Universal quantum operation of spin-3/2 Blume-Capel chains

Silas Hoffman, Yiyuan Chen, Hai-Ping Cheng, and X.-G. Zhang
Department of Physics and Quantum Theory Project, University of Florida, Gainesville, Florida 32611, USA and Center for Magnetic Molecular Quantum Materials, University of Florida, Gainesville, Florida 32611, USA

(Received 26 April 2021; accepted 17 February 2022; published 8 March 2022)

We propose a logical qubit based on the Blume-Capel model: A higher-spin generalization of the Ising chain which allows for an on-site anisotropy-preserving rotational invariance around the Ising axis. We show that such a spin-3/2 Blume-Capel model can also support localized Majorana zero modes at the ends of the chain. Inspired by known braiding protocols of these Majorana zero modes, upon appropriate manipulation of the system parameters, we demonstrate a set of universal gate operations which act on qubits encoded in the doubly degenerate ground states of the chain.

DOI: 10.1103/PhysRevA.105.032604

I. INTRODUCTION

A single spin-1/2 particle provides a natural two-level system to support a qubit. According to the Loss-DiVincenzo realization of such a qubit [1], pulsed control of electric and magnetic fields enables one-qubit and multiqubit quantum gates which are sufficient for universal quantum computation. In this setup, and most other setups for large-scale quantum computing, pulsing the controllable parameters imparts a dynamical phase which enables conventional gate control. Alternatively, adiabatic and cyclic control of the magnetic field imparts a geometric phase, i.e., a non-Abelian Berry phase or a holonomy [2], in addition to the dynamical phase. Exploiting the holonomies to perform a quantum operation is known as holonomic quantum computing [3]. As compared to dynamic gates, geometric gates may (but do not necessarily [4]) benefit from tolerance to fluctuations [5]. For a spin 1/2, the group generated by such holonomies is the Abelian U(1) group [6] and is insufficient for universal quantum computation.

To achieve universal quantum computation the holonomy group must be non-Abelian and necessitates both (i) two or more degenerate states and (ii) auxiliary states; these are known as dark and bright states, respectively, in the trapped ion literature [7]. Consequently, realization of a universal set of quantum gates generated by the holonomy necessitates a system with more than two levels. Although there are several systems in which geometric control of qubits has been realized, we restrict our interest to spin-based qubits.

One route to realizing this geometric control in spin systems is higher-spin particles [8]. In particular, it is known that adiabatically controlling the anisotropy of spin-3/2 particles generates an SU(2) holonomy group [9,10]. In this case, because the anisotropies guarantee time-reversal symmetry, there are two pairs of degenerate states, either of which can be used to furnish the physical qubit basis while the others are auxiliary and facilitate operation. These states can be realized as heavy holes in quantum dots whereby an electric field, which couples to the anisotropy via the Stark effect or the quadrupole moment of the field, can control the anisotropies and thereby perform any single-qubit quantum gate [11,12]. Moreover, it has recently been shown that entanglement of such holes can be geometrically generated when they are simultaneously coupled to an electromagnetic cavity [13].

Rather than considering larger spins, auxiliary states can be provided by more spins as in, for instance, the Ising spin-1/2 chain. In such a system, it is convenient to express the spins as nonlocal fermions [14] which, for a chain of finite length, host Majorana zero modes (MZMs) in the fermionic representation. As exchanging two MZMs generates a quantum operation [15], an analogous manipulation of the Ising chain provides a nonuniversal set of holonomic quantum gates [16] which must be aided by dynamic operations to realize universal quantum control. For instance, in Ref. [17] the authors considered a spin-1/2 Ising chain in which the accumulation of geometric phase, upon manipulating the direction of the anisotropic exchange interaction, was supplemented by an applied magnetic field to perform single-qubit quantum operations. A pulsed exchange interaction between spin chains imparted a dynamic phase, sufficient to entangle two qubits.

In an effort to realize an entirely geometric manipulation of a spin chain, we study a higher-spin generalization of a spin-1/2 Ising chain known as the Blume-Capel model which includes an on-site anisotropy along the Ising axis. Like the spin-1/2 Ising chain, we find that spin-3/2 chains with rotational symmetry also support MZMs localized to the ends of the chain. Moreover, we find that a single-site chain, i.e., an isolated spin-3/2 state, also hosts zero-energy MZMs which can be effectively braided by adiabatic control of the on-site anisotropy. By exploiting this Majorana representation, we discover an entirely geometric protocol for single- and two-qubit operations using the spin-3/2 states. These protocols can be extended from qubits furnished by one spin-3/2 particle to chains of spin-3/2 particles, thus providing an entirely geometric protocol for universal quantum computation of high-spin chains.

We emphasize that, while the Majorana representation is convenient for constructing the sufficient adiabatic gates to perform universal quantum computation, the topological protection is not guaranteed. First, similar to the spin-1/2 chain but unlike p-wave superconductors [18], magnetic-field fluctuations along the Ising axis can spoil the state of the qubit. Second, topological protection of effective braiding is guaranteed so long as the anisotropy can be reasonably set to zero, i.e., the anisotropies depend exponentially on the external parameters that control them [19]. While the coupling of MZMs implemented in a p-wave superconductor depends exponentially on their separation, this is unlikely the case for the spin-3/2 implementation of MZMs.

Our paper is organized as follows. In Sec. II we show how a single spin-3/2 state can be used as a qubit and, moreover, that it has a convenient description in terms of MZMs. We continue by showing this idea can be generalized to a chain of spin-3/2 states in Sec. III. In Sec. IV we discuss the quantum operation. We consider the necessary ingredients to extend our system to higher spins in Sec. V. We briefly summarize in Sec. VI.

II. SPIN-3/2 QUBIT

Before discussing a chain of spins, we consider a single spin 3/2 and the means to utilize it as a qubit. Our approach will be to decompose the generators of rotation of spin into tensor products of Pauli matrices S^i and σ^i, respectively, for $i = x, y, z$. The latter can be written in terms of Majorana operators. Using braiding or partially braiding Majorana zero-energy modes realizes a non-Abelian geometric phase which can be used as a qubit gate and reinterpreted in the spin language.

In order to operate and initialize a spin 3/2 as a qubit, we suppose there exists a controllable on-site anisotropy which takes the rather general form $H_A = -\sum_{a=1}^{5}(d^a\Gamma^a)$. Here the anisotropies, quadratic in the spin-3/2 generators of rotation, can also be understood as tensor products of spin-1/2 matrices

$$\begin{aligned}
\Gamma^1 &= \frac{1}{4\sqrt{3}}\{S_1^y, S_1^z\} = \sigma_1^z\sigma_2^x,\\
\Gamma^2 &= \frac{1}{4\sqrt{3}}\{S_1^z, S_1^x\} = \sigma_1^z\sigma_2^y,\\
\Gamma^3 &= \frac{1}{4\sqrt{3}}\{S_1^x, S_1^y\} = \sigma_1^y,\\
\Gamma^4 &= \frac{1}{4\sqrt{3}}\left[\left(S_1^x\right)^2 - \left(S_1^y\right)^2\right] = \sigma_1^x,\\
\Gamma^5 &= \frac{1}{4}\left[\left(S_1^z\right)^2 - \left(\frac{5}{4}\right)\mathbb{1}_{4\times 4}\right] = \sigma_1^z\sigma_2^z,
\end{aligned} \tag{1}$$

where d^a are the effective weights of the anisotropies. Because the anisotropy couples to the electric field through the Stark effect [11] or the quadrupole component [12], we henceforth assume time-dependent control of d^a, which is essential to the operation of the qubit. To initialize our system, we require a magnetic field with magnitude h along the x axis, $H_B = -(hS^x) = -h(\sqrt{3}\sigma_2^x + \sigma_1^x\sigma_2^x + \sigma_1^y\sigma_2^y)/2$. Thus,

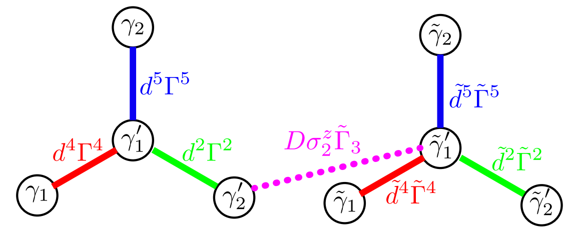

FIG. 1. Schematic of two spin-3/2 particles represented as Majorana fermions. Coupling of Majorana fermions within the same particle is controlled by the on-site anisotropy (highlighted in blue, green, and red), while the coupling between interspin Majorana fermions (highlighted in magenta) is controlled by the product of spin and anisotropy operators.

the full combined Hamiltonian describing the system is $H_{3/2} = H_A + H_B$.

The four eigenstates of S^z, $|\Uparrow\rangle$, $|\Downarrow\rangle$, $|\upharpoonleft\rangle$, and $|\downharpoonleft\rangle$, with eigenvalues 3/2, −3/2, 1/2, and −1/2, respectively, are simultaneously eigenstates of Γ^5. In particular, $\langle\Gamma^5\rangle = 1$ ($\langle\Gamma^5\rangle = -1$) with the expectation values taken with respect to the state $|\Uparrow\rangle$ or $|\Downarrow\rangle$ ($|\upharpoonleft\rangle$ and $|\downharpoonleft\rangle$). The spin-3/2 qubit basis is furnished by $|\pm_{3/2}\rangle = (|\Uparrow\rangle \pm |\Downarrow\rangle)/\sqrt{2}$. To initialize our system to $|+\rangle$, we begin with a large magnetic field and zero on-site anisotropy $d^a = 0$ so that the spin relaxes to the positive x axis, $|\rightarrow\rangle = (|\Uparrow\rangle + |\sqrt{3}\upharpoonleft\rangle + \sqrt{3}|\downharpoonleft\rangle + |\Downarrow\rangle)/2\sqrt{2}$. Upon decreasing the magnetic field, $h \to 0$, and increasing the z-axis anisotropy, $d^5 > 0$, the spin becomes a symmetric superposition of the high spin aligned parallel and antiparallel to the z axis, $(|\Uparrow\rangle + |\Downarrow\rangle)/\sqrt{2}$. We emphasize that a magnetic field is only necessary to initialize the system but is otherwise absent, i.e., when changing d^a. Consequently, when operating the qubit, because H_A preserves time-reversal symmetry, the Kramers theorem guarantees two pairs of degenerate states split by $\sqrt{\sum_a |d^a|^2}$.

We perform a Jordan-Wigner mapping [14] and decompose the resultant complex fermions into Majorana fermions wherein $\sigma_j^x = i\gamma_j\gamma_j'$, $\sigma_j^y = \gamma_j'\prod_{k<j}(i\gamma_k\gamma_k')$, and $\sigma_j^z = \gamma_j\prod_{k<j}(i\gamma_k\gamma_k')$. Consequently, the anisotropies [Eq. (1)] can likewise be fermionized to obtain

$$\begin{aligned}
&\Gamma^1 = -i\gamma_1\gamma_2\gamma_2', \quad \Gamma^2 = -i\gamma_1'\gamma_2', \quad \Gamma^3 = \gamma_1',\\
&\Gamma^4 = -i\gamma_1\gamma_1', \quad \Gamma^5 = -i\gamma_1'\gamma_2.
\end{aligned} \tag{2}$$

Clearly, when there is only z axis anisotropy, i.e., $d^5 \neq 0$ and $d^a = 0$ for $a \neq 5$, $[H_A, \gamma_1] = [H_A, \gamma_2'] = 0$ and γ_1 and γ_2' are MZMs. Moreover, $|\pm_{3/2}\rangle$ are eigenstates of $i\gamma_1\gamma_2'$ with eigenvalues ±1.

Focusing on Γ^2, Γ^4, and Γ^5, the anisotropy has the structure of an inner MZM coupled to three outer MZMs, known in the literature as a Y junction (Fig. 1) [20]. Borrowing the protocol from Ref. [19], one can braid two uncoupled MZMs: Consider $d^1 = d^3 = 0$ and parametrizing the remaining magnitudes of anisotropy by $\vec{d} = (d^2, d^4, d^5) = d[\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta)]$. Consider an initial Hamiltonian with $\theta = \phi = 0$, i.e., H_A with $d^5 \neq 0$ and the remaining $d^a = 0$, and an initial state $|\psi\rangle = \alpha|+_{3/2}\rangle + \beta|-_{3/2}\rangle$ which is a ground state of that Hamiltonian. We proceed in three steps: (i) Rotate $\vec{d}$ about the y axis so that $\theta = 0 \to \theta =$

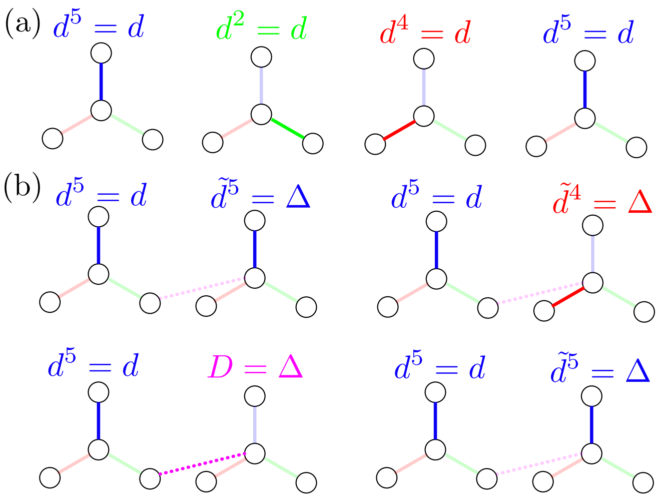

FIG. 2. Schematic of the changes in coupling necessary to braid MZMs. (a) For a single spin-3/2 qubit and starting from $d^5 = d$ and the rest of the anisotropies zero, we adiabatically perform the following changes when going from the leftmost panel to the rightmost panel: (i) $d^5 \to 0$ and $d^2 \to d$, (ii) $d^2 \to 0$ and $d^3 \to d$, and (iii) $d^3 \to 0$ and $d^5 \to d$. (b) Analogously to braid MZMs residing in different spin-3/2 qubits and starting from $d^5 = d$, $\tilde{d}^5 = \Delta$, and the rest of the anisotropies zero, we adiabatically perform the following changes when going from the upper leftmost panel to the lower rightmost panel: (i) $d^5 \to 0$ and $d^3 \to \Delta$, (ii) $d^3 \to 0$ and $D \to \Delta$, and (iii) $D \to 0$ and $\tilde{d}^5 \to \Delta$.

$\pi/2$, (ii) rotate $\vec{d}$ about the z axis so that $\phi = 0 \to \phi = \varphi$, and (iii) rotate $\vec{d}$ so that it once again points along the z axis. Here $\hat{d} = \vec{d}/d$ traces out a solid angle on the unit sphere, $\Omega_Z = \varphi$ [Fig. 2(a)]. This results in the geometric phase of $\Omega_Z/2$ imprinted on the state: $|\psi\rangle \to |\psi\rangle = \exp(i\Omega_Z/2)\alpha|+_{3/2}\rangle + \exp(-i\Omega_Z/2)\beta|-_{3/2}\rangle$. That is, this operation corresponds to a rotation by angle Ω_Z around the z axis of the Bloch sphere, $R_z(\Omega_Z)$, of a qubit defined on the basis $|\pm_{3/2}\rangle$. Although we have chosen a specific path that $\hat{d}$ traces out, the operation is independent of the path for any solid angle traced out and rate independent so long as the inverse time of operation is much smaller than $|\vec{d}|$. For $\Omega_Z = \pi/2$, this operation corresponds precisely to braiding MZMs γ_1 and γ_2'.

It is convenient to continue to use this MZM picture when attempting to entangle states using a similar protocol. Consider an additional spin-3/2 site also described by H_A; as a point of notation, we use a tilde to distinguish parameters and operators of the second spin from the first spin. Because of the nonlocal string operator, the anisotropies on the second site, after the Jordan-Wigner transformation, take the following form:

$$\tilde{\Gamma}^1 = i\gamma_1\gamma_1'\gamma_2\gamma_2'\tilde{\gamma}_1\tilde{\gamma}_2\tilde{\gamma}_2', \quad \tilde{\Gamma}^2 = -i\tilde{\gamma}_1'\tilde{\gamma}_2',$$
$$\tilde{\Gamma}^3 = -\gamma_1\gamma_1'\gamma_2\gamma_2'\tilde{\gamma}_1', \quad \tilde{\Gamma}^4 = -i\tilde{\gamma}_1\tilde{\gamma}_1', \quad \tilde{\Gamma}^5 = -i\tilde{\gamma}_1'\tilde{\gamma}_2. \quad (3)$$

Because $\tilde{\Gamma}^2$, $\tilde{\Gamma}^4$, and $\tilde{\Gamma}^5$ take that same form as their counterparts without tildes, it is clear that the single-qubit operation $R_z(\Omega_Z)$ can be performed on the second spin 3/2.

To entangle two qubits, one can braid two MZMs originating from different spins, which requires a coupling between them. While there are a number of couplings that could enable entanglement, we find that the coupling between γ_2' and $\tilde{\gamma}_1'$, i.e., $H_{\text{Ising}} = iD\tilde{\gamma}_1'\gamma_2' = D[S_1^z - (S_1^z)^3]\tilde{\Gamma}^3$, in addition to anisotropy control is sufficient. Consider $d^1 = d^2 = d^3 = d^4 = \tilde{d}^1 = \tilde{d}^3 = 0$ and $d^5 \neq 0$. The remaining anisotropies and D are dynamically controlled and parametrized according to $\vec{\Delta} = (D, \tilde{d}^4, \tilde{d}^5) = \Delta[\cos(\phi)\sin(\theta), \sin(\phi)\sin(\theta), \cos(\theta)]$. Starting with $\theta = \phi = 0$ and proceeding analogously to the single-spin operation we (i) rotate $\vec{\Delta}$ about the y axis so that $\theta = 0 \to \theta = \pi/2$, (ii) rotate $\vec{\Delta}$ about the z axis so that $\phi = 0 \to \phi = \varphi$, and (iii) rotate $\vec{\Delta}$ so that it once again points along the z axis. Here $\hat{\Delta} = \vec{\Delta}/\Delta$ traces out a solid angle Ω_I over the unit sphere embedded in the space of parameters D, $\tilde{d}^4$, and $\tilde{d}^5$; this change in coupling is schematically shown Fig. 2(b). We find that this operates as an Ising ZZ gate in the basis of $|\uparrow\rangle$ and $|\downarrow\rangle$,

$$\begin{pmatrix} |\uparrow\rangle|\tilde{\uparrow}\rangle \\ |\uparrow\rangle|\tilde{\downarrow}\rangle \\ |\downarrow\rangle|\tilde{\uparrow}\rangle \\ |\downarrow\rangle|\tilde{\downarrow}\rangle \end{pmatrix} \to \begin{pmatrix} e^{-i\Omega_\text{I}/2} & 0 & 0 & 0 \\ 0 & e^{-i\Omega_\text{I}/2} & 0 & 0 \\ 0 & 0 & e^{-i\Omega_\text{I}/2} & 0 \\ 0 & 0 & 0 & e^{i\Omega_\text{I}/2} \end{pmatrix} \begin{pmatrix} |\uparrow\rangle|\tilde{\uparrow}\rangle \\ |\uparrow\rangle|\tilde{\downarrow}\rangle \\ |\downarrow\rangle|\tilde{\uparrow}\rangle \\ |\downarrow\rangle|\tilde{\downarrow}\rangle \end{pmatrix}, \quad (4)$$

or as an Ising XX gate in the $|\pm_{3/2}\rangle$ basis:

$$\begin{pmatrix} |+_{3/2}\rangle|\tilde{+}_{3/2}\rangle \\ |+_{3/2}\rangle|\tilde{-}_{3/2}\rangle \\ |-_{3/2}\rangle|\tilde{+}_{3/2}\rangle \\ |-_{3/2}\rangle|\tilde{-}_{3/2}\rangle \end{pmatrix} \to \begin{pmatrix} \cos(\Omega_\text{I}) & 0 & 0 & -i\sin(\Omega_\text{I}) \\ 0 & \cos(\Omega_\text{I}) & -i\sin(\Omega_\text{I}) & 0 \\ 0 & -i\sin(\Omega_\text{I}) & \cos(\Omega_\text{I}) & 0 \\ -i\sin(\Omega_\text{I}) & 0 & 0 & \cos(\Omega_\text{I}) \end{pmatrix} \begin{pmatrix} |+_{3/2}\rangle|\tilde{+}_{3/2}\rangle \\ |+_{3/2}\rangle|\tilde{-}_{3/2}\rangle \\ |-_{3/2}\rangle|\tilde{+}_{3/2}\rangle \\ |-_{3/2}\rangle|\tilde{-}_{3/2}\rangle \end{pmatrix}. \quad (5)$$

Although these operations are conveniently encoded in partially braiding MZMs, they are not sufficient for universal control of a qubit encoded by $|\pm_{3/2}\rangle$. Nonetheless, one may utilize H_a to encode employing the protocol introduced in Ref. [21]. In particular, to generate a rotation around the y axis of the Bloch sphere by an angle φ, $d^3 = d^5 = 0$ while

$$d^2 = -\cos(\varphi)\sin(\Phi),$$
$$d^3 = \sin(2\varphi)\sin(\Phi/2)^2,$$
$$d^5 = \sin(\varphi)^2 + 2\cos(2\varphi)\cos(\Phi) + 2\cos(\Phi). \quad (6)$$

Here Φ is a function of time which changes from 0 to 2π. Again, because this is a geometric phase, the details of how Φ changes are unimportant so long as it changes adiabatically. Note that this does not have a convenient description in terms of MZMs as $\Gamma^3 = \gamma_1'$. This operation with the single- and two-qubit operations described previously is sufficient for universal quantum computation.

Alternatively, one could redefine a single qubit to be two spin-3/2 particles in which a qubit is defined on the subspace $|\uparrow\rangle|\tilde{\downarrow}\rangle$ and $|\downarrow\rangle|\tilde{\uparrow}\rangle$. The two-qubit protocol described above would allow a geometric navigation to any point on the Bloch sphere. A two-qubit protocol in the new basis could be achieved by coupling two spin-3/2 particles constituted in two different qubits.

III. SPIN-3/2 BLUME-CAPEL MODEL

In this section we generalize the encoding of a qubit from a single spin 3/2 to a chain. Our starting point is a generalization to the well-known Blume-Capel model [22],

$$H_{\rm BC} = -J\left(\sum_{i=1}^{N-1} S_i^z S_{i+1}^z\right) - K\sum_{i=1}^{N}\left(S_i^z\right)^2 - \sum_{i=1}^{N}\left(h_i S_i^x\right) + \sum_{a=1}^{5}(d^a\Gamma^a), \tag{7}$$

which generalizes the Ising chain by including an on-site anisotropy, parametrized by K, in addition to anisotropic exchange, parametrized by J. Moreover, we have included a site-dependent transverse magnetic field of magnitude h_i and generic anisotropy on the first site of the chain, similar to Sec. II, where the Γ^a are defined analogously by Eq. (1). While the latter two terms in Eq. (7) will be convenient for quantum operation (Sec. IV), we consider $h_i = d^a = 0$ for the remainder of this section.

In Eq. (7), S_i^z is the generator of rotation around the z axis of the spin at site i and, for concreteness, we focus on the $S = 3/2$ case wherein

$$S^z = \begin{pmatrix} 3/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & -3/2 \end{pmatrix}. \tag{8}$$

The spin-3/2 case is convenient because it can be understood in terms of Pauli matrices: One spin-3/2 operator can be mapped into a tensor product of two spin-1/2 generators of rotation, $S^z = \sigma^z \otimes \mathbb{1}/2 + \mathbb{1}\otimes\sigma^z$ or, for the chain, $S_i^z = \sigma_{2i-1}^z/2 + \sigma_{2i}^z$. Accordingly, Eq. (7) transforms from a chain of spin-3/2 particles to a ladder of spin-1/2 particles (Fig. 3),

$$H = -K\left(\sum_{i=1}^{N}\sigma_{2i-1}^z\sigma_{2i}^z\right) - \frac{J}{4}\left(\sum_{i=1}^{N-1}\sigma_{2i-1}^z\sigma_{2i+1}^z + 2\sigma_{2i}^z\sigma_{2i+1}^z + 2\sigma_{2i-1}^z\sigma_{2i+2}^z + 4\sigma_{2i}^z\sigma_{2i+2}^z\right). \tag{9}$$

This equation can be rewritten as an interacting spinless fermion system using the Jordan-Wigner mapping [14] $\sigma_j^z = [\prod_{i<j}(1-2n_i)](c_j + c_j^\dagger)$ with $n_j = c_j^\dagger c_j$. Because each term in Eq. (9) is in general a nonlocal product of Pauli matrices, they can be written as

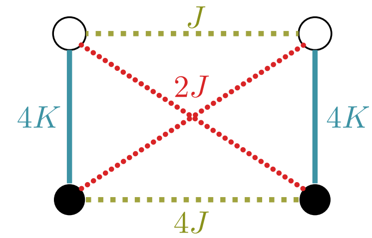

FIG. 3. A spin-3/2 Blume-Capel model with on-site anisotropy K and exchange interaction J can be mapped onto a ladder of exchange-coupled spins 1/2, i.e., a quantum Ising model coupling spins up to three sites apart. The odd sites (white) couple to even sites (black) on the same rung with strength $4K$ and to sites one apart with strength $2J$. Odd sites couple to nearest-neighbor odd (even) sites with strength J ($4J$).

$$\sigma_i^z\sigma_j^z = (c_i^\dagger - c_i)\left[\prod_{i<k<j}(1-2n_k)\right](c_j^\dagger + c_j), \tag{10}$$

where we have assumed without loss of generality that $j > i$. We can further decompose the complex fermions into Majoranas according to the definition $c_j = (\gamma_j + i\gamma_j')/2$ wherein $c_j + c_j^\dagger = \gamma_j$, $c_j - c_j^\dagger = i\gamma_j'$, and $1 - 2n_j = i\gamma_j\gamma_j'$. The products of Pauli matrices take the form $\sigma_z^i\sigma_z^j = -i\gamma_i'\gamma_j\prod_{i<k<j}(i\gamma_k\gamma_k')$. Because the terms in Eq. (9) follow this form with $1 \leqslant i < j \leqslant 2N$, the first and last Majoranas γ_1 and γ_{2N}', respectively, are absent from the the Hamiltonian and are therefore zero-energy operators that are localized to the ends of the chain; these are the MZMs.

The degenerate ground states of this system are when all the spins point parallel or antiparallel to the z axis, which we denote by $|\Uparrow\rangle$ and $|\Downarrow\rangle$, respectively. The degenerate states can be characterized by the occupation of the zero-energy complex fermion composed of the MZMs, $f = (\gamma_1 + i\gamma_N')/2$. The physical meaning of these states is elucidated by noting that

$$1 - 2f^\dagger f = i\gamma_1\gamma_N' = \sigma_1^z\left[\prod_{j<N}\left(-\sigma_j^x\right)\right]\left(-i\sigma_N^y\right) = \sigma_1^z\left[\prod_{j\leqslant N}\left(-\sigma_j^x\right)\right]\left(-\sigma_N^z\right) = \mathcal{P}\sigma_1^z\sigma_N^z, \tag{11}$$

with $\mathcal{P}$ a π rotation at each site in the chain. That is, this operator transforms between the ground states, $(1-2f^\dagger f)|\Uparrow\rangle = |\Downarrow\rangle$ and $(1-2f^\dagger f)|\Downarrow\rangle = |\Uparrow\rangle$, i.e., the eigenstates of this operator are $|\pm\rangle = (|\Uparrow\rangle \pm |\Downarrow\rangle)/\sqrt{2}$ with $f^\dagger f|\pm\rangle = \pm|\pm\rangle$.

While the MZMs γ_1 and γ_N' are constructed in the absence of a magnetic field, we find that for a sufficiently small homogeneous magnetic field $h_i = h \ll J, K$ there exist MZMs dressed by the magnetic field (see Appendix A). An analogous chain with a periodic boundary condition is likewise doubly degenerate; however, there exists no such fermionic modes f and $f^\dagger$ whose occupancy can be used to characterize the ground state.

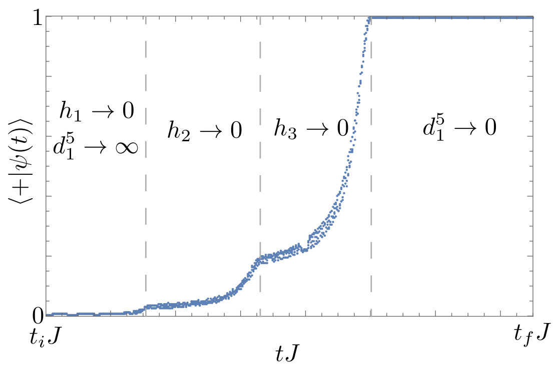

FIG. 4. Numerical simulation of our initialization procedure. Beginning in a high magnetic field on all the sites, $h_i = 10J$, and $K = d^5 = 0$, the anisotropy (magnetic field on the first site) is linearly ramped up (down) to $d^5 = 10J$ ($h^1 = 0$). Linearly ramping down the magnetic field on subsequent sites initializes the state into $|+\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$. Here $(t_f - t_i)J = 1500$, where we have taken $\hbar = 1$.

IV. QUANTUM OPERATION

In this section we detail the initialization and operation of a qubit furnished by the states $|\pm\rangle$. Although our formulation of quantum operation is similar to the spin-1/2 Ising chain [17], we take advantage of the on-site anisotropy available in higher-spin systems rather than relying on the anisotropy in the exchange interaction. Critical to the initialization and operation of the qubit is our ability to decouple the spin at the first site from the remainder of the chain. This is achieved by systematically increasing the transverse magnetic field as we describe below [17]. While we focus on a three-spin chain below, the procedure can be straightforwardly generalized to a chain of any length.

A. Initialization

To initialize our system to $|+\rangle$, we begin with a large magnetic field on all the sites, $h_i/J \gg 1$ with zero on-site anisotropy $d^a = 0$, so all the spins point along the positive x axis, $|\Rightarrow\rangle = |\rightarrow\rightarrow\rightarrow\rangle$ with $S_i^x|\rightarrow\rangle = (3/2)|\rightarrow\rangle$. Upon decreasing the magnetic field on the first site, $h_1 \to 0$, and increasing the z axis anisotropy, $d^5 > 0$, the first spin becomes a symmetric superposition of the spin aligned parallel and antiparallel to the z axis, $(|\uparrow\rangle + |\downarrow\rangle)|\rightarrow\rightarrow\rangle/\sqrt{2}$. Note that because h_i is large, $\langle S_2^z\rangle = 0$, so the first site is effectively decoupled from the rest of the chain; in anticipation of subsequent operations, we note that this will be a convenient position in parameter space in which to perform gate operations. Upon decreasing the magnetic field on the second site, anisotropic exchange interaction aligns that site parallel to the first site, $(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)|\rightarrow\rangle/\sqrt{2}$. Proceeding analogously on subsequent sites, the system reaches the state $|+\rangle = (|\uparrow\uparrow\uparrow\rangle + |\downarrow\downarrow\downarrow\rangle)/\sqrt{2}$. Indeed, upon numerically simulating this manipulation of parameters, the state $|\rightarrow\rangle$ which is initially orthogonal to $|+\rangle$, $\langle +|\psi(t_0)\rangle = 0$, is manipulated into the state $|+\rangle$, $\langle +|\psi(t_f)\rangle = 1$ (Fig. 4).

B. Single-qubit gates

Operation by a quantum gate can be likewise obtained by adiabatic control of our parameters. The essence of our operation relies on the ability to dynamically control the anisotropies of the first site to exploit the non-Abelian holonomy of the spin-3/2 particle as in Sec. II.

Consider a state that has been initialized to $|+\rangle$ and $d^a = h_i = 0$. Similar to the initialization procedure, let us slowly ramp up the magnetic field to a large positive value on all the sites except the first site, $h_i \gg J$ for $i > 1$. As the acquired Berry phase is zero, the state adiabatically transforms to $(|\uparrow\rangle + |\downarrow\rangle)|\rightarrow\rightarrow\rangle/\sqrt{2}$. As the first site is now decoupled from the rest of the chain, we focus only on that state and, for notational convenience, only indicate the state of the first site. Exploiting the single-qubit operations of a single spin-3/2, we transform $|+_{3/2}\rangle$ to any superposition of the basis states $\alpha|+_{3/2}\rangle + \beta|-_{3/2}\rangle$. It will be convenient to write this state equivalently as $\lambda|\uparrow\rangle + \kappa|\downarrow\rangle$ for $\lambda = (\alpha+\beta)/\sqrt{2}$ and $\kappa = (\alpha-\beta)/\sqrt{2}$. Analogous to the initialization of the qubit, we slowly ramp down the magnetic field while ramping up the anisotropy on the second site so that $(\lambda|\uparrow\rangle + \kappa|\downarrow\rangle)|\uparrow\uparrow\rangle \to (\lambda|\uparrow\uparrow\rangle + \kappa|\downarrow\downarrow\rangle)|\rightarrow\rangle$. Applying the same procedure to the third site results in the state $\lambda|\uparrow\uparrow\uparrow\rangle + \kappa|\downarrow\downarrow\downarrow\rangle = \alpha|+\rangle + \beta|-\rangle$. That is, because we have access to any state on the Bloch sphere defined by $|\pm_{3/2}\rangle$, we have access to any state on the Bloch sphere defined by $|\pm\rangle$.

C. Two-qubit gate

Consider two spin chains described by the generalized Blume-Capel model in a transverse magnetic field, described by H defined in Sec. III. As a point of notation, we use a tilde to distinguish parameters and operators of the second chain from the first chain. It will be convenient to consider a product state of the two chains $|\Uparrow\rangle|\tilde{\Uparrow}\rangle = (|+\rangle| + |-\rangle)(|\tilde{+}\rangle + |\tilde{-}\rangle)/2$ which is related to the parity eigenstates by a $\pi/2$ rotation around the y axis.

Similar to the single-qubit operations, we ramp up the transverse magnetic field on all but the first spin sites to decouple the former from the remainder of the chain. This leaves the system described by two copies of H_A with $d^a = \tilde{d}^a = 0$ for $a \neq 0$ and $d^5 = \tilde{d}^5$ in general not zero and in the product state $|\uparrow\rangle|\tilde{\uparrow}\rangle$. In order to couple the two spins, we use the interaction between the first two sites given by H_{Ising} and the protocol for the spin-3/2 qubit to apply the Ising XX gate on two spin-3/2 states so that $|\uparrow\rangle|\tilde{\uparrow}\rangle \to \exp(-i\Omega_I/2)|\uparrow\rangle|\tilde{\uparrow}\rangle$. Ramping down the magnetic field on sites $i > 1$ and ramping up the anisotropy d^5 in both chains, the state transforms to $\exp(-i\Omega_I/2)|\Uparrow\rangle|\tilde{\Uparrow}\rangle$. Similar to the transformation made between Eqs. (4) and (5), this is an Ising XX gate on the basis of products states of $|\pm\rangle$ and $|\tilde{\pm}\rangle$.

V. DISCUSSION

Because the Ising XX gate along with rotations about any axis of the Bloch sphere of individual qubits allows for universal quantum computation [23], our outlined procedure offers universal quantum control of qubits defined in the parity sectors of the generalized Blume-Capel model. Although one can obtain a similar manipulation of the spin-1/2 Ising chain, the spin-3/2 Blume-Capel model offers of quantum operation by geometric manipulation. The potential advantage of geometric gates is that they do not suffer from timing errors to which dynamical phases induced by pulsed gates are susceptible. However, within the literature (see Ref. [4] and references therein) there are conflicting reports as to the degree by which geometric operation benefits over its dynamical counterpart. Simulating error in our system could compare the operational advantage of geometrically controlled gates (Sec. III) versus some set of pulsed gates, e.g., those described in Appendix B. A full-scale simulated comparison of these methods is beyond the scope of our analysis but could be characterized in a follow-up study.

A. Physical realization

One system that is often targeted with adiabatic control is spin-3/2 holes in semiconductors. Adiabatically controlled electric fields have been theoretically proposed to span the Bloch sphere [11,12] of a spin-3/2 qubit, in the sense of Sec. II, and some coherent electrical control of high-spin nuclei has been experimentally realized [24]. Moreover, coherent photons can mediate adiabatic entanglement of two spin-3/2 holes in a microwave cavity [13]. These tools suggest that H_A and H_{Ising} could be realized using electric fields acting on spin-3/2 holes in semiconductors and this system would be a good candidate for a spin-3/2 qubit as described in Sec. II. However, constructing a chain of such spins could prove technically difficult.

Perhaps a more natural candidate to realize a high-spin chain is a molecular magnet. Recently, a few molecules have been proposed to host MZMs [25]. These molecules are chains of high spins with a large on-site anisotropy. While they are not predicted to have anisotropic exchange, we expect that targeting these molecules for simulation could yield molecular chains that are well described by the Blume-Capel Hamiltonian or its generalization and should host MZMs as per the discussion in Sec. III. Moreover, in a different class of molecular magnets, several promising experiments have shown a spectral dependence on the electric field [26–28], which implies that the anisotropy can be controlled by careful application of the electric field. More recent experiments have shown that such an electric field directly modifies the anisotropy in a holmium-based nanomagnet [29]. This coupling to the electric field can be as strong as holes in semiconductors [26,29]. Consequently, we propose a setup similar to Ref. [13] wherein molecules hosting chains of spin-3/2 rather than spin-3/2 holes are coupled via a microwave cavity.

Because our spin-3/2 chain can be transformed into a spin-1/2 ladder (Fig. 3), a quantum computer or quantum simulator could support H_{BC} and H_{Ising}. While braiding of MZMs in spin-1/2 Ising chains was shown on commercial quantum computers [30] and in photonic systems [31], braiding of spin in our system could likewise be realized.

B. Higher-spin chain

Although we have restricted ourselves to the spin-3/2 chain, fermionization of the Blume-Capel model can be generalized for larger spins in which $|S| = (2^M - 1)/2$ for any natural number M in which S_z can written as M spin-1/2 states:

$$S^z = 2^M \sum_{i=1}^{M} 2^{i-1} (\mathbb{1}\otimes)^{i-1} \sigma^z (\otimes\mathbb{1})^{M-i}. \tag{12}$$

Consequently, the Blume-Capel model described by Eq. (7) generalizes to a ladder with M legs. For a chain of N sites there are equivalently $M \times N$ spin-1/2 sites. Notice that because S_z is linear in Pauli matrices, the product of any two spin operators, on either the same site or adjacent sites, will be quadratic in spin-1/2 operators on differing sites, i.e., of the form $\sigma_z^i \sigma_z^j$ for $i \neq j$, which commutes with both σ_z^1 and $\mathcal{P}\sigma_z^{NM}$ using the generalization $\mathcal{P} = \prod_{j=1}^{NM}(-\sigma_x^j)$. Thus, we find the persistence of MZMs at zero energy for higher-spin generalizations of the Blume-Capel model which satisfy $|S| = (2^M - 1)/2$. Defining a complex fermion analogous to the previous section, we can similarly define the ground states $|\pm\rangle = |\uparrow\rangle \pm |\downarrow\rangle$. Moreover, any anisotropy of even order in S^z preserves the energy of the MZMs. As a direct result, a single large spin with anisotropy even in S^z will support two zero-energy MZMs. Although generalizing the braiding of these MZMs and subsequent geometric manipulation of the quantum states is beyond the scope of our present analysis, should a single spin support quantum operation, an analogous manipulation of the chain parameters should naturally endow a universal set of quantum operation in the parity basis of the chain.

VI. CONCLUSION

In consideration of the Blume-Capel model, we have found that some higher-spin chains are capable of supporting MZMs whose occupation naturally defines the basis for a qubit. Guided by the braiding of MZMs, we developed a protocol for universal quantum computation of these qubits by holonomic quantum computing. Having identified spin-3/2 states with MZMs, the vast literature of MZMs can now be directly applied to single spin-3/2 states and their chains. Specifically, methods to encode or read MZMs and apply error correction can now be mapped into the spin-3/2 description. We anticipate this to guide the utilization of higher-spin chains for hosting quantum information.

ACKNOWLEDGMENTS

This work was supported as part of the Center for Molecular Magnetic Quantum Materials, an Energy Frontier Research Center funded by the U.S. Department of Energy, Office of Science, Basic Energy Sciences under Award No. DE-SC0019330. Computations were done using the utilities of National Energy Research Scientific Computing Center and University of Florida Research Computing systems.

APPENDIX A: MZMS IN THE MAGNETIC FIELD

Similar to the the spin-1/2 Ising chain, we show that the MZMs persist for a sufficiently small transverse magnetic field in the Blume-Capel model. Consider the spin-3/2

Blume-Capel Hamiltonian in a transverse magnetic field,

$$H_h = -J\left(\sum_{i=1}^{N-1} S_i^z S_{i+1}^z\right) - \sum_{i=1}^{N} K(S_i^z)^2 - h\sum_{i=1}^{N} S_i^x. \quad \text{(A1)}$$

In the absence of a magnetic field, the eigenstates of the system can be enumerated by the tensor product of the spin states localized to each site. The ground states $|\uparrow\rangle$ have energy $E_0/S^2 = -(N-1)J - NK$. The lowest-energy excitation is a domain wall with maximal spin, e.g., of the form $\cdots\uparrow\uparrow\downarrow\downarrow\cdots$, with energy $E_0/S^2 = -(N-3)J - NK$. That is, the spectrum is gapped by an energy of $2JS^2$. Moreover, for a small magnetic field, the ground energy and first excited states are unchanged by the application of a magnetic field to order h/K and h/J. For a sufficiently large magnetic field, the system is paramagnetically ordered along the direction of the applied magnetic field. Therefore, there exists a critical magnetic field h_c at which the gap vanishes [32]. Below h_c, we can construct dressed localized edge operators commuting with the Hamiltonian, i.e., the many-body generalization of γ_1 and γ_N', that can be constructed explicitly using quasiadiabatic continuation.

Following the recipe given in Ref. [33], to leading order in h, the dressed MZM is $\gamma_1 + ih[\mathcal{D}_0, \gamma_1]$, where

$$\mathcal{D}_0 = i\int_{-\infty}^{\infty} dt\, F(2JS^2 t)e^{iH_0 t}\left(\sum_{j=1}^{N} S_j^x\right)e^{-iH_0 t} \quad \text{(A2)}$$

and $F(t)$ is a filter function defined by

$$\int_{-\infty}^{\infty} dt\, e^{i\Omega t}F(2JS^2 t) = -\frac{1}{\Omega} \quad \text{for } |\Omega| \geqslant 1. \quad \text{(A3)}$$

Because $[\gamma_1, H_0] = 0$, we first evaluate

$$i[S_1^x, \gamma_1] = i\begin{pmatrix} 0 & \sqrt{3} & 0 & 0 \\ -\sqrt{3} & 0 & -2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & -\sqrt{3} & 0 \end{pmatrix} = \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{M}_3 + \text{H.c.},$$

$$\mathcal{M}_1 = \begin{pmatrix} 0 & \sqrt{3}i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{M}_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -2i & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathcal{M}_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3}i & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad \text{(A4)}$$

The problem amounts to finding the time evolution of these matrices with respect H_0. We find that

$$\mathcal{M}_j(t) = \exp(i\omega_j t)\mathcal{M}_j \otimes \exp(iJS^z t), \quad \text{(A5)}$$

where $\omega_1 = 2K$, $\omega_2 = 0$, $\omega_3 = -2K$, and M_j and S^z act on the spin of the first and second sites, respectively. We can evaluate the integral in Eq. (A2),

$$\begin{aligned} i[\mathcal{D}_0, \gamma_1] &= i\int_{-\infty}^{\infty} dt\, F(2JS^2 t)\left(\sum_{i=1}^{3} e^{i\omega_i t}\mathcal{M}_i \otimes \exp(iJS^z t) + \text{H.c.}\right) \\ &= -\left[\sum_{i=1}^{3}\sum_{j=1}^{4} \frac{1}{\omega_i + m_j J}\mathcal{N}_i \otimes \mathbb{1}_j\right], \end{aligned} \quad \text{(A6)}$$

where $\mathcal{N}_j = i(\mathcal{M}_j - \mathcal{M}_j^\dagger)$ and $(\mathbb{1}_i)_{mn} = 1$ for $m = n = i$ and zero otherwise. We can rewrite these matrices in terms of Kronecker products of Pauli matrices,

$$\begin{aligned} &\mathcal{N}_1 = \frac{\sqrt{3}}{2}\sigma^y \otimes (\mathbb{1} + \sigma^z), \quad \mathcal{N}_2 = \sigma^x \otimes \sigma^y - \sigma^y \otimes \sigma^x, \quad \mathcal{N}_3 = \frac{\sqrt{3}}{2}\sigma^y \otimes (\mathbb{1} - \sigma^z), \\ &\mathbb{1}_1 = \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z + \sigma_z\otimes\sigma_z), \quad \mathbb{1}_2 = \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} - \sigma_z\otimes\mathbb{1} + \mathbb{1}\otimes\sigma_z - \sigma_z\otimes\sigma_z), \\ &\mathbb{1}_3 = \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z - \sigma_z\otimes\sigma_z), \quad \mathbb{1}_4 = \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} - \sigma_z\otimes\mathbb{1} - \mathbb{1}\otimes\sigma_z + \sigma_z\otimes\sigma_z. \end{aligned} \quad \text{(A7)}$$

Collecting terms and simplifying, we can rewrite Eq. (A6) as

$$\begin{aligned} i[\mathcal{D}_0, \gamma_1] = &\frac{1}{16\left[(2K)^2 - \left(\frac{3J}{2}\right)^2\right]}\left[2K\sigma_1^z\sigma_2^y\left(1 + \sigma_3^y\sigma_4^z\right) + \frac{3J}{2}\sigma_2^y\left(\sigma_3^z + \sigma_4^z\right)\right] + \frac{1}{3J}\left[\sigma_1^x\sigma_2^y\left(\sigma_4^z - 2\sigma_3^z\right) - \sigma_1^y\sigma_2^x\left(2\sigma_3^z - \sigma_4^z\right)\right)\right] \\ &+ \frac{1}{16\left[(2K)^2 - \left(\frac{J}{2}\right)^2\right]}\left[2K\sigma_1^z\sigma_2^y\left(1 - \sigma_3^y\sigma_4^z\right) + \frac{J}{2}\sigma_2^y\left(\sigma_3^z - \sigma_4^z\right)\right]. \end{aligned} \quad \text{(A8)}$$

Upon converting the Pauli matrices to Majoranas, we find that the dressed MZM is

$$\gamma_1 + \frac{h}{16\left[(2K)^2 - \left(\frac{3J}{2}\right)^2\right]}\left[2iK\gamma_1'\gamma_2'(1+i\gamma_3'\gamma_4) - \frac{3iJ}{2}\gamma_2\gamma_3(1+i\gamma_3'\gamma_4)\right] - \frac{ih}{3J}\gamma_1\gamma_3(1-i\gamma_1'\gamma_2')(2-i\gamma_3'\gamma_4)$$
$$+ \frac{h}{16\left[(2K)^2 - \left(\frac{J}{2}\right)^2\right]}\left[2iK\gamma_1'\gamma_2'(1-i\gamma_3'\gamma_4) - \frac{iJ}{2}\gamma_2\gamma_3(1-i\gamma_3'\gamma_4)\right]. \quad \text{(A9)}$$

While this expression is complicated, it shows the weight of the dressed MZM on the first and second sites. An analogous expression can be obtained for the dressed γ_N'.

APPENDIX B: PULSED GATE IMPLEMENTATION

Alternatively to geometric control, some operations can be performed by pulsing external parameters to apply quantum gates. In particular, consider one chain in the state $|\uparrow\rangle$ whose first site is in a controllable magnetic field along the Ising axis, $H_B = BS_1^z$. By pulsing this magnetic field for a time t_Z, the initial state transforms as $|\uparrow\rangle \to \exp(i3Bt_z/2)|\uparrow\rangle$. Similarly, $|\downarrow\rangle \to \exp(-i3Bt_z/2)|\downarrow\rangle$, so this is a rotation about the z axis of the Bloch sphere by an angle $3Bt_Z/2$ in the basis of $|\uparrow\rangle$ and $|\downarrow\rangle$. In the parity basis, this a rotation about the x axis by the same angle, $3Bt_z/2$.

Likewise, a two-qubit gate can be incorporated by pulsing an Ising-like exchange $H_{\rm ex} = -\mathcal{J}S_1^z\tilde{S}_1^z$ between the first two spins of two spin-3/2 chains. If the chains are initially in the state $|\uparrow\rangle|\tilde{\Uparrow}\rangle$, the product state is transformed to $\exp(i9\mathcal{J}t_{\rm ex}/4)|\uparrow\rangle|\tilde{\Uparrow}\rangle$ after pulsing the interaction $H_{\rm ex}$ for a time $t_{\rm ex}$. Proceeding analogously in this basis, one can show that such an operation leads to an Ising ZZ gate in the $|\uparrow\rangle$ and $|\downarrow\rangle$ basis and an Ising XX gate in the $|+\rangle$ and $|-\rangle$ basis.

[1] D. Loss and D. P. DiVincenzo, Phys. Rev. A **57**, 120 (1998).
[2] F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111 (1984).
[3] P. Zanardi and M. Rasetti, Phys. Lett. A **264**, 94 (1999).
[4] R. K. Colmenar, U. Güngördü, and J. Kestner, Invalidating the robustness conjecture for geometric quantum gates, arXiv:2105.02882.
[5] S. Berger, M. Pechal, A. A. Abdumalikov, C. Eichler, L. Steffen, A. Fedorov, A. Wallraff, and S. Filipp, Phys. Rev. A **87**, 060303(R) (2013).
[6] M. V. Berry, Proc. R. Soc. London Ser. A Math. Phys. Sci. **392**, 45 (1984).
[7] L.-M. Duan, J. I. Cirac, and P. Zoller, Science **292**, 1695 (2001).
[8] A. Zee, Phys. Rev. A **38**, 1 (1988).
[9] J. E. Avron, L. Sadun, J. Segert, and B. Simon, Phys. Rev. Lett. **61**, 1329 (1988).
[10] J. E. Avron, L. Sadun, J. Segert, and B. Simon, Commun. Math. Phys. **124**, 595 (1989).
[11] B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. **96**, 106802 (2006).
[12] J. C. Budich, D. G. Rothe, E. M. Hankiewicz, and B. Trauzettel, Phys. Rev. B **85**, 205425 (2012).
[13] M. M. Wysokiński, M. Płodzień, and M. Trif, Phys. Rev. B **104**, L041402 (2021).
[14] E. Lieb, T. Schultz, and D. Mattis, Ann. Phys. (NY) **16**, 407 (1961).
[15] D. A. Ivanov, Phys. Rev. Lett. **86**, 268 (2001).
[16] S. Backens, A. Shnirman, Y. Makhlin, Y. Gefen, J. E. Mooij, and G. Schön, Phys. Rev. B **96**, 195402 (2017).
[17] Y. Tserkovnyak and D. Loss, Phys. Rev. A **84**, 032333 (2011).
[18] J. Alicea, Rep. Prog. Phys. **75**, 076501 (2012).
[19] T. Karzig, Y. Oreg, G. Refael, and M. H. Freedman, Phys. Rev. X **6**, 031019 (2016).
[20] J. Alicea, Y. Oreg, G. Refael, F. von Oppen, and M. P. A. Fisher, Nat. Phys. **7**, 412 (2011).
[21] J. C. Budich, S. Walter, and B. Trauzettel, Phys. Rev. B **85**, 121405(R) (2012).
[22] M. Blume, Phys. Rev. **141**, 517 (1966); H. Capel, Physica **32**, 966 (1966).
[23] S. Debnath, N. M. Linke, C. Figgatt, K. A. Landsman, K. Wright, and C. Monroe, Nature (London) **536**, 63 (2016).
[24] S. Asaad, V. Mourik, B. Joecker, M. A. I. Johnson, A. D. Baczewski, H. R. Firgau, M. T. Mądzik, V. Schmitt, J. J. Pla, F. E. Hudson, K. M. Itoh, J. C. McCallum, A. S. Dzurak, A. Laucht, and A. Morello, Nature (London) **579**, 205 (2020).
[25] S. Hoffman, J.-X. Yu, S.-L. Liu, C. Brantley, G. D. Stroscio, R. G. Hadt, G. Christou, X.-G. Zhang, and H.-P. Cheng, Majorana zero modes emulated in a magnetic molecule chain, arXiv:2110.12019.
[26] J. Liu, J. Mrozek, W. K. Myers, G. A. Timco, R. E. P. Winpenny, B. Kintzel, W. Plass, and A. Ardavan, Phys. Rev. Lett. **122**, 037202 (2019).
[27] M. Fittipaldi, A. Cini, G. Annino, A. Vindigni, A. Caneschi, and R. Sessoli, Nat. Mater. **18**, 329 (2019).
[28] J. Robert, N. Parizel, P. Turek, and A. K. Boudalis, J. Am. Chem. Soc. **141**, 19765 (2019).
[29] J. Liu, J. Mrozek, A. Ullah, Y. Duan, J. Baldoví, E. Coronado, A. Gaita-Ariño, and A. Ardavan, Nat. Phys. **17**, 1205 (2021).
[30] J. P. T. Stenger, N. T. Bronn, D. J. Egger, and D. Pekker, Phys. Rev. Res. **3**, 033171 (2021).
[31] J.-S. Xu, K. Sun, Y.-J. Han, C.-F. Li, J. K. Pachos, and G.-C. Guo, Nat. Commun. **7**, 13194 (2016).
[32] S. Sachdev, *Quantum Phase Transitions*, 2nd ed. (Cambridge University Press, 2011).
[33] A. Alexandradinata, N. Regnault, C. Fang, M. J. Gilbert, and B. A. Bernevig, Phys. Rev. B **94**, 125103 (2016).