Supersensitivity of Kerr phase estimation with two-mode squeezed vacuum states

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We analytically investigate the sensitivity of Kerr nonlinear phase estimation in a Mach-Zehnder interferometer with two-mode squeezed vacuum states. We find that such a metrological scheme could access a sensitivity scaling over the Boixo *et al.*'s generalized sensitivity limit [S. Boixo *et al.*, Phys. Rev. Lett. **98**, 090401 (2007)], which is saturable with celebrated NOON states. We also show that parity detection is a quasioptimal measurement which can nearly saturate the quantum Cramér-Rao bound in the aforementioned situation. Moreover, we further clarify the supersensitive performance observed in the above scheme is due to the restriction of Boixo *et al.*'s generalized sensitivity limit (BGSL) to probe states with fixed photon numbers. To conquer this problem, we generalize the BGSL into the case with probe states of a fluctuating number of photons, to which our scheme belongs.

I. INTRODUCTION

The interferometer is a widely used optical device allowing one to implement precision measurements ranging from the first measurements of the speed of light to modern microscopic imaging and gravitational wave detection [1–4]. The optical interferometer has an atomic analog [5], too. Based on these devices, the problem of measurement of an unknown physical quantity is converted to the problem of estimating the relative phase shift between the two modes of the interferometer. Hence the sensitivity of phase estimation is a crucial factor to determine the performance of specific applications of precision measurement.

For linear phase estimations, the optical interferometer with N uncorrelated photons is highly possible to attain a phase uncertainty scaling as $N^{-1/2}$, as a consequence of the quantum fluctuation of photons. This sensitivity scaling is also referred to as the shot-noise limit, which would be broken when quantum resources are taken into account. To obtain sub-shot-noise-limit sensitivities, using nonclassical states of light has been theoretically and experimentally confirmed as an effective way. A large number of nonclassical states have been proposed to enhance the estimation sensitivity in both optical and atomic interferometry [3,5], such as squeezed states [6–19], Fock states [20–23], entangled coherent (EC) states [24], and other robust quantum states [25–30], etc. Moreover, using NOON states is expected to attain the Heisenberg limit N^{-1}, which was known as the ultimate accessible sensitivity in the linear phase estimation [20,31,32].

In a seminal work [33], Boixo *et al.* developed generalized sensitivity limits scaling with N^{-k} for single-parameter estimation with k the nonlinearity order of the Hamiltonian governing the system dynamics and N the total number of particles in the system. By replacing k-body interactions by N-body interactions, Roy *et al.* showed that an exponential enhanced accuracy would be obtained [34]. According to Boixo *et al.*'s generalized sensitivity limits (BGSLs), the ultimate sensitivity for a second-order nonlinear phase estimation (i.e., $k = 2$) should scale as N^{-2}, a N factor improvement over the Heisenberg limit (see Sec. IV for detailed discussions). Due to its potential to suppress the conventional Heisenberg limit in contrast to the linear one [35], nonlinear phase estimation has been receiving increasing attention. Several works identified that a Heisenberg-limit-scaling sensitivity is attainable even without the use of entangled resources in the nonlinear cases with single- and two-field modes [36–41]. Recently, It has shown that using coherent state of light may provide better accuracy than the Heisenberg limit in Kerr phase estimation by a scaling factor of $\bar{N}^{-3/2}$ with $\bar{N}$ the total mean number of photons [42,43]. Moreover, Joo *et al.* demonstrated that EC states of small photon numbers can outperform NOON states in nonlinear settings, suggesting that the BGSL would be further overcome [44]. Besides the demonstrations in optical systems, the study of nonlinear metrology in the atomic area is growing rapidly, such as interaction-based measurement of ensemble magnetization [40,45], precision measurement of atomic scattering [38,46,47], etc.

In this manuscript, we address the problem of Kerr-medium-induced phase estimation in a two-mode optical interferometry with two-mode squeezed vacuum (TMSV) states (see Fig. 1), due to its high feasibility in both optical [2,48] and atomic [5,9,10,14,15] experiments. By invoking the phase averaged approach, we analytically derive the ultimate phase uncertainties set by quantum Cramér-Rao bound in the aforementioned situation. We also identify that parity detection is a nearly optimal measurement for saturating the phase

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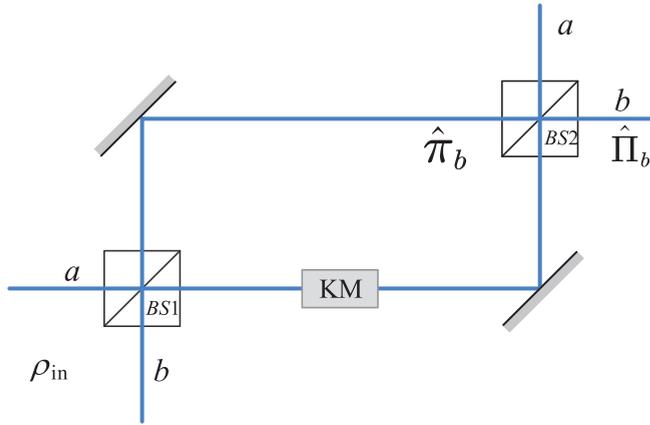
FIG. 1. Sketch of the Kerr nonlinear phase estimation in a standard Mach-Zehnder interferometer with a single-port parity detection.

uncertainties we derived. Our results suggest that the scheme with TMSV states is highly superior to the one with EC states [44], and also acquires a sensitivity beyond the BGSL [33]. More importantly, this sensitivity superiority effect can still be observed when the photon number becomes large, which is in sharp contrast with the previous result reported in Ref. [44]. Moreover, although widely used in various applications related to quantum teleportation [49], quantum dense coding [50], quantum illumination [51–53], quantum state tomography based on ultracold atomic ensemble [14], and especially linear phase estimation [8,15,54–56], the TMSV states have never been discussed in the literature on Kerr nonlinear phase estimation. Thus our work also serves to complement studies in this aspect.

This paper is organized as follows. In Sec. II, we introduce the Kerr phase estimation setup and derive the accessible phase uncertainties for both twin Fock (TF) and TMSV states with a single-port parity detection scheme. In Sec. III, we analytically compute the ultimate sensitivities for the above two states via the quantum Fisher information (QFI). In Sec. IV, we revisit the derivation of the BGSL and generalize it to the cases with probe states with variable photon numbers. Finally, our conclusions are given in Sec. V.

II. ACHIEVABLE SENSITIVITIES WITH PARITY DETECTION

The scheme of our Mach-Zehnder interferometer (MZI) setup is depicted in Fig. 1. A MZI is usually composed of two balanced beam splitters B_i ($i = 1, 2$) and a phase shifting U_φ with φ to be estimated. During the photon propagation between the BSs, an unknown phase of interest is accumulated. According to different propagation mechanisms, the operation of a phase shifter acting on the lower mode can be formally modeled by

$$U_\varphi = \exp[-i\varphi(a^\dagger a)^k], \tag{1}$$

where $a^\dagger (a)$ stands for the creation (annihilation) operator of the corresponding mode and the exponent k denotes the order of nonlinearity. In this expression, $k = 1$ corresponds to the linear phase shift and $k = 2$ to the Kerr nonlinear phase shift

[36,37]. Physically, they may describe the behavior of light propagating in free space and Kerr medium, respectively.

In what follows, we mainly focus on the case of $k = 2$, i.e., a Kerr phase shifting. Given B_i ($i = 1, 2$) (see below for specific expressions) and U_φ, the dynamics of the Mach-Zehnder interferometer is represented as a compound operation, i.e., $K_\varphi = B_2 U_\varphi B_1$. Let $\rho_{\rm in}$ denote the state of light entering at the input ports of the interferometer. Then, the state at the output ports reads $\rho_{\rm out} = K_\varphi \rho_{\rm in} K_\varphi^\dagger$. Finally, a measurement is performed at the output port of the interferometer and then the true value of the phase is extracted from the measurement outcomes. Given a measurement observable $\mathcal{O}$, the value of φ can be inferred from the average value of the observable $\langle \mathcal{O} \rangle$. The real accessible precision on φ is given by the error-propagation formula as follows [57]:

$$\Delta\varphi = \frac{1}{\sqrt{\upsilon}} \frac{\sqrt{\langle \mathcal{O}^2 \rangle - \langle \mathcal{O} \rangle^2}}{|\partial\langle \mathcal{O} \rangle / \partial\varphi|}, \tag{2}$$

with $\langle \cdot \rangle \equiv {\rm Tr}(\cdot \rho_{\rm out})$ being the expectation and υ the repetitions of the experiment. One should note that applying this method requires $\langle \mathcal{O} \rangle$ to be a monotonous function of the parameter φ at least in a local region of parameter values determined from prior knowledge [5].

We assume the TMSV states of light as the input states of the interferometer in the above setting. The TMSV states can be understood as a linear superposition of TF states $|n, n\rangle$ [22] (which is known as the Holland-Burnett state in the optical setting [20]) as

$$|\psi_{\rm TMSV}\rangle = \sum_{n=0}^{\infty} \sqrt{p_n} |n, n\rangle, \tag{3}$$

where

$$p_n = \left(1 - \frac{\bar{N}}{\bar{N} + 2}\right)\left(\frac{\bar{N}}{\bar{N} + 2}\right)^n, \tag{4}$$

with $\bar{N}$ the average photon number [58]. Hence the output state is given by $|\psi_{\rm out}\rangle = K_\varphi |\psi_{\rm TMSV}\rangle$. As shown in Fig. 1, a single-port parity detection is assumed to be carried out on the output mode b. The parity measurement was originally proposed to probe atomic frequency in trapped ions by Bollinger *et al.* [59] and later employed for optical interferometry by Gerry [60]. It accounts for distinguishing the states with even and odd numbers of photons in a given output port. Specifically, the parity is assigned as the value of $+1$ when the photon number of a state is even, and the value of -1 if odd. Hence it can be formulated as

$$\Pi_b = (-1)^{b^\dagger b} = \exp(i\pi b^\dagger b). \tag{5}$$

Due to the identity $\Pi_b^2 = \mathbb{1}$ with $\mathbb{1}$ being the identity matrix, the calculation of sensitivity from Eq. (2) can be simplified to the calculation of the expectation value of Π_b for the output state [30,61]. With Eq. (3), this expectation can be expressed as

$$\langle \Pi_b \rangle_{\rm TMSV} = \sum_{n=0}^{\infty} p_n \langle \Pi_b \rangle_{\rm TF}, \tag{6}$$

where

$$\langle \Pi_b \rangle_{\rm TF} \equiv \langle n, n | K_\varphi^\dagger \Pi_b K_\varphi | n, n \rangle \tag{7}$$

is the expectation value of Π_b for TF states. Note that the expression of Eq. (6) is a direct consequence of the commutation relation of the total photon number operator $a^\dagger a + b^\dagger b$ and the compound operator $K^\dagger \Pi_b K_\varphi$, i.e., $[a^\dagger a + b^\dagger b, K^\dagger \Pi_b K_\varphi] = 0$, which leads to a vanishing value of $\langle n', n'|K^\dagger \Pi_b K_\varphi |n, n\rangle$ when $n' \neq n$. To analytically derive Eq. (7), we here consider the second BS operation as a part of the measurement and hence the parity measurement through the BS is transformed into [28,62]

$$\pi_b = B_2 \Pi_b B_2^\dagger = \sum_{N=0}^{\infty} i^N \sum_{l=0}^{N} (-1)^l |l, N-l\rangle\langle N-l, l|, \quad (8)$$

where the operation of B_2 is formulated by $B_2 = \exp[-i\pi(a^\dagger b + ab^\dagger)/4]$ following the form adopted in [21]. It means that the measurement on the output mode b is equivalent to performing a projective measurement π_b to the state before the second BS B_2, i.e.,

$$|\psi_{2n}(\varphi)\rangle \equiv U_\varphi B_1 |n, n\rangle, \quad (9)$$

such that

$$\langle \Pi_b \rangle_{\text{TF}} = \langle \psi_{2n}(\varphi)|\pi_b|\psi_{2n}(\varphi)\rangle. \quad (10)$$

In the Schrödinger representation, the parametric TF states of Eq. (9) can be explicitly expressed as follows:

$$|\psi_{2n}(\varphi)\rangle = \sum_{k=0}^{n} C_{nk}\exp(-i4k^2\varphi)|2k, 2n-2k\rangle, \quad (11)$$

with

$$C_{nk} = (-1)^{n-k} \frac{1}{2^n}\left[\binom{2k}{k}\binom{2n-2k}{n-k}\right]^{1/2}. \quad (12)$$

Here we simply select the first BS operation in the form of $B_1 = \exp[\pi(a^\dagger b - a b^\dagger)/4]$ as in [21]. Although it does not satisfy the symmetric relation $\hat{B}_1 = \hat{B}_2^\dagger$ as usually assumed in previous studies [29,63], the final measurement results remain invariant, apart from a translation of $\pi/2$ in terms of φ. With Eqs. (8), (11), and (12), we explicitly derive the expectation of the parity operator for the TF states as

$$\langle \Pi_b \rangle_{\text{TF}} = \sum_{k=0}^{n} C_{nk}^2 \cos[4n(n-2k)\varphi]. \quad (13)$$

Inserting Eq. (13) into Eq. (6) then finally yields the signal of parity measurement $\langle \Pi_b \rangle_{\text{TMSV}}$ with respect to the TMSV states.

We plot in Fig. 2 the signals with parity measurement as a function of φ for both TF and TMSV states according to Eqs. (13) and (6). At first glance, they exhibit a completely different behavior. As seen in Fig. 2(a), the signal for the TF states has an oscillation with a period depending on the total photon number $2n$. In the cases with odd n, the oscillation amplitude varies within the range between -1 and 1 and, in the cases with even n, it varies within the range between -0.5 and 1. While, as shown in Fig. 2(b), the signal for the TMSV states does not have a behavior of the periodic oscillation as presented in Fig. 2(a) and varies only within the range between 0 and 1. Another key difference is that the signal for the TMSV states changes with the period of $\pi/2$ rad irrespective of the average photon number $\bar{N}$. In
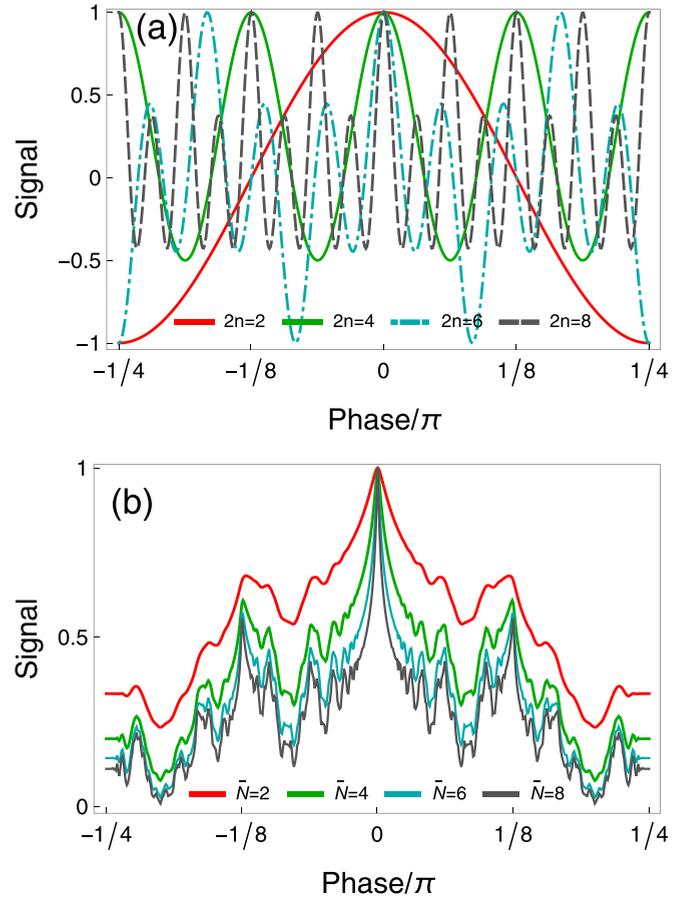
FIG. 2. Signals with parity detection versus phase for (a) TF states $|\psi_{\text{TF}}\rangle \equiv |n, n\rangle$ of total photon number $2n$ and (b) TMSV states $|\psi_{\text{TMSV}}\rangle$ of mean photon number $\bar{N}$. Different colors here refer to different numbers of $2n$ and $\bar{N}$, respectively. In (a) the frequency of signal oscillation becomes higher as n increases. In (b) the width of the peak at the zero phase point becomes narrower as $\bar{N}$ increases.

this case, the signal has a sharp peak at $\varphi = 0$ and the peak width becomes narrower as $\bar{N}$ increases, which will render a significant improvement in sensitivity as shown in Fig. 3. These distinctions can be understood from the expression of Eq. (6), which shows $\langle \Pi_b \rangle_{\text{TMSV}}$ is the weighted sum of $\langle \Pi_b \rangle_{\text{TF}}$ with the weights p_n given by Eq. (4).

Furthermore, with Eqs. (13) and (6) and according to Eq. (2), we numerically plot in Fig. 3 the phase uncertainties of Kerr phase measurement with parity detection for the TF and TMSV states around the zero point of φ. As a contrast, we take the EC state as a benchmark with the same detection strategy (see Appendix A for detailed derivation), and plot in Fig. 3 the phase uncertainty corresponding to Eq. (A9). Our results indicate that the TF state asymptotically approaches the BGSL $\bar{N}^{-2}$ as the number of photons decreases and saturates the limit at $\bar{N} = 2$ (see Sec. IV for a demonstration of saturation of the BGSL), where the probe state is a NOON state as a result of the Hong-Ou-Mandel effect [64]. We see that the TMSV state has a significant improvement in sensitivity over the EC state and both of them are able to overcome the uncertainty limit $\bar{N}^{-2}$. This seems to contradict the BGSL [33]. We note that such a counterintuitive behavior is attributed
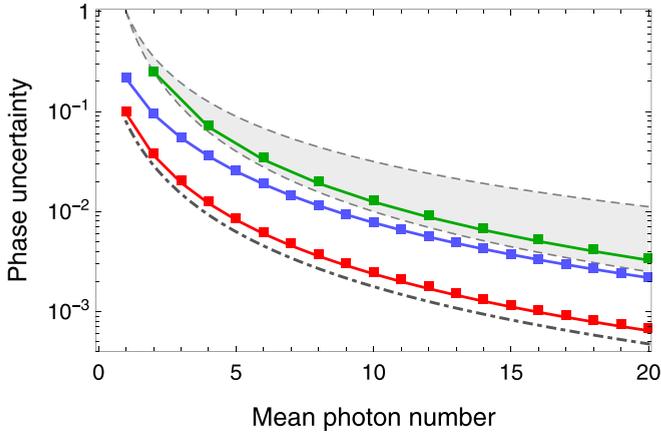
FIG. 3. Phase uncertainties as a function of mean photon number for TF (the green upper solid and squared-block line), TMSV (the red lower solid and squared-block line), and EC (the blue middle solid and squared-block line) states. The solid lines correspond to the ultimate sensitivities limited by QCR bound and the squared-block lines to those with parity measurement. The shaded area represents the uncertainty region bounded by $1/\bar{N}^{3/2}$ and $1/\bar{N}^2$ from [33]. The black dot-dashed line denotes the sensitivity limit defined by Eq. (39) for TMSV states.

to the problematic definition of the BGSL. It is true as the fundamental limit for the states of a definite photon number, but it is false for the cases with a fluctuating photon number. A similar phenomenon has also been observed in linear phase estimation [8,19]. In order to circumvent this problem, introducing a more general sensitivity limit valid for both cases has been of interest in several studies only related to linear phase estimation [32,65–67], but it is still an open question in the field of nonlinear Mach-Zehnder interferometry. We will further discuss this problem in Sec. IV by generalizing the BGSL into cases associating with the fluctuating number of photons.

III. ULTIMATE SENSITIVITIES DETERMINED BY QCR BOUND

In what follows, we wish to evaluate the ultimate sensitivities in the above scenarios based on quantum estimation theory, which states that, whatever measurement scheme is employed, the phase uncertainty of an unbiased estimator φ_{est} is determined by quantum Cramér-Rao (QCR) bound as

$$\delta\varphi_{\text{est}} \geqslant \frac{1}{\sqrt{\upsilon F}}, \tag{14}$$

where F is the so-called quantum Fisher information (QFI). It has been proven that the sensitivity by Eq. (2) could saturate the QCR bound with optimal measurement observables [68], of which the condition is, however, hardly satisfied in practical applications [69]. Thus it is often desirable to seek nearly optimal measurements which could closely approach the QCR bound.

To identify the effect of parity measurement on our case, we need to compare the sensitivities with parity detection as derived in the above section to the ultimate sensitivities by Eq. (14) under the same circumstance. For simplicity,

throughout this manuscript, we assume that the system is noiseless, in the sense that quantum states of consideration are pure. Hence, given $\rho_{\text{in}} = |\psi_{\text{in}}\rangle\langle\psi_{\text{in}}|$, the QFI in our quantum interferometry setting is given by

$$F = 4[\langle\psi_{\text{in}}|G^2|\psi_{\text{in}}\rangle - \langle\psi_{\text{in}}|G|\psi_{\text{in}}\rangle^2], \tag{15}$$

with $G \equiv B_1^\dagger (a^\dagger a)^2 B_1$. Note that a phase-averaging operation is required here in calculation of the QFI due to the lack of an external reference beam in our setting [63]. This is because the resolution of phase shift in interferometry may rely on coherence between states of different numbers of photon. However, this part of the resolution is generally not measurable when additional resources are lacking [63]. The issue under consideration here is related to this case as the TMSV state featuring a fluctuating photon number.

Under the phase-averaging operation, the TMSV state becomes a mixed state that consists of a statistical ensemble of TF states, that is,

$$\varrho_{\text{TMSV}} = \sum_{n=0}^{\infty} p_n |n, n\rangle\langle n, n|. \tag{16}$$

Although Eq. (15) is not valid for the state of Eq. (16), the QFI of this state can be directly obtained by

$$F_{\text{TMSV}} = \sum_{n=0}^{\infty} p_n F_{\text{TF}}(n), \tag{17}$$

as a consequence of the summability of the QFI [70,71]. Here $F_{\text{TF}}(n)$ refers to the QFI for the TF states, defined by Eq. (15) by replacing $|\psi_{\text{in}}\rangle$ with $|n, n\rangle$. Note that the above expression is valid for any order of nonlinearity given in Eq. (1). As shown in Eq. (17), the QFI is the sum of the QFI of the TF states with different n with probability p_n, which seems that the QFI of the TF state is essential contributing the QFI of the TMSV state, in the sense that one can acquire the same sensitivity reached with the TMSV state by sending a fixed number of photons in TF states with probability p_n. Although there is no essential difference mathematically between the TF and TMSV states in the current situation, the complications of preparing those states in experiments may be far more serious. In experiments, an effective way to prepare Fock states is to first produce pairs of light beams in the TMSV state from a pulsed parametric down-conversion source and project one of the beams onto a heralded Fock state by measuring another beam with a high-efficiency photon-number-resolving detector [72]. Obviously it is more complicated to create a heralded TF state because of twofold equipment for creating Fock states being involved [72]. Otherwise, it is generally a nontrivial task to produce Fock states of large photon numbers due to low probability of multiphoton events and low efficiency of the detector in resolving photons at high numbers [73,74]. Moreover, we learn from Eq. (17) that all pairs of Fock states contained in the TMSV state contribute to phase sensitivity. If we take the heralded TF state as the input state, those unheralded TF states contained in the entangled resources, which have been discarded during the state preparation, will not make any contribution to phase sensitivity. This causes a substantial waste of resources.

To calculate $F_{\text{TF}}(n)$, we need to first expand the G and G^2 defined in Eq. (15) in terms of a multiplication of operators consisting of creation and annihilation operators of the input modes with the help of

$$B_1^\dagger a^\dagger a B_1 = \frac{1}{2}(a^\dagger a + ab^\dagger + a^\dagger b + b^\dagger b), \qquad (18)$$

and then take the expectation over all TF states. This could be a daunting task involving a sum of hundreds of terms to calculate. But thanks to the orthogonality and normalization properties of Fock states, most of these terms are vanishing except for the ones with a (b) and $a^\dagger$ ($b^\dagger$) of equal count, for instance, $\langle aa^\dagger a^\dagger ab^\dagger bbb^\dagger \rangle_{\text{TF}} = (n^2 + n)^2$ but $\langle aaa^\dagger ab^\dagger bbb^\dagger \rangle_{\text{TF}} = 0$. Thus we get

$$\langle G \rangle_{\text{TF}} = \frac{1}{2}(3n^2 + n), \qquad (19)$$

$$\langle G^2 \rangle_{\text{TF}} = \frac{1}{8}(35n^4 + 30n^3 + n^2 - 2n). \qquad (20)$$

Using the above expressions, it is straightforward to obtain the QFI for Fock states:

$$F_{\text{TF}}(n) = \frac{17}{2}n^4 + 9n^3 - \frac{n^2}{2} - n. \qquad (21)$$

Our ultimate goal is to determine the sensitivities of Kerr phase measurement for the TMSV states. Combing Eq. (17) with Eq. (21) finally yields

$$F_{\text{TMSV}} = \frac{51}{4}\bar{N}^4 + 45\bar{N}^3 + 43\bar{N}^2 + 8\bar{N}, \qquad (22)$$

by utilizing the following equations:

$$\sum_{n=0}^{\infty} p_n n = \frac{\bar{N}}{2}, \qquad (23)$$

$$\sum_{n=0}^{\infty} p_n n^2 = \frac{1}{2}(\bar{N}^2 + \bar{N}), \qquad (24)$$

$$\sum_{n=0}^{\infty} p_n n^3 = \frac{1}{4}(3\bar{N}^3 + 6\bar{N}^2 + 2\bar{N}), \qquad (25)$$

$$\sum_{n=0}^{\infty} p_n n^4 = \frac{1}{2}(3\bar{N}^4 + 9\bar{N}^3 + 7\bar{N}^2 + \bar{N}). \qquad (26)$$

In addition, we also derive the accessible QFI for EC states in the present setting,

$$F_{\text{EC}} = 2\mathcal{N}_\alpha^2 \sum_{n=1}^{\infty} |c_n|^2 n^4, \qquad (27)$$

with $\mathcal{N}_\alpha = 1\sqrt{2(1 + e^{-|\alpha|^2})}$ the normalization factor of the EC state and $c_n = e^{-|\alpha|^2/2}\alpha^n/\sqrt{n!}$ the corresponding superposition coefficient in terms of NOON states of n photon numbers (see Appendix A for detailed derivation).

We plot in Fig. 3 the phase uncertainties corresponding to Eqs. (21), (22), and (27) for the three states: TF, TMSV, and EC, respectively. It is clearly shown that the phase uncertainty achieved with parity measurement for the EC state is identical with that determined by the QCR bound, in the sense that parity detection is an optimal measurement for EC states in Kerr phase estimation (see Appendix A for an explicit proof). While they are not identical for the TF and TMSV states, the difference between them is slightly small, in the sense that parity detection serves as a near-optimal measurement in Kerr
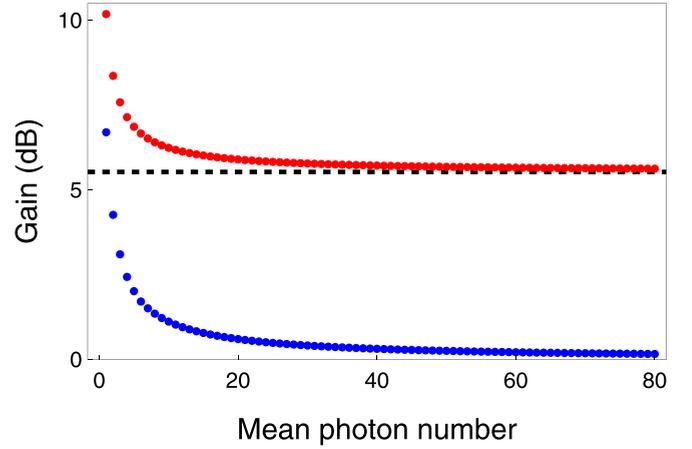
FIG. 4. Sensitivity gain defined by Eq. (28) for the TMSV and EC states as a function of mean photon numbers. The red upper dotted line corresponds to the TMSV state and the blue lower dotted line to the EC state. The black horizontal dashed line represents $g = -10 \log_{10}(\sqrt{4/51}) \sim 5.53$ dB in the infinite $\bar{N}$ limit.

phase estimation with these two states. It is also confirmed that, as suggested in the previous section, the BGSL $1/\bar{N}^2$ is overcome by both the TMSV and EC states. Unlike the EC states which lose their supersensitive advantage as the number of mean photons becomes sufficiently large, the TMSV states retain their capacity for overcoming the BGSL irrespective of the photon number.

In order to clearly show their difference, we plot in Fig. 4 the sensitivity gain which is defined with respect to the BGSL $1/\bar{N}^2$ as

$$g \equiv -10 \log_{10}(\bar{N}^2/\sqrt{F}). \qquad (28)$$

It is clear that the behavior of the TMSV states is in sharp contrast to the result of the EC states in that they display a supersensitive performance only in the region of a very modest photon number and perform equally well as the NOON states for larger $\bar{N}$ (see Sec. IV for demonstration of NOON states being able to saturate the BGSL of Kerr phase estimation). Similar results have been observed in [44] where a common reference beam is involved. Remarkably, the supersensitive advantage for the TMSV state is always maintained for all $\bar{N}$ and a gain of 5.53 dB is still expected for a large $\bar{N}$, while there is no potential gain for the EC state for sufficiently large $\bar{N}$.

IV. SENSITIVITY LIMITS FOR NONLINEAR MACH-ZEHNDER INTERFEROMETRY

As demonstrated in Sec. II, the supersensitive performance of the TMSV states over the BGSL is caused by the problematic definition of the BGSLs in the cases involving photon number fluctuation. Below, we address this problem by introducing a more general sensitivity limit for Kerr phase estimation with probe states of a fluctuating number of photons.

To solve this problem, we first revisit the method applied in [33] to derive the generalized sensitivity limits for single-parameter estimation with the k-order nonlinear coupling

Hamiltonian. Assume the phase accumulation is represented as a unitary operation $U_\varphi = \exp(-iH\varphi)$ where the generator H is the coupling Hamiltonian of N systems of the form [33,40],

$$H = \sum_{\{i_1, i_2, \cdots, i_k\}} h_{i_1} \otimes h_{i_2} \otimes \cdots \otimes h_{i_k}, \quad (29)$$

with the sum running over all subsets of k systems and h_{i_k} the dimensionless Hamiltonian of the i_kth subsystem. As shown in Eq. (14), the phase sensitivity is theoretically limited by the inverse of the QFI, which means that the larger value of the QFI is the higher sensitivity of phase estimation that could be acquired. Given a U_φ, the QFI is upper bounded by

$$\sqrt{F} \leqslant 2\Delta H \leqslant \|H\|, \quad (30)$$

where the first inequality is due to the fact that the QFI equals the variance for pure states and is less than the variance for mixed states [75] and $\|H\|$ is the operator seminorm of a Hermitian operator H defined as $\|H\| = \lambda_M - \lambda_m$ with $\lambda_M(\lambda_m)$ the maximum (minimum) eigenvalue of H [33]. These inequalities indicate that the estimation sensitivity limit is solely determined by the coupling Hamiltonian of the system. For the symmetric k-body coupling of Eq. (29), we have

$$\|H\| \leqslant \sum_{\{i_1, i_2, \ldots, i_k\}} \|h_{i_1} \otimes h_{i_2} \otimes \cdots \otimes h_{i_k}\|$$
$$\leqslant \binom{N}{k} \|h_1 \otimes h_2 \otimes \cdots \otimes h_k\|, \quad (31)$$

as a result of the triangle inequality property of the seminorm. Assuming $N \gg k$ and applying Stirling's approximation to the above expression finally yields the sensitivity limit that scales as [33,40]

$$\delta\varphi \sim \frac{k!}{N^k \|h_1 \otimes h_2 \otimes \cdots \otimes h_k\|} \sim \frac{1}{N^k}. \quad (32)$$

This limit was first proposed by Boixo *et al.* [33]. Note that the above expression simply provides a rough sensitivity limit for nonlinear phase estimation of a fixed number of particles N, but without assuming a specific form of H.

Now we apply the above method to analyze the sensitivity limit in nonlinear Mach-Zehnder interferometry. According to Eq. (1), one can identify $H = (a^\dagger a)^2$. Note that directly submitting this Hamiltonian into Eq. (30) may obtain an unachievable upper bound of sensitivity due to an immeasurable global phase. To derive a more tight sensitivity bound, we resort to the Schwinger representation as $J_x = (a^\dagger b + ab^\dagger)/2$, $J_y = (a^\dagger b - ab^\dagger)/2i$ and $J_z = (a^\dagger a - b^\dagger b)/2$. Under this representation, the Kerr Hamiltonian H can be divided into two parts as

$$H = \frac{\hat{N}^2}{4} + H_{\text{eff}}, \quad H_{\text{eff}} = J_z^2 + \hat{N}J_z, \quad (33)$$

with $\hat{N} = a^\dagger a + b^\dagger b$ the total photon number operator. Consider a probe state of fixed photon number N which can be written as follows:

$$|\psi_N\rangle = \sum_{n=0}^{N} C_n |n, N-n\rangle. \quad (34)$$

By changing into the basis space spanned by the common eigenstates $|j, m\rangle$ of the operators $J^2 = J_x^2 + J_y^2 + J_z^2$ and J_z, the expression of Eq. (34) can be rewritten as $|\psi_N\rangle = \sum_{m=-j}^{j} C_m |j, m\rangle$ with $j = N/2$. After the evolution with the Hamiltonian $\hat{N}^2/4$ the probe state $|\psi_N\rangle$ remains unchanged up to a global phase which cannot be measured. Hence the sensitivity limit is given by maximizing $F = 4\Delta^2 H_{\text{eff}}$ only dependent on the effective Hamiltonian H_{eff} given in Eq. (33). An optimal probe state $|\psi_N\rangle$ to maximize the variance of H_{eff} is the equally weighted superposition of $|j, -j\rangle$ and $|j, j\rangle$ up to an arbitrary relative phase, i.e., $(|j, -j\rangle + e^{i\phi}|j, j\rangle)/\sqrt{2}$, where $|j, -j\rangle$ and $|j, j\rangle$ correspond to the maximum and minimum eigenvalues of H_{eff}, respectively. It can be equivalently expressed in the Fock basis as NOON states $|\psi_{\text{NOON}}\rangle = (|N0\rangle + e^{i\phi}|0N\rangle)/\sqrt{2}$, with which the QFI takes the maximum value of $F_{\text{NOON}} = N^4$, in the sense that the sensitivity limit scales as $\delta\varphi \sim 1/(\sqrt{\upsilon}N^2)$, which is in agreement with the BGSL for second-order nonlinear phase estimation [33]. The same sensitivity limit and optimal probe states would be obtained if assuming $H = \hat{N}J_z$, which has been theoretically proposed and experimentally studied in atomic systems [38,40]. While it is different for $H = J_z^2$ [36,46,76,77] which is known as the one-axis twisting Hamiltonian in the atomic system. For this Hamiltonian, the sensitivity limit should scale with $\delta\varphi \sim 4/(\sqrt{\upsilon}N^2)$ and the optimal probe states for saturating the limit is $|\psi_N\rangle = (|j, 0\rangle + e^{i\phi}|j, j\rangle)/\sqrt{2}$.

Below we relax the constraint by allowing the total particle number to be fluctuating. Here, we simply follow the method used in [66] to derive the generalized sensitivity limit for nonlinear phase estimation with the variable particle number. The general states of the variable photon number can be represented in the form of

$$\varrho = \sum_{N=0}^{\infty} p_N |\psi_N\rangle\langle\psi_N|, \quad (35)$$

under the assumption of absence of a suitable phase reference beam [29,63,66,78,79]. This state can be obtained from a generic two-mode pure state $|\psi\rangle = \sum_{n,n'} C_{n,n'}|n, n'\rangle$ by taking the phase-averaged operation [29,63,79]. Correspondingly, the state of Eq. (35) is identified with $p_N = \sum_{n=0}^{N} |C_{n,N-n}|^2$ and

$$|\psi_N\rangle = \frac{1}{\sqrt{p_N}} \sum_{n=0}^{N} C_{n,N-n}|n, N-n\rangle. \quad (36)$$

In Eq. (35), it is an incoherent statistical ensemble of pure states of the form in Eq. (34) with a different number of photons. Based on the result derived for Eq. (34), the maximum QFI with respect to Eq. (35) is bounded by

$$F(\varrho) = \sum_N p_N F(|\psi_N\rangle) \leqslant \sum_N p_N N^4 = \langle \hat{N}^4 \rangle. \quad (37)$$

Thus the true sensitivity limit of nonlinear Mach-Zehnder interferometry should scale as

$$\delta\varphi \sim 1/(\sqrt{\upsilon\langle\hat{N}^4\rangle}), \quad (38)$$

when applying probe states with a fluctuating number of photons, such as the case encountered in our study.

Now let us come back to the initial question. In our case the input state is a phase-averaged TMSV state ϱ_{TMSV} given by Eq. (16). Thus the corresponding probe state is the state after applying the first beam splitter on the input state, i.e., $\varrho_{\text{TMSV}}^b = B_1 \varrho_{\text{TMSV}} B_1^\dagger$. The expectation value of the operator $\hat{N}^4$ with respect to ϱ_{TMSV}^b is equivalent to that with respect to ϱ_{TMSV}, i.e., $\langle \hat{N}^4 \rangle_{\varrho_{\text{TMSV}}^b} = \langle \hat{N}^4 \rangle_{\varrho_{\text{TMSV}}}$, due to the commutation of $[\hat{N}, B_i] = 0$ ($i = 1, 2$). It is thus straightforward to obtain

$$\langle \hat{N}^4 \rangle_{\varrho_{\text{TMSV}}} = 24\bar{N}^4 + 72\bar{N}^3 + 56\bar{N}^2 + 8\bar{N}, \qquad (39)$$

by using the results of $\langle n, n|\hat{N}^4|n, n\rangle = 16n^4$ and Eq. (26). We plot in Fig. 3 the true sensitivity limit in our situation by combing Eqs. (38) and (39), and learn that the limit is clearly not overcome by the TMSV states. This means that the sensitivity bound given by Eq. (38) is applicable to Kerr phase estimation with states of fluctuating particle number, but the BGSLs fail.

V. CONCLUSION

In this paper we have analytically discussed the phase enhancement of both TF and TMSV input states for a Kerr phase estimation using the QFI. We have shown that the TF states can approach the BGSL proposed by Boixo *et al.* [33], while the TMSV states can lead to a supersensitivity beyond the BGSL for any power of intensity of incident light, which is in sharp contrast to the EC states that display a supersensitive performance only in the region of a very modest photon number. With high power density a sensitivity gain of 5.53 dB with respect to the BGSL could be still acquired for the TMSV states. Meanwhile, on the basis of error propagation formula, we identify parity detection as a quasioptimal measurement for both TF and TMSV states and a genuine-optimal measurement for the EC state in the present Kerr nonlinear phase estimation settings.

Moreover, we elaborate that the supersensitive behavior observed with the TMSV state is attributed to the problematic definition of the BGSL for cases associating with a fluctuating number of photons. To address this problem, we propose a generalized BGSL which is applicable for these cases with probe states of a fluctuating number of photons, to which our scheme belongs. Our work may shine some light on quantum supersensitive measurements based on a Mach-Zehnder interferometer with nonlinear Kerr media.

APPENDIX: SENSITIVITY REACHED WITH ENTANGLED COHERENT STATES

In this Appendix, we calculate the phase sensitivities with EC states in our considered case. The EC state can be understood as a superposition of NOON states with different photons,

$$|\psi_{\text{EC}}\rangle = \mathcal{N}_\alpha \sum_{n=0}^{\infty} c_n[|n\rangle|0\rangle + |0\rangle|n\rangle], \qquad (A1)$$

with $\mathcal{N}_\alpha = 1\sqrt{2(1 + e^{-|\alpha|^2})}$ the normalization factor and $c_n = e^{-|\alpha|^2/2}\alpha^n/\sqrt{n!}$ the superposition coefficient. This state can be generated by powering a coherent state into one input port mode of a beam splitter and a coherent superposition of macroscopically distinct coherent states into another input port [80]. It has been demonstrated that the EC state of small photon numbers can overcome the sensitivity reached with NOON states [44]. According to the result given by Eq. (19) in Ref. [44], the QFI of the EC states is approximately expressed as

$$F_{\text{EC}}^r = \bar{N}^4 + 10\bar{N}^3 + 13\bar{N}^2 + 2\bar{N}, \qquad (A2)$$

for $|\alpha| \gg 1$ such that $\mathcal{N}_\alpha = 1/\sqrt{2}$. Obviously, the value of Eq. (22) is larger than above, in the sense that TMSV states outperform EC states in Kerr phase estimation.

However, the sensitivity given by Eq. (A2) cannot be generally saturated with rare photon-counting detection if without introducing additional resources [27,63,81], e.g., parity measurement applied in our manuscript where the reference beam is absent. In our case, a phase-averaged operation is required to derive an accessible sensitivity for the EC state. After the phase-averaged operation, the state of Eq. (A1) is straightforwardly expressed as

$$\varrho_{\text{EC}} = 2\mathcal{N}_\alpha^2 \sum_{n=0}^{\infty} |c_n|^2 |n :: 0\rangle\langle n :: 0|, \qquad (A3)$$

where we have introduced the notation $|n :: 0\rangle \equiv (|n\rangle|0\rangle + |0\rangle|n\rangle)/\sqrt{2}$ for simplicity. Reminding one that the QFI of NOON states is equal to $F_{\text{NOON}} = N^4$ [44], as demonstrated in Sec. IV, the QFI for the state of Eq. (A3) can be obtained by

$$F_{\text{EC}} = 2\mathcal{N}_\alpha^2 \sum_{n=1}^{\infty} |c_n|^2 F_{\text{noon}}(n) = 2\mathcal{N}_\alpha^2 \sum_{n=1}^{\infty} |c_n|^2 n^4. \quad (A4)$$

For larger amplitude $|\alpha| \gg 1$, the expression of Eq. (A4) approximately reduces to

$$F_{\text{EC}} = \bar{N}^4 + 6\bar{N}^3 + 7\bar{N}^2 + \bar{N}, \qquad (A5)$$

which is less than Eq. (A2) for the case where a common reference beam must be established.

In what follows, let us calculate the sensitivity attained by parity detection in the above considered scenario. Similar to the case with the TMSV state, the expectation value of Π_b for EC states can be expressed as the weighted linear combination of the expectations of Π_b for NOON states with different

photon numbers as

$$\langle \Pi_b \rangle_{\text{EC}} = 2\mathcal{N}_\alpha^2 \sum_{n=0}^{\infty} |c_n|^2 \langle \Pi_b \rangle_{\text{noon}}, \tag{A6}$$

where

$$\langle \Pi_b \rangle_{\text{noon}} \equiv \langle n :: 0 | U_\varphi^\dagger B_2^\dagger \Pi_b B_2 U_\varphi | n :: 0 \rangle$$

$$= \begin{cases} 2, & n = 0, \\ \cos(n^2 \varphi), & n \neq 0. \end{cases} \tag{A7}$$

Interestingly, with the help of Eq. (A7), we find that

$$\Delta\varphi = \frac{1}{\sqrt{\upsilon}} \frac{\sqrt{1 - \langle \Pi_b \rangle_{\text{NOON}}^2}}{\left| \frac{\partial \langle \Pi_b \rangle_{\text{NOON}}}{\partial \varphi} \right|} = \frac{1}{\sqrt{\upsilon} N^2}. \tag{A8}$$

This indicates that parity detection could saturate the sensitivity limit $1/N^2$ independent of the true value of φ, in the sense that it is a global optimal measurement for Kerr phase estimation with NOON states. A similar result has also been found in linear phase estimation [61,69,82]. With Eqs. (A6) and (A7), the sensitivity for EC states attained by parity detection is given by

$$\Delta\varphi = \frac{1}{\sqrt{\upsilon}} \frac{\sqrt{1 - \langle \Pi_b \rangle_{\text{EC}}^2}}{\left| \frac{\partial \langle \Pi_b \rangle_{\text{EC}}}{\partial \varphi} \right|}$$

$$= \frac{\sqrt{1 - \left(2\mathcal{N}_\alpha^2 \left[2|c_0|^2 + \sum_{n=1}^{\infty} |c_n|^2 \cos(n^2\varphi)\right]\right)^2}}{\sqrt{\upsilon} |2\mathcal{N}_\alpha^2 \sum_{n=1}^{\infty} |c_n|^2 n^2 \sin(n^2\varphi)|}. \tag{A9}$$

The above expression is explicitly simplified to $\Delta\varphi = 1/\sqrt{\upsilon F_{\text{EC}}}$ in the asymptotic limit $\varphi \to 0$, in the sense that parity detection is responsible for saturating the QCR bound for any power intensity of incident lights.

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