

## Discrete scale-invariant boson-fermion duality in one dimension

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We introduce models of one-dimensional  $n(\geq 3)$ -body problems that undergo phase transition from a continuous scale-invariant phase to a discrete scale-invariant phase. In this paper, we focus on identical spinless particles that interact only through two-body contacts. Without assuming any particular cluster-decomposition property, we first classify all possible scale-invariant two-body contact interactions that respect unitarity, permutation invariance, and translation invariance in one dimension. We then present a criterion for the breakdown of continuous scale invariance to discrete scale invariance. Under the assumption that the criterion is met, we solve the many-body Schrödinger equation exactly; we obtain the exact  $n$ -body bound-state spectrum as well as the exact  $n$ -body  $S$ -matrix elements for arbitrary  $n \geq 3$ , all of which enjoy discrete scale invariance or log periodicity. Thanks to the boson-fermion duality, these results can be applied equally well to both bosons and fermions. Finally, we demonstrate how the criterion is met in the case of  $n = 3$ ; we determine the exact phase diagram for the scale-invariance breaking in the three-body problem of identical bosons and fermions. The zero-temperature transition from the unbroken phase to the broken phase is the Berezinskii-Kosterlitz-Thouless-like transition discussed in the literature.

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### I. INTRODUCTION

Discrete scale invariance, or scale invariance with respect to one particular scale, has attracted considerable attention in many scientific disciplines [1,2] because of its unique yet universal predictions. For example, in quantum scattering theory, discrete scale invariance manifests itself in log-periodic oscillations [3] of  $S$  matrices and in geometric scaling of bound-state energies. Let us first take a brief look at these ideas by using a toy example.

Consider a  $1 \times 1$   $S$  matrix  $S(E)$  in a specific channel, where  $E$  stands for energy. The most general scaling law that respects the unitarity  $|S(E)| = 1$  would have the following form:

$$S(e^t E) = S(E), \quad (1)$$

where  $t$  is a real parameter. If this holds for any continuous  $t \in \mathbb{R}$ , the general solution to Eq. (1) must be independent of the modulus of  $E$ ; that is, the  $S$  matrix is a constant in continuous scale-invariant theory. On the other hand, if Eq. (1) holds only for some discrete  $t \in t_* \mathbb{Z} = \{0, \pm t_*, \pm 2t_*, \dots\}$ , where  $t_*$  defines one particular scale, the general solution becomes  $S(E) = f(\ln E)$ , where  $f$  is a periodic function with period  $t_*$ ; that is, if continuous scale invariance is broken to discrete scale invariance, the  $S$  matrix exhibits periodic oscillations as a function of  $\ln E$ .

In addition to this log periodicity, discrete scale invariance also leads to a striking consequence in bound-state problems. Suppose that the  $S$  matrix has a bound-state pole along the negative  $E$  axis; that is,  $S(E) \rightarrow \frac{N_*}{E+E_*}$  as  $E \rightarrow -E_*$ , where

$E_* > 0$  and  $N_*$  are some constants. Then, the scaling law  $S(E) = S(e^{m_*} E)$  implies that there in fact exist infinitely many poles of the form

$$S(E) \rightarrow \frac{N_* e^{-m_*}}{E + E_* e^{-m_*}}, \quad n \in \mathbb{Z}. \quad (2)$$

Hence, in bound-state problems, discrete scale invariance manifests itself as the onset of infinitely many bound states with the energies  $E_n = -E_* e^{-n m_*}$ , which satisfy the geometric scaling  $E_{n+1} = e^{-t_*} E_n$ . Notice that the residues of the  $S$  matrix (2), which are related to normalization constants of bound-state wave functions (see, e.g., [4]), also satisfy the same geometric scaling.

The above discussion, although simplified, captures the general impact of discrete scale invariance in quantum theory. To date, a number of quantum systems that enjoy discrete scale invariance, log periodicity, or geometric scaling have been discovered; see [2] for a nice review. Among the notable examples is the Efimov effect [5,6] in three-body problems under two-body short-range interactions, where the geometric series of three-body bound states emerge if scattering lengths diverge and dimensionful parameters apparently disappear. Note, however, that this Efimov effect is known to be highly susceptible to particle statistics and dimensionality. For example, for three identical bosons, it was shown that the Efimov effect is present only when the spatial dimension  $d$  is in the range  $2.3 < d < 3.8$  [7]. As discussed in [8], this was due to the absence of nontrivial scale-invariant two-body contact interactions (at least in the limit of infinite scattering length) in other dimensions.

One purpose of this paper is to show that—contrary to the conventional wisdom—there in fact exist a lot of scale-invariant two-body contact interactions in one dimension if

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the number of particles is greater than two. Another purpose is then to present concrete examples of one-dimensional  $n(\geq 3)$ -body problems that undergo phase transition from a continuous scale-invariant phase to a discrete scale-invariant phase. For the sake of simplicity, in this work we focus on identical spinless particles that interact only through two-body contacts [9]. Remarkably, any such many-body systems generally enjoy the *boson-fermion duality*—the one-to-one correspondence between isospectral bosonic and fermionic systems—which enables us to treat bosons and fermions on equal footing. In essence, the boson-fermion duality in one dimension is just the equivalence between the even-parity sector of the  $\delta$ -function potential system and the odd-parity sector of the  $\varepsilon$ -function potential system [10]. The simplest application of this equivalence to many-body problems is the well-known boson-fermion duality between the Lieb-Liniger model [11] of identical spinless bosons and the Cheon-Shigehara model [12] of identical spinless fermions. Recently, it was shown [13] that this duality can be further generalized because one-dimensional two-body contact interactions have much more variety than previously investigated. Most importantly, this generalization includes scale-invariant two-body contact interactions which—at least at the formal level—render the system invariant under continuous scale transformation. Such continuous scale invariance, however, can be broken down to discrete scale invariance just as in the Efimov effect. The goal of this paper is to show that this indeed happens for both bosons and fermions and to present the *exact*  $n$ -body bound-state spectrum as well as the *exact*  $n$ -body  $S$ -matrix elements that exhibit geometric scaling and log periodicity. The key to this achievement is the *configuration-space approach* to identical particles [14–17]. Before discussing scale-invariance breaking, let us first briefly review the boson-fermion duality in [13] from the viewpoint of the configuration-space approach.

## II. BOSON-FERMION DUALITY IN ONE DIMENSION

Roughly speaking, the configuration-space approach is an approach to identical particles in which permutation invariance is regarded as *gauge symmetry*; that is, invariance of physical observables under permutation of multiparticle coordinates is merely a redundancy in description [17]. As in any gauge theory, all gauge-equivalent configurations are physically equivalent such that the configuration space must be a collection of inequivalent gauge orbits. To be more precise, given a one-particle configuration space  $X$ , the  $n$ -particle configuration space of identical particles is generally given by the orbit space  $\mathcal{M}_n = \hat{X}^n / S_n$ , where  $\hat{X}^n = X^n - \Delta_n$  is the configuration space of  $n$  distinguishable particles and  $S_n$  is the symmetric group. Here  $X^n$  stands for the Cartesian product of  $n$  copies of  $X$ , and  $\Delta_n$  is the set of coincidence points at which two or more particles occupy the same place simultaneously. In general, such a set can be defined as the following locus:

$$\Delta_n = \left\{ (x_1, \dots, x_n) \in X^n : \prod_{1 \leq j < k \leq n} \|x_j - x_k\| = 0 \right\}, \quad (3)$$

where  $\|\cdot\|$  stands for the norm equipped with  $X^n$ . Note that many-body contact interactions are those that have support

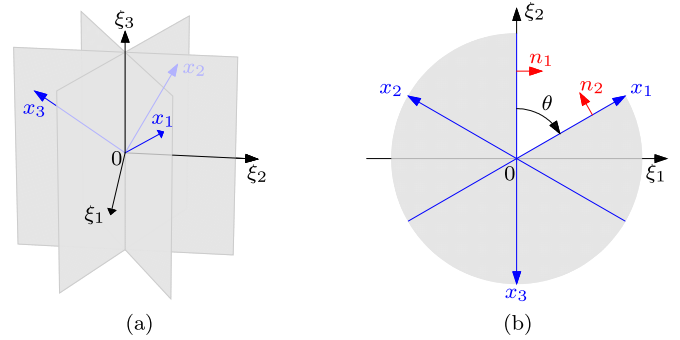


FIG. 1. Configuration space of three identical particles in one dimension. (a) The gray shaded regions represent the locus  $\Delta_3 = \{(x_1, x_2, x_3) : (x_1 - x_2)(x_1 - x_3)(x_2 - x_3) = 0\}$  which splits  $\mathbb{R}^3$  into  $3!$  disconnected regions. The three-body configuration space  $\mathcal{M}_3 = \{(x_1, x_2, x_3) : x_1 > x_2 > x_3\}$  is just one of those disconnected regions. The  $\xi_1$ ,  $\xi_2$ , and  $\xi_3$  axes point along the directions of the unit vectors  $\mathbf{e}_1 = \frac{1}{\sqrt{2}}(1, -1, 0)$ ,  $\mathbf{e}_2 = \frac{1}{\sqrt{6}}(1, 1, -2)$ , and  $\mathbf{e}_3 = \frac{1}{\sqrt{3}}(1, 1, 1)$ . (b) The blank white region represents the relative space  $\mathcal{R}_2 = \{(\xi_1, \xi_2) : 0 < \xi_1 < \sqrt{3}\xi_2\}$ , which is just the  $\xi_3 = \text{const}$  section of  $\mathcal{M}_3$ . The gray shaded region represents the impenetrable region for identical particles. The red arrows represent the inward-pointing unit normal vectors  $\mathbf{n}_1 = \frac{1}{\sqrt{2}}(1, -1, 0)$  and  $\mathbf{n}_2 = \frac{1}{\sqrt{2}}(0, 1, -1)$ .

only on  $\Delta_n$ , where wave functions become singular in general. Figure 1(a) shows  $\Delta_3$  for  $X = \mathbb{R}$ . [Note that, for  $X = \mathbb{R}$ ,  $\Delta_n$  can also be defined as the vanishing locus of the Vandermonde polynomial,  $\prod_{1 \leq j < k \leq n} (x_j - x_k) = 0$ .]

Now let us focus on the case  $X = \mathbb{R}$ , in which  $\hat{\mathbb{R}}^n \ni (x_1, \dots, x_n)$  consists of  $n!$  disconnected regions described by the inequality  $x_{\sigma(1)} > \dots > x_{\sigma(n)}$ , where  $\sigma \in S_n$  is a permutation of  $n$  indices. All of these  $n!$  regions are gauge equivalent for identical particles. Hence, the configuration space of  $n$  identical particles in one dimension can be identified with the following  $n$ -dimensional space:

$$\mathcal{M}_n = \{(x_1, \dots, x_n) : x_1 > \dots > x_n\}. \quad (4)$$

Note that this space has a number of nontrivial boundaries; see Fig. 1(b) for the case  $n = 3$ . Of particular importance are the following codimension-1 boundaries at which two out of  $n$  particles collide:

$$\begin{aligned} \partial \mathcal{M}_{n,j}^{2\text{-body}} &= \{(x_1, \dots, x_n) : x_1 > \dots > x_j \\ &= x_{j+1} > \dots > x_n\}, \end{aligned} \quad (5)$$

where  $j = 1, \dots, n - 1$ .

Let us now focus on the situation where identical particles freely propagate almost everywhere on the line yet interact only at the two-body coincidence points. Since all the coincidence points are excluded in  $\mathcal{M}_n$ , in the configuration-space approach the  $n$ -body Hamiltonian for such systems is just the following free Hamiltonian:

$$H_0 = -\frac{\hbar^2}{2m} \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}, \quad (6)$$

where  $m$  is the mass of the identical particles. The two-body contact interactions are then described by boundary conditions of wave functions at the codimension-1 boundaries (5).

Such boundary conditions must be chosen to fulfill unitarity (probability conservation). It is well known that such boundary conditions are generally given by the following Robin boundary conditions:

$$\frac{\partial \psi}{\partial n_j} - \frac{1}{a_j} \psi = 0 \quad \text{on} \quad \partial \mathcal{M}_{n,j}^{2\text{-body}}, \quad (7)$$

where  $\partial \psi / \partial n_j$  stands for the normal derivative to the boundary  $\partial \mathcal{M}_{n,j}^{2\text{-body}}$  given by

$$\frac{\partial \psi}{\partial n_j} = \mathbf{n}_j \cdot \nabla \psi = \frac{1}{\sqrt{2}} \left( \frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_{j+1}} \right) \psi. \quad (8)$$

Here  $\mathbf{n}_j = \nabla(x_j - x_{j+1}) / \|\nabla(x_j - x_{j+1})\| = \frac{1}{\sqrt{2}}(0, \dots, 0, 1, -1, 0, \dots, 0)$  is the inward-pointing unit normal vector [18] to the surface  $x_j - x_{j+1} = 0$ ,  $\nabla = (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n})$  is the derivative on  $\mathcal{M}_n$ , and  $a_j$  is a real parameter that can depend on the coordinates orthogonal to  $\mathbf{n}_j$ . In this way, in the configuration-space approach the free Hamiltonian (6) and the Robin boundary conditions (7) set the problem of identical spinless particles under two-body contact interactions.

Now, one may want to know how Eqs. (6) and (7) describe the boson-fermion duality in the conventional approach, where the configuration space is taken to be  $\mathbb{R}^n$  rather than  $\mathbb{R}^n/S_n$ . To see this, let us first construct conventional bosonic and fermionic wave functions on  $\mathbb{R}^n$ , which can easily be done by extending the domain of wave functions. Let  $\psi$  be a normalized wave function on  $\mathcal{M}_n$ , and let  $\mathbf{x} = (x_1, \dots, x_n)$  be in the region  $x_{\sigma(1)} > \dots > x_{\sigma(n)}$ . Then we define

$$\psi_B(\mathbf{x}) = \frac{1}{\sqrt{n!}} \psi(\sigma \mathbf{x}), \quad (9a)$$

$$\psi_F(\mathbf{x}) = \frac{1}{\sqrt{n!}} \text{sgn}(\sigma) \psi(\sigma \mathbf{x}), \quad (9b)$$

where  $\sigma \mathbf{x} = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$  and  $\text{sgn}(\sigma)$  stands for the signature of  $\sigma \in S_n$ . As  $\sigma$  runs through all the permutations, Eqs. (9a) and (9b) define the totally symmetric and antisymmetric functions on  $\mathbb{R}^n$ , thus providing wave functions of identical spinless bosons and fermions in the conventional approach. By construction, it is obvious that the identity  $\psi_F(\mathbf{x}) = \text{sgn}(\sigma) \psi_B(\mathbf{x})$  holds in the region  $x_{\sigma(1)} > \dots > x_{\sigma(n)}$ , which can be extended to  $\mathbb{R}^n$  in the following way:

$$\psi_F(\mathbf{x}) = \left( \prod_{1 \leq j < k \leq n} \text{sgn}(x_j - x_k) \right) \psi_B(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n, \quad (10)$$

where  $\text{sgn}(x) = x/|x|$  stands for the sign function. In this way, for identical spinless particles, the one-to-one correspondence holds between the bosonic and fermionic wave functions in the conventional approach. This is the celebrated boson-fermion mapping in one dimension [19].

Let us next construct the Hamiltonians for  $\psi_B$  and  $\psi_F$ , which can be achieved by studying connection conditions at the codimension-1 singularities in  $\mathbb{R}^n$ . To this end, let us first start with the following toy example:

$$f'(0_+) - \frac{1}{a} f(0_+) = 0, \quad (11)$$

where  $f(x)$  is some function on  $\mathbb{R}$  and the prime indicates the derivative with respect to  $x$ . If  $f(x)$  is an even function that satisfies  $f(x) = f(-x)$  and  $f'(x) = -f'(-x)$ , then  $f(0_+) = f(0_-)$  and  $f'(0_+) = -f'(0_-)$  automatically hold. Hence, for such even functions, the boundary condition (11) is equivalent to the connection condition  $f'(0_+) - f'(0_-) - \frac{1}{a}[f(0_+) + f(0_-)] = 0$  at  $x = 0$ . On the other hand, if  $f(x)$  is an odd function that satisfies  $f(x) = -f(-x)$  and  $f'(x) = f'(-x)$ , then  $f(0_+) = -f(0_-)$  and  $f'(0_+) = f'(0_-)$  automatically hold. Hence, for such odd functions, the boundary condition (11) is equivalent to the connection condition  $f'(0_+) + f'(0_-) - \frac{1}{a}[f(0_+) - f(0_-)] = 0$ .

The above discussion can easily be generalized to  $\psi_B$  and  $\psi_F$ . A careful analysis shows that, for totally symmetric functions, the Robin boundary condition (7) is equivalent to the following connection condition at the codimension-1 singularity  $\{x_{\sigma(1)} > \dots > x_{\sigma(j)} = x_{\sigma(j+1)} > \dots > x_{\sigma(n)}\}$  in  $\mathbb{R}^n$  [13]:

$$\left( \frac{\partial}{\partial x_{\sigma(j)}} - \frac{\partial}{\partial x_{\sigma(j+1)}} \right) \psi_B \Big|_{0_+} - \left( \frac{\partial}{\partial x_{\sigma(j)}} - \frac{\partial}{\partial x_{\sigma(j+1)}} \right) \psi_B \Big|_{0_-} - \frac{\sqrt{2}}{a_j} (\psi_B|_{0_+} + \psi_B|_{0_-}) = 0, \quad (12)$$

where  $|_{0_{\pm}}$  is shorthand for  $|_{x_{\sigma(j)} - x_{\sigma(j+1)} = 0_{\pm}}$  and  $\sigma$  is an even permutation. For totally antisymmetric functions, on the other hand, the Robin boundary condition (7) is equivalent to the following connection condition:

$$\left( \frac{\partial}{\partial x_{\sigma(j)}} - \frac{\partial}{\partial x_{\sigma(j+1)}} \right) \psi_F \Big|_{0_+} + \left( \frac{\partial}{\partial x_{\sigma(j)}} - \frac{\partial}{\partial x_{\sigma(j+1)}} \right) \psi_F \Big|_{0_-} - \frac{\sqrt{2}}{a_j} (\psi_F|_{0_+} - \psi_F|_{0_-}) = 0. \quad (13)$$

Note that, thanks to the symmetry property,  $\psi_B$  and the normal derivative of  $\psi_F$  are both continuous at the codimension-1 singularities.

Now, Eq. (12) together with  $\psi_B|_{0_+} = \psi_B|_{0_-}$  is nothing but the connection condition for the  $\delta$ -function potential  $\delta(x_{\sigma(j)} - x_{\sigma(j+1)}; \frac{\sqrt{2}}{a_j}) = \frac{\sqrt{2}}{a_j} \delta(x_{\sigma(j)} - x_{\sigma(j+1)})$  supported on the codimension-1 singularity  $\{x_{\sigma(1)} > \dots > x_{\sigma(j)} = x_{\sigma(j+1)} > \dots > x_{\sigma(n)}\}$  in  $\mathbb{R}^n$ , where the coupling constant is  $\sqrt{2}/a_j$ . On the other hand, Eq. (13) together with  $(\frac{\partial}{\partial x_{\sigma(j)}} - \frac{\partial}{\partial x_{\sigma(j+1)}}) \psi_F|_{0_+} = (\frac{\partial}{\partial x_{\sigma(j)}} - \frac{\partial}{\partial x_{\sigma(j+1)}}) \psi_F|_{0_-}$  is nothing but the connection conditions for the  $\varepsilon$ -function potential [20]  $\varepsilon(x_{\sigma(j)} - x_{\sigma(j+1)}; \frac{a_j}{\sqrt{2}})$ , where in this case the coupling constant is  $a_j/\sqrt{2}$ . Thus, we find the following Hamiltonians for  $\psi_B$  and  $\psi_F$ :

$$H_{B/F} = H_0 + V_{B/F}, \quad (14)$$

where

$$V_B = \frac{\hbar^2}{m} \sum_{j=1}^{n-1} \sum_{\sigma \in A_n} \left[ \prod_{k \in \{1, \dots, n-1\} \setminus \{j\}} \theta(x_{\sigma(k)} - x_{\sigma(k+1)}) \right] \times \delta(x_{\sigma(j)} - x_{\sigma(j+1)}; \frac{\sqrt{2}}{a_j}), \quad (15a)$$

$$V_F = \frac{\hbar^2}{m} \sum_{j=1}^{n-1} \sum_{\sigma \in A_n} \left[ \prod_{k \in \{1, \dots, n-1\} \setminus \{j\}} \theta(x_{\sigma(k)} - x_{\sigma(k+1)}) \right] \times \varepsilon(x_{\sigma(j)} - x_{\sigma(j+1)}; \frac{a_j}{\sqrt{2}}). \quad (15b)$$

Here  $A_n$  is the alternating group that consists of only even permutations. The factor  $\prod \theta(x_{\sigma(k)} - x_{\sigma(k+1)})$  is introduced in order to guarantee the ordering  $x_{\sigma(1)} > \dots > x_{\sigma(j)} = x_{\sigma(j+1)} > \dots > x_{\sigma(n)}$ , where  $\theta(x)$  is the step function. Note that, since the coupling constants of the two systems are inverse to each other, the one-to-one correspondence holds between the strong-coupling regime in one system and the weak-coupling regime in the other. This is a natural generalization of the celebrated strong-weak duality in [12]. Note also that, since the eigenvalue equations  $H_B \psi_B = E \psi_B$  and  $H_F \psi_F = E \psi_F$  both boil down to  $H_0 \psi = E \psi$  on  $\mathcal{M}_n$  with the same boundary conditions,  $H_B$  and  $H_F$  are completely isospectral.

To summarize, the  $n$ -body problem described by the free Hamiltonian (6) and the Robin boundary conditions (7) on  $\mathcal{M}_n$  is equivalent to the  $n$ -boson and  $n$ -fermion problems described by  $H_B$  and  $H_F$ . By construction, we have the spectral equivalence between  $H_B$  and  $H_F$ , the boson-fermion mapping between  $\psi_B$  and  $\psi_F$ , and the strong-weak duality. Notice that, if  $a_1 = \dots = a_{n-1} = \text{const}$ , Eq. (14) just reduces to the Lieb-Liniger model and the Cheon-Shigehara model. Note also that, since the  $n$ -body Hamiltonian (14) is of the form  $H = -\frac{\hbar^2}{2m} \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2} + V(x_1, \dots, x_n)$ , it in general does not admit any nontrivial cluster decomposition into the sum of cluster Hamiltonians and intercluster potentials. In other words, the  $n$ -boson and  $n$ -fermion systems in the present paper are generally  $n$ -body clusters that cannot be decomposed into subclusters. We will elaborate on this cluster property in Sec. V by using the three-body scattering problem.

Now, as noted before,  $a_j$  can depend on the coordinates orthogonal to  $\mathbf{n}_j$  without spoiling unitarity. This opens up a new vista for realizing scale invariance in one-dimensional  $n$ -body problems under two-body contact interactions. Let us next move on to study such a scale-invariant subfamily of the Robin boundary conditions (7).

### III. SCALE-INVARIANT TWO-BODY BOUNDARY CONDITIONS

To begin with, let us first introduce the normalized Jacobi coordinates  $(\xi_1, \dots, \xi_n)$  in  $\mathcal{M}_n$ , which can be defined through the orthogonal transformation  $x_j \mapsto \xi_j = \mathbf{e}_j \cdot \mathbf{x} = \sum_{k=1}^n e_{jk} x_k$ . Here  $\mathbf{e}_j = (e_{j1}, \dots, e_{jn})$  is the following

$n$ -dimensional orthonormal vector:

$$\mathbf{e}_j = \frac{1}{\sqrt{j(j+1)}} (1, \dots, 1, -j, 0, \dots, 0), \quad j \in \{1, \dots, n-1\}, \quad (16a)$$

$$\mathbf{e}_n = \frac{1}{\sqrt{n}} (1, \dots, 1), \quad (16b)$$

where  $-j$  in Eq. (16a) is in the  $(j+1)$ th component. Note that  $\mathbf{x}$  can be written as  $\mathbf{x} = \xi_1 \mathbf{e}_1 + \dots + \xi_n \mathbf{e}_n$ . Note also that  $\xi_n = \frac{x_1 + \dots + x_n}{\sqrt{n}}$ , which ranges from  $-\infty$  to  $\infty$ , corresponds to the center-of-mass coordinates. Hence, it is convenient to separate the one-dimensional subspace  $\mathbb{R} \mathbf{e}_n = \{\xi_n \mathbf{e}_n : -\infty < \xi_n < \infty\}$  from  $\mathcal{M}_n$ . It is also convenient to introduce the hyperspherical coordinates in the  $(n-1)$ -dimensional subspace spanned by the set of vectors  $\{\mathbf{e}_1, \dots, \mathbf{e}_{n-1}\}$ , which describes the relative space  $\mathcal{R}_{n-1}$  (the configuration space of relative motion). First, the hyperradius in the relative space  $\mathcal{R}_{n-1}$  is defined by

$$r = \sqrt{\xi_1^2 + \dots + \xi_{n-1}^2} = \sqrt{\frac{1}{n} \sum_{1 \leq j < k \leq n} (x_j - x_k)^2}. \quad (17)$$

The hyperangles in the relative space  $\mathcal{R}_{n-1}$  are then described by the unit vector  $\hat{\xi} = (\hat{\xi}_1, \dots, \hat{\xi}_{n-1}) := \frac{1}{r} (\xi_1, \dots, \xi_{n-1})$ , which, from the condition  $x_1 > \dots > x_n$ , must satisfy the condition  $0 < \hat{\xi}_1 < \dots < \sqrt{\frac{n(n-1)}{2}} \hat{\xi}_{n-1}$  [13]. The configuration space is then factorized as follows:

$$\mathcal{M}_n = \mathbb{R} \times \mathbb{R}_+ \times \Omega_{n-2}, \quad (18)$$

where  $\mathbb{R} = \{\xi_n : -\infty < \xi_n < \infty\}$  is the space of the center-of-mass motion,  $\mathbb{R}_+ = \{r : 0 < r < \infty\}$  is the space of the hyperradial motion, and  $\Omega_{n-2}$  is the space of the hyperangular motion given by

$$\Omega_{n-2} = \left\{ (\hat{\xi}_1, \dots, \hat{\xi}_{n-1}) : \hat{\xi}_1^2 + \dots + \hat{\xi}_{n-1}^2 = 1, \right. \\ \left. 0 < \hat{\xi}_1 < \dots < \sqrt{\frac{n(n-1)}{2}} \hat{\xi}_{n-1} \right\}. \quad (19)$$

As we will see shortly, this factorization plays a pivotal role in solving the  $n$ -body Schrödinger equation by the method of separation of variables. Note that, for  $n=2$ , the factor  $\Omega_{n-2}$  should be discarded in Eq. (18). Note also that the relative space  $\mathcal{R}_{n-1} = \mathbb{R}_+ \times \Omega_{n-2}$  can also be written as  $\mathcal{R}_{n-1} = \{(\xi_1, \dots, \xi_n) : 0 < \xi_1 < \dots < \sqrt{\frac{n(n-1)}{2}} \xi_{n-1}\}$ .

Now, in the coordinate system  $(\xi_n, r, \hat{\xi})$ , the gradient of a wave function  $\psi$  is written as follows:

$$\nabla \psi = \frac{\partial \psi}{\partial \xi_n} \mathbf{e}_{\xi_n} + \frac{\partial \psi}{\partial r} \mathbf{e}_r + \frac{1}{r} \nabla_{\Omega_{n-2}} \psi, \quad (20)$$

where  $\mathbf{e}_{\xi_n}$  and  $\mathbf{e}_r$  are the unit vectors pointing along the  $\xi_n$  and  $r$  directions and  $\nabla_{\Omega_{n-2}}$  is the gradient on  $\Omega_{n-2}$ . Notice that, for  $n \geq 3$ , all the normal vectors  $\mathbf{n}_j$  are orthogonal to  $\mathbf{e}_{\xi_n}$  and  $\mathbf{e}_r$ . (For  $n=2$ , the normal vector is equivalent to  $\mathbf{e}_r$ .)



Hence, for  $n \geq 3$ , the Robin boundary condition (7) is cast in the following form [21]:

$$\frac{1}{r} \mathbf{n}_j \cdot \nabla_{\Omega_{n-2}} \psi - \frac{1}{a_j} \psi = 0 \quad \text{on} \quad \partial \mathcal{M}_{n,j}^{2\text{-body}}. \quad (21)$$

Below we will focus on the case  $n \geq 3$ .

Now we are ready to identify the scale-invariant subfamily of the boundary conditions. First of all, under the scale transformation  $\mathcal{S}_\alpha : \mathbf{x} \mapsto \alpha \mathbf{x}$  ( $\alpha > 0$ ), one-dimensional  $n$ -body wave functions transform as follows:

$$\psi(\mathbf{x}) \mapsto (\mathcal{S}_\alpha \psi)(\mathbf{x}) := \alpha^{\frac{n}{2}} \psi(\alpha \mathbf{x}). \quad (22)$$

The boundary condition (21) is then said to be scale invariant if  $\mathcal{S}_\alpha \psi$  satisfies the same boundary condition as  $\psi$ . It is, however, obvious from the factor  $\frac{1}{r}$  that Eq. (21) does not remain unchanged under  $\mathcal{S}_\alpha$  unless  $a_j$  depends on the coordinates and transforms as  $a_j(\mathbf{x}) \mapsto a_j(\alpha \mathbf{x}) = \alpha a_j(\mathbf{x})$ . (Note that  $\nabla_{\Omega_{n-2}}$  is invariant under  $\mathcal{S}_\alpha$ .) Note also that coordinate-dependent  $a_j$  generally breaks translation invariance unless it satisfies  $a_j(x_1 + \beta, \dots, x_n + \beta) = a_j(x_1, \dots, x_n)$  for any real  $\beta$ . Hence, in order to realize scale- and translation-invariant boundary conditions,  $a_j(\mathbf{x})$  must satisfy the following conditions for the scaling law and translation invariance, respectively:

$$a_j(\alpha \mathbf{x}) = \alpha a_j(\mathbf{x}), \quad (23a)$$

$$a_j(\mathbf{x} + \beta \mathbf{e}_n) = a_j(\mathbf{x}). \quad (23b)$$

It is easy to see that the general solution to these conditions is given by

$$a_j(\mathbf{x}) = r g_j(\hat{\xi}), \quad (24)$$

where  $g_j$  is an arbitrary function of  $\hat{\xi}$ . The boundary condition (21) then becomes

$$\mathbf{n}_j \cdot \nabla_{\Omega_{n-2}} \psi - \frac{1}{g_j} \psi = 0 \quad \text{on} \quad \partial \mathcal{M}_{n,j}^{2\text{-body}}. \quad (25)$$

This describes the most general scale- and translation-invariant two-body contact interactions in the  $n (\geq 3)$ -body problems of identical spinless particles in one dimension. Since both the Hamiltonian and the boundary conditions are scale invariant, the  $n$ -body system described by Eqs. (6) and (25) is, at least formally, scale invariant. This continuous scale invariance, however, can be broken down to discrete scale invariance in exactly the same way as the Efimov effect. Let us next investigate the criterion for such symmetry breaking by using the  $n$ -body Schrödinger equation.

#### IV. FROM CONTINUOUS TO DISCRETE SCALE INVARIANCE

Let us study the time-independent  $n$ -body Schrödinger equation  $H_0 \psi = E \psi$ . To this end, we first note that, in the coordinate system  $(\xi_n, r, \hat{\xi})$ , the free Hamiltonian (6) can be written as  $H_0 = r^{-\frac{n-2}{2}} \tilde{H}_0 r^{\frac{n-2}{2}}$ , where [22]

$$\tilde{H}_0 = -\frac{\hbar^2}{2m} \left( \frac{\partial^2}{\partial r^2} + \frac{\Delta_{\Omega_{n-2}} - \frac{(n-2)(n-4)}{4}}{r^2} + \frac{\partial^2}{\partial \xi_n^2} \right). \quad (26)$$

Here  $\Delta_{\Omega_{n-2}}$  stands for the Laplacian on  $\Omega_{n-2}$ . Hence, under the assumption that the wave function has the form

$$\psi(\mathbf{x}) = r^{-\frac{n-2}{2}} \Psi(\xi_n) R(r) \Theta(\hat{\xi}), \quad (27)$$

the Schrödinger equation  $H_0 \psi = E \psi$  boils down to the following differential equations:

$$-\frac{\partial^2}{\partial \xi_n^2} \Psi(\xi_n) = \frac{2mE_{\text{cm}}}{\hbar^2} \Psi(\xi_n), \quad (28a)$$

$$-\Delta_{\Omega_{n-2}} \Theta(\hat{\xi}) = \lambda \Theta(\hat{\xi}), \quad (28b)$$

$$\left( -\frac{\partial^2}{\partial r^2} + \frac{\lambda + \frac{(n-2)(n-4)}{4}}{r^2} \right) R(r) = \frac{2mE_{\text{rel}}}{\hbar^2} R(r), \quad (28c)$$

where  $E_{\text{cm}} + E_{\text{rel}} = E$ . Note that the boundary condition (25) is only for the hyperangular wave function  $\Theta$ . In other words, all the information about the two-body contact interactions is encoded in the eigenvalue  $\lambda$ . Note also that, since  $\xi_n$  and  $r$  are permutation invariant, the totally symmetric and antisymmetric wave functions in  $\mathbb{R}^n$  are obtained by just extending the domain of  $\Theta$ . For example, in the region where  $\xi_n \in \mathbb{R}$ ,  $r \in \mathbb{R}_+$ , and  $\sigma \hat{\xi} \in \Omega_{n-2}$ , where  $\sigma \hat{\xi}$  stands for the action of the permutation  $\sigma \in S_n$  on the unit vector  $\hat{\xi}$ , we have

$$\psi_{\text{B}}(\xi_n, r, \hat{\xi}) = \frac{1}{\sqrt{n!}} r^{-\frac{n-2}{2}} \Psi(\xi_n) R(r) \Theta(\sigma \hat{\xi}), \quad (29a)$$

$$\psi_{\text{F}}(\xi_n, r, \hat{\xi}) = \frac{1}{\sqrt{n!}} \text{sgn}(\sigma) r^{-\frac{n-2}{2}} \Psi(\xi_n) R(r) \Theta(\sigma \hat{\xi}). \quad (29b)$$

As  $\sigma$  runs through all the permutations, the above equations define the  $n$ -body wave functions of identical bosons and fermions in  $\mathbb{R}^n$ . It should be emphasized that, if  $\Psi$ ,  $R$ , and  $\Theta$  are normalized solutions to Eqs. (28a)–(28c), then Eqs. (29a) and (29b) automatically become the normalized eigenfunctions of the Hamiltonians (14) with the eigenvalue  $E = E_{\text{cm}} + E_{\text{rel}}$ .

Now, it is well known in the context of the  $1/r^2$  potential that infinitely many discrete energy levels appear if  $\lambda + \frac{(n-2)(n-4)}{4} < -\frac{1}{4}$  [23]; that is, continuous scale invariance is broken down to discrete scale invariance if  $\lambda < \lambda_c$ , where  $\lambda_c = -\frac{(n-3)^2}{4}$  is the critical value [24]. Hence, the sufficient condition for the scale-invariance breaking is

$$\inf \sigma(-\Delta_{\Omega_{n-2}}) < \lambda_c, \quad (30)$$

where  $\sigma(-\Delta_{\Omega_{n-2}})$  stands for the spectrum of the operator  $-\Delta_{\Omega_{n-2}}$ . Let us, for the moment, assume that there exists at least one such negative eigenvalue and see the impact of discrete scale invariance by solving the hyperradial equation (28c) exactly.

##### A. Exact $n$ -body bound-state spectrum

Let us first consider the case  $E_{\text{rel}} < 0$ . In this case the normalized solution to the hyperradial equation (28c) for  $\lambda < \lambda_c$  is given by

$$\begin{aligned} R_{\kappa\lambda}(r) &= N_\kappa \sqrt{\frac{2\kappa r}{\pi}} K_{iv}(\kappa r) \\ &\rightarrow N_\kappa e^{-\kappa r} \quad \text{as} \quad r \rightarrow \infty, \end{aligned} \quad (31)$$

where

$$|N_\kappa| = \sqrt{\frac{\kappa \sinh(\nu\pi)}{\nu}}. \quad (32)$$

Here  $\kappa = \sqrt{2m|E_{\text{rel}}|/\hbar^2} > 0$ ,  $\nu = \sqrt{\lambda_c - \lambda} > 0$ , and  $K_{iv}$  is the modified Bessel function of the second kind. It is well known [23] that Eq. (31) provides orthogonal functions if  $\kappa$  is quantized as  $\kappa_\ell = \kappa_* e^{-\frac{\ell\pi}{\nu}}$ , where  $\kappa_* > 0$  is a newly emerged inverse length scale and  $\ell \in \mathbb{Z}$ . Thus, there exist infinitely many negative energy eigenvalues given by

$$E_{\text{rel}}^{(\ell)} = -\frac{\hbar^2 \kappa_*^2}{2m} \exp\left(-\frac{2\ell\pi}{\nu}\right), \quad \ell \in \mathbb{Z}. \quad (33)$$

These are the binding energies of  $n$ -body bound states of identical particles in the channel  $\lambda (< \lambda_c)$ . Note that Eq. (33) satisfies the geometric scaling  $E_{\text{rel}}^{(\ell+1)} = e^{-\frac{2\pi}{\nu}} E_{\text{rel}}^{(\ell)}$ , which—as discussed in the Introduction—is a manifestation of discrete scale invariance in the bound-state problem. It should also be noted that, under the full discrete scale invariance which forms the group  $\mathbb{Z}$ , the energy spectrum (33) cannot be bounded from below. In other words, in order to make the spectrum lower-bounded, we have to break this invariance under  $\mathbb{Z}$ . One easy way to achieve this is to cut off and regularize the inverse-square potential. Since the purpose of this paper is to demonstrate the full discrete scale invariance, we will not discuss this regularization procedure any further. For more details, we refer to the literature [25–32] (see also [33] for the related field-theory approach).

### B. Exact $n$ -body $S$ -matrix elements

Let us next consider the case  $E_{\text{rel}} > 0$ . In this case the solution to the hyperradial equation (28c) is given by the following linear combination:

$$R_{k\lambda}(r) = \sqrt{\frac{\pi kr}{2}} \left[ e^{-i\frac{(1+2iv)\pi}{4}} H_{iv}^{(2)}(kr) + S_\lambda(k) e^{+i\frac{(1+2iv)\pi}{4}} H_{iv}^{(1)}(kr) \right] \\ \rightarrow e^{-ikr} + S_\lambda(k) e^{+ikr} \quad \text{as } r \rightarrow \infty, \quad (34)$$

where  $k = \sqrt{2mE_{\text{rel}}/\hbar^2} > 0$ .  $H_{iv}^{(1)}$  and  $H_{iv}^{(2)}$  are the Hankel functions of the first and second kind, respectively. It is straightforward to show that Eqs. (31) and (34) become orthogonal if the coefficient  $S_\lambda(k)$  takes the following form [34]:

$$S_\lambda(k) = i \frac{\sinh[\nu\pi/2 - iv \ln(k/\kappa_*)]}{\sinh[\nu\pi/2 + iv \ln(k/\kappa_*)]}. \quad (35)$$

This is the  $S$ -matrix element of  $n$ -body scattering of identical particles in the channel  $\lambda (< \lambda_c)$ . Indeed, it satisfies unitarity,

$$\overline{S_\lambda(k)} S_\lambda(k) = 1, \quad \forall k > 0, \quad (36)$$

discrete scale invariance,

$$S_\lambda(e^{\frac{\pi}{\nu}} k) = S_\lambda(k), \quad \forall k > 0, \quad (37)$$

and possesses the following structure of bound-state poles and residues:

$$\lim_{k \rightarrow i\kappa_\ell} (k - i\kappa_\ell) S_\lambda(k) = i |N_{\kappa_\ell}|^2, \quad \forall \ell \in \mathbb{Z}, \quad (38)$$

all of which are desired properties of  $S$  matrix in discrete scale-invariant quantum theory [35]. Here the overline stands

for the complex conjugate. Note that Eq. (37) is equivalent to the log periodicity of  $S_\lambda(k)$  with the period  $\pi/\nu$  as a function of  $\ln k$ . As discussed in the Introduction, this log periodicity is a manifestation of discrete scale invariance in the scattering problem.

To summarize, we have seen that discrete scale invariance manifests itself in the geometric series of  $n$ -body bound states as well as in the log-periodic oscillation of  $n$ -body  $S$ -matrix elements. Note, however, that these exact results are based on the assumption that at least one negative eigenvalue  $\lambda (< \lambda_c)$  exists in the Laplace equation (28b). Let us finally investigate whether and when such a negative eigenvalue appears. To simplify the problem, below we will focus on the case  $n = 3$ , in which the critical value is  $\lambda_c = 0$ .

### V. EXAMPLE: EXACT PHASE DIAGRAM IN THE THREE-BODY PROBLEM

Now we wish to solve the Laplace equation on the three-body hyperangular space  $\Omega_1 = \{(\hat{\xi}_1, \hat{\xi}_2) : \hat{\xi}_1^2 + \hat{\xi}_2^2 = 1, 0 < \hat{\xi}_1 < \sqrt{3}\hat{\xi}_2\}$  with the scale-invariant boundary conditions. To this end, let us first introduce the following polar coordinates in  $\Omega_1$  [see Fig. 1(b)]:

$$(\hat{\xi}_1, \hat{\xi}_2) = (\sin \theta, \cos \theta), \quad (39)$$

where  $\theta \in (0, \frac{\pi}{3})$ . Then the Laplace equation (28b) simply becomes

$$-\frac{\partial^2}{\partial \theta^2} \Theta(\theta) = \lambda \Theta(\theta). \quad (40)$$

Note that the codimension-1 boundaries  $\partial \mathcal{M}_{3,1}^{2\text{-body}}$  and  $\partial \mathcal{M}_{3,2}^{2\text{-body}}$  correspond to  $\theta = 0$  and  $\theta = \frac{\pi}{3}$ , respectively. The inward-pointing unit normal vectors on these boundaries are  $\mathbf{n}_1 = \mathbf{e}_{\theta=0}$  and  $\mathbf{n}_2 = -\mathbf{e}_{\theta=\frac{\pi}{3}}$ , where  $\mathbf{e}_\theta$  stands for the unit vector in the  $\theta$  direction at the angle  $\theta$ . Since the gradient of scalar functions on  $\Omega_1$  is  $\nabla_{\Omega_1} = \mathbf{e}_\theta \frac{\partial}{\partial \theta}$ , the boundary conditions (25) read

$$+\frac{\partial}{\partial \theta} \Theta(\theta) - \frac{1}{g_1} \Theta(\theta) = 0 \quad \text{at } \theta = 0, \quad (41a)$$

$$-\frac{\partial}{\partial \theta} \Theta(\theta) - \frac{1}{g_2} \Theta(\theta) = 0 \quad \text{at } \theta = \frac{\pi}{3}, \quad (41b)$$

where  $g_1$  and  $g_2$  are real constants. [Note that, for  $n \geq 3$ ,  $g_j(\hat{\xi})$  in Eq. (24) becomes constant at the boundaries.]

Let us now solve the eigenvalue equation (40) with the above boundary conditions. First, the general solution for  $\lambda \neq 0$  is given by

$$\Theta_\lambda(\theta) = A(\lambda) e^{i\sqrt{\lambda}\theta} + B(\lambda) e^{i\sqrt{\lambda}(\frac{\pi}{3}-\theta)}, \quad (42)$$

where  $A(\lambda)$  and  $B(\lambda)$  are integration constants. By imposing the boundary conditions (41a) and (41b), we get the following quantization condition for  $\lambda$  [36]:

$$\tan\left(\frac{\pi}{3}\sqrt{\lambda}\right) = \frac{(g_1 + g_2)\sqrt{\lambda}}{g_1 g_2 \lambda - 1}, \quad (43)$$

or, equivalently,

$$[X - X_0(\lambda)]^2 - Y^2 = Z(\lambda), \quad (44)$$

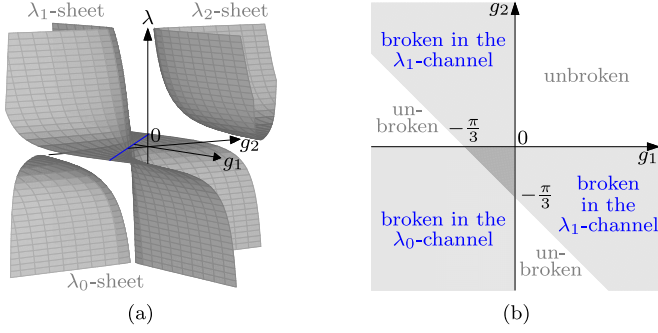


FIG. 2. (a) Parameter dependence of the eigenvalues  $\lambda_0 < \lambda_1 < \lambda_2 < \dots < \lambda_2$  is always positive.  $\lambda_1$  changes sign if we go across the blue line defined by  $g_1 + g_2 = -\pi/3$  on the  $\lambda = 0$  plane. The negative  $\lambda_1$  exists in the domain  $D_1 = \{(g_1, g_2) : -\pi/3 < g_1 + g_2 < |g_1 - g_2|\}$ .  $\lambda_0$  is always negative and exists in the domain  $D_0 = \{(g_1, g_2) : g_1 < 0, g_2 < 0\}$ . (b) Exact phase diagram. The light-gray shaded region represents the union  $D_0 \cup D_1$  in which continuous scale invariance is broken to discrete scale invariance in the  $\lambda_0$  or  $\lambda_1$  channel. The dark-gray region represents the intersection  $D_0 \cap D_1$  in which continuous scale invariance is broken in both the  $\lambda_0$  and  $\lambda_1$  channels.

where  $X = \frac{g_1 + g_2}{\sqrt{2}}$ ,  $Y = \frac{-g_1 + g_2}{\sqrt{2}}$ ,  $X_0(\lambda) = \sqrt{\frac{2}{\lambda}} \cot(\frac{\pi}{3}\sqrt{\lambda})$ , and  $Z(\lambda) = \frac{2}{\lambda} [1 + 2 \cot^2(\frac{\pi}{3}\sqrt{\lambda})]$ . Note that Eq. (44) defines infinitely many two-dimensional sheets in the  $(g_1, g_2, \lambda)$  space whose intersection with the  $\lambda = \text{const}$  plane is a hyperbola [see Fig. 2(a)]. As one can observe from Fig. 2(a), two distinct sheets—the  $\lambda_0$  sheet and the  $\lambda_1$  sheet—on which the eigenvalues  $\lambda_0$  and  $\lambda_1$  go below the critical value  $\lambda_c = 0$  exist. Close inspection shows that  $\lambda_0$  is always negative and exists only in the domain  $D_0 = \{(g_1, g_2) : g_1 < 0, g_2 < 0\}$ , while  $\lambda_1$  changes sign if it crosses the line  $g_1 + g_2 = -\pi/3$  and becomes negative in the domain  $D_1 = \{(g_1, g_2) : -\pi/3 < g_1 + g_2 < |g_1 - g_2|\}$ . Hence, in the region  $D_0 \cup D_1$ , continuous scale invariance is broken down to discrete scale invariance in the  $\lambda_0$  or  $\lambda_1$  channel. The exact phase diagram is depicted in Fig. 2(b). It should be noted that the zero-temperature transition from the unbroken phase to the broken phase is nothing but the Berezinskii-Kosterlitz-Thouless-like transition discussed in [37].

Before closing this section, it is worthwhile to revisit the three-body scattering by using the hyperangular wave function (42). To simplify the argument, let us consider the case  $g_1 = g_2 =: g \in (-\frac{\pi}{6}, 0)$ , in which two negative eigenvalues  $\lambda_0$  and  $\lambda_1$  appear. In this case, it is easy to show that the eigenfunctions take the following simple forms:

$$\Theta_{\lambda_0}(\theta) \propto e^{-\nu_0\theta} + e^{-\nu_0(\frac{\pi}{3}-\theta)}, \quad (45a)$$

$$\Theta_{\lambda_1}(\theta) \propto e^{-\nu_1\theta} - e^{-\nu_1(\frac{\pi}{3}-\theta)}, \quad (45b)$$

where  $\nu_0 = \sqrt{|\lambda_0|}$  and  $\nu_1 = \sqrt{|\lambda_1|}$  are the solutions to the conditions  $g = -\frac{1}{\nu} \coth(\frac{\pi}{6}\nu)$  and  $g = -\frac{1}{\nu} \tanh(\frac{\pi}{6}\nu)$ . Notice that the first terms in Eqs. (45a) and (45b) sharply localize to the boundary  $\theta = 0$  (i.e., the two-body coincidence point  $x_1 = x_2$ ) with the exponential decay rate  $1/\nu_{0,1}$ , while the second terms localize to the opposite boundary  $\theta = \pi/3$  (i.e.,  $x_2 = x_3$ ) with the same exponential decay rate. In other words, these eigenfunctions describe the superpositions of

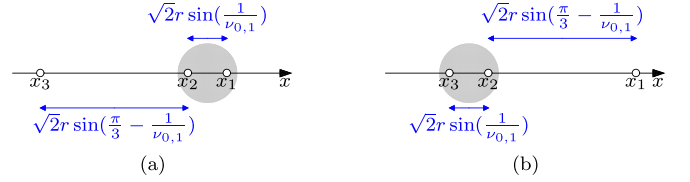


FIG. 3. Typical particle configurations for (a) the dimer on the right and (b) the dimer on the left. The sizes of the dimers are about  $\sqrt{2}r \sin(1/\nu_{0,1})$ , which follow from the exponential decay rates  $1/\nu_{0,1}$  in Eqs. (45a) and (45b) and the relations  $\frac{x_1 - x_2}{\sqrt{2}} = r \sin \theta$  and  $\frac{x_2 - x_3}{\sqrt{2}} = r \sin(\frac{\pi}{3} - \theta)$ . [Note that, in the three-body problem, the relations between the normalized Jacobi coordinates and the hyperspherical coordinates are  $\xi_1 = \frac{x_1 - x_2}{\sqrt{2}} = r \sin \theta$  and  $\xi_2 = \frac{x_1 + x_2 - 2x_3}{\sqrt{6}} = r \cos \theta$ , which lead to  $\frac{x_2 - x_3}{\sqrt{2}} = r \sin(\frac{\pi}{3} - \theta)$ .]

two “dimers” whose spatial extents are about  $\sqrt{2}r \sin(1/\nu_{0,1})$  (see Fig. 3). The physical meaning of the scattering solutions  $\Theta_{\lambda_{0,1}}(\theta)R_{k\lambda_{0,1}}(r)$  is now clear: they describe the superpositions of “atom-dimer” scatterings. Note, however, that these scatterings are not quite the same as the standard atom-dimer scattering in the three-dimensional Efimov effect [30] because the spatial extents of our dimers scale with the hyperradius  $r$ ; that is, the dimer’s size becomes smaller and smaller as the third particle approaches the dimer. This difference comes from the cluster properties of the models: in the standard atom-dimer scattering, the total Hamiltonian decomposes into the summation of the one-body cluster Hamiltonian, two-body cluster Hamiltonian, and intercluster potential between one- and two-body clusters, thereby determining the dimer’s size only through the two-body cluster Hamiltonian. In the present model, however, there is no such cluster decomposition such that the whole three-body system simply scales with  $r$ . This scaling is the characteristic feature of the three-body scattering in our model.

## VI. CONCLUSION

In this paper, we have introduced the most general scale-invariant model of  $n$  identical spinless particles in one dimension, where interparticle interactions are only two-body contacts. In this model, we have found that continuous scale invariance can be broken down to discrete scale invariance for any  $n \geq 3$ . The physical consequences of this scale-invariance breaking are the onset of geometric series of  $n$ -body bound states and the log-periodic oscillation of the  $n$ -body  $S$ -matrix elements. Thanks to the boson-fermion duality, our findings can be applied equally well to both bosons and fermions. We emphasize that our results are based on the assumption that the system fulfills (i) probability conservation, (ii) permutation invariance, (iii) translation invariance, and (iv) scale invariance. Hence, any  $n$ -body problems under two-body contact interactions that satisfy (i)–(iv) must fall into our model. It should also be emphasized that we did not require cluster separability in the present paper. If, in addition, one requires cluster separability, the available parameter space becomes a much smaller subspace. For example, if we required the three-body system to be decomposed into the one- and two-body clusters, then the scale invariance would be realized

only for the Dirichlet and Neumann boundary conditions at the two-body coincidence points, which correspond to  $g_j = 0$  and  $g_j = \infty$  in Eqs. (41a) and (41b), respectively. Hence, in this case the system cannot exhibit discrete scale invariance, which is consistent with the no-go result [7] that the Efimov effect cannot be realized in one dimension. Our nontrivial results are therefore applicable to  $n$ -body systems that cannot be decomposed into smaller subclusters.

Let us finally comment on the criterion (30). Our exact results (33) and (35) in the broken phase are based on the assumption that there exists at least one negative eigenvalue  $\lambda$  that satisfies the inequality  $\lambda < \lambda_c = -\frac{(n-3)^2}{4}$ . For  $n = 3$ , we have checked that this condition is indeed satisfied and

determined the exact phase diagram of scale-invariance breaking. However, the case  $n \geq 4$  is left open. From the physical viewpoint, it is quite reasonable to expect that, for sufficiently strong attractive interactions, at least one negative eigenvalue  $\lambda$  ( $< \lambda_c$ ) would always appear for any  $n$ . This is simply because, just as in the case of the ordinary  $\delta$ -function potential problem in one dimension, at least a single negative eigenvalue should appear and take an arbitrary large absolute value as we increase the strength of attractive coupling constants. Future studies should investigate whether and when the criterion is met for  $n \geq 4$  by employing the spectral analysis of the Laplace equation (28b) with the scale-invariant boundary conditions (25).

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