parameters was used in [22] to consider a qubit with degenerate energy levels. The adiabatic regime could also be attained by using the method presented in [37] which starts with a qubit-oscillator system described by the Jaynes-Cummings model and uses two classical fields to drive the qubit in the RWA to simulate the quantum Rabi model. Other methods for quantum simulation can be found in [38].

We now illustrate how the eigenvectors of $\hat{H}$ evolve with the driving, that is, how they evolve with $\hat{H}_{dq}(t)$. Assume that $\omega_q \ll \omega_r$. Since $|\omega_{qo}| \leqslant \omega_q$, it follows that one can apply the adiabatic approximation to both $\hat{H}_I^{\rm avg}$ and $\hat{H}$. Then, both Hamiltonians have the same eigenvectors (16) because they are independent of the qubit transition frequency.

Assume that the qubit-oscillator initial state is

$$\hat{\rho}(0) = |E_n, \pm\rangle\langle E_n, \pm|, \tag{32}$$

with $|E_n, \pm\rangle$ in (16).

Since the IP and SP coincide at $t = 0$ and $|E_n, \pm\rangle$ is an eigenvector of $\hat{H}_I^{\rm avg}$ with eigenvalue

$$E_{n\pm} = \hbar\omega_n \pm \frac{\hbar\omega_{qo}}{2} d_{nn} \quad (n = 0, 1, 2, \ldots), \tag{33}$$

it follows from (29) that the qubit-oscillator density operator in the IP at time t is

$$\hat{\rho}_I(t) = |\psi_I(t)\rangle\langle\psi_I(t)| \tag{34}$$

with

$$|\psi_I(t)\rangle = e^{-i(\omega_n \pm \frac{\omega_{qo}}{2} d_{nn})t} |E_n, \pm\rangle. \tag{35}$$

Recall that $\hat{\rho}_I(t)$ and $\hat{\rho}(t)$ are connected through the unitary transformation in (25). Then, the qubit-oscillator density operator in the SP is

$$\hat{\rho}(t) = |\psi(t)\rangle\langle\psi(t)| \tag{36}$$

with

$$\begin{aligned} |\psi(t)\rangle &= \hat{U}_{IS}(t)|\psi_I(t)\rangle \\ &= e^{-i(\omega_n \pm \frac{\omega_{qo}}{2} d_{nn})t} e^{-ib_0 t_2} \frac{1}{\sqrt{2}} \\ &\quad \times \left[e^{\frac{i}{2}\theta(t_1)} |\omega_n, +\rangle \pm e^{-\frac{i}{2}\theta(t_1)} |\omega_n, -\rangle \right]. \end{aligned} \tag{37}$$

Using the expression of the eigenvectors $|E_{n\pm}\rangle$ in terms of the kets $|\omega_n, \pm\rangle$ in (16) and omitting a global phase, one can write (37) as

$$|\psi(t)\rangle = \cos\left[\frac{\theta(t_1)}{2}\right] |E_{n\pm}\rangle + i \sin\left[\frac{\theta(t_1)}{2}\right] |E_{n\mp}\rangle. \tag{38}$$

It follows that the probability to find the system in the state $|E_{n\mp}\rangle$ is

$$P(t) = \sin^2\left[\frac{\theta(t_1)}{2}\right] = \sin^2\left[\frac{\Omega_d}{\omega_d} \sin(\omega_d t)\right]. \tag{39}$$

Observe from (39) that there is complete population transfer if and only if the driving parameters satisfy $\Omega_d/\omega_d \geqslant \pi/2$. Figure 1 illustrates $P(t)$ as a function of $t_1 = \omega_d t$ for $\Omega_d/\omega_d = \pi/2$ (red solid line) and $\pi/4$ (blue dashed line). For $\Omega_d/\omega_d = \pi/2$ one has $\omega_{qo}/\omega_q = -0.3042$, while $\omega_{qo}/\omega_q = 0.4720$ for $\Omega_d/\omega_d = \pi/4$. Notice the difference with the usual Rabi oscillations [13].

In particular, if $\Omega_d/\omega_d = \pi/2$ and one considers the neighborhood of a maximizer of $P(t)$ given by

$$\omega_d t = (2n - 1)\frac{\pi}{2} + \Delta, \quad |\Delta| \leqslant 0.66, \quad n = 1, 2, \ldots \tag{40}$$

then $P(t)$ in (39) is well approximated by a quartic polynomial [obtained by considering $P(t)$ as a function of $\omega_d t$ and approximating it by a fourth-order Taylor polynomial in Δ centered at $(2n - 1)\pi/2$]:

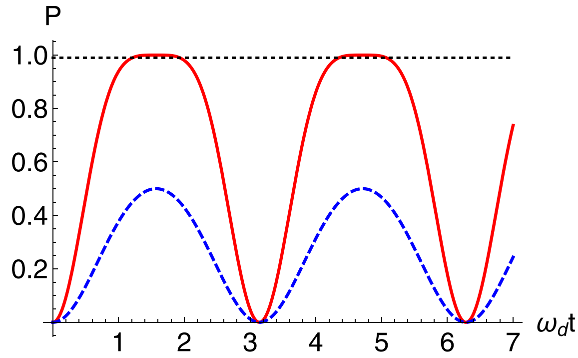

FIG. 1. The figure illustrates the transition probability $P(t)$ in (39) as a function of $t_1 = \omega_d t$ for the driving parameters $\Omega_d/\omega_d = \pi/2$ (red solid line) and $\pi/4$ (blue dashed line). It also illustrates the horizontal line 0.99 (black dotted line).

$$P(t) \simeq 1 - \frac{\pi^2}{16}\Delta^4. \tag{41}$$

The quartic is the reason for the *plateaus* shown in Fig. 1 (red solid line). It is an accurate approximation because the relative error is small for the values in (40): $|P(t) - [1 - (\pi^2\Delta^4/16)]|/P(t) < 0.01$. Moreover, using (41) one can determine the region where there is approximately complete population transfer:

$$P(t) \geqslant 0.99 \Leftrightarrow |\Delta| \leqslant \sqrt{\frac{2}{5\pi}} = 0.36. \tag{42}$$

Figure 1 shows the horizontal line 0.99 (black dotted line) indicating that the plateaus correspond to a probability $\geqslant 0.99$ for population transfer. The plateaus can also be explained within the model of a sinusoidally driven qubit in the large, blue detuned regime [36]. From (38) the driven qubit-oscillator system with Hamiltonian $\hat{H}_{dq}(t)$ behaves like a sinusoidally driven qubit with states $|E_{n+}\rangle$ and $|E_{n-}\rangle$, and with transition frequency $|E_{n+} - E_{n-}|/\hbar = |\omega_q d_{nn}| \leqslant \omega_q \ll \omega_d$ (ω_q is used because we are using $|E_{n\pm}\rangle$ as eigenvectors of $\hat{H}$). This corresponds to the large, blue detuned regime. According to [36], in this regime the trajectory of the Bloch vector of a sinusoidally driven qubit exhibits (approximate) cusps because the velocity (the time derivative) of the Bloch vector is (approximately) zero. If $\Omega_d/\omega_d = \pi/2$ and the Bloch vector starts at the south pole (which would correspond to the state $|E_{n,\pm}\rangle$), then the Bloch vector has a cusp at the north pole (which would correspond to the state $|E_{n,\mp}\rangle$) and the Bloch vector *spends more time* at the north pole because its velocity is (approximately) zero at that point. The plateaus correspond to the aforementioned cusp.

These results allow one to generalize the concepts of a π and $\pi/2$ pulse for resonant radiation in atomic physics [39] to the system we are considering. Recall that a π pulse results in the complete transfer of population from one state to the other, while a $\pi/2$ pulse puts the state of the qubit in an equal weights superposition of the excited and ground states if it is initially in the excited or ground state. π and $\pi/2$ pulses are used atomic physics [39], cavity QED [10], and circuit QED [32] to manipulate the state of a real or artificial two-level atom. In the system we are considering, they could be used to manipulate the state of the qubit-oscillator system in spite of the large detuning $\omega_q \ll \omega_r$.

For a pulse with driving parameters $\Omega_d/\omega_d = \pi/2$ and duration $\omega_d t_m = 3\pi/2 + 2\pi m$ with m a non-negative integer, the qubit-oscillator state immediately after the pulse is

$$|\psi(t_m)\rangle = -i|E_{n\mp}\rangle \tag{43}$$

[see (38)]. Hence, the qubit-oscillator is transferred from $|E_n, \pm\rangle$ to $|E_n, \mp\rangle$ and one has the analog of a π pulse. Notice that one can take advantage of the *plateaus* illustrated in Fig. 1 and characterized in (40)–(42) to have less stringent constraints on the duration of the pulse in order to have complete population transfer.

On the other hand, if the pulse has driving parameters $\Omega_d/\omega_d = \pi/4$ and a duration $\omega_d t_m = 3\pi/2 + 2\pi m$ with m a non-negative integer, then one has the analog of a $\pi/2$ pulse because from (38) the state of the system immediately after the pulse is

$$|\psi(t_m)\rangle = \frac{1}{\sqrt{2}}|E_{n\pm}\rangle - \frac{i}{\sqrt{2}}|E_{n\mp}\rangle. \tag{44}$$

Unfortunately, it seems that one cannot choose a value of the driving parameters Ω_d/ω_d so that the probability $P(t)$ associated with a $\pi/2$ pulse has plateaus similar to those of a π pulse.

Observe that both pulses have the same duration but the driving parameters change. These pulses could be used to manipulate the state of the qubit-oscillator by transferring the population of one dressed state to another or to prepare the qubit-oscillator in a superposition of dressed states.

IV. MASTER EQUATION

To deduce the master equation we first approximate $\hat{H}$ by $\hat{H}_{\rm AL}$ and make a spectral decomposition using the eigenvalues and eigenvectors in (15) and (16):

$$\hat{H} = \sum_{n=0}^{+\infty}(E_{n+}|E_{n+}\rangle\langle E_{n+}| + E_{n-}|E_{n-}\rangle\langle E_{n-}|). \tag{45}$$

We assume that the initial state of the complete system is separable and of the form

$$\hat{\rho}_T(0) = \hat{\rho}(0)\otimes\hat{\rho}_{\rm B_1}(0)\otimes\hat{\rho}_{\rm B_2}(0), \tag{46}$$

with $\hat{\rho}(0)$ and $\hat{\rho}_{{\rm B}_j}(0)$ the initial density operators of the qubit-oscillator and B_j, respectively. Also, $\hat{\rho}_{{\rm B}_j}(0)$ is a thermal state at a temperature $T > 0$:

$$\hat{\rho}_{{\rm B}_j}(0) = \prod_k \hat{\rho}_{jk}(0), \quad \hat{\rho}_{jk}(0) = \frac{1}{Z_{jk}}e^{-\beta_{jk}\hat{a}^\dagger_{jk}\hat{a}_{jk}},$$
$$\beta_{jk} = \frac{\hbar\omega_{jk}}{k_{\rm B}T}, \quad Z_{jk} = N(\omega_{jk}, T) + 1, \tag{47}$$

with $k_{\rm B}$ the Boltzmann constant and

$$N(\omega, T) = \frac{1}{e^{\hbar\omega/(k_{\rm B}T)} - 1}. \tag{48}$$

Note that $\hat{\rho}_{jk}(0)$ is the density operator of the kth oscillator of B_j and that it is a thermal state at temperature T. Also, let $\rho_{\rm Dj}(\omega)$ be the density of states where $\rho_{\rm Dj}(\omega)d\omega$ gives the number of oscillators of B_j with frequencies in the interval $[\omega, \omega + d\omega]$ $(j = 1, 2)$.

Applying a standard method [17] one can deduce the Born-Markov secular master equation in the first standard form governing the evolution of the qubit-oscillator density operator $\hat{\rho}(t)$. It is given by

$$\frac{d}{dt}\hat{\rho}(t) = -\frac{i}{\hbar}[\hat{H}', \hat{\rho}(t)] + \mathcal{D}[\hat{\rho}(t)]. \tag{49}$$

The Hamiltonian $\hat{H}'$ and the dissipator $\mathcal{D}$ appearing in the master equation are defined below in (50) and (55).

$\hat{H}'$ includes the original qubit-oscillator Hamiltonian in (45) and the *Lamb shift* Hamiltonian which introduces frequency shifts due to the interaction with the thermal baths:

$$\hat{H}' = \sum_{j=\pm}\sum_{n=0}^{+\infty}(E_{nj} + \hbar\delta_{nj})|E_{nj}\rangle\langle E_{nj}|. \tag{50}$$

The frequency shifts $\delta_{n\pm}$ are given by

$$\delta_{n\pm} = s_1(\pm\omega_q d_{nn}) + \left(\frac{g+g^*}{\omega_r}\right)^2 s_2(\pm\omega_q d_{nn}) + ns_2(\Delta^\pm_{n,n-1})$$
$$- (n+1)[S_2(\Delta^\pm_{n+1,n}) - S_{20}(-\Delta^\pm_{n+1,n})] \tag{51}$$

with the Bohr frequencies of $\hat{H}$,

$$\Delta^\pm_{n,m} = \frac{E_{n\pm} - E_{m\pm}}{\hbar} = \omega_r\left[(n-m) \pm \left(\frac{\omega_q}{\omega_r}\right)\left(\frac{d_{nn} - d_{mm}}{2}\right)\right],$$
$$\Delta_{n,m} = \frac{E_{n+} - E_{m-}}{\hbar} = \omega_r\left[(n-m) + \left(\frac{\omega_q}{\omega_r}\right)\left(\frac{d_{nn} + d_{mm}}{2}\right)\right], \tag{52}$$

and for $j = 1, 2$,

$$S_j(\omega') = -2\omega'{\rm P}\int_0^{+\infty} d\omega\,\rho_{\rm Dj}(\omega)|f_j(\omega)|^2\frac{N(\omega, T)}{\omega^2 - (\omega')^2},$$
$$S_{j0}(\omega') = -{\rm P}\int_0^{+\infty} d\omega\,\rho_{\rm Dj}(\omega)|f_j(\omega)|^2\frac{1}{\omega - \omega'},$$
$$s_j(\omega') = S_j(\omega') + S_{j0}(\omega'),$$
$$f_j(\omega) = \begin{cases} g_1(\omega) & \text{if } j = 1, \\ \kappa(\omega) & \text{if } j = 2. \end{cases} \tag{53}$$

Here, P denotes the principal value. Observe that $s_j(\omega')$ is a frequency shift induced by the interaction with B_j and that it has both a temperature-dependent contribution $S_j(\omega')$ and a temperature-independent part $S_{j0}(\omega')$. Moreover, notice from (51) that the frequency shifts induced by B_1 and B_2 depend on the dressed state and that the part corresponding to B_2 can be enhanced or decreased by the qubit-oscillator coupling g due to the factor $[(g+g^*)/\omega_r]^2$.

Using the operators

$$\hat{Q}_n = |E_{n-}\rangle\langle E_{n+}|, \quad \hat{R}_{n\pm} = |E_{n\pm}\rangle\langle E_{n+1,\pm}|, \tag{54}$$

and the anticommutator $\{\cdot,\cdot\}$, the dissipator $\mathcal{D}(\cdot)$ can be expressed as

$$
\begin{aligned}
\mathcal{D}(\hat{\rho}) = &\sum_{\substack{n,m=0\\ d_{nn}=-d_{mm}}}^{+\infty}[\Gamma_1(\omega_q d_{nn})\hat{Q}_n\hat{\rho}\hat{Q}_m+\Gamma_1(-\omega_q d_{nn})\hat{Q}_n^\dagger\hat{\rho}\hat{Q}_m^\dagger]-\sum_{n=0}^{+\infty}\frac{\Gamma_1(\omega_q d_{nn})}{2}\{\hat{Q}_n^\dagger\hat{Q}_n,\hat{\rho}\}\\
&+\sum_{\substack{n,m=0\\ d_{nn}=d_{mm}}}^{+\infty}[\Gamma_1(\omega_q d_{nn})\hat{Q}_n\hat{\rho}\hat{Q}_m^\dagger+\Gamma_1(-\omega_q d_{nn})\hat{Q}_n^\dagger\hat{\rho}\hat{Q}_m]-\sum_{n=0}^{+\infty}\frac{\Gamma_1(-\omega_q d_{nn})}{2}\{\hat{Q}_n\hat{Q}_n^\dagger,\hat{\rho}\}\\
&+\left(\frac{g+g^*}{\omega_r}\right)^2\left(\sum_{\substack{n,m=0\\ d_{nn}=-d_{mm}}}^{+\infty}[\Gamma_2(\omega_q d_{nn})\hat{Q}_n\hat{\rho}\hat{Q}_m+\Gamma_2(-\omega_q d_{nn})\hat{Q}_n^\dagger\hat{\rho}\hat{Q}_m^\dagger]-\sum_{n=0}^{+\infty}\frac{\Gamma_2(\omega_q d_{nn})}{2}\{\hat{Q}_n^\dagger\hat{Q}_n,\hat{\rho}\}\right)\\
&+\left(\frac{g+g^*}{\omega_r}\right)^2\left(\sum_{\substack{n,m=0\\ d_{nn}=d_{mm}}}^{+\infty}[\Gamma_2(\omega_q d_{nn})\hat{Q}_n\hat{\rho}\hat{Q}_m^\dagger+\Gamma_2(-\omega_q d_{nn})\hat{Q}_n^\dagger\hat{\rho}\hat{Q}_m]-\sum_{n=0}^{+\infty}\frac{\Gamma_2(-\omega_q d_{nn})}{2}\{\hat{Q}_n\hat{Q}_n^\dagger,\hat{\rho}\}\right)\\
&+\sum_{j=\pm}\sum_{\substack{n,m=0\\ \Delta^j_{n+1,n}=\Delta^j_{m+1,m}}}^{+\infty}\sqrt{n+1}\sqrt{m+1}\big[\Gamma_2\big(\Delta^j_{n+1,n}\big)\hat{R}_{nj}\hat{\rho}\hat{R}_{mj}^\dagger+\gamma_2\big(\Delta^j_{n+1,n}\big)\hat{R}_{nj}^\dagger\hat{\rho}\hat{R}_{mj}\big]\\
&+\sum_{\substack{n,m=0\\ \Delta^+_{n+1,n}=\Delta^-_{m+1,m}}}^{+\infty}\sqrt{n+1}\sqrt{m+1}[\Gamma_2(\Delta^+_{n+1,n})\hat{R}_{n+}\hat{\rho}\hat{R}_{m-}^\dagger+\gamma_2(\Delta^+_{n+1,n})\hat{R}_{n+}^\dagger\hat{\rho}\hat{R}_{m-}]\\
&+\sum_{\substack{n,m=0\\ \Delta^-_{n+1,n}=\Delta^+_{m+1,m}}}^{+\infty}\sqrt{n+1}\sqrt{m+1}[\Gamma_2(\Delta^-_{n+1,n})\hat{R}_{n-}\hat{\rho}\hat{R}_{m+}^\dagger+\gamma_2(\Delta^-_{n+1,n})\hat{R}_{n-}^\dagger\hat{\rho}\hat{R}_{m+}]\\
&-\sum_{j=\pm}\sum_{n=0}^{+\infty}\frac{(n+1)}{2}\big[\Gamma_2\big(\Delta^j_{n+1,n}\big)\{\hat{R}_{nj}^\dagger\hat{R}_{nj},\hat{\rho}\}+\gamma_2\big(\Delta^j_{n+1,n}\big)\{\hat{R}_{nj}\hat{R}_{nj}^\dagger,\hat{\rho}\}\big],
\end{aligned}
\tag{55}
$$

where sums over empty sets of indices are zero, $f_j(\omega)$ is defined in (53), and the relaxation rates are given by ($j=1,2$)

$$
\begin{aligned}
\gamma_j(\omega') &= 2\pi\int_0^{+\infty}d\omega\,\rho_{\mathrm{Dj}}(\omega)|f_j(\omega)|^2N(\omega,T)\\
&\quad\times[\delta(\omega+\omega')+\delta(\omega-\omega')],\\
\gamma_{j0}(\omega') &= 2\pi\int_0^{+\infty}d\omega\,\rho_{\mathrm{Dj}}(\omega)|f_j(\omega)|^2\delta(\omega-\omega'),\\
\Gamma_j(\omega') &= \gamma_j(\omega')+\gamma_{j0}(\omega'),
\end{aligned}
\tag{56}
$$

with $\delta(x)$ the Dirac delta function. Observe that $\Gamma_j(\omega')$ is a relaxation rate induced by the interaction with B_j. It has a temperature-dependent part $\gamma_j(\omega')$ and a temperature-independent contribution $\gamma_{j0}(\omega')$. Moreover, these depend on the dressed states involved in the transition due to the dependence on n of the frequency in which they are evaluated. Also, note that the qubit-oscillator coupling g can enhance or decrease some relaxation rates because they are multiplied by $[(g+g^*)/\omega_r]^2$. In fact, if g is pure imaginary, then those relaxation rates disappear.

It is important to state the approximations under which the master equation [Eqs. (49)–(56)] was deduced:

(1) The adiabatic approximation: This was used in the spectral decomposition (45) and is based on assumption (4).

(2) The Born approximation: This approximation has two requirements, weak coupling between the qubit-oscillator system and the thermal baths and a separable initial state of the complete system (qubit + oscillator + thermal baths) as in (46). It can be expressed by neglecting the influence of the qubit-oscillator system on the thermal baths so that the state of the complete system can be approximated by

$$\hat{\rho}_T(t)=\hat{\rho}(t)\otimes\hat{\rho}_{\mathrm{B}_1}(0)\otimes\hat{\rho}_{\mathrm{B}_2}(0). \tag{57}$$

(3) The Markov approximation: It requires that the thermal baths' self-correlation functions decay to zero over a time T_B much smaller than the time scale T_{qr} over which the interaction picture qubit-oscillator density operator $\hat{\rho}_I(t)$ evolves. The baths' self-correlation functions are

$$c_j(t)=\mathrm{Tr}_{\mathrm{B}_j}\big[e^{\frac{i}{\hbar}\hat{H}_{B_j}t}\hat{E}_je^{-\frac{i}{\hbar}\hat{H}_{B_j}t}\hat{E}_j\hat{\rho}_{\mathrm{B}_j}(0)\big], \tag{58}$$

with $\mathrm{Tr}_{\mathrm{B}_j}$ the trace with respect to the degrees of freedom of thermal bath B_j and

$$\hat{E}_j = \begin{cases} \sum_k (g_{1k}\hat{a}_{1k}^\dagger + g_{1k}^*\hat{a}_{1k}) & \text{if } j = 1, \\ \sum_k (\kappa_k \hat{a}_{2k}^\dagger + \kappa_k^* \hat{a}_{2k}) & \text{if } j = 2. \end{cases} \tag{59}$$

Also, $\hat{\rho}_I(t)$ is defined by

$$\hat{\rho}_I(t) = e^{\frac{i}{\hbar}\hat{H}t}\hat{\rho}(t)e^{-\frac{i}{\hbar}\hat{H}t}. \tag{60}$$

Now, let Γ_{qr} be the maximum relaxation rate, that is,

$$\Gamma_{qr} = \max\Bigg\{\Gamma_1(\pm\omega_q d_{nn}), (n+1)\Gamma_2(\Delta_{n+1,n}^\pm), \left(\frac{g+g^*}{\omega_r}\right)^2 \Gamma_2(\pm\omega_q d_{nn}) : n = 0, 1, 2, \dots, \bar{n}\Bigg\} \tag{61}$$

with $\bar{n}$ the maximum value of n such that $|E_{n\pm}\rangle$ has a non-negligible contribution in the evolution of the qubit-oscillator system. Given the spectral decomposition in (45), the expression for the eigenvalues $E_{n\pm}$ in (11) and (15), and the assumption $\omega_q \ll \omega_r$ in (4), one can use the initial expected value of the energy $\langle\hat{H}\rangle(0)$ and the initial root-mean-square deviation of the energy $\Delta\hat{H}(0)$ to estimate the value of $\bar{n}$:

$$\bar{n} = \frac{1}{\hbar\omega_r}[\langle\hat{H}\rangle(0) + 5\Delta\hat{H}(0)] + \frac{|g|^2}{\omega_r^2}. \tag{62}$$

Notice that a value of $\bar{n}$ would also appear in the maximum relaxation rate of the usual master equation for the damped harmonic oscillator [40] if one expresses the harmonic oscillator annihilation operator as $\hat{a} = \sum_{n=1}^{+\infty}\sqrt{n}|n-1\rangle\langle n|$.

Since the equation describing the evolution of $\hat{\rho}_I(t)$ is identical to the master equation in (49) with the replacement of $E_{n\pm}$ by zero in $\hat{H}'$ defined in (50), it follows that

$$T_{qr} \sim \frac{1}{\Gamma_{qr}}. \tag{63}$$

Due to (63), T_{qr} is called the relaxation time of the qubit-oscillator system. We have not taken into account the frequency shifts in (63) because they are usually small and are sometimes neglected.

The Markov approximation allows one to obtain a master equation that is local in time and in which the evolution of the qubit-oscillator system is described on a coarse-grained timescale (the dynamical behavior is not resolved over times $\sim T_B$).

(4) The secular approximation: The secular approximation is a RWA and neglects terms in the equation for $\hat{\rho}_I(t)$ that are multiplied by $e^{\pm i\omega t}$ where the frequencies $\omega \neq 0$ come from the sum or the difference of two Bohr frequencies of $\hat{H}$:

(a) $\omega = |\omega_q(d_{nn} \pm d_{mm})| \leqslant 2\omega_q$ with $d_{nn} \pm d_{mm} \neq 0$,

(b) $\omega = |\Delta_{n+1,n}^j - \Delta_{m+1,m}^k| \leqslant 2\omega_q$ with $j, k = \pm$,

(c) $\omega = |\Delta_{n+1,n}^j \pm \omega_q d_{mm}| \simeq \omega_r$ with $j = \pm$,

(d) $\omega = |\Delta_{n+1,n}^j + \Delta_{m+1,m}^k| \simeq 2\omega_r$ with $j, k = \pm$.

In writing $\leqslant$ we have used that $|d_{nn}| \leqslant 1$ for all $n = 0, 1, 2, \dots$. Also, $\simeq$ means that we have neglected ω_q with respect to ω_r based on the assumption $\omega_q \ll \omega_r$ in (4). The conditions appearing on some of the sums in the dissipator (55) eliminate those terms that in the interaction picture are multiplied by $e^{\pm i\omega t}$ with $\omega \neq 0$ in items (a) and (b). The terms associated with an ω in items (c) and (d) were explicitly eliminated in the dissipator.

The secular approximation assumes that the relaxation time T_{qr} is much larger than $1/\omega$ for all frequencies $\omega \neq 0$ in the items (a)–(d) above so that one can neglect terms in the equation for $\hat{\rho}_I(t)$ that are multiplied by $e^{\pm i\omega t}$ because they average to zero. Hence, the secular approximation requires that all the frequencies $\omega \neq 0$ in the items (a)–(d) above should satisfy $1/\omega \ll T_{qr} \sim 1/\Gamma_{qr}$. Since $\omega \simeq \omega_r$ or $2\omega_r$ in items (c) and (d) above, one requires $\Gamma_{qr} \ll \omega_r$. On the other hand, the frequencies ω in items (a) and (b) above can be small and $\Gamma_{qr} \ll \omega$ with ω in items (a) and (b) should be satisfied so that the secular approximation is valid. Once again, we have not taken into account the frequency shifts in the discussion of the secular approximation because we have assumed that they are very small.

The master equation (49)–(56) seems to be adequate only for numerical calculations. Also, of the approximations listed above, only the secular approximation poses a real problem on the validity of the master equation because it requires $\Gamma_{qr} \ll \omega$ with ω in items (a) and (b) above. One would have to restrict to a subspace spanned by $\{|E_{n\pm}\rangle : n = 0, 1, 2, \dots, N\}$ for some non-negative integer N so that this condition is satisfied. In the next section we show how one can eliminate this problem. The idea is similar to that used in [14] were one applies a *quasisecular approximation*, that is, one preserves all the terms associated with frequencies in items (a) and (b) above.

A. Simplified master equation

In this section we simplify the master equation in (49)–(56) so that it becomes manageable and accurately describes the evolution of $\hat{\rho}(t)$.

The first step is to use the conditions $d_{nn} = \pm d_{mm}$ on some of the sums in the dissipator (55) to rewrite the relaxation rates as follows ($j = 1, 2$):

If $d_{nn} = -d_{mm}$, then

$$\Gamma_j(\pm\omega_q d_{nn}) = \sqrt{\Gamma_j(\pm\omega_q d_{nn})\Gamma_j(\mp\omega_q d_{mm})}. \tag{64}$$

If $d_{nn} = d_{mm}$, then

$$\Gamma_j(\pm\omega_q d_{nn}) = \sqrt{\Gamma_j(\pm\omega_q d_{nn})\Gamma_j(\pm\omega_q d_{mm})}. \tag{65}$$

Then, make the following approximations:

(5) $\Delta_{n+1,n}^\pm \simeq \omega_r$: Since $|d_{nn}| \leqslant 1$ for $n = 0, 1, 2, \dots$, this approximation follows directly from the definition of $\Delta_{n+1,n}^\pm$ in (52) and from the assumption $\omega_q \ll \omega_r$ in (4). Explicitly, one would require $\omega_q \lesssim 0.1\omega_r$.

(6) $\Delta_{n+1,n}^j \simeq \Delta_{m+1,m}^k$ for all n, m and $j, k = \pm$: This approximation is a consequence of the one in the previous item.

(7) $d_{nn} \simeq \pm d_{mm}$ for all n, m: These terms come from $|\omega_q(d_{nn} \pm d_{mm})|$, the frequency in item (a) of the previous section. Since $|d_{nn}| \leqslant 1$ for $n = 0, 1, 2, \dots$, it follows from assumption (4) that $|\omega_q(d_{nn} \pm d_{mm})|/\omega_r \ll 1$. Based on this we make the approximation $d_{nn} \simeq \pm d_{mm}$ for all $n, m = 0, 1, 2, \dots$.

(8) $\Gamma_2(\Delta_{n+1,n}^\pm) \simeq \Gamma_2(\omega_r)$: One uses the approximation in item 5 above and one requires that the density of states $\rho_{D2}(\omega)$ be approximately constant in the interval $[\omega_r - \omega_q, \omega_r + \omega_q]$

containing $\Delta^{\pm}_{n+1,n}$ and ω_r. It is similar to the approximation made in the description of a two-level atom interacting with a single mode, classical electromagnetic field where the Rabi frequency is neglected with respect to the frequency of the field in the expressions for the relaxation rates [40].

Applying these approximations one can drop the requirements $d_{nn} = \pm d_{mm}$ and $\Delta^j_{n+1,n} = \Delta^k_{m+1,m}$ ($j,k=\pm$) from the sums in the dissipator (55) and one can approximate the relaxation rates $\Gamma_2(\Delta^{\pm}_{n+1,n})$ by $\Gamma_2(\omega_r)$. Notice that this means that the terms in the equation for $\hat{\rho}_I(t)$ that are multiplied by $e^{\pm i\omega t}$ with ω in items (a) and (b) of the previous section are preserved. The difference with respect to just preserving them is that the relaxation rates were rewritten as in (64) and (65). Hence, the secular approximation in item 4 of the previous section is changed to the following:

(4′) The quasisecular approximation: It neglects terms in the equation for $\hat{\rho}_I(t)$ that are multiplied by $e^{\pm i\omega t}$ where the frequencies $\omega \neq 0$ are of the form

(1) $\omega = |\Delta^j_{n+1,n} \pm \omega_q d_{mm}| \simeq \omega_r$ with $j=\pm$,

(2) $\omega = |\Delta^j_{n+1,n} + \Delta^k_{m+1,m}| \simeq 2\omega_r$ with $j,k=\pm$.

It only requires $\Gamma_{qr} \ll \omega_r$.

The approximations in items 5–8 above allow one to write the resulting master equation in the Lindblad form

$$\frac{d}{dt}\hat{\rho}(t) = \mathcal{L}_{\rm HO}\hat{\rho}(t) + \mathcal{L}_{12}\hat{\rho}(t), \tag{66}$$

where the superoperators $\mathcal{L}_{\rm HO}$ and $\mathcal{L}_{12}$ are defined by

$$\begin{aligned}\mathcal{L}_{\rm HO}\hat{\rho} &= -\frac{i}{\hbar}[\hbar\omega'_r\hat{b}^\dagger\hat{b}, \hat{\rho}] + \mathcal{D}_{\rm HO}(\hat{\rho}),\\ \mathcal{L}_{12}\hat{\rho} &= -\frac{i}{\hbar}[\hat{H}_{12}, \hat{\rho}] + \mathcal{D}_{12}(\hat{\rho}).\end{aligned} \tag{67}$$

The dissipators are given by

$$\begin{aligned}\mathcal{D}_{\rm HO}(\hat{\rho}) &= \gamma_0[N(\omega_r,T)+1]\left(\hat{b}\hat{\rho}\hat{b}^\dagger - \frac{1}{2}\{\hat{b}^\dagger\hat{b},\hat{\rho}\}\right)\\ &\quad + \gamma_0 N(\omega_r,T)\left(\hat{b}^\dagger\hat{\rho}\hat{b} - \frac{1}{2}\{\hat{b}\hat{b}^\dagger,\hat{\rho}\}\right),\\ \mathcal{D}_{12}(\hat{\rho}) &= \hat{J}_1\hat{\rho}\hat{J}_1^\dagger - \frac{1}{2}\{\hat{J}_1^\dagger\hat{J}_1,\hat{\rho}\}\\ &\quad + \left(\frac{g+g^*}{\omega_r}\right)^2\left[\hat{J}_2\hat{\rho}\hat{J}_2^\dagger - \frac{1}{2}\{\hat{J}_2^\dagger\hat{J}_2,\hat{\rho}\}\right],\end{aligned} \tag{68}$$

with $\{\cdot,\cdot\}$ the anticommutator, while the Hamiltonian $\hat{H}_{12}$ is defined by

$$\begin{aligned}\hat{H}_{12} = \hbar\sum_{n=0}^{+\infty}\Bigg[&\left(\frac{\omega_q}{2}d_{nn} + \delta'_{n+}\right)|E_{n+}\rangle\langle E_{n+}|\\ &-\left(\frac{\omega_q}{2}d_{nn} - \delta'_{n-}\right)|E_{n-}\rangle\langle E_{n-}|\Bigg].\end{aligned} \tag{69}$$

Note that $N(\omega_r,T)$ is defined in (48) and that it represents the mean number of B_2-thermal excitations at frequency ω_r (photons if B_2 is an electromagnetic field). Moreover, only ω_r appears in $N(\omega_r,T)$ due to the approximation in item 5 above.

The frequency shifts $\delta'_{n\pm}$ included in $\hat{H}_{12}$ and the shifted harmonic oscillator frequency ω'_r are given below, respectively, in (70) and (71). Also, the operators and relaxation rates appearing in the dissipators are defined below in (72) and (75)–(77), respectively.

Using $s_j(\omega')$ defined in (53), the frequency shifts appearing in $\hat{H}_{12}$ can be expressed as

$$\delta'_{n\pm} = s_1(\pm\omega_q d_{nn}) + \left(\frac{g+g^*}{\omega_r}\right)^2 s_2(\pm\omega_q d_{nn}). \tag{70}$$

Observe that the qubit-oscillator coupling g can enhance or decrease the frequency shift $\delta'_{n\pm}$ due to the factor $[(g+g^*)/\omega_r]^2$. Also, using $S_{j0}(\omega')$ defined in (53), the shifted harmonic oscillator frequency ω'_r is defined by

$$\begin{aligned}\omega'_r &= \omega_r + S_{20}(\omega_r) + S_{20}(-\omega_r)\\ &= \omega_r + \mathrm{P}\int_0^{+\infty} d\omega\,\rho_{\rm D2}(\omega)|\kappa(\omega)|^2\frac{2\omega}{\omega_r^2-\omega^2}.\end{aligned} \tag{71}$$

Observe that the frequency shift of ω_r does not have a thermal contribution just like the usual damped harmonic oscillator [40].

The linear operators appearing in the dissipators are defined by

$$\begin{aligned}\hat{b} &= \hat{a} - \frac{g}{\omega_r}\hat{\sigma}_x,\\ \hat{J}_{1\pm} &= \sum_{n=0}^{+\infty}\sqrt{\Gamma_1(\mp\omega_q d_{nn})}|E_{n,\pm}\rangle\langle E_{n,\mp}|,\\ \hat{J}_1 &= \hat{J}_{1+} + \hat{J}_{1-},\\ \hat{J}_{2\pm} &= \sum_{n=0}^{+\infty}\sqrt{\Gamma_2(\mp\omega_q d_{nn})}|E_{n,\pm}\rangle\langle E_{n,\mp}|,\\ \hat{J}_2 &= \hat{J}_{2+} + \hat{J}_{2-}.\end{aligned} \tag{72}$$

It is very important to note that $\hat{b}^\dagger$ and $\hat{b}$ are the respective creation and annihilation operators of a harmonic oscillator where the eigenvectors $|\omega_n,\pm\rangle$ of $\hat{H}_{\rm DL}$ play the role of the number states:

$$\begin{aligned}&[\hat{b},\hat{b}^\dagger] = 1, \quad \hat{b}^\dagger|\omega_n,\pm\rangle = \sqrt{n+1}|\omega_{n+1},\pm\rangle,\\ &\hat{b}^\dagger\hat{b}|\omega_n,\pm\rangle = n|\omega_n,\pm\rangle, \quad \hat{b}|\omega_n,\pm\rangle = \sqrt{n}|\omega_{n-1},\pm\rangle.\end{aligned} \tag{73}$$

In addition, from (16) and (73) one finds that the states $|E_{n,\pm}\rangle$ also behave as number states:

$$\begin{aligned}\hat{b}^\dagger|E_{n\pm}\rangle &= \sqrt{n+1}|E_{n+1,\pm}\rangle,\\ \hat{b}^\dagger\hat{b}|E_{n\pm}\rangle &= n|E_{n\pm}\rangle,\\ \hat{b}|E_{n\pm}\rangle &= \sqrt{n}|E_{n-1,\pm}\rangle.\end{aligned} \tag{74}$$

Observe that the operators $\hat{J}_{k\pm}$ ($k=1,2$) act as raising and lowering operators in the basis of dressed states (eigenvectors of $\hat{H}$) and include a state-dependent relaxation rate $\sqrt{\Gamma_k(\mp\omega_q d_{nn})}$. In addition, the qubit-oscillator coupling g can enhance or decrease the relaxation rates included in $\hat{J}_2$ and $\hat{J}_2^\dagger$ due to the factor $[(g+g^*)/\omega_r]^2$ in the dissipator $\mathcal{D}_{12}(\hat{\rho})$. If g is pure imaginary, then those relaxation rates are zero and the terms involving $\hat{J}_2$ and $\hat{J}_2^\dagger$ disappear from $\mathcal{D}_{12}(\hat{\rho})$.

Finally, the relaxation rate γ_0 is associated with thermal bath B_2 and is given by

$$\gamma_0 = \gamma_{20}(\omega_r) = 2\pi \rho_{D2}(\omega_r)|\kappa(\omega_r)|^2, \tag{75}$$

while the other relaxation rates are evaluated from (56) as

$$\Gamma_j(\mp\omega_q d_{nn}) = 2\pi \rho_{Dj}(|\omega_q d_{nn}|)|f_j(|\omega_q d_{nn}|)|^2 \times [N(|\omega_q d_{nn}|, T) + \Theta(\mp\omega_q d_{nn})], \tag{76}$$

with $f_j(\omega')$ in (53) and $\Theta(\omega')$ the step function given by

$$\Theta(\omega') = \begin{cases} 1 & \text{if } \omega' > 0, \\ \frac{1}{2} & \text{if } \omega' = 0, \\ 0 & \text{if } \omega' < 0. \end{cases} \tag{77}$$

Recall that $N(|\omega_q d_{nn}|, T)$ is defined in (48) and that it represents the mean number of B_j-thermal excitations at frequency $|\omega_q d_{nn}|$ (photons if B_j is an electromagnetic field). Also, notice that one requires the densities of states $\rho_{Dj}(\omega')$ and the couplings $f_j(\omega')$ to be such that (76) are defined for $\omega' = \mp\omega_q d_{nn} = 0$.

We have separated the right-hand side of the master equation (66) into two addends because $\mathcal{L}_{HO}$ is exactly the same as the generator of a damped harmonic oscillator [17,40]. Moreover, $\mathcal{D}_{HO}$ describes transitions from one group of dressed states $\{|E_{n,+}\rangle, |E_{n-}\rangle\}$ to another, while $\mathcal{D}_{12}$ describes transitions within each group of dressed states $\{|E_{n,+}\rangle, |E_{n-}\rangle\}$. In addition, $\mathcal{L}_{HO}$ involves the relaxation rate γ_0 at frequency ω_r, which is usually larger than the relaxation rates at frequencies $\pm\omega_q d_{nn}$ appearing in $\mathcal{L}_{12}$ because $|\omega_q d_{nn}| \leqslant \omega_q \ll \omega_r$ [see (4)].

Before proceeding we summarize the conditions for the validity of the simplified master equation in (66)–(70): the adiabatic approximation (that is, $\omega_q \lesssim 0.1\omega_r$) and the Born-Markov approximations in items 1–3 of the previous section, the quasisecular approximation (that is ω_r much larger than all relaxation rates) in item 4′ of this section, and near equality of the relaxation rates $\Gamma_2(\Delta^{\pm}_{n+1,n})$ and $\Gamma_2(\omega_r)$ in item 8 of this section. The approximations in items 5–7 of this section are contained in the adiabatic approximation and the quasisecular approximation. It is very important to notice that the simplified master equation holds for all values of the qubit-oscillator coupling g and in all the qubit-oscillator state space. Also, it is valid for relaxation rates smaller than, of the order of, and larger than ω_q.

To end this section, note that, in the limit $\omega_q \to 0$, the simplified master equation (66)–(72) reduces to

$$\frac{d}{dt}\hat{\rho}(t) = \mathcal{L}_{HO}\hat{\rho}(t) + \mathcal{L}_{120}\hat{\rho}(t) \tag{78}$$

with

$$\mathcal{L}_{120}\hat{\rho} = \gamma_{00}\big(\hat{\sigma}_x\hat{\rho}\hat{\sigma}_x - \tfrac{1}{2}\{\hat{\sigma}_x\hat{\sigma}_x, \hat{\rho}\}\big), \tag{79}$$

and the relaxation rate

$$\gamma_{00} = \gamma_1(0) + \gamma_{10}(0) + \left(\frac{g+g^*}{\omega_r}\right)^2[\gamma_2(0) + \gamma_{20}(0)]. \tag{80}$$

The master equation (78)–(80) was deduced in [22] and correctly describes the evolution of the system in the case $\omega_q = 0$. More specifically, Sec. V of [22] presented the following result: If one considers exactly the same system as this article but with the complete system Hamiltonian $\hat{H}_T - \hbar\Omega_d\cos(\omega_d t)\hat{\sigma}_x$ where $\hat{H}_T$ is defined in (1) and $\omega_q = 0$, then the Born-Markov secular master equation in the Lindblad form describing the evolution of the qubit-oscillator density operator $\hat{\rho}(t)$ is given by

$$\frac{d}{dt}\hat{\rho}(t) = \mathcal{L}_{HO}\hat{\rho}(t) + \mathcal{L}_{120}\hat{\rho}(t) - \frac{i}{\hbar}[-\hbar\Omega_d\cos(\omega_d t)\hat{\sigma}_x, \hat{\rho}(t)], \tag{81}$$

with $\mathcal{L}_{HO}$ and $\mathcal{L}_{120}$ in (67) and (79) and with all relaxation rates and frequency shifts as defined above. In addition, one only needs to change γ_{00} to γ_{000} defined in Eqs. (123) and (124) of [22] if $B_1 = B_2$, that is, if B_1 and B_2 are the same thermal bath. We mention that Sec. V of [22] only considered the case $\Omega_d/\omega_d = 1.2025$ and ω_d large with respect to the rest of the parameters of $\hat{H}_T$ in (1) due to the context of that article. Nevertheless, the calculations used to deduce the master equation (81) were completely general and (81) holds for arbitrary real values of the qubit driving parameters Ω_d and ω_d. Taking $\Omega_d = 0$ in (81), one recovers (78).

In the next sections we use the simplified master equation to describe the dynamics of the system.

B. Comparison with another master equation

In this section we compare the simplified master equation presented in the previous section with that deduced in [21], which considered exactly the same system as this article. In order to do this, in this section and only this section we neglect all frequency shifts and take g real and $N(\omega, T) = 0$ for all ω because [21] considered these conditions. First, we rewrite the simplified master equation.

Define the linear operators and superoperators

$$\begin{aligned} \hat{\mathbb{I}}_\pm &= \sum_{n=0}^{+\infty} |E_{n,\pm}\rangle\langle E_{n,\pm}|, \\ \hat{a}_\pm &= \hat{\mathbb{I}}_\pm \hat{b}, \\ \mathcal{D}[\hat{A}]\hat{\rho} &= \hat{A}\hat{\rho}\hat{A}^\dagger - \frac{1}{2}\{\hat{A}^\dagger\hat{A}, \hat{\rho}\}, \\ \mathcal{G}_{HO}(\hat{\rho}) &= \hat{a}_+\hat{\rho}\hat{a}_-^\dagger + \hat{a}_-\hat{\rho}\hat{a}_+^\dagger - \frac{1}{2}\{\hat{a}_+^\dagger\hat{a}_- + \hat{a}_-^\dagger\hat{a}_+, \hat{\rho}\}, \\ \mathcal{G}_1(\hat{\rho}) &= \mathcal{G}_{11}(\hat{\rho}) + 4\left(\frac{g}{\omega_r}\right)^2\mathcal{G}_{12}(\hat{\rho}), \end{aligned} \tag{82}$$

where $\hat{A}$ is a linear operator and

$$\mathcal{G}_{1k}(\hat{\rho}) = \sum_{n=0}^{+\infty}\sum_{\substack{m=0\\ m\neq n}}^{+\infty}\sqrt{\gamma_{k0}(\omega_q d_{nn})\gamma_{k0}(\omega_q d_{mm})}|E_{n-}\rangle\langle E_{n+}|\hat{\rho}|E_{m+}\rangle\langle E_{m-}| - \frac{1}{2}\sum_{n=0}^{+\infty}\gamma_{k0}(-\omega_q d_{nn})\{|E_{n-}\rangle\langle E_{n-}|, \hat{\rho}\}$$
$$+ \sum_{n,m=0}^{+\infty}\{\sqrt{\gamma_{k0}(-\omega_q d_{nn})\gamma_{k0}(-\omega_q d_{mm})}|E_{n+}\rangle\langle E_{n-}|\hat{\rho}|E_{m-}\rangle\langle E_{m+}| + \sqrt{\gamma_{k0}(-\omega_q d_{nn})\gamma_{k0}(\omega_q d_{mm})}|E_{n+}\rangle\langle E_{n-}|\hat{\rho}|E_{m+}\rangle\langle E_{m-}|$$
$$+ \sqrt{\gamma_{k0}(\omega_q d_{nn})\gamma_{k0}(-\omega_q d_{mm})}|E_{n-}\rangle\langle E_{n+}|\hat{\rho}|E_{m-}\rangle\langle E_{m+}|\}. \tag{83}$$

Notice that $\hat{a}_+^\dagger\hat{a}_- = \hat{a}_-^\dagger\hat{a}_+ = 0$ and that $\mathcal{G}_{HO}(\hat{\rho}) \neq 0$. Observe that $\gamma_k(\omega') = 0$ (a stimulated transition rate) and $\Gamma_k(\omega') = \gamma_{k0}(\omega')$ (a spontaneous transition rate) if $N(\omega', T) = 0$ for all ω' [see (56)]. Also, $\gamma_{k0}(\omega') > 0$ if and only if $\omega' > 0$ and $\gamma_0 = \gamma_{20}(\omega_r)$ [see (75)].

Using $N(\omega', T) = 0$ for all ω', a real qubit-oscillator coupling g, and the operators and superoperators in (82) and (83) and neglecting all frequency shifts, the simplified master equation (66)–(70) takes the form

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = &-\frac{i}{\hbar}[\hat{H}_{\text{AL}}, \hat{\rho}(t)]\\ &+\gamma_{20}(\omega_r)\{\mathcal{D}[\hat{a}_+]\hat{\rho}(t) + \mathcal{D}[\hat{a}_-]\hat{\rho}(t)\}\\ &+\sum_{n=0}^{+\infty}\left[\gamma_{10}(\omega_q d_{nn}) + 4\left(\frac{g}{\omega_r}\right)^2\gamma_{20}(\omega_q d_{nn})\right]\\ &\times\mathcal{D}[|E_{n-}\rangle\langle E_{n+}|]\hat{\rho}(t)\\ &+\mathcal{G}_1[\hat{\rho}(t)] + \gamma_{20}(\omega_r)\mathcal{G}_{\text{HO}}[\hat{\rho}(t)].\end{aligned} \tag{84}$$

Using the notation of this article, one can write the master equation presented in Eqs. (22) and (23) of [21] as follows:

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = &-\frac{i}{\hbar}[\hat{H}_{\text{AL}}, \hat{\rho}(t)]\\ &+\gamma_{20}(\omega_+)\mathcal{D}[\hat{a}_+]\hat{\rho}(t) + \gamma_{20}(\omega_-)\mathcal{D}[\hat{a}_-]\hat{\rho}(t)\\ &+\sum_n\left[\gamma_{10}(\tilde{\omega}_n) + 4\left(\frac{g}{\omega_r}\right)^2\gamma_{20}(\tilde{\omega}_n)\right]\\ &\times\mathcal{D}[|E_{n-}\rangle\langle E_{n+}|]\hat{\rho}(t)\end{aligned} \tag{85}$$

with

$$\omega_\pm = \omega_r \mp 2\omega_q\left(\frac{g}{\omega_r}\right)^2,$$
$$\tilde{\omega}_n = \omega_q\left[1 - 2\left(\frac{g}{\omega_r}\right)^2 - 4n\left(\frac{g}{\omega_r}\right)^2\right]. \tag{86}$$

The connection between the notation of [21] (left-hand side) and that of this article (right-hand side) is

$$\begin{aligned}&\omega_0 = \omega_q, \quad \omega_\pm = \omega_\pm, \quad \gamma(\omega') = \gamma_{10}(\omega'),\\ &\omega = \omega_r, \quad \tilde{\omega}_N = \tilde{\omega}_N, \quad \Gamma(\omega') = \gamma_{20}(\omega'),\\ &\beta = -\frac{g}{\omega_r}, \quad \mathcal{E}_N = \hbar\omega_N, \quad E_N^\pm = E_{N\pm}\end{aligned} \tag{87}$$

for the parameters (recall that $\beta\omega = -g$ is real in [21]),

$$|m, N_m\rangle = |\omega_N, m\rangle \ (m = \pm), \quad |\Psi_N^\pm\rangle = |E_{N\pm}\rangle \tag{88}$$

for the kets, $\mathcal{D}(\cdot) = \mathcal{D}(\cdot)$, and

$$H_{\text{AD}} = \hat{H}_{\text{AL}}, \quad 1_\pm = \hat{\mathbb{I}}_\pm, \quad a_\pm = \hat{a}_\pm. \tag{89}$$

We now list the conditions under which (85) was deduced (see Sec. 7 of [21]): the adiabatic approximation with $\omega_q \leqslant 0.3\omega_r$, the Born-Markov secular approximations, a qubit-oscillator coupling $|g/\omega_r| \leqslant 0.2$, terms of order $|g/\omega_r|^n$ with $n \geqslant 4$ were neglected, dynamics restricted to the subspace spanned by $\{|E_{n\pm}\rangle : n \ll |\omega_r/(2g)|^2\}$, zero temperature, and all frequency shifts were neglected.

It is important to mention that $|g/\omega_r| \leqslant 0.2$ and the restriction to the aforementioned subspace allowed [21] to expand d_{nn} in a Taylor series in $|g/\omega_r|$ centered at 0 and to neglect terms of order $|g/\omega_r|^n$ with $n \geqslant 4$, so

$$\begin{aligned}&d_{nn} = 1 - 2\left(\frac{g}{\omega_r}\right)^2 - 4n\left(\frac{g}{\omega_r}\right)^2, \quad \omega_q d_{nn} = \tilde{\omega}_n,\\ &\Delta_{n+1,n}^\pm = \omega_\pm, \quad 0.976\omega_r \leqslant \omega_\pm \leqslant 1.024\omega_r.\end{aligned} \tag{90}$$

The condition $\omega_q \leqslant 0.3\omega_r$ was also used in the inequalities involving $\omega_\pm$. Notice that $\omega_\pm$ are very close to ω_r with the conditions under which (85) was deduced.

We now compare the simplified master equation in (84) with the master equation (85) presented in [21]. First observe that both equations are identical except for three things: (i) the two relaxation rates $\gamma_{20}(\omega_\pm)$ in (85) compared to the single relaxation rate $\gamma_{20}(\omega_r)$ appearing in (84); (ii) the relaxation rates $\gamma_{j0}(\tilde{\omega}_n)$ in (85) compared to the relaxation rates $\gamma_{j0}(\omega_q d_{nn})$ in (84); (iii) the two addends in the last line of the right-hand side of (84). We now comment on these differences and the approximations used to deduce each master equation:

(C1) Both master equations were deduced under the adiabatic and Born-Markov approximations.

(C2) The master equation in (85) used the secular approximation, which requires the frequency difference between any two dressed states to be much larger than the involved relaxation rates. This imposes a very severe limitation to how small ω_q can be because the frequency difference between the dressed states $|E_{n+}\rangle$ and $|E_{n-}\rangle$ is $|\omega_q d_{nn}|$, which is smaller than ω_q and, as a function of n, behaves as an oscillating function whose amplitude decreases to zero as $n \to +\infty$. The way that [21] avoided this problem was restricting both the qubit-oscillator coupling to values $|g/\omega_r| \leqslant 0.2$ and the qubit-oscillator dynamics to the subspace spanned by $\{|E_{n\pm}\rangle : n \ll |\omega_r/(2g)|^2\}$. Nevertheless, this does not eliminate the problem completely. For example, the master equation (85) leads to the following equation for the matrix element of $\hat{\rho}(t)$ between $|E_{1\pm}\rangle$:

$$\begin{aligned}\frac{d}{dt}\langle E_{1+}|\hat{\rho}(t)|E_{1-}\rangle \\ = \Bigg\{-i\omega_q d_{11} - \frac{1}{2}\left[\gamma_{10}(\tilde{\omega}_1) + 4\left(\frac{g}{\omega_r}\right)^2\gamma_{20}(\tilde{\omega}_1)\right] \\ -\frac{1}{2}\gamma_{20}(\omega_+) - \frac{1}{2}\gamma_{20}(\omega_-)\Bigg\}\langle E_{1+}|\hat{\rho}(t)|E_{1-}\rangle. \quad (91)\end{aligned}$$

Observe that the right-hand side of (91) involves the relaxation rates $\gamma_{20}(\omega_\pm)/2$, so one requires $\gamma_{20}(\omega_\pm)/2 \ll |\omega_q d_{11}| < \omega_q \leqslant 0.3\omega_r$ for the secular and adiabatic approximations to hold. Moreover, the conditions under which (85) was deduced lead to $\omega_\pm \simeq \omega_r$ [see (90)], so $\gamma_{20}(\omega_\pm)/2$ can be quite large in some experiments. For example, the circuit QED experiment in [41] has an oscillator frequency $\omega_r = 2\pi \times (70\text{–}81) \times 10^9$ 1/s and relaxation rates in the range $(21\text{–}43) \times 10^9$ 1/s. In any case, the master equation (85) cannot describe the whole adiabatic regime because the qubit frequency cannot be arbitrarily small. In addition, restricting to the aforementioned subspace can be very limiting. For example, if $|g/\omega_r| = 0.2$, then $\{|E_{n\pm}\rangle : n \ll |\omega_r/(2g)|^2 = 6.25\}$ and one would be limited to the first few dressed states.

On the other hand, the simplified master equation (84) used the quasisecular approximation, which only requires the relaxation rates to be much smaller than the oscillator frequency ω_r. This allows one to consider arbitrary values of the qubit-oscillator coupling, qubit frequencies $\omega_q \lesssim 0.1\omega_r$, and one can use the whole qubit-oscillator state space. A consequence of the quasisecular approximation is the appearance of the two addends in the last line of the right-hand side of (84) because they are associated with the small frequencies in items 4(a) and 4(b) before Sec. IV A. These terms do not appear in (85) because it used the secular approximation, restricted the dynamics to the subspace spanned by $\{|E_{n\pm}\rangle : n \ll |\omega_r/(2g)|^2\}$, and considered $|g/\omega_r| \leqslant 0.2$. The last two conditions eliminate all terms multiplied by $\gamma_{j0}(-\omega_q d_{nn})$ in the aforementioned addends because $-\omega_q d_{nn} < 0$ for $n \ll |\omega_r/(2g)|^2$ and $|g/\omega_r| \leqslant 0.2$, while the secular approximation eliminates the rest.

(C3) The simplified master equation (84) requires near equality of the relaxation rates $\gamma_{20}(\Delta^\pm_{n+1,n})$ and $\gamma_{20}(\omega_r)$. This is reasonable because $|\Delta^\pm_{n+1,n} - \omega_r| \leqslant \omega_q$ and the simplified master equation considers the adiabatic regime ($\omega_q \lesssim 0.1\omega_r$). On the other hand, the master equation in (85) does not require the aforementioned near equality of the relaxation rates because $\gamma_{20}(\omega_\pm)$ appear in (85). Nevertheless, observe from (90) that $\omega_\pm$ are very close to ω_r when $|g/\omega_r| \leqslant 0.2$ and $\omega_q \leqslant 0.3\omega_r$, the conditions for the master equation (85). Also, this difference decreases for smaller values of ω_q because $|\Delta^\pm_{n+1,n} - \omega_r| \leqslant \omega_q$ and $\omega_\pm$ come from $\Delta^\pm_{n+1,n}$ [see (90)].

The relaxation rates $\gamma_{j0}(\omega_q d_{nn})$ appearing in the simplified master equation (84) are valid for arbitrary values of the qubit-oscillator coupling $|g/\omega_r|$. If one restricts to $|g/\omega_r| \leqslant 0.2$, then it follows from (90) that $\gamma_{j0}(\omega_q d_{nn}) = \gamma_{j0}(\tilde{\omega}_n)$.

(C4) It was shown in the previous section that the simplified master equation does reduce to the correct equation when $\omega_q = 0$. Nevertheless, the master equation in (85) does not reduce to the correct equation when $\omega_q = 0$. If one takes $\omega_q = 0$, one finds that the master equation (85) is identical to the simplified master equation in (84) except for the two addends in the last line of the right-hand side of (84). As a consequence, the master equation in (85) cannot correctly describe the dynamics of the qubit-oscillator system when $\omega_q = 0$. For example, when $\omega_q = 0$ the qubit-oscillator Hamiltonian is given by $\hat{H}_{\rm DL}$ in (6) and its lowest energy $\hbar\omega_0$ is two degenerate. Two linearly independent eigenvectors of $\hat{H}_{\rm DL}$ associated with $\hbar\omega_0$ are $|\omega_0, \pm\rangle$. Hence, $|\omega_0, \pm\rangle\langle\omega_0, \pm|$ should be stationary states when $\omega_q = 0$. Nevertheless, they are not stationary states of the master equation in (85) when $\omega_q = 0$ unless $\gamma_{10}(0) = \gamma_{20}(0) = 0$. Even if $\gamma_{10}(0) = \gamma_{20}(0) = 0$, the master equation in (85) does not reduce to the correct equation because it is still missing the second addend in the last line of the right-hand side of (84).

(C5) The simplified master equation presented in the previous section allows one to consider $T > 0$, while [21] only reported the zero-temperature case. This is important because ω_q can be in the radio-frequency or microwave regime, so the terms multiplied by $N(\omega_q d_{nn}, T)$ in the simplified master equation may be non-negligible.

In conclusion, the simplified master equation allows one to consider arbitrary values of the qubit-oscillator coupling $|g/\omega_r|$, the whole state space of the qubit-oscillator system, qubit frequencies $\omega_q \lesssim 0.1\omega_r$, a temperature $T \geqslant 0$, and only requires relaxation rates much smaller than the oscillator frequency ω_r and the near equality of the relaxation rates mentioned above. In addition, it also allows one to consider the case $\omega_q \to 0$.

C. Addition of qubit driving

The purpose of this section is to determine the effects of qubit driving on the qubit-oscillator open system. Previous experiments in circuit QED [42–46] have studied qubit-driving effects when the qubit-oscillator coupling strength is small compared to the oscillator frequency $|g/\omega_r| \ll 0.1$ and have observed both the resonant [43] and dispersive [42] photon-blockade phenomenon, the Autler-Townes doublet [45], the Mollow triplet in the resonance fluorescence spectrum [43–45], and have demonstrated amplification of a microwave signal [46]. This section differs from those works in that USC and DSC values of $|g|$ can be considered and the qubit-oscillator system is in the adiabatic regime ($\omega_q \lesssim 0.1\omega_r$).

In the rest of the article assume that the temperature T of the thermal baths is sufficiently low so that

$$N(\omega_r, T) \simeq 0. \quad (92)$$

Then, one can neglect all terms in the simplified master equation (66)–(69) multiplied by $N(\omega_r, T)$. All of these are contained in $\mathcal{D}_{\rm HO}(\hat{\rho})$, which takes the form of the dissipator for the zero-temperature, damped harmonic oscillator (recall that $\hat{b}^\dagger$ and $\hat{b}$ are creation and annihilation operators of a harmonic oscillator):

$$\mathcal{D}_{\rm HO}(\hat{\rho}) = \gamma_0\big(\hat{b}\hat{\rho}\hat{b}^\dagger - \tfrac{1}{2}\{\hat{b}^\dagger\hat{b}, \hat{\rho}\}\big). \quad (93)$$

It follows that the stationary solution of the simplified master equation with (93) is given by a statistical mixture involving the two lowest-energy eigenstates of $\hat{H}$:

$$\hat{\rho}_{\rm ST} = \frac{N(\omega_q d_{00},T)}{2N(\omega_q d_{00},T)+1}|E_{0,+}\rangle\langle E_{0,+}| + \frac{N(\omega_q d_{00},T)+1}{2N(\omega_q d_{00},T)+1}|E_{0,-}\rangle\langle E_{0,-}|. \tag{94}$$

Observe that $\hat{\rho}_{\rm ST}$ tends to a maximum mixed state for $\frac{1}{2} \ll N(\omega_q d_{00}, T)$, the mean number of thermal photons at frequency $\omega_q d_{00}$ if the thermal baths represent electromagnetic fields. Also, if $N(\omega_q d_{00}, T) \simeq 0$, then $\hat{\rho}_{\rm ST}$ tends to $|E_{0,-}\rangle\langle E_{0,-}|$ which corresponds to the lowest-energy eigenstate of $\hat{H}$.

We now add sinusoidal qubit driving to the system. The master equation takes the form

$$\frac{d}{dt}\hat{\rho}(t) = -\frac{i}{\hbar}[-\hbar\Omega_d \cos(\omega_d t+\phi)\hat{\sigma}_x, \hat{\rho}(t)] + \mathcal{L}_{\rm HO}\hat{\rho}(t) + \mathcal{L}_{12}\hat{\rho}(t), \tag{95}$$

where $\omega_d, \Omega_d > 0$ are the driving angular frequency and strength, respectively, and ϕ is a real number representing the initial phase of the driving. Notice that (95) is identical to the simplified master equation (66)–(69) except for two things: it uses the dissipator in (93) because we are assuming $N(\omega_r, T) \simeq 0$ and the first addend on the right-hand side of (95) was added to describe the driving. Here we are assuming that the driving does not alter dissipation. This happens at least when the driving is weak. We state this assumption explicitly further below.

Before proceeding, it is important to identify three facts. First, in the orthonormal basis of adiabatic eigenstates $\{|E_{n,\pm}\rangle : n = 0, 1, 2, \dots\}$ the driving $-\hbar\Omega_d\cos(\omega_d t+\phi)\hat{\sigma}_x$ has nonzero matrix elements only between the states $|E_{n,+}\rangle$ and $|E_{n,-}\rangle$ because

$$\hat{\sigma}_x|E_{n,\pm}\rangle = |E_{n,\mp}\rangle. \tag{96}$$

Second, the superoperators $\mathcal{L}_{\rm HO}$ and $\mathcal{L}_{12}$ in the master equation (95) do not provoke transitions from $|E_{0,\pm}\rangle$ to any other states $|E_{n,\pm}\rangle$ with $n \neq 0$. Third, if there is no driving, the stationary solution (94) of the qubit-oscillator system involves only the two lowest-energy eigenstates $|E_{0,\pm}\rangle$ of $\hat{H}$. Hence, if one first lets the qubit-oscillator system relax to the steady-state solution (94) and then one applies the driving, the dynamics of the qubit-oscillator system is going to involve only the two lowest-energy eigenstates $|E_{0,\pm}\rangle$ of $\hat{H}$. Therefore, one can assume that the qubit-oscillator density operator has the form

$$\hat{\rho}(t) = A_{11}(t)|E_{0,+}\rangle\langle E_{0,+}| + [1-A_{11}(t)]|E_{0,-}\rangle\langle E_{0,-}| + A_{12}(t)|E_{0,+}\rangle\langle E_{0,-}| + A_{21}(t)|E_{0,-}\rangle\langle E_{0,+}|. \tag{97}$$

In fact, since the dynamics only involves the two lowest-energy eigenstates $|E_{0,\pm}\rangle$ of $\hat{H}$, the driven qubit-oscillator system behaves as a *new qubit* Q: $|E_{0,+}\rangle$ is the excited state, $|E_{0,-}\rangle$ is the ground state, and from (13) and (15) one finds that the angular transition frequency of Q is

$$\frac{E_{0+}-E_{0-}}{\hbar} = \omega_q d_{00} = \omega_q e^{-2|g/\omega_r|^2}. \tag{98}$$

Notice that the qubit-oscillator coupling g alters the transition frequency of the new qubit Q. In particular, USC and DSC values $|g/\omega_r| \geqslant 0.1$ considerably decrease its value with respect to the transition frequency ω_q.

From the discussion in the previous paragraph the driven qubit-oscillator open system behaves like a driven qubit open system, so one can obtain evolution equations that have a form similar to those of the *Bloch vector* obtained from the *optical Bloch equations* in a frame rotating about the z axis at the driving frequency ω_d [13]. In order to do this, it is convenient to use the quantities

$$\begin{aligned} u(t) &= \frac{1}{2}[A_{21}(t)e^{-i(\omega_d t+\phi)} + A_{12}(t)e^{i(\omega_d t+\phi)}],\\ v(t) &= -\frac{i}{2}[A_{21}(t)e^{-i(\omega_d t+\phi)} - A_{12}(t)e^{i(\omega_d t+\phi)}],\\ w(t) &= A_{11}(t) - \frac{1}{2}. \end{aligned} \tag{99}$$

Also, define *effective relaxation rates* Γ_{12} and γ_{12} and the *detuning* δ_d between the driving frequency ω_d and the shifted new qubit frequency $(\omega_q d_{00} + \delta'_{0+} - \delta'_{0-})$:

$$\begin{aligned} \Gamma_{12} &= \left[\gamma_{10}(\omega_q d_{00}) + \left(\frac{g+g^*}{\omega_r}\right)^2 \gamma_{20}(\omega_q d_{00})\right] \times[1+2N(\omega_q d_{00},T)],\\ \gamma_{12} &= \Gamma_{12}\frac{\sqrt{N(\omega_q d_{00},T)[N(\omega_q d_{00},T)+1]}}{1+2N(\omega_q d_{00},T)},\\ \delta_d &= \omega_d - (\omega_q d_{00} + \delta'_{0+} - \delta'_{0-}). \end{aligned} \tag{100}$$

Recall that $\gamma_{j0}(\omega')$ are defined in (56) and correspond to spontaneous transition rates coming from thermal bath B_j. Also, $\delta'_{0\pm}$ are defined in (70) and denote frequency shifts due to the interactions with the thermal baths. Observe that Γ_{12} is composed of relaxation rates coming from both thermal baths B_1 and B_2 and that it describes both spontaneous and thermal induced transitions due to the factor $[1 + 2N(\omega_q d_{00}, T)]$. In addition, notice that the qubit-oscillator coupling g alters the relaxation rate coming from B_2 because it is multiplied by $[(g + g^*)/\omega_r]^2$. If g is real, the relaxation rate coming from B_2 is reduced for $0 < |g/\omega_r| < 0.5$ and it is enhanced for $|g/\omega_r| > 0.5$. Moreover, it is suppressed entirely if g is pure imaginary. The same behavior is observed in γ_{12}.

Substituting (97) into (95) and using (99) one obtains

$$\begin{aligned} \frac{d}{dt}u(t) &= -\frac{\Gamma_{12}}{2}u(t) + \delta_d v(t) + \frac{\gamma_{12}}{2}e^{-i2(\omega_d t+\phi)}[u(t)-iv(t)]\\ &\quad + \frac{\gamma_{12}}{2}e^{i2(\omega_d t+\phi)}[u(t)+iv(t)]\\ &\quad + \Omega_d \sin[2(\omega_d t+\phi)]w(t),\\ \frac{d}{dt}v(t) &= -\delta_d u(t) - \frac{\Gamma_{12}}{2}v(t) + \Omega_d w(t)\\ &\quad - i\frac{\gamma_{12}}{2}e^{-i2(\omega_d t+\phi)}[u(t)-iv(t)]\\ &\quad + i\frac{\gamma_{12}}{2}e^{i2(\omega_d t+\phi)}[u(t)+iv(t)]\\ &\quad + \Omega_d\cos[2(\omega_d t+\phi)]w(t), \end{aligned}$$

$$\frac{d}{dt}w(t) = -\Omega_d v(t) - \Gamma_{12}w(t) - \frac{\Gamma_{12}}{2[1+2N(\omega_q d_{00}, T)]} + i\frac{\Omega_d}{2}e^{i2(\omega_d t+\phi)}[u(t)+iv(t)] - i\frac{\Omega_d}{2}e^{-i2(\omega_d t+\phi)}[u(t)-iv(t)]. \tag{101}$$

In the rest of the article assume that

$$|\delta_d|, \Gamma_{12}, \frac{\gamma_{12}}{2}, \Omega_d \ll \omega_d. \tag{102}$$

Observe that we are assuming quasiresonance between the driving frequency and the shifted new qubit transition frequency and weak dissipation and driving.

Then, one can apply the RWA to the equations in (101), that is, one can average them in an interval of length π/ω_d to obtain equations describing accurately the evolution of the quantities in (99):

$$\begin{aligned}\frac{d}{dt}w(t) &= -\Omega_d v(t) - \Gamma_{12}w(t) - \frac{\Gamma_{12}}{2[1+2N(\omega_q d_{00}, T)]},\\ \frac{d}{dt}v(t) &= -\delta_d u(t) - \frac{\Gamma_{12}}{2}v(t) + \Omega_d w(t),\\ \frac{d}{dt}u(t) &= -\frac{\Gamma_{12}}{2}u(t) + \delta_d v(t).\end{aligned} \tag{103}$$

These are identical in form to the equations for the Bloch vector obtained from the optical Bloch equations in a frame rotating about the z axis at the frequency ω_d [13], except for the factor $1/[1+2N(\omega_q d_{00}, T)]$ in the last addend of the right-hand side of the first equation in (103). In particular, they have exactly the same form if $N(\omega_q d_{00}, T) = 0$.

The steady-state solution of (103) is

$$\begin{aligned}u_{ss} &= \frac{\delta_d}{-\Omega_d}\left(\frac{s}{1+s}\right)\frac{1}{1+2N(\omega_q d_{00}, T)},\\ v_{ss} &= \frac{\Gamma_{12}}{-2\Omega_d}\left(\frac{s}{1+s}\right)\frac{1}{1+2N(\omega_q d_{00}, T)},\\ w_{ss} &= -\left[\frac{1}{2(1+s)}\right]\frac{1}{1+2N(\omega_q d_{00}, T)},\end{aligned} \tag{104}$$

where s is the *saturation parameter*:

$$s = \frac{\Omega_d^2/2}{\delta_d^2 + \frac{\Gamma_{12}^2}{4}}. \tag{105}$$

From (104) one finds the steady-state behavior of the coefficients $A_{ij}(t)$:

$$\begin{aligned}A_{11}^{ss} &= \left[\frac{s}{2(1+s)} + N(\omega_q d_{00}, T)\right]\frac{1}{1+2N(\omega_q d_{00}, T)},\\ A_{12}^{ss}(t) &= -\left(\frac{s}{1+s}\right)\left(\frac{\delta_d}{\Omega_d} - i\frac{\Gamma_{12}}{2\Omega_d}\right)\frac{e^{-i(\omega_d t+\phi)}}{1+2N(\omega_q d_{00}, T)},\\ A_{21}^{ss}(t) &= A_{12}^{ss}(t)^*.\end{aligned} \tag{106}$$

First observe that A_{11}^{ss} is the steady-state probability to find the qubit-oscillator in the state $|E_{0,+}\rangle$ and that its maximum value is $\frac{1}{2}$. It is approximately reached when

$$s \gg 1 \quad \text{or} \quad N(\omega_q d_{00}, T) \gg \tfrac{1}{2}. \tag{107}$$

Notice that $s \gg 1$ implies $\Omega_d \gg |\delta_d|, \Gamma_{12}$.

If $1 \ll s$, then $A_{12}^{ss}(t) = A_{21}^{ss}(t)^* \simeq 0$ and the qubit-oscillator steady-state density operator is approximately the maximum mixed state involving the lowest-energy eigenstates of $\hat{H}$:

$$\hat{\rho}_{ss} \simeq \tfrac{1}{2}|E_{0,+}\rangle\langle E_{0,+}| + \tfrac{1}{2}|E_{0,-}\rangle\langle E_{0,-}|. \tag{108}$$

If $N(\omega_q d_{00}, T) \gg \frac{1}{2}$, then (108) is approximately the stationary state (94) and the effect of the driving is negligible. The driving plays a most significant role when $N(\omega_q d_{00}, T) \simeq 0$ because (108) is very different from the stationary state (94), which reduces to $|E_{0-}\rangle\langle E_{0-}|$. At this point it is convenient to discuss if $N(\omega_q d_{00}, T) \simeq 0$ is possible with current parameters in experiments. Circuit QED experiments typically use resonators with a frequency $\omega_r/(2\pi) = 5$–15 GHz where microwave electronics are well developed [8]. Nevertheless, some have higher frequencies $\omega_r/(2\pi) \sim 100$ GHz [41], while others can have a fundamental frequency in the radio-frequency regime $\omega_r/(2\pi) = 92$ MHz [47]. We use the typical values. The adiabatic regime requires a small qubit frequency $\omega_q \ll \omega_r$, say, $\omega_q = 0.1\omega_r$. Then, one would have $\omega_q/(2\pi) = 0.5$–1.5 GHz. If, in addition, one considers a DSC value $|g/\omega_r| = 1$, then the new qubit frequency in (98) satisfies $\omega_q d_{00}/(2\pi) = 0.07$–0.2 GHz. Therefore, $N(\omega_q d_{00}, T) \leqslant 0.1$ requires a temperature $T \leqslant 1$–4 mK. Current circuit QED setups can operate at a temperature $T < 100$ mK [3,8,31,41,45–48] with some [41,45–48] working in the temperature range $T = 10$–20 mK and being able to detect radio-frequency photons [48]. If temperatures $T \sim 1$ mK are reached, then $N(\omega_q d_{00}) \leqslant 0.1$ can become feasible. On the other hand, the setup in [41] has $\omega_r/(2\pi) = 70$–81 GHz, so $\omega_q = 0.1\omega_r$ and $|g/\omega_r| = 1$ give a new qubit frequency $\omega_q d_{00}/(2\pi) = 0.95$–1.1 GHz. In this case, $N(\omega_q d_{00}) \leqslant 0.1$ requires a temperature $T \leqslant 19$–22 mK. This is feasible in some circuit QED setups such as [41] which works at $T = 15$ mK.

Using (106) one obtains the steady-state behavior of the components of the Bloch vector

$$\begin{aligned}\langle\hat{\sigma}_x\rangle^{ss}(t) &= \left[\frac{\delta_d}{\Omega_d}\cos(\omega_d t+\phi) - \frac{\Gamma_{12}}{2\Omega_d}\sin(\omega_d t+\phi)\right] \times\left(\frac{s}{1+s}\right)\frac{-2}{1+2N(\omega_q d_{00}, T)},\\ \langle\hat{\sigma}_y\rangle^{ss}(t) &= \left[\frac{\delta_d}{\Omega_d}\sin(\omega_d t+\phi) + \frac{\Gamma_{12}}{2\Omega_d}\cos(\omega_d t+\phi)\right] \times\left(\frac{s}{1+s}\right)\frac{-2e^{-2|g/\omega_r|^2}}{1+2N(\omega_q d_{00}, T)},\\ \langle\hat{\sigma}_z\rangle^{ss}(t) &= -\left(\frac{1}{1+s}\right)\frac{e^{-2|g/\omega_r|^2}}{1+2N(\omega_q d_{00}, T)}.\end{aligned} \tag{109}$$

Observe that all of the expected values in (109) are approximately equal to zero if $1 \ll s$. Also, notice that one has $\langle\hat{\sigma}_y\rangle^{ss}(t) \simeq 0$ and $\langle\hat{\sigma}_z\rangle^{ss}(t) \simeq 0$ for DSC values $|g/\omega_r| \gg 1$, although $\langle\hat{\sigma}_x\rangle^{ss}(t)$ is not affected. Therefore, the qubit-oscillator coupling g has three effects: it reduces the effective transition frequency [see (98)], it decreases or enhances the effective relaxation rate Γ_{12} [see (100)], and it decreases the steady-state expected values $\langle\hat{\sigma}_y\rangle^{ss}(t)$ and $\langle\hat{\sigma}_z\rangle^{ss}(t)$. Also, since the driven qubit-oscillator open system behaves as a driven qubit open system, one should have phenomena similar to the resonance fluorescence Mollow triplet [40,43–45].

To end this section, we observe that one must be careful with the interpretation of $u(t)$, $v(t)$, and $w(t)$ defined in (99) due to the relationship between the coefficients $A_{jk}(t)$ and the expected values of the Pauli operators $\hat{\sigma}_j$ $(j = x, y, z)$. In the case of the optical Bloch equations $u(t)$, $v(t)$, and $w(t)$ are the average values of $\hat{\sigma}_x$, $\hat{\sigma}_y$, and $\hat{\sigma}_z$ in a frame rotating about the z axis at frequency ω_d and one has

$$\begin{aligned} A_{21}(t) &= \frac{1}{2}[\langle\hat{\sigma}_x\rangle(t) + i\langle\hat{\sigma}_y\rangle(t)],\\ A_{12}(t) &= \frac{1}{2}[\langle\hat{\sigma}_x\rangle(t) - i\langle\hat{\sigma}_y\rangle(t)],\\ A_{11}(t) &= \frac{1}{2} + \frac{\langle\hat{\sigma}_z\rangle(t)}{2}. \end{aligned} \tag{110}$$

In the system we are considering the relationships in (110) do not hold because one has

$$\begin{aligned} A_{21}(t) &= \frac{1}{2}\left[\langle\hat{\sigma}_x\rangle(t) + \frac{i}{d_{00}}\langle\hat{\sigma}_y\rangle(t)\right],\\ A_{12}(t) &= \frac{1}{2}\left[\langle\hat{\sigma}_x\rangle(t) - \frac{i}{d_{00}}\langle\hat{\sigma}_y\rangle(t)\right],\\ A_{11}(t) &= \frac{1}{2} + \frac{\langle\hat{\sigma}_z\rangle(t)}{2d_{00}}. \end{aligned} \tag{111}$$

The quantities in (111) reduce to the ones of the optical Bloch equations in (110) if and only if $d_{00} = 1$, that is, if and only if there is no qubit-oscillator coupling g.

V. CONCLUSIONS

This article considered an open quantum system composed of a qubit and a harmonic oscillator coupled to two independent thermal baths of harmonic oscillators. The case where the transition frequency ω_q of the qubit is much smaller than the oscillator frequency ω_r was considered and the master equation in the Lindblad form describing the evolution of the qubit-oscillator system was deduced. The regime of validity of this master equation is the following: the Born-Markov approximations must hold, $\omega_q \lesssim 0.1\omega_r$, the relaxation rates must be much smaller than ω_r, and there should be near equality of the relaxation rates at frequencies in the interval $[\omega_r - \omega_q, \omega_r + \omega_q]$. The master equation is valid for all values of the qubit-oscillator coupling g and in all the qubit-oscillator state space. Moreover, it reduces to the correct master equation when $\omega_q \to 0$. Also, it was shown that the coupling g can enhance or decrease both the relaxation rates and the frequency shifts induced by the thermal baths.

The effects of weak, sinusoidal qubit driving on the open system were also considered. It was found that the driven qubit-oscillator open system behaves like a sinusoidally driven qubit whose excited and ground states are the two lowest-energy states of the qubit-oscillator system and whose transition frequency decreases with increasing qubit-oscillator coupling strength $|g|$. The evolution of the driven qubit-oscillator open system is governed by equations similar to those of the Bloch vector in the optical Bloch equations. It was found that increasing the magnitude of the qubit-oscillator coupling $|g|$ decreases the steady-state expected values of the Pauli operators $\hat{\sigma}_y$ and $\hat{\sigma}_z$, although that of $\hat{\sigma}_x$ is not affected.

Finally, it was shown how one can reach the adiabatic and dispersive regimes by using sinusoidal qubit driving with large driving frequency and the concepts of π and $\pi/2$ pulses from atomic physics were generalized to manipulate transitions between dressed states of the qubit-oscillator system.

The results of the article are especially useful in areas where the ultrastrong coupling and deep strong coupling regimes can be reached, such as in circuit quantum electrodynamics.

APPENDIX

In this Appendix we deduce the adiabatic limit Hamiltonian $\hat{H}_{\mathrm{AL}}$ presented in (12). It is obtained by applying the RWA, that is, it is obtained by averaging the Schrödinger equation in an appropriate interaction picture (IP). This is an application of the *averaging theorem* of dynamical systems [24].

It is convenient to start with expression (5) for $\hat{H}$ because the eigenvectors and eigenvalues of $\hat{H}_{\mathrm{DL}}$ are used to define the appropriate IP. The first step is to pass to the IP defined by the unitary transformation

$$\hat{U}_{\mathrm{AL}}(t) = e^{-\frac{i}{\hbar}\hat{H}_{\mathrm{DL}}t}. \tag{A1}$$

Recall that $\hat{H}_{\mathrm{DL}}$ is defined in (6) and note that the subindex AL stands for *adiabatic limit*. For clarity, quantities in the IP defined by (A1) have a subindex AL: if $|\psi(t)\rangle$ is the state of the system and $\hat{A}(t)$ is a linear operator in the Schrödinger picture (SP), then the state of the system and the linear operator in the IP defined by (A1) are, respectively, given by

$$\begin{aligned} |\psi_{\mathrm{AL}}(t)\rangle &= \hat{U}^\dagger_{\mathrm{AL}}(t)|\psi(t)\rangle,\\ \hat{A}_{\mathrm{AL}}(t) &= \hat{U}^\dagger_{\mathrm{AL}}(t)\hat{A}(t)\hat{U}_{\mathrm{AL}}(t). \end{aligned} \tag{A2}$$

Using the orthonormal basis β for the qubit-oscillator state space composed of eigenvectors of $\hat{H}_{\mathrm{DL}}$ defined in (8)–(11), it follows that the Schrödinger equation in the IP defined by (A1) takes the form

$$\frac{d}{dt}|\psi_{\mathrm{AL}}(t)\rangle = -i\frac{\omega_q}{2}(\hat{\sigma}_z)_{\mathrm{AL}}(t)|\psi_{\mathrm{AL}}(t)\rangle, \tag{A3}$$

where

$$(\hat{\sigma}_z)_{\mathrm{AL}}(t) = \sum_{m,n=0}^{+\infty} e^{i\omega_r(n-m)t}(d_{nm}|\omega_n,-\rangle\langle\omega_m,+| + d^*_{mn}|\omega_n,+\rangle\langle\omega_m,-|) \tag{A4}$$

and

$$d_{mn} = \langle m|\hat{D}\left(2\frac{g}{\omega_r}\right)|n\rangle = e^{-2|\frac{g}{\omega_r}|^2}\sqrt{\frac{\min(m,n)!}{\max(m,n)!}}L^{|m-n|}_{\min(m,n)}\left(4\left|\frac{g}{\omega_r}\right|^2\right)v^{|m-n|}. \tag{A5}$$

Here $L^m_n(x)$ is the nth generalized Laguerre polynomial corresponding to the parameter m [25] and

$$v = \begin{Bmatrix} 2g/\omega_r & \text{if } m \geqslant n\\ -2g^*/\omega_r & \text{if } m < n \end{Bmatrix}. \tag{A6}$$

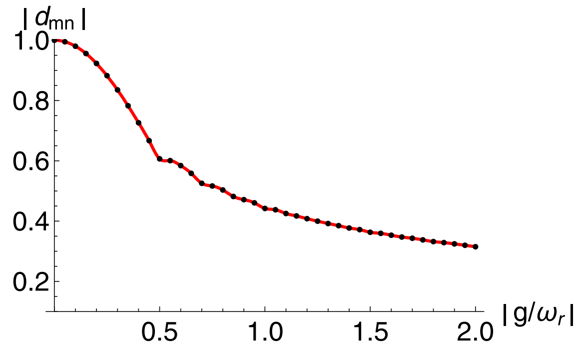

FIG. 2. The figure shows the maximum value of $|d_{mn}|$ for $m, n = 0, 1, 2, \ldots, 10^3$ as a function of $|g/\omega_r|$. The black points are the maximum value of $|d_{mn}|$ for $|g/\omega_r| = 5k \times 10^{-2}$ with $k = 0, 1, 2, \ldots, 40$. The red line corresponds to the cubic spline interpolant with a not-a-knot condition of the points.

Figure 2 shows the maximum value of $|d_{mn}|$ for $m, n = 0, 1, 2, \ldots, 10^3$ as a function of $|g/\omega_r|$. Observe that $|d_{mn}| \leqslant 1$. The data were obtained as follows: for each value $|g/\omega_r| = 5k \times 10^{-2}$ with $k = 0, 1, 2, \ldots, 40$, the maximum value of $|d_{mn}|$ was calculated for $m, n = 0, 1, 2, \ldots, 10^3$. The maximum values appear as black points in Fig. 2. Afterwards, a cubic spline interpolant with a not-a-knot condition [49] was constructed (see the red, solid line in Fig. 2).

In order to apply the RWA, it is better to measure time in units of 1 over the oscillator frequency ω_r. Let

$$\tau = \omega_r t, \quad |\Psi_{\mathrm{AL}}(\tau)\rangle = |\psi_{\mathrm{AL}}(\tau/\omega_r)\rangle. \tag{A7}$$

Notice that τ is a nondimensional time and that the IP Schrödinger equation (A3) takes the form

$$\begin{aligned}\frac{d}{d\tau}|\Psi_{\mathrm{AL}}(\tau)\rangle &= -i\left(\frac{\omega_q}{2\omega_r}\right)\sum_{m,n=0}^{+\infty} e^{i(n-m)\tau}(d_{nm}|\omega_n, -\rangle\langle\omega_m, +| \\ &\quad + d_{mn}^*|\omega_n, +\rangle\langle\omega_m, -|)|\Psi_{\mathrm{AL}}(\tau)\rangle.\end{aligned} \tag{A8}$$

Observe that $e^{i(n-m)\tau}$ are 2π periodic as a function of τ. Moreover, according to Fig. 2 the nondimensional coefficients appearing on the right-hand side of (A8) are bounded at least for $m, n = 0, 1, 2, \ldots, 10^3$ and $|g/\omega_r| \leqslant 2$:

$$\left|-i\left(\frac{\omega_q}{2\omega_r}\right)e^{i(n-m)\tau}\begin{Bmatrix} d_{nm} \\ d_{mn}^*\end{Bmatrix}\right| = \frac{\omega_q}{2\omega_r}|d_{mn}| \leqslant \frac{\omega_q}{2\omega_r}. \tag{A9}$$

Hence, one can apply the averaging theorem if $\omega_q/(2\omega_r) \ll 1$ and the dynamics of the qubit-oscillator system are restricted to $\beta_N = \{|\omega_n, \pm\rangle : n = 0, 1, \ldots, N\}$ with $N = 10^3$ and $|g/\omega_r| \leqslant 2$. If the bound in (A9) holds for all values of m, n and of the coupling strength $|g/\omega_r|$, then one could consider values of N and $|g/\omega_r|$ as large as one wants. We now show how the averaging theorem works.

The adiabatic regime is defined by the assumption in (4): $\omega_q \ll \omega_r$. Under this condition (and at least for $m, n = 0, 1, 2, \ldots, 10^3$ and $|g/\omega_r| \leqslant 2$), the modulus of each coefficient in (A9) is much smaller than 1. Hence, in the adiabatic regime it follows from (A8) that $|\Psi_{\mathrm{AL}}(\tau)\rangle$ is approximately constant in a τ interval of length 2π:

$$|\Psi_{\mathrm{AL}}(\tau')\rangle \simeq |\Psi_{\mathrm{AL}}(\tau)\rangle \quad (\tau \leqslant \tau' \leqslant \tau + 2\pi), \tag{A10}$$

while the exponentials $e^{i(n-m)\tau}$ with $n \neq m$ in (A9) perform one complete oscillation. The consequence of this slow evolution of $|\Psi_{\mathrm{AL}}(\tau)\rangle$ and fast evolution of the exponentials $e^{i(n-m)\tau}$ with $n \neq m$ is that the effect of the corresponding coefficients in (A8) averages to zero. Explicitly, averaging (A8) on a τ interval of length 2π and using the approximation in (A10) one obtains

$$\begin{aligned}\frac{d}{d\tau}|\Psi_{\mathrm{AL}}(\tau)\rangle &\simeq \frac{1}{2\pi}\int_{\tau}^{\tau+2\pi} d\tau' \frac{d}{d\tau'}|\Psi_{\mathrm{AL}}(\tau')\rangle \\ &\simeq \frac{1}{2\pi}\int_{\tau}^{\tau+2\pi} d\tau' \left(-i\frac{\omega_q}{2\omega_r}\right)\sum_{m,n=0}^{+\infty} e^{i(n-m)\tau'} \\ &\quad \times (d_{nm}|\omega_n, -\rangle\langle\omega_m, +| + d_{mn}^*|\omega_n, +\rangle\langle\omega_m, -|)|\Psi_{\mathrm{AL}}(\tau)\rangle \\ &= -i\frac{\omega_q}{2\omega_r}\sum_{n=0}^{+\infty} d_{nn}(|\omega_n, -\rangle\langle\omega_n, +| + |\omega_n, +\rangle\langle\omega_n, -|) \\ &\quad \times |\Psi_{\mathrm{AL}}(\tau)\rangle.\end{aligned} \tag{A11}$$

Observe that all terms multiplied by $e^{i(n-m)\tau'}$ with $n \neq m$ averaged to zero. Also, notice that (A10) was used in the first line of (A11) to consider $(d/d\tau')|\Psi_{\mathrm{AL}}(\tau')\rangle$ approximately constant in the τ interval of length 2π and to replace $|\Psi_{\mathrm{AL}}(\tau')\rangle$ by $|\Psi_{\mathrm{AL}}(\tau)\rangle$ inside the integral in the fourth line of (A11).

Using (A7) to introduce units and (A2) to return to the SP, one obtains

$$i\hbar\frac{d}{dt}|\psi(t)\rangle = \hat{H}_{\mathrm{AL}}|\psi(t)\rangle, \tag{A12}$$

where $\hat{H}_{\mathrm{AL}}$ is defined in (12).

It is important to observe that the averaging only requires $\omega_q/(2\omega_r) \ll 1$ and that the bound in (A9) holds for the values $|g/\omega_r|$ of the coupling strength and the value of N in $\beta_N = \{|\omega_n, \pm\rangle : n = 0, 1, \ldots, N\}$ that are considered. Moreover, $\hat{H}_{\mathrm{AL}}$ is going to be a better approximation to $\hat{H}$ for smaller values of $|\omega_q/\omega_r|$ and for larger values of the coupling strength $|g/\omega_r|$ because the bound on the coefficients in (A9) is smaller (see in Fig. 2 how $|d_{mn}|$ decreases as a function of $|g/\omega_r|$) and $|\Psi_{\mathrm{AL}}(\tau)\rangle$ is going to evolve more slowly.

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