Quantifying correlations via local channels

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Quantum channels, in general, will disturb quantum states and will lead to decoherence (i.e., information leakage to the environment). For bipartite systems, local channels often cause more decoherence for global states than for local states due to correlations. Intuitively, the difference between these two kinds of decoherence (i.e., global and local ones) may be regarded as a quantifier characterizing certain aspects of correlations in the global bipartite states. Based on this idea, we introduce a quantifier of correlations (relative to a local channel) as the coherence difference and investigate its basic properties. We probe and quantify correlations relative to various channels including the unitary channels, the twirling channels, the projective measurements, and the weak measurements. In particular, we show that both product states and some natural classical-quantum states can be operationally characterized in terms of local channels: The product states are just those states with vanishing coherence difference relative to the twirling channel induced by the unitary group, whereas these natural classical-quantum states are just those states with vanishing coherence difference relative to the twirling channel induced by the subgroup of the unitary group that does not disturb the local states. We further illustrate the results by various examples.

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I. INTRODUCTION

Quantum coherence and quantum correlations, arising from the superposition principle, are intrinsic and intriguing features of quantum mechanics with profound implications and applications. As important physical resources, they have been widely applied to various quantum tasks, such as quantum computation, quantum communication, quantum key distribution, and so on [1]. In recent years, characterizations and quantifications of coherence and correlations have attracted much attention. The relationships between quantum coherence and various quantum correlations, such as quantum entanglement, quantum discord, and quantum nonlocality have been studied extensively [2–19].

Quantum channel is a fundamental concept in quantum mechanics and an important tool for extracting and transmitting information. Many quantum resources can be characterized and quantified in terms of quantum channels. For example, quantum discord [20–26], measurement-induced disturbance [27–29], and measurement-induced nonlocality [30–35] are all defined in terms of local von Neumann measurements, which may be regarded as particular instances of channels. Effects on correlations caused by perturbation of local unitary operations have been studied in Refs. [36–40]. Local quantum uncertainty as another kind of nonclassical correlations is defined by virtue of local observables with the nondegenerate spectrum [41]. For continuous-variable quantum systems, the corresponding issues have been investigated in Refs. [29,42–44].

In this paper we aim to probe and quantify correlations via channels from the perspective of coherence. For this purpose, we will exploit the coherence of a state relative to a channel [14,18,19], which naturally extends the coherence relative to an orthonormal basis of the system Hilbert space [4].

To gain a quick preliminary understanding of our approach and idea, consider a bipartite state ρ^{ab} shared between two parties a and b with reduced states $\rho^a = \mathrm{Tr}_b\rho^{ab}$ on party a and $\rho^b = \mathrm{Tr}_a\rho^{ab}$ on party b, respectively. A local channel $\mathcal{E}^a$ on party a induces decoherence on both the local state ρ^a and the global state ρ^{ab}. In a dual fashion, we may regard the decoherence after the action of the channel as the coherence of the state relative to the channel (before the action of the channel). In general, as a consequence of correlations in the global state, the degree of coherence of ρ^{ab} is larger than that of ρ^a. Therefore, it is natural to quantify correlations (relative to the local channel $\mathcal{E}^a$) in the bipartite state ρ^{ab} in terms of the difference between the coherence of ρ^{ab} and that of ρ^a (both relative to $\mathcal{E}^a$). This motivates us to introduce

$$D(\rho^{ab}, \mathcal{E}^a) = Q(\rho^{ab}, \mathcal{E}^a \otimes \mathcal{I}^b) - Q(\rho^a, \mathcal{E}^a), \tag{1}$$

as a quantifier of correlations in ρ^{ab} (relative to the local channel $\mathcal{E}^a$). Here $\mathcal{I}^b$ stands for the identity channel on party b, and $Q(\rho, \mathcal{E})$ can be any suitable measure of coherence of state ρ (relative to the channel $\mathcal{E}$) satisfying the monotonicity relation $Q(\rho^{ab}, \mathcal{E}^a \otimes \mathcal{I}^b) \geqslant Q(\rho^a, \mathcal{E}^a)$ for any state ρ^{ab} and any local

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channel $\mathcal{E}^a$. For example, we may take the measure of coherence based on the Wigner-Yanase skew information due to its significant information-theoretical meaning and properties [14].

The remainder of this paper is arranged as follows. In Sec. II, we review basic features of coherence of a state relative to a channel (which extends that relative to an orthonormal basis) and introduce the notion of correlations relative to a local channel in terms of coherence difference. In Sec. III, we investigate correlations in bipartite states relative to some typical channels (measurements), such as the unitary channels, the twirling channels, the projective measurements, and the weak measurements. We further discuss basic properties of these unconventional quantifiers of correlations. By the way, we also show that both product states and some natural classical-quantum states can be operationally characterized in terms of correlations relative to specific local channels. In Sec. IV, we evaluate the correlations of some prototypical states, such as the Bell-diagonal states, the Werner states, and the isotropic states. Finally, we conclude with a summary in Sec. V.

II. CORRELATIONS IN TERMS OF COHERENCE DIFFERENCE

In this section, we first review a measure of coherence of a state relative to a channel studied in Ref. [14]. Then we introduce a quantifier of correlations relative to a local channel in terms of coherence difference and reveal its basic properties.

A. Coherence relative to a channel

Consider a quantum system with Hilbert space H of dimension d. Let ρ be a state (density operator) on H and

$$\mathcal{E}(\rho) = \sum_i E_i \rho E_i^\dagger \tag{2}$$

be a channel (completely positive and trace-preserving map) on the system with the Kraus operators $E_i, i = 1, 2, \ldots, m$, which satisfy $\sum_i E_i^\dagger E_i = \mathbf{1}$ (identity operator). The coherence of ρ (relative to the channel $\mathcal{E}$) via the generalized Wigner-Yanase skew information was defined as [14]

$$I(\rho, \mathcal{E}) = \sum_i I(\rho, E_i), \tag{3}$$

where

$$I(\rho, E) = \tfrac{1}{2}\mathrm{Tr}[\sqrt{\rho}, E][\sqrt{\rho}, E]^\dagger \tag{4}$$

is the generalized Wigner-Yanase skew information of ρ with respect to any operator E (not necessarily self-adjoint) [14,45,46], and $[X, Y] = XY - YX$ denotes the commutator between the operators X and Y.

For later convenience, we list some basic properties of the coherence measure $I(\rho, \mathcal{E})$ [14]:

(a) (Non-negativity) $I(\rho, \mathcal{E}) \geqslant 0$, and the equality holds if and only if $\mathcal{E}^\dagger(\sqrt{\rho}) = \sqrt{\rho}$ and $\mathcal{E}^\dagger(\rho) = \rho$. Here the dual channel is defined as $\mathcal{E}^\dagger(X) = \sum_i E_i^\dagger X E_i$ for any operator X on the system Hilbert space.

(b) (Convexity) $I(\rho, \mathcal{E})$ is convex in ρ.

(c) (Affineness) $I(\rho, \mathcal{E})$ is affine in channel $\mathcal{E}$ in the sense that

$$I(\rho, c_1\mathcal{E}_1 + c_2\mathcal{E}_2) = c_1 I(\rho, \mathcal{E}_1) + c_2 I(\rho, \mathcal{E}_2) \tag{5}$$

for any channel $\mathcal{E}_i$ and constants $c_i \geqslant 0, c_1 + c_2 = 1$.

(d) (Unitary covariance) $I(U\rho U^\dagger, U\mathcal{E}U^\dagger) = I(\rho, \mathcal{E})$, where $U\mathcal{E}U^\dagger(\rho) = \sum_j (UE_jU^\dagger)\rho(UE_jU^\dagger)^\dagger$ for $\mathcal{E}(\rho) = \sum_j E_j \rho E_j^\dagger$.

(e) (Decreasing under partial trace)

$$I(\rho^{ab}, \mathcal{E}^a \otimes \mathcal{I}^b) \geqslant I(\rho^a, \mathcal{E}^a), \tag{6}$$

where ρ^{ab} is a bipartite state shared by parties a and b, and $\mathcal{I}^b$ denotes the identity channel on party b. In particular, when $\rho^{ab} = \rho^a \otimes \rho^b$ is a product state, we have

$$I(\rho^a \otimes \rho^b, \mathcal{E}^a \otimes \mathcal{I}^b) = I(\rho^a, \mathcal{E}^a), \tag{7}$$

which may be interpreted as the ancillary-independence property of coherence.

(f) (Contractivity) $I(\rho^{ab}, \mathcal{E}^a \otimes \mathcal{I}^b)$ is contractive under any channel $\mathcal{E}^b$ on party b in the sense that

$$I((\mathcal{I}^a \otimes \mathcal{E}^b)(\rho^{ab}), \mathcal{E}^a \otimes \mathcal{I}^b) \leqslant I(\rho^{ab}, \mathcal{E}^a \otimes \mathcal{I}^b). \tag{8}$$

We remark that $I(\rho, \mathcal{E})$ may also be used to quantify quantum uncertainty of the channel $\mathcal{E}$ in the state ρ [47]. For any purification $|\Psi\rangle\langle\Psi|$ of ρ with $\mathrm{Tr}_b|\Psi\rangle\langle\Psi| = \rho$, we have

$$I(|\Psi\rangle\langle\Psi|, \mathcal{E} \otimes \mathcal{I}^b) = V(|\Psi\rangle\langle\Psi|, \mathcal{E} \otimes \mathcal{I}^b), \tag{9}$$

where the variance of ρ with respect to $\mathcal{E}$,

$$V(\rho, \mathcal{E}) = \frac{1}{2}\sum_i \mathrm{Tr}\rho(E_{i0}E_{i0}^\dagger + E_{i0}^\dagger E_{i0}) \tag{10}$$

is introduced to quantify total uncertainty of the channel $\mathcal{E}$ in state ρ [47]. Here $E_{i0} = E_i - \mathrm{Tr}\,\rho E_i$ for any i. From Eq. (9), we know that even though in general quantum uncertainty is less than or equal to total uncertainty of $\mathcal{E} \otimes \mathcal{I}^b$ in ρ^{ab}, i.e., $I(\rho^{ab}, \mathcal{E} \otimes \mathcal{I}^b) \leqslant V(\rho^{ab}, \mathcal{E} \otimes \mathcal{I}^b)$, for a bipartite pure state $|\Psi\rangle$, the equality holds. An intrinsic relation between coherence and quantum uncertainty is discussed in Ref. [48].

We emphasize that although we have employed the Wigner-Yanase skew information in defining the quantifier of correlations in Eq. (3), one may also consider general quantum Fisher information, which has properties very similar to that of the Wigner-Yanase skew information [49]. In particular, the quantum Fisher information defined via the symmetric logarithmic derivative is a good candidate [49,50].

B. Correlations as coherence difference relative to a local channel

Consider a bipartite system ab with Hilbert space $H^a \otimes H^b$. Let ρ^{ab} be a bipartite state on $H^a \otimes H^b$ with reduced states ρ^a on party a and ρ^b on party b, respectively. Let $\mathcal{E}^a$ be a local channel on party a, which naturally induces a channel $\mathcal{E}^a \otimes \mathcal{I}^b$ on the composite system ab. By property (e) of $I(\rho, \mathcal{E})$, we know that, in general, the amount of coherence of the global state ρ^{ab} (relative to $\mathcal{E}^a \otimes \mathcal{I}^b$) is larger than or equal to that of the local state ρ^a (relative to $\mathcal{E}^a$). The difference between them indicates correlations in ρ^{ab} relative to $\mathcal{E}^a$. Hence we introduce the coherence difference,

$$D(\rho^{ab}, \mathcal{E}^a) = I(\rho^{ab}, \mathcal{E}^a \otimes \mathcal{I}^b) - I(\rho^a, \mathcal{E}^a), \tag{11}$$

as a quantifier of correlations in ρ^{ab} (relative to $\mathcal{E}^a$). By Eq. (7), $D(\rho^{ab}, \mathcal{E}^a)$ can be rewritten as

$$D(\rho^{ab}, \mathcal{E}^a) = I(\rho^{ab}, \mathcal{E}^a \otimes \mathcal{I}^b) - I(\rho^a \otimes \rho^b, \mathcal{E}^a \otimes \mathcal{I}^b), \tag{12}$$

which implies that it can also be reinterpreted as the coherence difference between ρ^{ab} and the corresponding product state $\rho^a \otimes \rho^b$ relative to the same product channel $\mathcal{E}^a \otimes \mathcal{I}^b$. Alternatively, from Eq. (11), we know that the coherence of ρ^{ab} relative to a local channel $\mathcal{E}^a$ can be decomposed into two parts: the coherence of the local state ρ^a and the correlations in ρ^{ab}, i.e.,

$$I(\rho^{ab}, \mathcal{E}^a \otimes \mathcal{I}^b) = I(\rho^a, \mathcal{E}^a) + D(\rho^{ab}, \mathcal{E}^a). \tag{13}$$

It is easy to obtain the following properties of $D(\rho^{ab}, \mathcal{E}^a)$ from the corresponding properties of $I(\rho, \mathcal{E})$:

(i) $D(\rho^{ab}, \mathcal{E}^a) \geqslant 0$, and the equality holds when $\rho^{ab} = \rho^a \otimes \rho^b$ is a product state.

(ii) For any channel $\mathcal{E}^b$ on party b,

$$D(\mathcal{I}^a \otimes \mathcal{E}^b(\rho^{ab}), \mathcal{E}^a) \leqslant D(\rho^{ab}, \mathcal{E}^a), \tag{14}$$

where $\mathcal{I}^a$ denotes the identity channel on party a.

(iii) For any local unitary operators U^a and U^b on parties a and b, respectively, it holds that

$$D((U^a \otimes U^b)\rho^{ab}(U^a \otimes U^b)^\dagger, \mathcal{E}^a) = D(\rho^{ab}, U^{a\dagger}\mathcal{E}^a U^a), \tag{15}$$

where $U^{a\dagger}\mathcal{E}^a U^a(\rho) = \sum_i (U^{a\dagger} E_i U^a)\rho(U^{a\dagger} E_i U^a)^\dagger$ for $\mathcal{E}^a(\rho) = \sum_i E_i \rho E_i^\dagger$.

We note that for a pure bipartite state $\rho^{ab} = |\Psi^{ab}\rangle\langle\Psi^{ab}|$ with $\mathrm{Tr}_b|\Psi^{ab}\rangle\langle\Psi^{ab}| = \rho^a$, we have

$$\begin{aligned} D(|\Psi^{ab}\rangle\langle\Psi^{ab}|, \mathcal{E}^a) &= I(|\Psi^{ab}\rangle\langle\Psi^{ab}|, \mathcal{E}^a \otimes \mathcal{I}^b) - I(\rho^a, \mathcal{E}^a) \\ &= V(|\Psi^{ab}\rangle\langle\Psi^{ab}|, \mathcal{E}^a \otimes \mathcal{I}^b) - I(\rho^a, \mathcal{E}^a) \\ &= V(\rho^a, \mathcal{E}^a) - I(\rho^a, \mathcal{E}^a) \\ &= C(\rho^a, \mathcal{E}^a), \end{aligned} \tag{16}$$

where

$$C(\rho^a, \mathcal{E}^a) = V(\rho^a, \mathcal{E}^a) - I(\rho^a, \mathcal{E}^a) \tag{17}$$

quantifies the classical uncertainty of $\mathcal{E}^a$ in ρ^a [47]. Equation (16) means that the classical uncertainty of a channel in a state is equal to the amount of correlations in the corresponding purified state relative to the local channel, which is consistent with the idea illustrated in Ref. [51].

III. APPLICATIONS

In this section, we use the quantification method for correlations introduced in the previous section to study correlations relative to several important channels and investigate their basic properties. In particular, we characterize the set of bipartite states without correlations (product states) and some natural states without quantum discord (classical-quantum states). We also investigate correlations in bipartite states relative to weak measurements and prove that the amount of correlations is increasing with the measurement strength, achieves the maximum when the weak measurement reduces to a projective measurement, and becomes zero when it reduces to the identity channel.

A. Correlations relative to a unitary channel

Let us begin by considering the correlations in bipartite states relative to a unitary channel. Despite its simplicity, unitary channels are extremely important since they not only describe the evolution of a closed quantum system, but also can be used as quantum gates in quantum computation [1]. It is desirable and natural to investigate correlations in bipartite states relative to a unitary channel.

Consider a bipartite system shared by two parties a and b. Let U^a be a unitary operator on party a, which naturally induces a unitary channel $\mathcal{U}^a(\sigma) = U^a\sigma U^{a\dagger}$ for any state σ on party a. For any bipartite state ρ^{ab}, we define a quantifier of correlations in ρ^{ab} relative to the local unitary channel $\mathcal{U}^a$ as

$$\begin{aligned} D(\rho^{ab}, \mathcal{U}^a) &= I(\rho^{ab}, \mathcal{U}^a \otimes \mathcal{I}^b) - I(\rho^a, \mathcal{U}^a) \\ &= I(\rho^{ab}, U^a \otimes \mathbf{1}^b) - I(\rho^a, U^a), \end{aligned} \tag{18}$$

where $\mathbf{1}^b$ is the identity operator on party b.

When $I(\rho^a, U^a) = 0$, which is equivalent to $[\rho^a, U^a] = 0$, we know that the local unitary channel $\mathcal{U}^a$ does not induce any decoherence of the local state ρ^a relative to $\mathcal{U}^a$. Let $U^a = \sum_{i=1}^m e^{\sqrt{-1}\theta_i}\Pi_i$ be the spectral decomposition of U^a, where $\theta_i \in [0, 2\pi)$ for $i = 1, 2, \ldots, m$, $\theta_i \neq \theta_j$ for $i \neq j$, and the projectors Π_i constitute an orthogonal decomposition of the identity operator $\mathbf{1}^a$ on H^a, i.e., $\sum_i \Pi_i = \mathbf{1}^a$. For the projective measurement $\Pi^a = \{\Pi_i : i = 1, 2, \ldots, m\}$, if $I(\rho^a, U^a) = 0$, then $I(\rho^a, \Pi^a) = I(\rho^a, U^a) = 0$. This is because that the equation $U^a\rho^a U^{a\dagger} = \rho^a$ is equivalent to

$$\sum_{i,j=1}^m e^{\sqrt{-1}(\theta_i - \theta_j)}\Pi_i\rho^a\Pi_j = \sum_{i,j=1}^m \Pi_i\rho^a\Pi_j, \tag{19}$$

and, thus, is also equivalent to

$$\Pi^a(\rho^a) = \sum_{i=1}^m \Pi_i\rho^a\Pi_i = \rho^a. \tag{20}$$

This implies that the local unitary channel $\mathcal{U}^a$ does not induce any decoherence of ρ^a relative to $\mathcal{U}^a$ if and only if the corresponding projective measurement Π^a does not induce any decoherence of ρ^a relative to Π^a. It should be emphasized that when $I(\rho^a, U^a) > 0$, in general, $I(\rho^a, \Pi^a) \neq I(\rho^a, U^a)$.

In the case of $I(\rho^a, U^a) = 0$, in contrast to the fact that $\mathcal{U}^a$ does not induce any decoherence of the local state ρ^a relative to $\mathcal{U}^a$, it may induce decoherence of the global state ρ^{ab} relative to $\mathcal{U}^a$, i.e., $I(\rho^{ab}, U^a \otimes \mathbf{1}^b) > 0$ due to the correlations contained in ρ^{ab}. In this case the amount of correlations in ρ^{ab} relative to $\mathcal{U}^a$ does not vanish, i.e., $D(\rho^{ab}, \mathcal{U}^a) = I(\rho^{ab}, U^a \otimes \mathbf{1}^b) > 0$. For example, let $\rho^{ab} = |\Psi^{ab}\rangle\langle\Psi^{ab}|$ be the maximally entangled state on a two-qubit system with $|\Psi^{ab}\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$, and $U^a = |0\rangle\langle 0| + e^{\sqrt{-1}\theta}|1\rangle\langle 1|$ with $\theta \in (0, \pi)$, then $\rho^a = \mathbf{1}^a/2$. Consequently, $I(\rho^a, U^a) = 0$, but

$$I(\rho^{ab}, U^a \otimes \mathbf{1}^b) = \sin^2 \tfrac{\theta}{2} > 0. \tag{21}$$

It is easy to verify that under the condition $I(\rho^a, U^a) = 0$, $D(\rho^{ab}, U^a \otimes \mathbf{1}^b) = I(\rho^{ab}, U^a \otimes \mathbf{1}^b) = 0$ if and only if

$$\rho^{ab} = (\Pi^a \otimes \mathcal{I}^b)(\rho^{ab}) \equiv \sum_{i=1}^{m} (\Pi_i \otimes \mathbf{1}^b)\rho^{ab}(\Pi_i \otimes \mathbf{1}^b), \tag{22}$$

where Π^a is the projective measurement on party a corresponding to the spectral decomposition of U^a, and $\mathcal{I}^b$ denotes the identity channel on party b.

In order to get rid of the dependence on unitary operators and obtain some intrinsic quantities capturing correlations in ρ^{ab}, we may take the maximum, minimum, or integration over all unitary operators on party a. Obviously,

$$\min_{U^a} D(\rho^{ab}, \mathcal{U}^a) = I(\rho^{ab}, \mathbf{1}^a \otimes \mathbf{1}^b) - I(\rho^a, \mathbf{1}^a) = 0, \tag{23}$$

which is trivial. Now by taking the maximum, we define

$$\bar{D}_{\max}(\rho^{ab}) = \max_{U^a} D(\rho^{ab}, \mathcal{U}^a). \tag{24}$$

This quantity has the following properties:

(i) $\bar{D}_{\max}(\rho^{ab}) \geqslant 0$, and the equality holds if and only if $\rho^{ab} = \rho^a \otimes \rho^b$.

(ii) $\bar{D}_{\max}$ is invariant under local unitary operations, i.e., $\bar{D}_{\max}((V^a \otimes V^b)\rho^{ab}(V^a \otimes V^b)^\dagger) = \bar{D}_{\max}(\rho^{ab})$ for any unitary operators V^a and V^b on parties a and b, respectively.

(iii) $\bar{D}_{\max}$ decreases under any local quantum operation $\mathcal{E}^b$ on party b, i.e., $\bar{D}_{\max}(\mathcal{I}^a \otimes \mathcal{E}^b(\rho^{ab})) \leqslant \bar{D}_{\max}(\rho^{ab})$.

Now we sketch the proof of the above properties.

For item (i), if $\rho^{ab} = \rho^a \otimes \rho^b$ is a product state, then $D(\rho^{ab}, \mathcal{U}^a) = 0$ for any unitary operator U^a, which implies $\bar{D}_{\max}(\rho^{ab}) = 0$. Conversely, the desired result follows from the facts $0 \leqslant D(\rho^{ab}, \mathcal{T}_{U(H^a)}) \leqslant \bar{D}_{\max}(\rho^{ab})$ and $D(\rho^{ab}, \mathcal{T}_{U(H^a)}) = 0$ if and only if $\rho^{ab} = \rho^a \otimes \rho^b$ is a product state (see the next subsection).

Item (ii) follows from

$$\begin{aligned}
\bar{D}_{\max}((V^a \otimes V^b)\rho^{ab}(V^a \otimes V^b)^\dagger) &= \max_{U^a} D((V^a \otimes V^b)\rho^{ab}(V^a \otimes V^b)^\dagger, \mathcal{U}^a) \\
&= \max_{U^a}\{I((V^a \otimes V^b)\rho^{ab}(V^a \otimes V^b)^\dagger, U^a \otimes \mathbf{1}^b) - I(V^a\rho^a V^{a\dagger}, U^a)\} \\
&= \max_{U^a}\{I(\rho^{ab}, (V^{a\dagger}U^aV^a) \otimes \mathbf{1}^b) - I(\rho^a, V^{a\dagger}U^aV^a)\} \\
&= \max_{U^a} D(\rho^{ab}, \mathcal{U}^a) \\
&= \bar{D}_{\max}(\rho^{ab}).
\end{aligned} \tag{25}$$

Item (iii) follows from the contractivity of $D(\rho^{ab}, \mathcal{E}^a)$ under local operations on party b.

B. Correlations relative to twirling channel induced by unitary group

For a bipartite quantum system shared between two parties a and b, the unitary group $U(H^a)$ on party a with a d_a-dimensional system space H^a naturally induces a twirling channel,

$$\mathcal{T}_{U(H^a)}(\rho) = \int_{U(H^a)} U^a\rho U^{a\dagger} dU^a, \tag{26}$$

where dU^a is the normalized Haar measure on $U(H^a)$. For any bipartite state ρ^{ab}, the coherence of the partial state $\rho^a = \mathrm{Tr}_b\rho^{ab}$ and that of the global state ρ^{ab} relative to the local twirling channel $\mathcal{T}_{U(H^a)}$ are defined as

$$I\big(\rho^a, \mathcal{T}_{U(H^a)}\big) = \int_{U(H^a)} I(\rho^a, U^a) dU^a, \tag{27}$$

and

$$I(\rho^{ab}, \mathcal{T}_{U(H^a)} \otimes \mathcal{I}^b) = \int_{U(H^a)} I(\rho^{ab}, U^a \otimes \mathbf{1}^b) dU^a, \tag{28}$$

respectively. The corresponding quantifier of correlations in ρ^{ab} relative to $\mathcal{T}_{U(H^a)}$ is defined as

$$D\big(\rho^{ab}, \mathcal{T}_{U(H^a)}\big) = \int_{U(H^a)} [I(\rho^{ab}, U^a \otimes \mathbf{1}^b) - I(\rho^a, U^a)] dU^a. \tag{29}$$

By results in Ref. [52], we know that

$$\int_{U(H^a)} U^aXU^{a\dagger} dU^a = \mathrm{Tr}\, X \frac{\mathbf{1}^a}{d_a} \tag{30}$$

for any operator X on party a, and

$$\int_{U(H^a)} (U^a \otimes \mathbf{1}^b)T^{ab}(U^{a\dagger} \otimes \mathbf{1}^b) dU^a = \frac{\mathbf{1}^a}{d_a} \otimes \mathrm{Tr}_a T^{ab} \tag{31}$$

for any operator T^{ab} on the bipartite system ab. From the above equations we get

$$D(\rho^{ab}, \mathcal{T}_{U(H^a)}) = \frac{1}{d_a}[(\mathrm{Tr}\sqrt{\rho^a})^2 - \mathrm{Tr}_b(\mathrm{Tr}_a\sqrt{\rho^{ab}})^2]. \tag{32}$$

Let $\{X_i: i = 1, 2, \ldots, d_a^2\}$ be an orthonormal basis of the real Hilbert space $L(H^a)$ of all observables with the Hilbert-Schmidt inner product $\langle A|B\rangle = \mathrm{Tr}\, AB$, then $\sum_{i=1}^{d_a^2} X_i^2 = d_a\mathbf{1}^a$, and the set of operators $\{X_i/\sqrt{d_a}: i = 1, 2, \ldots, d_a^2\}$ naturally constitutes a Kraus representation of the completely depolarizing channel on party a, which is denoted as

$\Lambda^a(\rho) = \sum_{i=1}^{d_a^2} X_i\rho X_i/d_a$ and coincides with the twirling channel $\mathcal{T}_{U(H^a)}$. Alternatively, it is easy to directly verify that

$$\begin{aligned} D(\rho^{ab}, \Lambda^a) &= \sum_{i=1}^{d_a^2} I\left(\rho^{ab}, \frac{X_i}{\sqrt{d_a}} \otimes \mathbf{1}^b\right) - \sum_{i=1}^{d_a^2} I\left(\rho^a, \frac{X_i}{\sqrt{d_a}}\right) \\ &= \frac{1}{d_a}\sum_{i=1}^{d_a^2}[I(\rho^{ab}, X_i \otimes \mathbf{1}^b) - I(\rho^a, X_i)] \\ &= D(\rho^{ab}, \mathcal{T}_{U(H^a)}). \end{aligned} \tag{33}$$

Here $\sum_{i=1}^{d_a^2}[I(\rho^{ab}, X_i \otimes \mathbf{1}^b) - I(\rho^a, X_i)]$ is precisely the quantifier of correlations in terms of the Wigner-Yanase skew information introduced in Ref. [53]. By virtue of the result therein, we have $D(\rho^{ab}, \mathcal{T}_{U(H^a)}) = D(\rho^{ab}, \Lambda^a) = 0$ if and only if $\rho^{ab} = \rho^a \otimes \rho^b$.

So far we have proven that the amount of correlations in ρ^{ab} relative to the local twirling channel $\mathcal{T}_{U(H^a)}$ vanishes if and only if there does not exist any correlations in ρ^{ab}. This provides a characterization of product states in terms of the local twirling channel. In the following, we will show that by choosing a proper channel, the method of quantifying correlations we introduced here can also be applied to characterize some natural states without quantum discord. Before doing that, we recall that a bipartite state ρ^{ab} has no quantum discord if it can be expressed as

$$\rho^{ab} = \sum_{i=1}^{d_a} p_i|i\rangle\langle i| \otimes \rho_i^b, \tag{34}$$

with p_i a probability distribution (i.e., $p_i \geqslant 0$, $\sum_i p_i = 1$), $\{|i\rangle: i = 1, 2, \dots, d_a\}$ an orthonormal basis of the system Hilbert space H^a, and $\{\rho_i^b: i = 1, 2, \dots, d_a\}$ a set of local states on party b. In this instance, the state ρ^{ab} is also called classical-quantum [27].

In the next subsection, we will characterize some natural classical-quantum states, that is, the states of the form

$$\rho^{ab} = \sum_{i=1}^{m} \lambda_i\Pi_i \otimes \rho_i^b, \tag{35}$$

such that $\rho^a = \sum_{i=1}^m \lambda_i\Pi_i$ is the canonical spectral decomposition of ρ^a, i.e., $\lambda_i \neq \lambda_j$ for $i \neq j$.

C. Correlations relative to twirling channel induced by invariant unitary group

Consider a bipartite quantum system $H^a \otimes H^b$ shared between parties a and b. Let ρ^{ab} be a bipartite state on $H^a \otimes H^b$ with reduced states $\rho^a = \mathrm{Tr}_b\rho^{ab}$ and $\rho^b = \mathrm{Tr}_a\rho^{ab}$. Let $\rho^a = \sum_{i=1}^m \lambda_i\Pi_i$ be the canonical spectral decomposition of the local state ρ^a. Then the induced projective measurement $\Pi^a = \{\Pi_i: i = 1, 2, \dots, m\}$ satisfies $\Pi_i\Pi_j = \delta_{ij}\Pi_i$, $\sum_{i=1}^m \Pi_i = \mathbf{1}^a$. Let

$$H^{a_i} = \Pi_i H^a, \quad n_i = \dim H^{a_i}, \quad i = 1, 2, \dots, m, \tag{36}$$

then $\sum_{i=1}^m n_i = d_a = \dim H^a$. Let

$$U_0(H^a) = \{V \in U(H^a): V\rho^a V^\dagger = \rho^a\} \tag{37}$$

be the set of unitary operators which do not disturb ρ^a. It is easy to verify that it is a subgroup of $U(H^a)$. The corresponding twirling channel of $U_0(H^a)$ is defined as

$$\mathcal{T}_{U_0(H^a)}(\rho) = \int_{U_0(H^a)} V\rho V^\dagger dV, \tag{38}$$

where dV is the Haar measure on the group $U_0(H^a)$. By the definition of $U_0(H^a)$, we know that $I(\rho^a, V) = 0$ for any $V \in U_0(H^a)$, which, in turn, implies that

$$I\big(\rho^a, \mathcal{T}_{U_0(H^a)}\big) = \int_{U_0(H^a)} I(\rho^a, V)dV = 0. \tag{39}$$

Hence, the amount of correlations in ρ^{ab} relative to the local twirling channel $\mathcal{T}_{U_0(H^a)}$ is

$$\begin{aligned} D\big(\rho^{ab}, \mathcal{T}_{U_0(H^a)}\big) &= I\big(\rho^{ab}, \mathcal{T}_{U_0(H^a)} \otimes \mathcal{I}^b\big) - I\big(\rho^a, \mathcal{T}_{U_0(H^a)}\big) \\ &= \int_{U_0(H^a)} I(\rho^{ab}, V \otimes \mathbf{1}^b)dV. \end{aligned} \tag{40}$$

Actually, an explicit expression of $D(\rho^{ab}, \mathcal{T}_{U_0(H^a)})$ can be derived as follows. Since for any $V \in U_0(H^a)$, $[\rho^a, V] = 0$, we know that for any spectral projector Π_i of ρ^a, $[\Pi_i, V] = 0$, $i = 1, 2, \dots, m$. Then taking $V_i = \Pi_i V = \Pi_i V\Pi_i$, $i = 1, 2, \dots, m$, we have $V_iV_j = \delta_{ij}V_i$. Thus for any $V \in U_0(H^a)$, we have the direct sum decompositions,

$$V = \bigoplus_{i=1}^m V_i, \quad U_0(H^a) = \bigoplus_{i=1}^m U\big(H^{a_i}\big), \tag{41}$$

and the decomposition of the Haar measure $dV = dV_1dV_2\cdots dV_m$ with dV_i the Haar measure on $U(H^{a_i})$ (the full unitary group on H^{a_i}). Consequently,

$$\begin{aligned} D\big(\rho^{ab}, \mathcal{T}_{U_0(H^a)}\big) &= 1 - \mathrm{Tr}\sqrt{\rho^{ab}}\int_{U_0(H^a)}(V \otimes \mathbf{1}^b)\sqrt{\rho^{ab}}(V^\dagger \otimes \mathbf{1}^b)dV \\ &= 1 - \sum_{i,j}\mathrm{Tr}\left(\sqrt{\rho^{ab}}\int_{U(H^{a_1})}\cdots\int_{U(H^{a_m})}(V_i \otimes \mathbf{1}^b)\sqrt{\rho^{ab}}(V_j^\dagger \otimes \mathbf{1}^b)dV\right) \\ &= 1 - \sum_i \mathrm{Tr}\sqrt{\rho^{ab}}\int_{U(H^{a_i})}(V_i \otimes \mathbf{1}^b)\sqrt{\rho^{ab}}(V_i^\dagger \otimes \mathbf{1}^b)dV_i. \end{aligned} \tag{42}$$

The last equality follows from [52]

$$\int_{U(H^{a_i})} V_i dV_i = 0, \quad i = 1, 2, \ldots, m. \tag{43}$$

Furthermore, let $\{X_l : l = 1, 2, \ldots, d_a^2\}$ and $\{Y_m : m = 1, 2, \ldots, d_b^2\}$ be orthonormal bases of $L(H^a)$ and $L(H^b)$, respectively, then by letting

$$\sqrt{\rho^{ab}} = \sum_{l,m} a_{lm} X_l \otimes Y_m, \tag{44}$$

we get an explicit expression of $D(\rho^{ab}, \mathcal{T}_{U_0(H^a)})$ as

$$\begin{aligned} D\big(\rho^{ab}, \mathcal{T}_{U_0(H^a)}\big) &= 1 - \sum_{i,l,m} a_{lm} \mathrm{Tr}\sqrt{\rho^{ab}} \int_{U(H^{a_i})} V_i X_l V_i^\dagger \otimes Y_m dV_i \\ &= 1 - \sum_{i,l,m} a_{lm} \mathrm{Tr}\sqrt{\rho^{ab}} \frac{\mathrm{Tr}(\Pi_i X_l \Pi_i)}{\mathrm{Tr}\,\Pi_i} \Pi_i \otimes Y_m \\ &= 1 - \sum_i \frac{\mathrm{Tr}_b\{\mathrm{Tr}_a[(\Pi_i \otimes \mathbf{1}^b)\sqrt{\rho^{ab}}(\Pi_i \otimes \mathbf{1}^b)]\}^2}{\mathrm{Tr}\,\Pi_i}. \end{aligned} \tag{45}$$

In the following, we show that $D(\rho^{ab}, \mathcal{T}_{U_0(H^a)})$ can be used to characterize some natural classical-quantum states. Indeed, $D(\rho^{ab}, \mathcal{T}_{U_0(H^a)})$ has the following properties:

(i) $D(\rho^{ab}, \mathcal{T}_{U_0(H^a)}) \geqslant 0$, and the equality holds if and only if $\rho^{ab} = \sum_{i=1}^m \lambda_i \Pi_i \otimes \rho_i^b$ is a classical-quantum state with $\rho^a = \sum_{i=1}^m \lambda_i \Pi_i$ being the canonical spectral decomposition of ρ^a.

(ii) $D(\rho^{ab}, \mathcal{T}_{U_0(H^a)})$ is invariant under local unitary operations that do not disturb the local state ρ^a, i.e.,

$$D\big((V^a \otimes U^b)\rho^{ab}(V^a \otimes U^b)^\dagger, \mathcal{T}_{U_0(H^a)}\big) = D\big(\rho^{ab}, \mathcal{T}_{U_0(H^a)}\big) \tag{46}$$

for any unitary operator $V^a \in U_0(H^a)$ and $U^b \in U(H^b)$.

(iii) $D(\rho^{ab}, \mathcal{T}_{U_0(H^a)})$ decreases under local quantum operation $\mathcal{E}^b$ on party b, i.e.,

$$D\big(\mathcal{I}^a \otimes \mathcal{E}^b(\rho^{ab}), \mathcal{T}_{U_0(H^a)}\big) \leqslant D(\rho^{ab}, \mathcal{T}_{U_0(H^a)}). \tag{47}$$

We proceed to sketch the proof of the above properties.

For item (i) in order to prove the sufficiency, suppose that $\rho^{ab} = \sum_{i=1}^m \lambda_i \Pi_i \otimes \rho_i^b$ is a classical-quantum state with $\rho^a = \sum_{i=1}^m \lambda_i \Pi_i$ being the canonical spectral decomposition of ρ^a, then substituting $\sqrt{\rho^{ab}} = \sum_i \sqrt{\lambda_i}\Pi_i \otimes \sqrt{\rho_i^b}$ into Eq. (45), we have

$$\begin{aligned} D(\rho^{ab}, \mathcal{T}_{U_0(H^a)}) &= 1 - \sum_i \frac{\mathrm{Tr}_b\big[\mathrm{Tr}_a\big(\sqrt{\lambda_i}\Pi_i \otimes \sqrt{\rho_i^b}\big)\big]^2}{\mathrm{Tr}\,\Pi_i} \\ &= 1 - \sum_i \lambda_i \mathrm{Tr}\,\Pi_i \\ &= 0. \end{aligned} \tag{48}$$

The sufficiency is obtained. For the necessity, note that $D(\rho^{ab}, \mathcal{T}_{U_0(H^a)}) = 0$ implies that

$$I(\rho^{ab}, V \otimes \mathbf{1}^b) = I(\rho^a, V) = 0, \quad \forall V \in U_0(H^a). \tag{49}$$

Choosing $V_0 = \sum_{i=1}^m e^{\sqrt{-1}\theta_i}\Pi_i \in U_0(H^a)$ with $\theta_i \in [0, 2\pi)$, $i = 1, 2, \ldots, m$, $\theta_i \neq \theta_j$ for $i \neq j$, and Π_i the spectral projectors of ρ^a, then by Eq. (22), we know that

$$\rho^{ab} = \Pi^a \otimes \mathcal{I}^b(\rho^{ab}) = \sum_{i=1}^m (\Pi_i \otimes \mathbf{1}^b)\rho^{ab}(\Pi_i \otimes \mathbf{1}^b). \tag{50}$$

Let $\rho^{a_i b} = (\Pi_i \otimes \mathbf{1}^b)\rho^{ab}(\Pi_i \otimes \mathbf{1}^b)$, then $\rho^{a_i b} \in L(H^{a_i}) \otimes L(H^b)$ are non-negative definite operators satisfying $\sum_{i=1}^m \mathrm{Tr}\,\rho^{a_i b} = 1$, and ρ^{ab} can be decomposed as

$$\rho^{ab} = \bigoplus_{i=1}^m \rho^{a_i b}. \tag{51}$$

Since for any $V \in U_0(H^a)$, V has a direct sum decomposition $V = \bigoplus_{i=1}^m V_i \in U_0(H^a)$ with $V_i \in U(H^{a_i})$ being any unitary operator on H^{a_i}, substituting this into $\rho^{ab} = (V \otimes \mathbf{1}^b)\rho^{ab}(V^\dagger \otimes \mathbf{1}^b)$, we have

$$\bigoplus_{i=1}^m \rho^{a_i b} = \bigoplus_{i=1}^m (V_i \otimes \mathbf{1}^b)\rho^{a_i b}(V_i^\dagger \otimes \mathbf{1}^b). \tag{52}$$

Therefore,

$$\rho^{a_i b} = (V_i \otimes \mathbf{1}^b)\rho^{a_i b}(V_i^\dagger \otimes \mathbf{1}^b) \tag{53}$$

for any $V_i \in U(H^{a_i})$, $i = 1, 2, \ldots, m$. Consequently,

$$\rho^{a_i b} = \mathbf{1}_i \otimes Z_i^b, \quad i = 1, 2, \ldots, m, \tag{54}$$

where $\mathbf{1}_i$ are the identity operators on H^{a_i}, and $Z_i^b \in L(H^b)$ are non-negative definite operators satisfying $\sum_{i=1}^m n_i \mathrm{Tr}\, Z_i^b = 1$. Finally, we get

$$\rho^{ab} = \bigoplus_{i=1}^m \rho^{a_i b} = \bigoplus_{i=1}^m \mathbf{1}_i \otimes Z_i^b = \sum_{i=1}^m \lambda_i \Pi_i \otimes \rho_i^b, \tag{55}$$

where $\lambda_i = \mathrm{Tr}\, Z_i^b$, $\rho_i^b = Z_i^b/\lambda_i$. This completes the proof of item (i).

Item (ii) can be obtained by direct manipulation of Eq. (45).

Item (iii) follows from the contractivity of the coherence measure $I(\rho, \mathcal{E})$ under local operations on party b.

It should be remarked that in Sec. III A, for a bipartite state ρ^{ab} and a local unitary channel $\mathcal{U}^a$ satisfying $I(\rho^{ab}, \mathcal{U}^a \otimes \mathcal{I}^b) = I(\rho^a, \mathcal{U}^a) = 0$, we can only conclude that

$$\rho^{ab} = \bigoplus_{i=1}^m \rho^{a_i b}, \tag{56}$$

where $\rho^{a_i b} = (\Pi_i \otimes \mathbf{1}^b)\rho^{ab}(\Pi_i \otimes \mathbf{1}^b) \in L(H^{a_i}) \otimes L(H^b)$ with Π_i the spectral projectors of U^a. In contrast, for the local twirling channel $\mathcal{T}_{U_0(H^a)}$, $I(\rho^{ab}, \mathcal{T}_{U_0(H^a)} \otimes \mathcal{I}^b) = I(\rho^a, \mathcal{T}_{U_0(H^a)}) = 0$ implies that

$$\rho^{ab} = \bigoplus_{i=1}^m \rho^{a_i b} = \sum_{i=1}^m \lambda_i \Pi_i \otimes \rho_i^b. \tag{57}$$

In other words, the condition that for all $V \in U_0(H^a)$, $I(\rho^{ab}, V \otimes \mathbf{1}^b) = 0$ is necessary to decouple all the correlations in $\rho^{a_i b}$, i.e., to make $\rho^{a_i b}$ a product state (up to a scale) and thereby to make ρ^{ab} a classical-quantum state.

D. Correlations relative to a Lüders measurement

Consider a bipartite system shared between two parties a and b. Let $\Pi_{\mathrm{vN}}^a = \{\Pi_i : i = 1, 2, \ldots, d_a\}$ be a local von Neumann measurement on party a. We quantify correlations in ρ^{ab} relative to the von Neumann measurement Π^a_{vN} by

$$D(\rho^{ab}, \Pi^a_{\mathrm{vN}}) = I(\rho^{ab}, \Pi^a_{\mathrm{vN}} \otimes \mathcal{I}^b) - I(\rho^a, \Pi^a_{\mathrm{vN}}) = \sum_i [I(\rho^{ab}, \Pi_i \otimes \mathbf{1}^b) - I(\rho^a, \Pi_i)], \quad (58)$$

which is just the correlations quantifier relative a von Neumann measurement discussed in Ref. [11]. Furthermore, the minimal coherence difference,

$$D_{\min}(\rho^{ab}) = \min_{\Pi^a_{\mathrm{vN}}} D(\rho^{ab}, \Pi^a_{\mathrm{vN}}), \quad (59)$$

and the maximal coherence difference,

$$D_{\max}(\rho^{ab}) = \max_{\Pi^a_{\mathrm{vN}}} D(\rho^{ab}, \Pi^a_{\mathrm{vN}}) \quad (60)$$

are studied. In particular, $D_{\min}(\rho^{ab}) = 0$ if ρ^{ab} is a classical-quantum state and $D_{\max}(\rho^{ab}) = 0$ if and only if ρ^{ab} is a product state.

Now consider a Lüders measurement $\Pi^a = \{\Pi_i : i = 1, 2, \ldots, m\}$ on party a with the projective operators Π_i (not necessarily one dimensional) satisfying $\Pi_i \Pi_j = \delta_{ij} \Pi_i$ and $\sum_i \Pi_i = \mathbf{1}^a$. The correlations in ρ^{ab} relative to the local Lüders measurement Π^a may be quantified by

$$D(\rho^{ab}, \Pi^a) = I(\rho^{ab}, \Pi^a \otimes \mathcal{I}^b) - I(\rho^a, \Pi^a) = \sum_i [I(\rho^{ab}, \Pi_i \otimes \mathbf{1}^b) - I(\rho^a, \Pi_i)]. \quad (61)$$

Similar to the case of von Neumann measurements by taking the minimum and the maximum over all Lüders measurements, we may further define

$$\tilde{D}_{\min}(\rho^{ab}) = \min_{\Pi^a} D(\rho^{ab}, \Pi^a), \quad (62)$$

$$\tilde{D}_{\max}(\rho^{ab}) = \max_{\Pi^a} D(\rho^{ab}, \Pi^a). \quad (63)$$

Obviously,

$$\tilde{D}_{\min}(\rho^{ab}) \leqslant D_{\min}(\rho^{ab}), \quad \tilde{D}_{\max}(\rho^{ab}) \geqslant D_{\max}(\rho^{ab}). \quad (64)$$

Hence, we know that $\tilde{D}_{\max}(\rho^{ab}) = 0$ if and only if ρ^{ab} is a product state, whereas $\tilde{D}_{\min}(\rho^{ab}) = 0$ if ρ^{ab} is a classical-quantum state.

When Π^a is a Lüders measurement that does not disturb the local state ρ^a, i.e., $\Pi^a(\rho^a) = \rho^a$, we have

$$D(\rho^{ab}, \Pi^a) = I(\rho^{ab}, \Pi^a \otimes \mathcal{I}^b) = \sum_i I(\rho^{ab}, \Pi_i \otimes \mathbf{1}^b). \quad (65)$$

Then $D(\rho^{ab}, \Pi^a) = 0$ if and only if $\rho^{ab} = \Pi^a \otimes \mathcal{I}^b(\rho^{ab})$.

By taking the minimum and the maximum over all Lüders measurements that do not disturb the local state ρ^a, we may define

$$\hat{D}_{\min}(\rho^{ab}) = \min_{\Pi^a:\, \Pi^a(\rho^a) = \rho^a} D(\rho^{ab}, \Pi^a), \quad (66)$$

and

$$\hat{D}_{\max}(\rho^{ab}) = \max_{\Pi^a:\, \Pi^a(\rho^a) = \rho^a} D(\rho^{ab}, \Pi^a). \quad (67)$$

It can be shown that $\hat{D}_{\max}(\rho^{ab}) = 0$ if and only if $\rho^{ab} = \sum_{i=1}^m \lambda_i \Pi_i \otimes \rho_i^b$ is a classical-quantum state with $\rho^a = \sum_{i=1}^m \lambda_i \Pi_i$ being the canonical spectral decomposition of ρ^a, whereas $\hat{D}_{\min}(\rho^{ab}) = 0$ if and only if there exists a Lüders measurement that does not disturb local state ρ^a such that $\rho^{ab} = \Pi^a \otimes \mathcal{I}^b(\rho^{ab})$.

Consider the diagonal unitary group,

$$G(\Pi) = \left\{ U_\theta = \sum_{i=1}^m e^{\sqrt{-1}\theta_i} \Pi_i : \theta = (\theta_1, \ldots, \theta_m) \in [0, 2\pi)^m \right\} \quad (68)$$

induced by the Lüders measurement $\Pi = \{\Pi_i : i = 1, 2, \ldots, m\}$. Then the twirling channel generated by the group $G(\Pi)$ is

$$\mathcal{T}_{G(\Pi)}(\rho) = \int_0^{2\pi} \cdots \int_0^{2\pi} U_\theta \rho U_\theta^\dagger \frac{d\theta}{(2\pi)^m}, \quad (69)$$

with $d\theta = d\theta_1 d\theta_2 \cdots d\theta_m$. It turns out that

$$\mathcal{T}_{G(\Pi)}(\rho) = \Pi(\rho), \quad (70)$$

with $\Pi(\rho) = \sum_{i=1}^m \Pi_i \rho \Pi_i$ the decohering channel. Thus,

$$D(\rho^{ab}, \mathcal{T}_{G(\Pi)}) = D(\rho^{ab}, \Pi), \quad (71)$$

which shows that the amount of correlations induced by the twirling channel $\mathcal{T}_{G(\Pi)}$ (associated with the diagonal unitary group $G(\Pi)$) is equal to that caused by the Lüders measurement Π (as generators of the diagonal unitary group $G(\Pi)$).

E. Correlations relative to a weak measurement

In this subsection, we investigate the correlations in bipartite states relative to weak measurements. Recall that weak measurements were introduced by Aharonov *et al.* [54], and are universal in the sense that any generalized measurement can be realized as a sequence of weak measurements [55]. For $x \in [0, \infty)$, consider the weak measurement $M_x = \{E_x, E_{-x}\}$ with the Kraus operators,

$$E_x = \alpha \Pi_1 + \beta \Pi_2, \quad E_{-x} = \beta \Pi_1 + \alpha \Pi_2, \quad (72)$$

where

$$\alpha = \sqrt{\frac{1 - \tanh x}{2}}, \quad \beta = \sqrt{\frac{1 + \tanh x}{2}}. \quad (73)$$

Here x denotes the measurement strength, Π_1 and Π_2 are orthogonal projectors satisfying $\Pi_1 + \Pi_2 = \mathbf{1}$ with $\mathbf{1}$ the identity operator on the system space. In particular, when $x = 0$, M_x reduces to the identity channel $\mathcal{I}$ and when $x \to \infty$, M_x tends to the projective measurement $\Pi = \{\Pi_1, \Pi_2\}$.

For a bipartite state ρ^{ab} and a weak measurement M_x on party a, we have a quantifier of correlations in ρ^{ab} relative to M_x as

$$D(\rho^{ab}, M_x) = I(\rho^{ab}, M_x \otimes \mathcal{I}^b) - I(\rho^a, M_x). \quad (74)$$

Noting that for any operator X on party a, it holds that

$$X = (\Pi_1 + \Pi_2) X (\Pi_1 + \Pi_2) = \sum_{i,j=1}^2 \Pi_i X \Pi_j, \quad (75)$$

and we have

$$\begin{aligned} D(\rho^{ab}, M_x) &= I(\rho^{ab}, M_x \otimes \mathcal{I}^b) - I(\rho^a, M_x) \\ &= (\alpha^2+\beta^2)\sum_{i=1}^{2}[\mathrm{Tr}\sqrt{\rho^a}\Pi_i\sqrt{\rho^a}\Pi_i - \mathrm{Tr}\sqrt{\rho^{ab}}(\Pi_i\otimes \mathbf{1}^b)\sqrt{\rho^{ab}}(\Pi_i\otimes\mathbf{1}^b)] + 4\alpha\beta \\ &\quad \times\ [\mathrm{Tr}\sqrt{\rho^a}\Pi_1\sqrt{\rho^a}\Pi_2 - \mathrm{Tr}\sqrt{\rho^{ab}}(\Pi_1\otimes\mathbf{1}^b)\sqrt{\rho^{ab}}(\Pi_2\otimes\mathbf{1}^b)] \\ &= (\alpha-\beta)^2[I(\rho^{ab},\Pi\otimes\mathcal{I}^b) - I(\rho^a,\Pi)] \\ &= (\alpha-\beta)^2 D(\rho^{ab},\Pi) \\ &= (1-\sqrt{1-\tanh^2 x})D(\rho^{ab},\Pi). \end{aligned} \tag{76}$$

It implies that the amount of correlations in ρ^{ab} relative to a local weak measurement M_x is increasing with the measurement strength x. Therefore, for a given bipartite state ρ^{ab}, the amount of correlations in ρ^{ab} relative to the identity channel $\mathcal{I}$ is minimal, whereas that relative to the projective measurement Π is maximal, among all weak measurements M_x. This is consistent with our intuition.

IV. ILLUSTRATING CORRELATIONS RELATIVE TO VARIOUS CHANNELS

In this section, we evaluate explicitly correlations of various states relative to the amplitude damping channel, the phase damping channel, and the depolarizing channel, respectively. In particular, for the two-qubit Werner states and the isotropic states, we discuss the behaviors of correlations with respect to the parameters of states and channels. Besides, we also calculate the correlations relative to the unitary channel, the twirling channel induced by the full unitary group and the twirling channel induced by the unitary group that does not disturb the local state for the general Werner states and isotropic states.

For convenience, we first recall some two-qubit states. Any Bell diagonal state can be represented in the spectral decomposition form as

$$\begin{aligned} \rho_{\mathrm{B}} &= \lambda_1|\Phi^+\rangle\langle\Phi^+| + \lambda_2|\Phi^-\rangle\langle\Phi^-| \\ &\quad + \lambda_3|\Psi^+\rangle\langle\Psi^+| + \lambda_4|\Psi^-\rangle\langle\Psi^-| \end{aligned} \tag{77}$$

with $|\Phi^\pm\rangle = (|00\rangle \pm |11\rangle)/\sqrt{2}$, $|\Psi^\pm\rangle = (|01\rangle \pm |10\rangle)/\sqrt{2}$, $\lambda_i \geqslant 0$, $\sum_{i=1}^4 \lambda_i = 1$. In particular, when

$$\lambda_1=\lambda_2=\lambda_3=\frac{1-x}{4},\quad \lambda_4=\frac{1+3x}{4},\quad x\in\left[-\frac{1}{3},1\right], \tag{78}$$

ρ_{B} are the two-qubit Werner states [56],

$$\mathbf{w} = \frac{1-x}{4}\mathbf{1}\otimes\mathbf{1} + x|\Psi^-\rangle\langle\Psi^-|. \tag{79}$$

When

$$\lambda_1 = y,\quad \lambda_2=\lambda_3=\lambda_4=\frac{1-y}{3},\quad y\in[0,1], \tag{80}$$

ρ_{B} are the two-qubit isotropic states [57],

$$\tau = \frac{1-y}{3}\mathbf{1}\otimes\mathbf{1} + \frac{4y-1}{3}|\Phi^+\rangle\langle\Phi^+|. \tag{81}$$

As a comparison, we also consider the family of pure states,

$$|S\rangle = \sum_{i=0}^{1}\sqrt{s_i}|i\rangle|i\rangle. \tag{82}$$

Example 1. For the amplitude damping channel $\mathcal{E}_{\mathrm{AD}}(\rho) = \sum_{i=1}^2 K_i\rho K_i^\dagger$ with the Kraus operators,

$$K_1 = \begin{pmatrix}1 & 0\\ 0 & \sqrt{1-p}\end{pmatrix},\quad K_2 = \begin{pmatrix}0 & \sqrt{p}\\ 0 & 0\end{pmatrix},\quad 0\leqslant p\leqslant 1, \tag{83}$$

let $q = 1-\sqrt{1-p}$, we have

$$D(|S\rangle, \mathcal{E}_{\mathrm{AD}}) = s_0s_1q^2 + \sqrt{s_0s_1}p, \tag{84}$$

$$\begin{aligned} D(\rho_{\mathrm{B}}, \mathcal{E}_{\mathrm{AD}}) &= \tfrac{1}{2}q[1-2(\sqrt{\lambda_1\lambda_2}+\sqrt{\lambda_3\lambda_4})] \\ &\quad + \tfrac{p}{4}(\sqrt{\lambda_1}+\sqrt{\lambda_2}-\sqrt{\lambda_3}-\sqrt{\lambda_4})^2. \end{aligned} \tag{85}$$

Specifying to the Werner states $\mathbf{w}$ and the isotropic states τ, we have

$$D(\mathbf{w}, \mathcal{E}_{\mathrm{AD}}) = \tfrac{1}{8}(p+2q)[1+x-\sqrt{(1-x)(1+3x)}], \tag{86}$$

$$D(\tau, \mathcal{E}_{\mathrm{AD}}) = \tfrac{1}{12}(p+2q)[1+2y-2\sqrt{3y(1-y)}]. \tag{87}$$

Example 2. Consider the phase damping channel $\mathcal{E}_{\mathrm{PD}}(\rho) = \sum_{i=1}^2 K_i\rho K_i^\dagger$ with

$$K_1 = \begin{pmatrix}1 & 0\\ 0 & \sqrt{1-p}\end{pmatrix},\quad K_2 = \begin{pmatrix}0 & 0\\ 0 & \sqrt{p}\end{pmatrix},\quad 0\leqslant p\leqslant 1. \tag{88}$$

Let $q = 1-\sqrt{1-p}$. By straightforward calculation, we have

$$D(|S\rangle, \mathcal{E}_{\mathrm{PD}}) = 2qs_0s_1, \tag{89}$$

$$D(\rho_{\mathrm{B}}, \mathcal{E}_{\mathrm{PD}}) = q\left(\frac{1}{2} - \sqrt{\lambda_1\lambda_2} - \sqrt{\lambda_3\lambda_4}\right). \tag{90}$$

The amounts of correlations of the Werner states $\mathbf{w}$ and the isotropic states τ relative to the phase damping channel $\mathcal{E}_{\mathrm{PD}}$ can be readily obtained as

$$D(\mathbf{w}, \mathcal{E}_{\mathrm{PD}}) = \tfrac{1}{4}q[1+x-\sqrt{(1-x)(1+3x)}], \tag{91}$$

$$D(\tau, \mathcal{E}_{\mathrm{PD}}) = \tfrac{1}{6}q[1+2y-2\sqrt{3y(1-y)}]. \tag{92}$$

Example 3. Consider the depolarizing channel,

$$\mathcal{E}_{\mathrm{De}}(\rho) = (1-3p)\rho + p\sum_{i=1}^{3}\sigma_i\rho\sigma_i,\quad 0\leqslant p\leqslant\frac{1}{3}, \tag{93}$$

where σ_i are the Pauli operators. For the pure state $|S\rangle$ and the Bell diagonal states $\rho_{\rm B}$, we have

$$D(|S\rangle, \mathcal{E}_{\rm De}) = 4\sqrt{s_0 s_1}(1 + \sqrt{s_0 s_1})p, \tag{94}$$

$$D(\rho_{\rm B}, \mathcal{E}_{\rm De}) = p[4 - (\sqrt{\lambda_1} + \sqrt{\lambda_2} + \sqrt{\lambda_3} + \sqrt{\lambda_4})^2]. \tag{95}$$

In particular, for the Werner states **w** and the isotropic states τ, we have

$$D(\mathbf{w}, \mathcal{E}_{\rm De}) = \frac{3p}{2}[1 + x - \sqrt{(1-x)(1+3x)}], \tag{96}$$

$$D(\tau, \mathcal{E}_{\rm De}) = p[1 + 2y - 2\sqrt{3y(1-y)}]. \tag{97}$$

Comparing the above three examples, we see that the amounts of correlations of both **w** and τ have similar behaviors for these three channels. Specifically, the amount of correlations of **w** is decreasing with x in $[-1/3, 0]$ and increasing with x in [0,1]. The amount of correlations of τ is decreasing with y in $[0, 1/4]$ and increasing with y in $[1/4, 1]$. On the other hand, $D(\mathbf{w}, \mathcal{E})$ and $D(\tau, \mathcal{E})$ are all increasing functions of the noise parameter p for the three channels.

Example 4. Consider the general Werner states,

$$\mathbf{w} = \frac{d-x}{d^3-d}\mathbf{1}_d \otimes \mathbf{1}_d + \frac{dx-1}{d^3-d}F, \quad x \in [-1, 1], \tag{98}$$

on $H \otimes H$ with $\{|\mu\rangle : \mu = 1, 2, \ldots, d\}$ an orthonormal basis of the d-dimensional Hilbert space H, and $F = \sum_{\mu,\nu=1}^d |\mu\rangle\langle\nu| \otimes |\nu\rangle\langle\mu|$ the swap operation. In the following, we evaluate the amounts of correlations of the general Werner states relative to various channels.

For the unitary channel $\mathcal{U}(\rho) = U\rho U^\dagger$ with U an arbitrary unitary operator on H, we have

$$\begin{aligned} D(\mathbf{w}, \mathcal{U}) &= I(\mathbf{w}, U \otimes \mathbf{1}) \\ &= 1 - [d^2 s_+^2 + 2\,d s_+ s_- + s_-^2 |\mathrm{Tr}\,U|^2], \end{aligned} \tag{99}$$

with

$$s_\pm = \frac{\sqrt{1+x}}{2\sqrt{d(d+1)}} \pm \frac{\sqrt{1-x}}{2\sqrt{d(d-1)}}. \tag{100}$$

For the twirling channel,

$$\mathcal{T}_{U(H)}(\rho) = \int_{U(H)} U\rho U^\dagger dU \tag{101}$$

induced by the unitary group $U(H)$, by Eq. (33) and the result in Ref. [53], we have

$$D(\mathbf{w}, \mathcal{T}_{U(H)}) = \tfrac{1}{2d}[d - x - \sqrt{(d^2-1)(1-x^2)}]. \tag{102}$$

For the twirling channel $\mathcal{T}_{U_0(H)}$ induced by the unitary group $U_0(H)$ that does not disturb the local state $\mathbf{1}/d$ of the Werner states **w**, since in this case $U_0(H) = U(H)$, we have

$$D(\mathbf{w}, \mathcal{T}_{U_0(H)}) = D(\mathbf{w}, \mathcal{T}_{U(H)}) = \tfrac{1}{2d}[d - x - \sqrt{(d^2-1)(1-x^2)}]. \tag{103}$$

Example 5. Consider the general isotropic states

$$\tau = \frac{1-x}{d^2-1}\mathbf{1}_d \otimes \mathbf{1}_d + \frac{d^2x-1}{d^2-1}|\Psi\rangle\langle\Psi|, \quad x \in [0, 1], \tag{104}$$

on $H \otimes H$, where $|\Psi\rangle = \frac{1}{\sqrt{d}}\sum_{\mu=1}^d |\mu\mu\rangle$ with $\{|\mu\rangle : \mu = 1, 2, \ldots, d\}$ an orthonormal basis of the d-dimensional Hilbert space H. For any unitary channel $\mathcal{U}(\rho) = U\rho U^\dagger$ with U as an arbitrary unitary operator on H, we have

$$D(\tau, \mathcal{U}) = 1 - \alpha^2 d^2 - 2\alpha\beta - \beta^2 |\mathrm{Tr}\,U|^2, \tag{105}$$

with

$$\alpha = \sqrt{\frac{1-x}{d^2-1}}, \quad \beta = \sqrt{x} - \sqrt{\frac{1-x}{d^2-1}}. \tag{106}$$

For the twirling channel $\mathcal{T}_{U(H)}$ induced by the unitary group $U(H)$, by Eq. (33) and the result in Ref. [53], we have

$$D(\tau, \mathcal{T}_{U(H)}) = \frac{1}{d^2}[d^2 x - 2x + 1 - 2\sqrt{x(1-x)(d^2-1)}]. \tag{107}$$

For the twirling channel $\mathcal{T}_{U_0(H)}$ induced by the unitary group $U_0(H)$ that does not disturb the local state $\mathbf{1}/d$ of the isotropic states τ, since in this case $U_0(H) = U(H)$, we have

$$\begin{aligned} D(\tau, \mathcal{T}_{U_0(H)}) &= D(\tau, \mathcal{T}_{U(H)}) \\ &= \frac{1}{d^2}[d^2 x - 2x + 1 - 2\sqrt{x(1-x)(d^2-1)}]. \end{aligned} \tag{108}$$

V. SUMMARY

Correlations are ubiquitous and multifaceted in quantum information theory and have been extensively studied from many perspectives. Some widely used quantifiers of correlations include mutual information, entanglement, quantum discord, etc. In this paper, we have investigated correlations by exploiting the coherence difference of the global and local states relative to a local channel. Since a local channel on a bipartite state usually causes more decoherence on the global state than on the local state due to the correlations therein, we quantify these correlations as the coherence difference between the global state and the local state relative to the local channel. Furthermore, we have shown that two typical sets of states, the set of product states and a natural subset of the set of classical-quantum states, can be simply and operationally characterized in this way. It is desirable to find applications of the channel approach in characterizing other kinds of correlations, such as entanglement and quantum steering by choosing proper channels.

We have also investigated correlations in bipartite states relative to weak measurements and have shown that the amount of correlations is increasing with the measurement strength, achieves the maximum when the weak measurement turns to a projective measurement and becomes zero when it reduces to the identity channel. This sheds further insights into the physical meaning of the strength of the weak measurements.

Probing correlations via channels has intrinsic relations to the metrological power of quantum states, and it may be interesting to seek operational meaning of our quantifiers of correlations in quantum metrology. This is left for further study.

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