

# Computable multipartite multimode Gaussian quantum correlation measure and the monogamy relations for continuous-variable systems

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In this paper, definitions of the unification condition, the hierarchy condition, and three kinds of monogamy relations for multipartite quantum correlation measures are given and discussed. A computable multipartite multimode Gaussian quantum correlation measure  $\mathcal{M}^{(k)}$  is proposed for any  $k$ -partite multimode continuous-variable systems with  $k \geq 2$ . The value of  $\mathcal{M}^{(k)}$  only depends on the covariance matrices of continuous-variable states, is invariant under any permutation of subsystems, has no ancilla problem, is nonincreasing under  $k$ -partite local Gaussian channels (particularly, invariant under  $k$ -partite locally Gaussian unitary operations), and vanishes on  $k$ -partite product states. For a  $k$ -partite Gaussian state  $\rho$ ,  $\mathcal{M}^{(k)}(\rho) = 0$  if and only if  $\rho$  is a  $k$ -partite product state. Moreover,  $\mathcal{M}^{(k)}$  satisfies the unification condition and the hierarchy condition that a multipartite quantum correlation measure should obey. We also show that  $\mathcal{M}^{(k)}$  is not strongly monogamous but completely monogamous and tightly monogamous.

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## I. INTRODUCTION

An amazing feature of quantum mechanics is the presence of quantum correlations in composite quantum systems. It is proved that quantum correlations beyond entanglement can also be exploited in quantum information tasks as physical resources. Various methods have been proposed to describe bipartite quantum correlations such as quantum discord [1], geometric quantum discord [2–4], measurement-induced non-locality (MIN) [5], and measurement-induced disturbance (MID) [6] for discrete-variable systems. Notice that in many quantum protocols the systems considered are continuous-variable (CV) systems. Therefore it is also important and interesting to study quantum correlations in CV systems.

Denote by  $\mathcal{GS}^{m+n}(H_A \otimes H_B)$  the set of all  $(m+n)$ -mode Gaussian states in the CV system described by a Hilbert space  $H_A \otimes H_B$ . Let  $\mathcal{G}_{A/B} : \mathcal{GS}^{m+n} \rightarrow [0, +\infty)$  be a functional. Following the idea from [7–10],  $\mathcal{G}_{A/B}$  is a Gaussian quantum correlation measure (GQCM) for a Gaussian quantum correlation (GQC) with respect to subsystem A/B if it satisfies the following four conditions:

(i) For any Gaussian state  $\rho_{AB}$ ,  $\mathcal{G}_{A/B}(\rho_{AB}) = 0$  if and only if  $\rho_{AB}$  contains no GQC.

(ii) (Locally Gaussian unitary invariant)  $\mathcal{G}_{A/B}(W \otimes V \rho_{AB} (W^\dagger \otimes V^\dagger)) = \mathcal{G}_{A/B}(\rho_{AB})$  holds for any Gaussian unitary operators  $W$  on  $H_A$ ,  $V$  on  $H_B$ , and any Gaussian state  $\rho_{AB}$ .

(iii) (Nonincreasing under local Gaussian channels performed on B/A)  $\mathcal{G}_A(I \otimes \Phi_B) \rho_{AB} \leq \mathcal{G}_A(\rho_{AB})$  [resp.  $\mathcal{G}_B((\Phi_A \otimes I) \rho_{AB}) \leq \mathcal{G}_B(\rho_{AB})$ ] holds for any Gaussian channel  $\Phi_B$  (resp.  $\Phi_A$ ) performed on subsystem B (resp. on subsystem A) and any Gaussian state  $\rho_{AB}$ .

(iv)  $\mathcal{G}_{A/B}$  describes the entanglement on pure Gaussian states; that is, if  $|\psi\rangle\langle\psi|$  is a pure Gaussian state, then  $\mathcal{G}_{A/B}(|\psi\rangle\langle\psi|) = 0$  if and only if  $|\psi\rangle$  is a product state.

Furthermore,  $\mathcal{G}_A$  is a nice GQCM if it satisfies (i)–(iii) and the following:

(iv') (Reducing to an entanglement measure for pure states) There exists an entanglement monotone  $\mathcal{E}$  such that  $\mathcal{G}_A(|\psi\rangle\langle\psi|) = \mathcal{E}(|\psi\rangle\langle\psi|)$  for any bipartite Gaussian pure state  $|\psi\rangle\langle\psi|$ .

Several GQCMs have been proposed for bipartite CV systems. Giorda and Paris [11] and Adesso and Datta [12] independently gave the definition of Gaussian quantum discord  $D$  for Gaussian states. Adesso and Girolami in Ref. [13] proposed the concept of Gaussian geometric discord  $D_G$ . It was shown that for a Gaussian state  $\rho_{AB}$ ,  $D(\rho_{AB}) = 0$  ( $D_G(\rho_{AB}) = 0$ ) if and only if  $\rho_{AB}$  is a product state; that is,  $\rho_{AB}$  has no quantum correlation if and only if it is a product state. Since then, remarkable efforts have been made to find simpler ways to quantify this Gaussian correlation. For instance, the MID of Gaussian states was proposed [14] and MINs  $Q$ ,  $Q_{\mathcal{P}}$  for Gaussian states were studied [15]. Gaussian discriminating strengths based on the minimum or maximum change induced on the state by a locally Gaussian unitary operation were investigated in [16–18]. Based on Gaussian unitary operation and the fidelity, several kinds of Gaussian response of discord (for example,  $\mathcal{GD}_R^x$ ,  $N_F$ ,  $N_{\mathcal{F}}$ ) were proposed and discussed in [19,20]. For other related results, see [21–25] and the references therein. All quantifications mentioned above describe the same GQC as that described by Gaussian quantum discord.

However, not one of the bipartite GQCMs mentioned above is easily accessible. It is very difficult to calculate the values for all  $(n+m)$ -mode Gaussian states except  $(1+1)$ -mode Gaussian states or some special Gaussian states, since these GQCMs involve some measurements on a subsystem

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and some optimization process. Also note that these GQCMs are not symmetric about the subsystems, though the corresponding GQCs are. The second point is that these GQCMs cannot be extended to multipartite systems evidently.

Thus, two problems arise.

*Problem 1.* Are there some ways of quantifying GQCs for bipartite CV systems that are easily accessible?

*Problem 2.* What are the rules that every multipartite multimode GQCM (beyond entanglement) should obey and are there such GQCMs for multipartite CV systems that are easily accessible and monogamous in some sense?

For the first problem, an effort is made in [26] where a computable GQCM  $\mathcal{M}$  for  $(n+m)$ -mode CV systems is proposed. It is shown that  $\mathcal{M}$  satisfies the following nicer properties:

- (1) For any  $(n+m)$ -mode Gaussian state  $\rho_{AB}$ ,  $\mathcal{M}(\rho_{AB}) = 0$  if and only if  $\rho_{AB}$  is a product state;
- (2)  $\mathcal{M}$  is locally Gaussian unitary invariant;
- (3)  $\mathcal{M}$  is nonincreasing under local Gaussian channels in the sense that  $\mathcal{M}((\Phi_A \otimes \Phi_B)\rho_{AB}) \leq \mathcal{M}(\rho_{AB})$  holds for any Gaussian channel  $\Phi_{A/B}$  performed on subsystem A/B and any Gaussian state  $\rho_{AB}$ ;
- (4)  $\mathcal{M}$  is independent of the mean, is symmetric about subsystems, and has no ancilla problem.

But for the second problem, by our knowledge, no results of quantifying GQCs (beyond entanglement) for multipartite multimode CV systems were known. The purpose of this paper is to give an answer to the second problem.

Not like the bipartite GQCM, as a multipartite multimode GQCM, it should obey some additional rules. For multipartite entanglement measures, these additional rules were discussed first in [27]. It is pointed out in [27] that a multipartite entanglement measure should meet the unification condition and the hierarchy condition. For a quantum correlation beyond entanglement, as a physical resource it is reasonable to require that the unification condition and the hierarchy condition should also be obeyed by their multipartite quantum correlation measures. The unification condition is easily understood, but the hierarchy condition is not defined clearly in [27].

In this paper we give an exact definition of the hierarchy condition which declares that the whole correlation of lower partition is not greater than the whole correlation of higher partition; the partial correlation is not greater than the whole correlation; and the correlation after kicking some parties out of subgroups is not greater than the correlation between the subgroups. We also propose a multipartite multimode GQCM  $\mathcal{M}^{(k)}$  and discuss its properties for any  $k$ -partite CV systems ( $k \geq 2$ ). The definition of  $\mathcal{M}^{(k)}$  only depends on the covariance matrix of CV states and thus is more easily calculated for any CV state with finite second moments.  $\mathcal{M}^{(k)}$  is a multipartite extension of  $\mathcal{M}$  as  $\mathcal{M}^{(2)} = \mathcal{M}$ . We show that  $\mathcal{M}^{(k)}$  has almost all expected good properties:  $\mathcal{M}^{(k)}$  vanishes on  $k$ -partite product states and, for a  $k$ -partite Gaussian state  $\rho$ ,  $\mathcal{M}^{(k)}(\rho) = 0$  if and only if  $\rho$  is a  $k$ -partite product state;  $\mathcal{M}^{(k)}$  is invariant under any permutation of subsystems; has no ancilla problem; and is nonincreasing under  $k$ -partite local Gaussian channels (particularly, is invariant under  $k$ -partite locally Gaussian unitary operations) in the sense that  $\mathcal{M}^{(k)}[(\Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_k)\rho_{A_1, A_2, \dots, A_k}] \leq \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k})$  for any Gaussian channel  $\Phi_j$  in subsystem

$H_{A_j}$  and any Gaussian state  $\rho_{A_1, A_2, \dots, A_k}$ . More importantly, we show that  $\mathcal{M}^{(k)}$  satisfies the unification condition and the hierarchy condition that a multipartite quantum correlation measure should obey. Therefore it is true that  $\mathcal{M}^{(k)}$  is an accessible multipartite multimode Gaussian quantum correlation measure for CV systems.

Finally, the monogamy relation for  $\mathcal{M}^{(k)}$  is investigated. Recall that a bipartite entanglement measure  $E$  is monogamous if  $E(\rho_{A|BC}) \geq E(\rho_{AB}) + E(\rho_{AC})$  (CKW inequality) holds for any  $\rho_{ABC}$  [28]. Many bipartite entanglement measures are monogamous (see Refs. [29–31] and the references therein). In [31], Guo and Gour proposed the monogamy without inequalities. It seems that the monogamy relation (i.e., CKW inequality) is a natural feature for quantum entanglement, because entanglement is a kind of physical resource and thus the amount of part entanglements cannot exceed the amount of total entanglement, which is almost equivalent to the statement that if two parties A and B are maximally entangled, then neither of them can share entanglement with a third party C [27,31]. The monogamy relations for quantum correlations beyond entanglement have also been investigated [32,33]. But it is surprising that all bipartite quantum correlation measures beyond entanglement for discrete systems, including quantum discord, are not monogamous in general [33], which is a contradiction to the fact that many quantum correlations beyond entanglement are physical resources. The trouble may come from the definition of monogamy relations. The monogamy relation discusses relationships between three parties by using a bipartite measure, focuses only on the relation between the parties A, BC, the parties A, B, as well as the parties A, C and ignores the relation contained in the parties ABC and the relation between parties B, C, which seems incomplete. In fact, by the hierarchy condition,  $E(\rho_{A|BC})$  is still a part of the total entanglement  $E^{(3)}(\rho_{ABC})$  shared by A, B, and C, where  $E^{(3)}$  is a tripartite entangled measure and is consistent with  $E$ . So, to understand the monogamy relation better one should consider the question in framework of multipartite entanglement measures. Guo and Zhang gave a strict framework for defining multipartite entanglement measures, based on which, the complete monogamy relation and the tight monogamy relation were established [27].

We give exactly the definitions of the monogamy relations for a multipartite quantum correlation measure in this paper. It is revealed that there are three kinds of monogamy relations: (1) the tight monogamy relation, which claims that the correlation between subgroups attains the total correlation will imply that the parties in the same subgroup are not correlated to each other; (2) the complete monogamy relation, which claims that the correlation of a subgroup attains the total correlation, will imply that the parties out of the subgroup are not correlated with any other parties of the system; and (3) the strong monogamy relation, which claims that the correlation between subgroups after “kicking some parties out of” each subgroup keeps invariant will imply that the remaining parties are not correlated with the parties kicked out of them. Then, we prove that  $\mathcal{M}^{(k)}$  is completely monogamous and tightly monogamous. However,  $\mathcal{M}^{(k)}$  is not strongly monogamous.

The paper is organized as follows. In Sec. II we recall briefly some notions and notations from CV systems and propose the quantity  $\mathcal{M}^{(k)}$ . Section III is devoted to studying

the basic properties of  $\mathcal{M}^{(k)}$ . In Sec. IV we show that  $\mathcal{M}^{(k)}$  satisfies the unification condition and the hierarchy condition. The monogamy relations for  $\mathcal{M}^{(k)}$  are studied in Sec. V. Finally, a short conclusion is given in Sec. VI.

## II. DEFINITION OF $\mathcal{M}^{(k)}$

Before giving our definition of the quantity  $\mathcal{M}^{(k)}$ , we need to briefly recall some notions and notations concerning Gaussian states (for more details, see Ref. [34]). Recall that an  $n$ -mode continuous-variable system (CV system) is a system determined by  $2n$ -tuple  $(\hat{Q}_1, \hat{P}_1, \dots, \hat{Q}_n, \hat{P}_n)$  of self-adjoint operators with state space  $H = H_1 \otimes H_2 \otimes \dots \otimes H_n$ , where  $\hat{P}_r, \hat{Q}_r$  are respectively the position and momentum operators of the  $r$ th mode, which acts on the separable infinite-dimensional complex Hilbert space  $H_r$ . As it is well known,  $\hat{Q}_r = (\hat{a}_r + \hat{a}_r^\dagger)/\sqrt{2}$  and  $\hat{P}_r = -i(\hat{a}_r - \hat{a}_r^\dagger)/\sqrt{2}$  ( $r = 1, 2, \dots, n$ ), with  $\hat{a}_r^\dagger$  and  $\hat{a}_r$  being the creation and annihilation operators in the  $r$ th mode  $H_r$ , which satisfy the canonical commutation relation (CCR),

$$[\hat{a}_r, \hat{a}_s^\dagger] = \delta_{rs}I \quad \text{and} \quad [\hat{a}_r^\dagger, \hat{a}_s^\dagger] = [\hat{a}_r, \hat{a}_s] = 0, \\ r, s = 1, 2, \dots, n.$$

Denote by  $\mathcal{S}(H)$  the set of all quantum states in a system described by  $H$  (the positive operators on  $H$  with trace 1). The characteristic function  $\chi_\rho$  for any state  $\rho \in \mathcal{S}(H)$  is defined as

$$\chi_\rho(z) = \text{tr}(\rho W(z)),$$

where  $z = (x_1, y_1, \dots, x_n, y_n)^T \in \mathbb{R}^{2n}$ ,  $W(z) = \exp(iRz)$  is the Weyl displacement operator,  $R = (\hat{R}_1, \hat{R}_2, \dots, \hat{R}_{2n}) = (\hat{Q}_1, \hat{P}_1, \dots, \hat{Q}_n, \hat{P}_n)$ .

Let  $\mathcal{FS}(H)$  be the set of all quantum states with finite second moments, that is,  $\rho \in \mathcal{FS}(H)$  if  $\text{Tr}(\rho \hat{R}_r) < \infty$  and  $\text{Tr}(\rho \hat{R}_r^2) < \infty$  for all  $r = 1, 2, \dots, 2n$ . For  $\rho \in \mathcal{FS}(H)$ , its first moment vector

$$\mathbf{d} = \mathbf{d}_\rho = (\langle \hat{R}_1 \rangle, \langle \hat{R}_2 \rangle, \dots, \langle \hat{R}_{2n} \rangle)^T \\ = (\text{tr}(\rho \hat{R}_1), \text{tr}(\rho \hat{R}_2), \dots, \text{tr}(\rho \hat{R}_{2n}))^T \in \mathbb{R}^{2n}$$

and the second moment matrix

$$\Gamma = \Gamma_\rho = (\gamma_{kl}) \in M_{2n}(\mathbb{R}),$$

defined by  $\gamma_{kl} = \text{tr}[\rho(\Delta \hat{R}_k \Delta \hat{R}_l + \Delta \hat{R}_l \Delta \hat{R}_k)]$ , with  $\Delta \hat{R}_k = \hat{R}_k - \langle \hat{R}_k \rangle$  ([35]) is called, respectively, the mean (or the displacement vector) of  $\rho$  and the covariance matrix (CM) of  $\rho$ . Here  $M_k(\mathbb{R})$  stands for the algebra of all  $k \times k$  matrices over the real field  $\mathbb{R}$ . Note that a CM  $\Gamma$  must be real symmetric and satisfy the condition  $\Gamma + i\Delta \geq 0$ , where  $\Delta = \bigoplus_{r=1}^n \Delta_r$  with  $\Delta_r = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$  for each  $r$ . A Gaussian state  $\rho \in \mathcal{FS}(H)$  is such a state of which the characteristic function  $\chi_\rho(z)$  is of the form

$$\chi_\rho(z) = \exp\left(-\frac{1}{4}z^T \Gamma_\rho z + i\mathbf{d}_\rho^T z\right).$$

A quantum channel (trace preserving complete positive map)  $\Phi$  is called a Gaussian channel if  $\Phi$  sends every Gaussian state into a Gaussian state. A unitary operator  $U$  acting on  $H$  is said to be Gaussian if the unitary operation  $\rho \mapsto U\rho U^\dagger$  is a Gaussian channel.

Let  $\rho_{A_1, A_2, \dots, A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$  be a CV state in a  $k$ -partite ( $n_1 + n_2 + \dots + n_k$ )-mode CV system. Then its CM can be represented as

$$\Gamma_{\rho_{A_1, A_2, \dots, A_k}} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{pmatrix}, \quad (1)$$

where  $A_{jj} \in M_{2n_j}(\mathbb{R})$  is the CM of the reduced state  $\rho_{A_j} = \text{Tr}_{A_j^c}(\rho_{A_1, A_2, \dots, A_k})$  of  $\rho_{A_1, A_2, \dots, A_k}$ ,  $A_j^c = \{A_1, \dots, A_{j-1}, A_{j+1}, \dots, A_k\}$ , namely,  $A_{jj} = \Gamma_{\rho_{A_j}}$ , and  $A_{ij} = A_{ji}^T \in M_{2n_i, 2n_j}(\mathbb{R})$  for any  $i, j \in \{1, 2, \dots, k\}$ , which reveals quantum correlation between subsystems  $A_i$  and  $A_j$ .

*Definition 1.* For any  $(n_1 + n_2 + \dots + n_k)$ -mode  $k$ -partite state  $\rho_{A_1, A_2, \dots, A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$ , the quantity  $\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k})$  is defined by

$$\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) = 1 - \frac{\det(\Gamma_{\rho_{A_1, A_2, \dots, A_k}})}{\prod_{j=1}^k \det(\Gamma_{\rho_{A_j}})},$$

where  $\Gamma_{\rho_{A_1, A_2, \dots, A_k}}$  and  $\Gamma_{\rho_{A_j}}$  are respectively the covariance matrices of  $\rho_{A_1, A_2, \dots, A_k}$  and  $\rho_{A_j}$ .

Obviously, the function  $\mathcal{M}^{(k)} : \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k}) \rightarrow [0, +\infty)$  satisfies  $0 \leq \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) < 1$  and is independent of the mean. Particularly, for bipartite case,  $\mathcal{M}^{(2)}$  is just the same as  $\mathcal{M}$  proposed in [26].

## III. BASIC PROPERTIES OF $\mathcal{M}^{(k)}$

By Definition 1, it is clear that for any  $\rho_{A_1, A_2, \dots, A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$ , the value  $\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k})$  is easily calculated, avoiding performing any measurement and any optimization procedure. Furthermore,  $\mathcal{M}^{(k)}$  has the following properties:

(1)  $\mathcal{M}^{(k)}$  vanishes on product states.

(2)  $\mathcal{M}^{(k)}$  is invariant under any permutation of subsystems, that is, for any permutation  $\pi$  of  $(1, 2, \dots, k)$ , denoting by  $\rho_{A_{\pi(1)}, A_{\pi(2)}, \dots, A_{\pi(k)}}$  the state obtained from the state  $\rho_{A_1, A_2, \dots, A_k}$  by changing the order of the subsystems according to the permutation  $\pi$ , we have

$$\mathcal{M}^{(k)}(\rho_{A_{\pi(1)}, A_{\pi(2)}, \dots, A_{\pi(k)}}) = \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}).$$

(3)  $\mathcal{M}^{(k)}$  has no ancilla problem:

$$\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k} \otimes \rho_C) = \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k})$$

when considering the  $k$ -partition  $A_1|A_2| \dots |A_{k-1}|A_k C$  of the  $(k+1)$ -partite system  $A_1 A_2 \dots A_k C$ .

(4)  $\mathcal{M}^{(k)}$  is invariant under  $k$ -partite locally Gaussian unitary operations.

(5) For any  $(n_1 + n_2 + \dots + n_k)$ -mode  $k$ -partite state  $\rho_{A_1, A_2, \dots, A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$  with CM  $\Gamma = (A_{ij})_{k \times k}$  as in Eq. (1),  $\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) = 0$  if and only if  $A_{ij} = 0$  whenever  $i \neq j$ . Particularly, if  $\rho_{A_1, A_2, \dots, A_k}$  is a Gaussian state, then  $\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) = 0$  if and only if  $\rho_{A_1, A_2, \dots, A_k}$  is a  $k$ -partite product Gaussian state, that is,  $\rho_{A_1, A_2, \dots, A_k} = \rho_{A_1} \otimes \rho_{A_2} \otimes \dots \otimes \rho_{A_k}$ .

(6) (Nonincreasing under local Gaussian channels) For any Gaussian state  $\rho_{A_1, A_2, \dots, A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$  and

any local Gaussian channel  $\Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_k$ , we have  $\mathcal{M}^{(k)}[(\Phi_1 \otimes \Phi_2 \otimes \cdots \otimes \Phi_k)\rho_{A_1, A_2, \dots, A_k}] \leq \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k})$ .

Particularly,  $\mathcal{M}^{(k)}$  is locally Gaussian unitary invariant.

Proofs of (4)–(6) will be given in Appendix A.

Thus  $\mathcal{M}^{(k)}$  ( $k \geq 2$ ) is a possible candidate of computable quantification of the multipartite multimode GQC for  $k$ -partite CV systems, which describes the natural quantum correlation for Gaussian states that a state contains no correlation if and only if it is a product state.

#### IV. UNIFICATION CONDITION AND HIERARCHY CONDITION FOR $\mathcal{M}^{(k)}$

To show that  $\mathcal{M}^{(k)}$  is a multipartite multimode GQCM, we have to check the unification condition and the hierarchy condition further for  $\mathcal{M}^{(k)}$ . The unification condition and the hierarchy condition were first proposed in [27] for the multipartite entanglement measure. Recall that a bipartite entanglement measure  $E$  is a non-negative functional on bipartite states which vanishes on separable states and is nonincreasing under local operations and classical communication (LOCC). However, a multipartite entanglement measure should satisfy some additional conditions, such as the *unification condition* and the *hierarchy condition*. For example, for a tripartite entanglement measure  $E^{(3)} : \mathcal{S}(H_A \otimes H_B \otimes H_C) \rightarrow [0, +\infty)$ , apart from the usual requirements that  $E^{(3)}$  vanishes on full separable states and cannot increase under 3-partite LOCC,  $E^{(3)}$  should satisfy further the unification condition (i.e.,  $E^{(3)}$  is invariant under the permutations of subsystems, and a bipartite entanglement measure  $E^{(2)}$  can be defined which is consistent with  $E^{(3)}$  and the hierarchy condition [i.e.,  $E^{(3)}(\rho_{ABC}) \geq E^{(2)}(\rho_{X|YZ}) \geq E^{(2)}(\rho_{XY})$ , where  $XYZ$  is any permutation of  $ABC$ ]. Generally, for a  $k$ -partite entanglement measure  $E^{(k)}$ , the unification condition ensures that one can restrict  $E^{(k)}$  to any subsystems and any subpartitions without causing any trouble; the hierarchy condition mainly requires that, as a kind of physical resource, the partial entanglement is never greater than the whole entanglement. Therefore the unification condition and the hierarchy condition are natural requirements for  $E^{(k)}$  to be a  $k$ -partite entanglement measure. But the situation is much more complicated for  $k > 3$ , particularly, no exact definition for the hierarchy condition is known.

We remark here that the hierarchy condition lives also in the bipartite entanglement measure. In fact, the inequality  $E(\rho_{A|BC}) \geq E(\rho_{AB})$  may be regarded as the hierarchy condition and should be satisfied by the bipartite entanglement measure  $E$ .

As many quantum correlations beyond entanglement are also physical resources, naturally, when quantifying these multipartite quantum correlations, the unification condition and the hierarchy condition should be basic requirements. Consider a multipartite quantum correlation (MQC) and assume that for any  $k \geq 2$ ,  $\mathcal{C}^{(k)}$  is a  $k$ -partite quantum correlation measure for MQC. The meaning of the unification condition is well understood. We say that  $\mathcal{C}^{(k)}$  satisfies the unification condition if, for any  $2 \leq l \leq k$ , one has a uniform way to introduce the  $l$ -partite quantum correlation measure  $\mathcal{C}^{(l)}$  for any  $l$  partition so that the elements in the sequence  $\{\mathcal{C}^{(l)}\}_{l=2}^k$

get along well with each other. For the hierarchy condition we should consider at least three situations. Roughly speaking,  $\mathcal{C}^{(k)}$  satisfies the hierarchy condition means that for any multipartite state, the correlation of subpartition is not greater than the whole correlation; the correlation of part is not greater than the correlation of the whole; and “kicking some parties out of” each subgroup will not increase the correlation between subgroups of subsystems. In the present paper, as  $\mathcal{M}^{(k)}$  is symmetric, we mainly consider *the symmetric multipartite quantum correlations* (SMQCs), namely, the multipartite quantum correlations which are invariant under any permutations of subsystems.

To consider the hierarchy condition for  $\mathcal{M}^{(k)}$  with  $k > 2$ , one has to check several kinds of inequalities. To make the question more clear, let us consider the case of  $k = 4$ . For a 4-partite system  $ABCD$ , it has three kinds of two subpartitions:  $WX|YZ$ ,  $W|XYZ$ ,  $WXY|Z$ ; and three kinds of 3-partitions:  $W|X|YZ$ ,  $W|XY|Z$ ,  $WX|Y|Z$ , where  $WXYZ$  is any permutation of  $ABCD$ . So the hierarchy condition requires that  $\mathcal{M}^{(4)}$  should satisfy the natural property “total correlation”  $\geq$  “partial correlation,” that is,

$$\begin{aligned} \mathcal{M}^{(4)}(\rho_{ABCD}) &\geq \mathcal{M}^{(2)}(\rho_{WX|YZ}) \\ &\geq \mathcal{M}^{(2)}(\rho_{W|XYZ}) \geq \mathcal{M}^{(2)}(\rho_{X|Y}), \end{aligned} \quad (2)$$

$$\begin{aligned} \mathcal{M}^{(4)}(\rho_{ABCD}) &\geq \mathcal{M}^{(2)}(\rho_{W|XYZ}) \\ &\geq \mathcal{M}^{(2)}(\rho_{W|XY}) \geq \mathcal{M}^{(2)}(\rho_{W|X}), \end{aligned} \quad (3)$$

$$\begin{aligned} \mathcal{M}^{(4)}(\rho_{ABCD}) &\geq \mathcal{M}^{(3)}(\rho_{W|X|YZ}) \\ &\geq \mathcal{M}^{(3)}(\rho_{WXY}) \geq \mathcal{M}^{(2)}(\rho_{W|XY}), \end{aligned} \quad (4)$$

and

$$\mathcal{M}^{(4)}(\rho_{ABC|DE}) \geq \mathcal{M}^{(4)}(\rho_{ABCD}). \quad (5)$$

Inequalities (2)–(5) motivate in general what kinds of inequalities should be checked for  $\mathcal{M}^{(4)}$  to obey the hierarchy condition. Obviously, the truth of Inequalities (2)–(4) also implies that both  $\mathcal{M}^{(3)}$  and  $\mathcal{M}^{(2)}$  meet the hierarchy condition.

Since the unification condition is usually clear from the definition of a multipartite quantum correlation measure, we give here only an exact definition of the hierarchy condition for quantifying SMQCs. Let  $k \geq 3$ . For any  $l$  subpartition  $\mathcal{P}_l(A_1 A_2 \dots A_k)$  ( $2 \leq l \leq k$ ) of a  $k$ -partite system  $A_1 A_2 \dots A_k$ , it is obvious that there exists a permutation  $\pi$  of  $(1, 2, \dots, k)$  and positive integers  $i_1, i_2, \dots, i_l$  with  $i_1 + \dots + i_l = k$  such that

$$\begin{aligned} \mathcal{P}_l(A_1 A_2 \dots A_k) &= A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} \dots A_{\pi(i_1+i_2)} | A_{\pi(i_1+i_2+1)} \dots \\ &A_{\pi(i_1+\dots+i_{l-1})} | A_{\pi(i_1+\dots+i_{l-1}+1)} \dots A_{\pi(k)}. \end{aligned} \quad (6)$$

*Definition 2.* For  $k \geq 2$ , let  $\mathcal{C}^{(k)}$  be a candidate as a  $k$ -partite quantum correlation measure of a symmetric multipartite quantum correlation which satisfies the unification condition on  $k$ -partite composite system  $A_1 A_2 \dots A_k$ . Let  $\mathcal{P}_l(A_1 A_2 \dots A_k)$  be any  $l$ -subpartition of  $A_1 A_2 \dots A_k$  determined by  $2 \leq l \leq k$  and the permutation  $\pi$  of  $(1, 2, \dots, k)$  as in Eq. (6).

(1)  $\mathcal{C}^{(k)}$  is nonincreasing under subpartition if, for any  $2 \leq l \leq k$  and the permutation  $\pi$ ,

$$\mathcal{C}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) \geq \mathcal{C}^{(l)}(\rho_{\mathcal{P}_l(A_1 A_2 \dots A_k)})$$

holds for any state  $\rho_{A_1 A_2 \dots A_k}$ .

(2)  $\mathcal{C}^{(k)}$  is nonincreasing under taking subgroup if, for any  $2 \leq l \leq k$ , the permutation  $\pi$  and each  $h \in \{0, 1, 2, \dots, l\}$  with  $i_0 = 0$ ,

$$\mathcal{C}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) \geq \mathcal{C}^{(i_h)}(\rho_{A_{\pi(i_0+i_1+\dots+i_{h-1}+1)} \dots A_{\pi(i_0+i_1+\dots+i_h)}})$$

holds for any state  $\rho_{A_1 A_2 \dots A_k}$ .

(3)  $\mathcal{C}^{(l)}$  ( $l \geq 2$ ) is nonincreasing under kickout, if, for any  $k > l$ , any permutation  $\pi$  of  $(1, 2, \dots, k)$  and for each  $h = 0, 1, 2, \dots, l$ , letting  $C_h$  be a nonempty subset of  $B_h = \{A_{\pi(\sum_{j=0}^{h-1} i_j + 1)}, A_{\pi(\sum_{j=0}^{h-1} i_j + 2)}, \dots, A_{\pi(\sum_{j=0}^h i_j)}\}$ ,

$$\mathcal{C}^{(l)}(\rho_{\mathcal{P}_l(A_1 A_2 \dots A_k)}) \geq \mathcal{C}^{(l)}(\rho_{C_1 | C_2 | \dots | C_{l-1} | C_l})$$

holds for any state  $\rho_{A_1 A_2 \dots A_k}$ .

We say that  $\mathcal{C}^{(k)}$  satisfies the hierarchy condition if  $\mathcal{C}^{(k)}$  is nonincreasing under the subpartition, nonincreasing under taking a subgroup and nonincreasing under kickout. In the case that  $\mathcal{C}^{(k)}$  is a  $k$ -partite Gaussian quantum correlation measure,  $\mathcal{C}^{(k)}$  satisfies the hierarchy condition if it is nonincreasing under subpartition, nonincreasing under taking a subgroup, and nonincreasing under kickout at least for all  $k$ -partite Gaussian states.

The hierarchy condition for bipartite quantum correlation measures beyond entanglement is less studied. Generally speaking, many known bipartite Gaussian correlations are not hierarchied. This is often the case for those quantum correlations with the ancilla problem. For example, considering the Gaussian nonlocality  $\mathcal{N}$  proposed in [18], with  $\rho_{ABC} = \rho_{AB} \otimes \rho_C$ , we have

$$\mathcal{N}(\rho_{A|BC}) = \mathcal{N}(\rho_{AB}) \text{tr} \rho_C^2 < \mathcal{N}(\rho_{AB}),$$

whenever  $\rho_C$  is not pure. This means that partial correlation may be bigger than the whole correlation, that is, the hierarchy condition is broken by  $\mathcal{N}$ . But  $\mathcal{M}(=\mathcal{M}^{(2)})$  proposed in [26] satisfies the following hierarchy condition:

For any  $(m+n+l)$ -mode tripartite state  $\rho_{ABC} \in \mathcal{FS}(H_A \otimes H_B \otimes H_C)$ , we have  $\mathcal{M}(\rho_{A|BC}) \geq \mathcal{M}(\rho_{AB})$ .

We claim that  $\mathcal{M}^{(4)}$  satisfies Inequalities (2)–(5) and thus meets the hierarchy condition. In fact, this is a special case of the following general result.

*Theorem 1.* For  $k \geq 2$ ,  $\mathcal{M}^{(k)}$  in Definition 1 satisfies the unification condition and the hierarchy condition. Thus  $\mathcal{M}^{(k)}$  is a  $k$ -partite multimode Gaussian quantum correlation measure.

It is clear from Definition 1 that  $\mathcal{M}^{(k)}$  satisfies the unification condition. To show that  $\mathcal{M}^{(k)}$  meets the hierarchy condition, one has to check that, by Definition 2,  $\mathcal{M}^{(k)}$  is nonincreasing under subpartition, nonincreasing under taking a subgroup, and nonincreasing under kickout for all  $k$ -partite  $\mathcal{FS}$  states. A proof will be given in Appendix B.

### V. MONOGAMY RELATIONS FOR $\mathcal{M}^{(k)}$

An important feature of many bipartite entanglement measures is that they are monogamous. Here we accept a slightly more general concept of monogamy relation of

entanglement without inequalities from [27,31] rather than the CKW inequality, which says that if two parties A and B are maximally entangled, then neither A nor B can be entangled with the third party C. In [27] multipartite entanglement measures and multipartite monogamy relations (mainly tripartite systems) were discussed. For a bipartite entanglement measure  $E : \mathcal{S}(H_A \otimes H_B) \rightarrow [0, +\infty)$ , the monogamy of  $E$  implies  $E(\rho_{A|BC}) \geq E(\rho_{AB})$ , and  $E(\rho_{A|BC}) = E(\rho_{AB})$  will force  $E(\rho_{AC}) = E(\rho_{BC}) = 0$  (which is equivalent to the statement that there exists some  $\alpha > 0$  such that  $E(\rho_{A|BC})^\alpha \geq E(\rho_{AB})^\alpha + E(\rho_{AC})^\alpha$  [27]). Clearly the monogamy relation of entanglement accords with the resource allocation theory: if the first part and the second part share all resources, then the third part can share any resource with neither the first part nor the second part. However, as pointed out in [27], the monogamy relation of the above form discusses the entanglement allocation among three parties by a bipartite entanglement measure  $E$  and thus is not complete, because  $E(\rho_{A|BC})$  is only a part of the whole entanglement contained in  $\rho_{ABC}$  (or shared by A,B,C). Therefore, to discuss the monogamy relation of entanglement, one needs the help of multipartite entanglement measures.

Different from the bipartite monogamy relation, there are three kinds of monogamy relations for a tripartite entanglement measure:  $E^{(3)}$  is bipartitelike monogamous if  $E^{(3)}(\rho_{A|B|(CD)}) = E^{(3)}(\rho_{ABC})$  implies  $E^{(2)}(\rho_{AD}) = E^{(2)}(\rho_{BD}) = E^{(2)}(\rho_{CD}) = 0$  (this monogamy relation is not proposed and discussed in [27]);  $E^{(3)}$  is completely monogamous if  $E^{(3)}(\rho_{ABC}) = E^{(2)}(\rho_{AB})$  implies  $E^{(2)}(\rho_{AC}) = E^{(2)}(\rho_{BC}) = 0$ ;  $E^{(3)}$  is tightly monogamous if  $E^{(3)}(\rho_{ABC}) = E^{(2)}(\rho_{A|BC})$  implies  $E^{(2)}(\rho_{BC}) = 0$ . The bipartitelike monogamy relation is more stringent or stronger than the complete monogamy relation, as we always have  $E^{(4)}(\rho_{ABCD}) \geq E^{(3)}(\rho_{A|B|(CD)})$  by the hierarchy condition. So we may call it the *strong monogamy relation*. Thus the monogamous bipartite entanglement measures are in fact strongly monogamous.

Note that many multipartite quantum correlations are physical resources. Naturally, when discussing multipartite quantum correlation measures, the three kinds of monogamy relations similar to those mentioned in the previous paragraph for  $E^{(3)}$  should be explored. Let us give a precise definition of the monogamy relations as follows.

*Definition 3.* For  $k \geq 2$ , let  $\mathcal{C}^{(k)}$  be a  $k$ -partite quantum correlation measure of a symmetric multipartite quantum correlation on a  $k$ -partite composite system  $A_1 A_2 \dots A_k$ . Let  $\mathcal{P}_l(A_1 A_2 \dots A_k)$  be a  $l$ -subpartition of  $A_1 A_2 \dots A_k$  determined by  $2 \leq l \leq k$  and the permutation  $\pi$  of  $(1, 2, \dots, k)$  as in Eq. (6).

(1) (Tight monogamy relation)  $\mathcal{C}^{(k)}$  ( $k \geq 3$ ) is tightly monogamous if, for any  $l$ , any  $\pi$ , and any  $k$ -partite state  $\rho_{A_1 A_2 \dots A_k}$ ,

$$\mathcal{C}^{(k)}(\rho_{A_1 A_2 \dots A_k}) = \mathcal{C}^{(l)}(\rho_{\mathcal{P}_l(A_1 A_2 \dots A_k)})$$

will imply that, for every  $h = 1, 2, \dots, l$ ,  $\mathcal{C}^{(i_h)}(\rho_{B_h}) = 0$  whenever  $i_h \geq 2$ , where  $B_h = A_{\pi(i_0+i_1+\dots+i_{h-1}+1)} A_{\pi(i_0+i_1+\dots+i_{h-1}+2)} \dots A_{\pi(i_0+i_1+\dots+i_{h-1}+i_h)}$  with  $i_0 = 0$ .

(2) (Complete monogamy relation)  $\mathcal{C}^{(k)}$  ( $k \geq 3$ ) is completely monogamous if, for any  $l$ , any  $\pi$  with  $i_1 \geq 2$  and any

$k$ -partite state  $\rho_{A_1 A_2 \dots A_k}$ ,  $\mathcal{C}^{(k)}(\rho_{A_1 A_2 \dots A_k}) = \mathcal{C}^{(i_1)}(\rho_{A_{\pi(1)} A_{\pi(2)} \dots A_{\pi(i_1)}})$  will imply that

$$\mathcal{C}^{(2)}(\rho_{A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} A_{\pi(i_1+2)} \dots A_{\pi(k)}}) = 0$$

and

$$\mathcal{C}^{(k-i_1)}(\rho_{A_{\pi(i_1+1)} A_{\pi(i_1+2)} \dots A_{\pi(k)}}) = 0.$$

(3) (Strong monogamy relation)  $\mathcal{C}^{(l)}$  ( $l \geq 2$ ) is strongly monogamous if, for any  $k > l$  and for each  $h = 1, 2, \dots, l$ , letting  $C_h$  be a nonempty subset of  $B_h = \{A_{\pi((\sum_{j=0}^{h-1} i_j)+1)}, A_{\pi((\sum_{j=0}^{h-1} i_j)+2)}, \dots, A_{\pi(\sum_{j=0}^h i_j)}\}$ ,

$$\mathcal{C}^{(l)}(\rho_{\mathcal{P}_l(A_1 A_2 \dots A_k)}) = \mathcal{C}^{(l)}(\rho_{C_1 | C_2 | \dots | C_{l-1} | C_l})$$

will imply that  $\mathcal{C}^{(r_h)}(\rho_{C_h}) = 0$  whenever  $r_h \geq 2$ , with  $r_h$  the number of subsystems contained in  $C_h$ , and that  $\mathcal{C}^{(2)}(\rho_{A_i A_j}) = 0$  whenever one of  $A_i$  and  $A_j$  is not in  $\cup_{h=1}^l C_h$ .

If  $\mathcal{C}^{(k)}$  is a multipartite multimode Gaussian quantum correlation measure for CV systems, we require that  $\mathcal{C}^{(k)}$  meets the definition at least on Gaussian states.

Roughly speaking, *the tight monogamy relation* means that if the correlation of subpartition attains the total correlation, then the parties in the same subgroup are not correlated to each other; *the complete monogamy relation* means that if the correlation of a subgroup of subsystems attains the total correlation, then the parties out of the subgroup are not correlated with any other parties in the system; *the strong monogamy relation* claims that if the correlation of subpartition keeps invariant after “kicking some parties out of” each subgroups, then the remain parties are not correlated with the parties kicked out.

The following bipartitelike monogamy relation is a special case of the strong monogamy relation.

(4) (*Special case of strong monogamy relation*)  $\mathcal{C}^{(k)}$  ( $k \geq 2$ ) is strongly monogamous if, for any  $(k+1)$ -partite state  $\rho_{A_1 A_2 \dots A_k A_{k+1}}$ ,  $\mathcal{C}^{(k)}(\rho_{A_1 A_2 \dots A_{k-1} (A_k A_{k+1})}) = \mathcal{C}^{(k)}(\rho_{A_1 A_2 \dots A_{k-1} A_k})$  will imply that  $\mathcal{C}^{(2)}(\rho_{A_1 A_{k+1}}) = \mathcal{C}^{(2)}(\rho_{A_2 A_{k+1}}) = \dots = \mathcal{C}^{(2)}(\rho_{A_k A_{k+1}}) = 0$ .

Many bipartite GQCMs beyond entanglement are not monogamous. For example, the Gaussian nonclassicality  $\mathcal{N}$  proposed in [18] is obviously not monogamous since it breaks the hierarchy condition, it is not monogamous by the next theorem. Hence it is reasonable that a good multipartite (Gaussian) quantum correlation measure should be at least completely monogamous and tightly monogamous.

**Theorem 2.** The bipartite Gaussian quantum correlation measure  $\mathcal{M}$  is not (strongly) monogamous.

*Proof.* Let  $\rho_{ABC}$  be an  $(m+n+l)$ -mode tripartite state with CM  $\Gamma_{ABC} = \begin{pmatrix} A & X & Z \\ X^T & B & Y \\ Z^T & Y^T & C \end{pmatrix}$ . Then, by Theorem B3 in Appendix B,  $\mathcal{M}(\rho_{A|BC}) = \mathcal{M}(\rho_{AB})$  if and only if  $XB^{-1}Y = Z$ , which may not be zero. However,  $\mathcal{M}(\rho_{AC}) = \mathcal{M}(\rho_{BC}) = 0$  if and only if  $Z = 0$  and  $Y = 0$ . Thus  $\mathcal{M}(\rho_{A|BC}) = \mathcal{M}(\rho_{AB})$  does not imply that  $\mathcal{M}(\rho_{AC}) = \mathcal{M}(\rho_{BC}) = 0$ .

To make it clearer, we give an example which reveals that there does exist a tripartite Gaussian state  $\rho_{ABC}$  so that  $\mathcal{M}(\rho_{A|BC}) = \mathcal{M}(\rho_{AB})$  but  $\mathcal{M}(\rho_{AC}) \neq 0$  and  $\mathcal{M}(\rho_{BC}) \neq 0$ . Therefore  $\mathcal{M}$  is not monogamous.

Let

$$\Gamma = \left( \begin{array}{cc|cc|cc} 2 & 0 & 1 & 0 & \frac{1}{3} & 0 \\ 0 & 2 & 0 & 1 & 0 & \frac{1}{3} \\ \hline 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 & 1 \\ \hline \frac{1}{3} & 0 & 1 & 0 & 2 & 0 \\ 0 & \frac{1}{3} & 0 & 1 & 0 & 2 \end{array} \right) \text{ and}$$

$$\Delta = \left( \begin{array}{cc|cc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{array} \right).$$

Since  $\Gamma + i\Delta \geq 0$ ,  $\Gamma$  is a CM of some  $(1+1+1)$ -mode Gaussian state  $\rho_{ABC}$ . Note that

$$Z = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = XB^{-1}Y.$$

However, it is easily calculated that  $\mathcal{M}(\rho_{A|BC}) = \mathcal{M}(\rho_{AB}) \approx 0.3056$ ,  $\mathcal{M}(\rho_{AC}) \approx 0.0548 \neq 0$ , and  $\mathcal{M}(\rho_{BC}) \approx 0.3056 \neq 0$ . So  $\mathcal{M}$  breaks the monogamy relation at  $\rho_{ABC}$ . ■

The fact that  $\mathcal{M}$  is not monogamous is not surprising because, by the hierarchy condition,  $\mathcal{M}(\rho_{A|BC}) = 0.3056$  is just a part of the total quantum correlation  $\mathcal{M}^{(3)}(\rho_{ABC}) = 0.5144$  shared by three parties A, B, C. So  $\mathcal{M}(\rho_{A|BC}) = \mathcal{M}(\rho_{AB})$  cannot always force that both  $\mathcal{M}(\rho_{AC})$  and  $\mathcal{M}(\rho_{BC})$  are zero. However, for some special tripartite Gaussian states  $\sigma_{ABC}$ ,  $\mathcal{M}(\sigma_{A|BC}) = \mathcal{M}(\sigma_{AB})$  do imply that  $\mathcal{M}(\sigma_{AC}) = \mathcal{M}(\sigma_{BC}) = 0$ .

*Example 1.* For any tripartite fully symmetric Gaussian state  $\sigma_{ABC}$ ,  $\mathcal{M}(\sigma_{A|BC}) = \mathcal{M}(\sigma_{AB})$  implies that  $\mathcal{M}(\sigma_{AC}) = \mathcal{M}(\sigma_{BC}) = 0$ . In fact, we have  $\mathcal{M}(\sigma_{A|BC}) = \mathcal{M}(\sigma_{AB})$  if and only if  $\sigma_{ABC} = \sigma_A \otimes \sigma_B \otimes \sigma_C$ .

Recall that  $\sigma_{ABC}$  is fully symmetric if it is an  $(n+n+n)$ -mode Gaussian state with CM having the form

$$\Gamma_{\sigma_{ABC}} = \begin{pmatrix} A & X & X \\ X^T & A & X \\ X^T & X^T & A \end{pmatrix},$$

where  $A, X \in M_{2n}(\mathbb{R})$ . We have to show that  $\mathcal{M}(\sigma_{A|BC}) = \mathcal{M}(\sigma_{AB})$  if and only if  $X = 0$ . This will be done in Appendix C.

To illustrate the meaning of three kinds of monogamy relations in Definition 3 for multipartite multimode GQCM  $\mathcal{C}^{(k)}$ , we consider the case  $k = 4$ . For a 4-partite composite system  $ABCD$ , let  $WXYZ$  be any permutation of  $ABCD$ . The three kinds of monogamy relations for  $\mathcal{C}^{(4)}$  can be stated as follows:

- (a) (Tight monogamy relation)  $\mathcal{C}^{(4)}$  is tightly monogamous if
  - (a<sub>1</sub>)  $\mathcal{C}^{(4)}(\rho_{ABCD}) = \mathcal{C}^{(2)}(\rho_{WX|YZ})$  implies that  $\mathcal{C}^{(2)}(\rho_{WX}) = \mathcal{C}^{(2)}(\rho_{YZ}) = 0$ ,
  - (a<sub>2</sub>)  $\mathcal{C}^{(4)}(\rho_{ABCD}) = \mathcal{C}^{(2)}(\rho_{W|XYZ})$  implies that  $\mathcal{C}^{(3)}(\rho_{XYZ}) = 0$ ,
  - and
  - (a<sub>3</sub>)  $\mathcal{C}^{(4)}(\rho_{ABCD}) = \mathcal{C}^{(3)}(\rho_{W|X|YZ})$  implies that  $\mathcal{C}^{(2)}(\rho_{YZ}) = 0$ .

(b) (Complete monogamy relation)  $\mathcal{C}^{(4)}$  is completely monogamous if

(b<sub>1</sub>)  $\mathcal{C}^{(4)}(\rho_{ABCD}) = \mathcal{C}^{(3)}(\rho_{WXY})$  implies that  $\mathcal{C}^{(2)}(\rho_{WZ}) = \mathcal{C}^{(2)}(\rho_{XZ}) = \mathcal{C}^{(2)}(\rho_{YZ}) = 0$

and

(b<sub>2</sub>)  $\mathcal{C}^{(4)}(\rho_{ABCD}) = \mathcal{C}^{(2)}(\rho_{WX})$  implies that  $\mathcal{C}^{(2)}(\rho_{WX|YZ}) = \mathcal{C}^{(2)}(\rho_{YZ}) = 0$ .

(c) (A special case of the strong monogamy relation)  $\mathcal{C}^{(4)}$  is strongly monogamous if  $\mathcal{M}^{(4)}(\rho_{ABC|DE}) = \mathcal{M}^{(4)}(\rho_{ABCD})$  implies that  $\mathcal{M}^{(2)}(\rho_{AE}) = \mathcal{M}^{(2)}(\rho_{BE}) = \mathcal{M}^{(2)}(\rho_{CE}) = \mathcal{M}^{(2)}(\rho_{DE}) = 0$ .

We claim that  $\mathcal{M}^{(4)}$  is completely monogamous and tightly monogamous but is not strongly monogamous. In fact, this is a special case of the following general result, which will be proved in Appendix C.

*Theorem 3.* Let  $\mathcal{M}^{(k)}: \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \cdots \otimes H_{A_k}) \rightarrow [0, +\infty)$  be the  $k$ -partite  $(n_1 + n_2 + \cdots + n_k)$ -mode Gaussian quantum correlation in Definition 1.

(1) For  $k \geq 3$ ,  $\mathcal{M}^{(k)}$  is tightly monogamous.

(2) For  $k \geq 3$ ,  $\mathcal{M}^{(k)}$  is completely monogamous.

(3) For  $k \geq 2$ ,  $\mathcal{M}^{(k)}$  is not strongly monogamous.

*Example 2.* Consider the  $(k+1)$ -partite  $(1 + \cdots + 1)$ -mode case. Let  $\Gamma = (A_{ij})$  be a real symmetric matrix, where  $A_{jj} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$  for  $j = 1, 2, \dots, k-1, k+1$ ,  $A_{kk} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ ,  $A_{k-1,k} = A_{k,k+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $A_{k-1,k+1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ , and otherwise,  $A_{ij} = 0$  for  $i < j$ . It is easily checked that  $\Gamma = \Gamma_\rho$  is a CM for some Gaussian state  $\rho = \rho_{A_1 A_2 \dots A_k A_{k+1}}$  since  $\Gamma + i \bigoplus_{j=1}^{k+1} \Delta_j \geq 0$  with  $\Delta_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . For this state  $\rho_{A_1 A_2 \dots A_{k-1} A_k A_{k+1}}$ , we have  $\mathcal{M}^{(k)}(\rho_{A_1 A_2 \dots A_{k-1} | A_k A_{k+1}}) = \mathcal{M}^{(k)}(\rho_{A_1 A_2 \dots A_{k-1} A_k}) = 0.3056$ , but  $\mathcal{M}^{(2)}(\rho_{A_{k-1} A_{k+1}}) \approx 0.0548 \neq 0$ . Hence  $\mathcal{M}^{(k)}$  is not strongly monogamous.

## VI. CONCLUSION

For a  $k$ -partite  $(n_1 + n_2 + \cdots + n_k)$ -mode Gaussian state  $\rho = \rho_{A_1 A_2 \dots A_k}$ , we say that  $\rho$  is not quantum correlated if it is a  $k$ -partite product state, that is,  $\rho = \rho_{A_1} \otimes \rho_{A_2} \otimes \cdots \otimes \rho_{A_k}$ . In this paper we propose a computable multipartite multimode Gaussian quantum correlation measure  $\mathcal{M}^{(k)}$  for any  $k$ -partite multimode continuous-variable (CV) system. The value of  $\mathcal{M}^{(k)}$  only depends on the covariance matrices of CV states, is invariant under any permutation of subsystems, has no ancilla problem, is nonincreasing under  $k$ -partite local Gaussian channels (particularly, invariant under  $k$ -partite local Gaussian unitary operations), and vanishes on  $k$ -partite product states. For a  $k$ -partite Gaussian state  $\rho$ ,  $\mathcal{M}^{(k)}(\rho) = 0$  if and only if  $\rho$  is a  $k$ -partite product state. Moreover, as a multipartite Gaussian quantum correlation measure,  $\mathcal{M}^{(k)}$  satisfies the unification condition and the hierarchy condition that a multipartite quantum correlation measure should obey (which means that  $\mathcal{M}^{(k)}$  is consistent with  $\mathcal{M}^{(l)}$  for any  $2 \leq l \leq k$ , the correlation of the subpartition is not greater than the whole correlation, the correlation of part is not greater than the correlation of whole, and the correlation after kicking some parties out of subgroups is not greater than the correlation between the subgroups). Finally, the monogamy relations for multipartite quantum correlation measures are discussed. Generally speaking, there are three kinds of monogamy relations for a

multipartite correlation measure: the strong monogamy relation (the correlation between subgroups after ‘‘kicking some parties out of’’ each subgroups keeps invariant will imply that the remain parties are not correlated with the parties kicked out), the complete monogamy relation (the correlation of a subgroup attains the total correlation will imply that the parties out of the subgroup are not correlated with any other parties of the system) and the tight monogamy condition (the correlation between subgroups attains the total correlation will imply that the parties in the same subgroup are not correlated to each other). Though  $\mathcal{M}^{(k)}$  is not strongly monogamous,  $\mathcal{M}^{(k)}$  is completely monogamous and tightly monogamous. Thus  $\mathcal{M}^{(k)}$  is a nice multipartite multimode Gaussian quantum correlation measure. As  $\mathcal{M}^{(k)}$  is easily calculated, it is more convenient to be applied in other scenarios of quantum information.

By now we think that  $\mathcal{M}^{(k)}$  is the only known multipartite multimode Gaussian quantum correlation measure beyond entanglement. It is interesting to find other multipartite multimode Gaussian quantum correlation measures.

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## APPENDIX A: PROOFS OF BASIC PROPERTIES OF $\mathcal{M}^{(k)}$

This Appendix section is devoted to proving the properties (4)–(6) of  $\mathcal{M}^{(k)}$  in Sec. III.

*Property 4.*  $\mathcal{M}^{(k)}$  is invariant under  $k$ -partite local Gaussian unitary operation.

*Proof.* For an  $n$ -mode CV system determined by  $R = (\hat{R}_1, \hat{R}_2, \dots, \hat{R}_{2n}) = (\hat{Q}_1, \hat{P}_1, \dots, \hat{Q}_n, \hat{P}_n)$ , it is known that a unitary operator  $U$  is Gaussian if and only if there is a vector  $\mathbf{m}$  in  $\mathbb{R}^{2n}$  and a matrix  $\mathbf{S} \in \text{Sp}(2n, \mathbb{R})$  such that  $U^\dagger R U = \mathbf{S}R + \mathbf{m}$  [36,37], where  $\text{Sp}(2n, \mathbb{R})$  is the symplectic group of all  $2n \times 2n$  real matrices  $\mathbf{S}$  that satisfy  $\mathbf{S} \in \text{Sp}(2n, \mathbb{R}) \Leftrightarrow \mathbf{S} \Delta \mathbf{S}^T = \Delta$ . Thus every Gaussian unitary operator  $U$  is determined by some affine symplectic map  $(\mathbf{S}, \mathbf{m})$  acting on the phase space and can be parameterized as  $U = U_{\mathbf{S}, \mathbf{m}}$ . It follows that if  $U_{\mathbf{S}, \mathbf{m}}$  is a Gaussian unitary operator, then for any  $n$ -mode state  $\rho$  with CM  $\Gamma_\rho$  and mean  $\mathbf{d}_\rho$ , the state  $\sigma = U_{\mathbf{S}, \mathbf{m}} \rho U_{\mathbf{S}, \mathbf{m}}^\dagger$  has the CM  $\Gamma_\sigma = \mathbf{S} \Gamma_\rho \mathbf{S}^T$  and the mean  $\mathbf{d}_\sigma = \mathbf{m} + \mathbf{S} \mathbf{d}_\rho$ . Particularly, if  $\rho$  is also Gaussian, then the characteristic function of the Gaussian state  $\sigma$  is of the form  $\exp(-\frac{1}{4} \mathbf{z}^T \Gamma_\sigma \mathbf{z} + i \mathbf{d}_\sigma^T \mathbf{z})$ .

Now, assume that  $\rho = \rho_{A_1, A_2, \dots, A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \cdots \otimes H_{A_k})$  is an  $(n_1 + n_2 + \cdots + n_k)$ -mode  $k$ -partite state, and  $U_{\mathbf{S}, \mathbf{m}} = U_1 \otimes U_2 \otimes \cdots \otimes U_k$  is a Gaussian unitary operator with  $U_j = U_{\mathbf{S}_j, \mathbf{m}_j}$  being Gaussian unitary on  $H_{A_j}$ . Clearly,  $\mathbf{S} = \bigoplus_{j=1}^k \mathbf{S}_j$  and  $\mathbf{m} = \bigoplus_{j=1}^k \mathbf{m}_j$ . Let  $\sigma = \sigma_{A_1, A_2, \dots, A_k} = U_{\mathbf{S}, \mathbf{m}} \rho_{A_1, A_2, \dots, A_k} U_{\mathbf{S}, \mathbf{m}}^\dagger$ . Then  $\Gamma_\sigma = \mathbf{S} \Gamma_\rho \mathbf{S}^T$  and  $\Gamma_{\sigma_j} = \mathbf{S}_j \Gamma_{\rho_j} \mathbf{S}_j^T$ . As  $\det(\mathbf{S}) = \prod_{j=1}^k \det(\mathbf{S}_j)$ , it follows

from Definition 1 that

$$\begin{aligned} \mathcal{M}^{(k)}(\sigma_{A_1, A_2, \dots, A_k}) &= 1 - \frac{\det(\Gamma_\sigma)}{\prod_{j=1}^k \det(\Gamma_{\sigma_{A_j}})} \\ &= 1 - \frac{\det(\mathbf{S}\Gamma_\rho \mathbf{S}^T)}{\prod_{j=1}^k \det(\mathbf{S}_j \Gamma_{\rho_{A_j}} \mathbf{S}_j^T)} \\ &= 1 - \frac{\det(\mathbf{S}) \det(\Gamma_\rho) \det(\mathbf{S}^T)}{\prod_{j=1}^k \det(\mathbf{S}_j) \det(\Gamma_{\rho_{A_j}}) \det(\mathbf{S}_j^T)} \\ &= 1 - \frac{\det(\Gamma_\rho)}{\prod_{j=1}^k \det(\Gamma_{\rho_{A_j}})} \\ &= \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}), \end{aligned}$$

as desired.  $\blacksquare$

*Property 5.* For any  $(n_1 + n_2 + \dots + n_k)$ -mode  $k$ -partite state  $\rho_{A_1, A_2, \dots, A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$  with CM  $\Gamma = (A_{ij})_{k \times k}$  as in Eq. (1),  $\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) = 0$  if and only if  $A_{ij} = 0$  whenever  $i \neq j$ . Particularly, if  $\rho_{A_1, A_2, \dots, A_k}$  is a Gaussian state, then  $\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) = 0$  if and only if  $\rho_{A_1, A_2, \dots, A_k}$  is a  $k$ -partite product Gaussian state, that is,  $\rho_{A_1, A_2, \dots, A_k} = \rho_{A_1} \otimes \rho_{A_2} \otimes \dots \otimes \rho_{A_k}$ .

*Proof.* This is an immediate consequence of Lemma A1 below.  $\blacksquare$

*Lemma A1.* Assume that

$$\Gamma_k = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1k} \\ A_{21} & A_{22} & \dots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \dots & A_{kk} \end{pmatrix}$$

is a positive-definite block matrix over the complex field  $\mathbb{C}$ . Then  $\det(\Gamma_k) = \prod_{j=1}^k \det(A_{jj})$  if and only if  $A_{ij} = 0$  whenever  $i \neq j$ .

To prove Lemma A1, we need the following lemma proved in [26], which is also useful in the other part of the present paper:

*Lemma A2.* For  $S, T \in M_n(\mathbb{C})$  with  $S \geq T > 0$ ,  $\det(S) = \det(T)$  if and only if  $S = T$ .

*Proof of Lemma A1.* The “if” part is obvious. We prove the “only if” part by induction on  $k$ . For the case  $k = 2$ , denote  $\Gamma_2 = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}$ . It is well known that  $\det(\Gamma_2) = \det(A) \det(B - C^T A^{-1} C) = \det(B) \det(A - C B^{-1} C^T) > 0$ . If  $\det(\Gamma_2) = \det(A) \det(B)$ , then  $\det(B - C^T A^{-1} C) = \det(B)$ . Let  $D = B - C^T A^{-1} C$ . As  $\Gamma_2 > 0$ , we have  $0 < D \leq B$ . Thus, by Lemma A2 we must have  $B = D$  as  $\det(D) = \det(B)$ , which entails that  $C = 0$ .

Now, assume that the assertion is true for  $k - 1 \geq 2$ . Denote by  $\Gamma_{k-1}$  the principal submatrix of  $\Gamma_k$ , that is,

$$\Gamma_{k-1} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1, k-1} \\ A_{21} & A_{22} & \dots & A_{2, k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k-1, 1} & A_{k-1, 2} & \dots & A_{k-1, k-1} \end{pmatrix},$$

which is positive definite, too. Note that the condition  $\det(\Gamma_k) = \prod_{j=1}^k \det(A_{jj})$  implies

$$\begin{aligned} \prod_{j=1}^k \det(A_{jj}) &= \det(\Gamma_k) \\ &= \det(A_{kk}) \det \left( \Gamma_{k-1} - \begin{pmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{k-1, k} \end{pmatrix} \right. \\ &\quad \left. \times A_{kk}^{-1} \begin{pmatrix} A_{1k}^\dagger & A_{2k}^\dagger & \dots & A_{k-1, k}^\dagger \end{pmatrix} \right) \\ &\leq \det(A_{kk}) \det(\Gamma_{k-1}) \\ &\leq \det(A_{kk}) \prod_{j=1}^{k-1} \det(A_{jj}) = \prod_{j=1}^k \det(A_{jj}). \end{aligned} \tag{A1}$$

It follows that  $\prod_{j=1}^{k-1} \det(A_{jj}) = \det(\Gamma_{k-1})$ . By the inductive assumption,  $A_{ij} = 0$  whenever  $i \neq j$  and  $i, j \in \{1, 2, \dots, k-1\}$ . What remains is to check that  $A_{jk} = 0$  for all  $j = 1, 2, \dots, k-1$ . By Eq. (A1) again, one gets

$$\begin{aligned} \det \left( \Gamma_{k-1} - \begin{pmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{k-1, k} \end{pmatrix} A_{kk}^{-1} \begin{pmatrix} A_{1k}^\dagger & A_{2k}^\dagger & \dots & A_{k-1, k}^\dagger \end{pmatrix} \right) \\ = \det(\Gamma_{k-1}), \end{aligned}$$

which forces

$$\begin{pmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{k-1, k} \end{pmatrix} A_{kk}^{-1} = 0,$$

and so  $A_{jk} = 0$  for all  $j = 1, 2, \dots, k-1$ . Hence the “only if” part is also true, completing the proof.  $\blacksquare$

*Property 6.* (Nonincreasing under local Gaussian channels) For any Gaussian state  $\rho_{A_1, A_2, \dots, A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$  and any local Gaussian channel  $\Phi_1 \otimes \Phi_2 \otimes \dots \otimes \Phi_k$ , we have

$$\mathcal{M}^{(k)}[(\Phi_1 \otimes \Phi_2 \otimes \dots \otimes \Phi_k) \rho_{A_1, A_2, \dots, A_k}] \leq \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}).$$

Particularly,  $\mathcal{M}^{(k)}$  is locally Gaussian unitary invariant.

To prove Property 6, we need a lemma on matrices from [26]:

*Lemma A3.* Let  $B, K, M \in M_n(\mathbb{C})$  with  $B$  and  $M$  positive semidefinite. If both  $B$  and  $KBK^\dagger + M$  are invertible, then  $K^\dagger(KBK^\dagger + M)^{-1}K \leq B^{-1}$ . The equality holds if and only if  $M = 0$  and  $K$  is invertible.

*Proof of Property 6.* As Gaussian state  $\rho$  is characterized by its CM  $\Gamma$  and mean  $\mathbf{d}$ , and we can parameterize it as  $\rho = \rho(\Gamma, \mathbf{d})$ . Recall that if  $\Phi$  is a Gaussian channel of  $n$ -mode Gaussian systems, then for any  $n$ -mode Gaussian state  $\rho = \rho(\Gamma, \mathbf{d})$ ,  $\Phi(\rho(\Gamma, \mathbf{d})) = \rho(\Gamma', \mathbf{d}')$  with

$$\mathbf{d}' = K\mathbf{d} + \bar{\mathbf{d}} \quad \text{and} \quad \Gamma' = K\Gamma K^T + M \tag{A2}$$

for some real matrices  $M, K \in M_{2n}(\mathbb{R})$  satisfying  $M = M^T \geq 0$  and  $\det M \geq (\det(K) - 1)^2$  and some vector  $\bar{\mathbf{d}} \in \mathbb{R}^{2n}$ . So we can parameterize the Gaussian channel  $\Phi$  as  $\Phi = \Phi(K, M, \bar{\mathbf{d}})$ .



Let  $\rho = \rho_{A_1, A_2, \dots, A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$  be a Gaussian state whose CM is presented as in Eq. (1) and  $\Phi_1 \otimes \Phi_2 \otimes \dots \otimes \Phi_k$  be a local Gaussian channel with  $\Phi_j = \Phi_j(K_j, M_j, \bar{\mathbf{d}}_j)$ . We first show that for any  $j \in \{1, 2, \dots, k\}$ ,

$$\mathcal{M}^{(k)}((I_1 \otimes \dots \otimes I_{j-1} \otimes \Phi_j \otimes I_{j+1} \otimes \dots \otimes I_k)\rho_{A_1, A_2, \dots, A_k}) \leq \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}), \tag{A3}$$

where  $\Phi_j$  is a Gaussian channel performed on subsystem  $A_j$ . Since  $\mathcal{M}^{(k)}$  is invariant under permutation of subsystems, we may assume that  $j = k$ . Denote

$$\rho' = \rho'_{A_1, A_2, \dots, A_k} = (I_1 \otimes I_2 \otimes \dots \otimes I_{k-1} \otimes \Phi_k)\rho_{A_1, A_2, \dots, A_k}.$$

Then the CM  $\Gamma_{\rho'}$  of  $\rho'_{A_1, A_2, \dots, A_k}$  has the form

$$\Gamma_{\rho'} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1, k-1} & A_{1k}K_k^T \\ A_{21} & A_{22} & \dots & A_{2, k-1} & A_{2k}K_k^T \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{k-1, 1} & A_{k-1, 2} & \dots & A_{k-1, k-1} & A_{k-1, k} \\ K_k A_{k1} & K_k A_{k2} & \dots & K_k A_{k, k-1} & K_k A_{kk}K_k^T + M_k \end{pmatrix},$$

and thus, by Lemma A3, one gets

$$\begin{aligned} \mathcal{M}^{(k)}(\rho'_{A_1, A_2, \dots, A_k}) &= 1 - \frac{\det(\Gamma_{\rho'})}{\det(K_k A_{kk}K_k^T + M_k)\prod_{j=1}^{k-1} \det(A_{jj})} \\ &= 1 - \frac{\det\left(\Gamma_{k-1} - \begin{pmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{k-1, k} \end{pmatrix} K_k^T (K_k A_{kk}K_k^T + M_k)^{-1} K_k \begin{pmatrix} A_{1k}^T & A_{2k}^T & \dots & A_{k-1, k}^T \end{pmatrix}\right)}{\prod_{j=1}^{k-1} \det(A_{jj})} \\ &\leq 1 - \frac{\det\left(\Gamma_{k-1} - \begin{pmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{k-1, k} \end{pmatrix} A_{kk}^{-1} \begin{pmatrix} A_{1k}^T & A_{2k}^T & \dots & A_{k-1, k}^T \end{pmatrix}\right)}{\prod_{j=1}^{k-1} \det(A_{jj})} = \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}). \end{aligned}$$

Therefore we have proved that the inequality in Eq. (A3) is true.

Then, applying the inequality in Eq. (A3), we have

$$\begin{aligned} \mathcal{M}^{(k)}((\Phi_1 \otimes \Phi_2 \otimes \dots \otimes \Phi_k)\rho_{A_1, A_2, \dots, A_k}) &= \mathcal{M}^{(k)}[(\prod_{j=1}^k (I_1 \otimes \dots \otimes I_{j-1} \otimes \Phi_j \otimes I_{j+1} \otimes \dots \otimes I_k))\rho_{A_1, A_2, \dots, A_k}] \\ &\leq \mathcal{M}^{(k)}[(\prod_{j=2}^k (I_1 \otimes \dots \otimes I_{j-1} \otimes \Phi_j \otimes I_{j+1} \otimes \dots \otimes I_k))\rho_{A_1, A_2, \dots, A_k}] \\ &\leq \dots \dots \\ &\leq \mathcal{M}^{(k)}((I_1 \otimes \dots \otimes I_{k-1} \otimes \Phi_k)\rho_{A_1, A_2, \dots, A_k}) \\ &\leq \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}). \end{aligned}$$

Hence  $\mathcal{M}^{(k)}$  is nonincreasing under  $k$ -partite local Gaussian channels.

Particularly, if the  $k$ -partite local Gaussian channel  $\Phi = \Phi_1 \otimes \Phi_2 \otimes \dots \otimes \Phi_k$  is invertible and  $\Phi^{-1}$  is still a Gaussian channel (it is the case when  $\Phi$  is a  $k$ -partite locally Gaussian unitary operation), then

$$\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) = \mathcal{M}^{(k)}(\Phi^{-1}\Phi(\rho_{A_1, A_2, \dots, A_k})) \leq \mathcal{M}^{(k)}(\Phi(\rho_{A_1, A_2, \dots, A_k})) \leq \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}),$$

and consequently,  $\mathcal{M}^{(k)}(\Phi(\rho_{A_1, A_2, \dots, A_k})) = \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k})$ . This reveals that  $\mathcal{M}^{(k)}$  is invariant under  $k$ -partite Gaussian unitary operations. ■

### APPENDIX B: PROOF OF THEOREM 1, THE HIERARCHY CONDITION FOR $\mathcal{M}^{(k)}$

In this Appendix section we show that  $\mathcal{M}^{(k)}$  satisfies the hierarchy condition and thus completes the proof of Theorem 1 in Sec. IV. We begin with considering the bipartite case. The following lemmas are needed.

*Lemma B1.* Let  $\begin{pmatrix} I & D & F \\ D^\dagger & I & E \\ F^\dagger & E^\dagger & I \end{pmatrix}$  be a positive-definite block matrix over the complex field  $\mathbb{C}$ . Then  $\max\{\|D\|, \|E\|, \|F\|\} < 1$ ,

$$\begin{aligned} (D \ F) \begin{pmatrix} I & E \\ E^\dagger & I \end{pmatrix}^{-1} \begin{pmatrix} D^\dagger \\ F^\dagger \end{pmatrix} &\geq DD^\dagger, & (D \ F) \begin{pmatrix} I & E \\ E^\dagger & I \end{pmatrix}^{-1} \begin{pmatrix} D^\dagger \\ F^\dagger \end{pmatrix} &\geq FF^\dagger, \\ (F^\dagger \ E^\dagger) \begin{pmatrix} I & D \\ D^\dagger & I \end{pmatrix}^{-1} \begin{pmatrix} F \\ E \end{pmatrix} &\geq F^\dagger F \quad \text{and} \quad (F^\dagger \ E^\dagger) \begin{pmatrix} I & D \\ D^\dagger & I \end{pmatrix}^{-1} \begin{pmatrix} F \\ E \end{pmatrix} &\geq E^\dagger E. \end{aligned}$$

Furthermore, the equality holds for any one of the above four inequalities if and only if the equality holds for all of the above four inequalities, and in turn, if and only if  $F = DE$ .

*Proof.* By the assumption,  $\begin{pmatrix} I & D & F \\ D^\dagger & I & E \\ F^\dagger & E^\dagger & I \end{pmatrix} \geq 0$  and is invertible. So we must have  $\max\{\|D\|, \|E\|, \|F\|\} < 1$ . We only give a proof of the first inequality in detail, and the others are checked similarly by noting that

$$E(I - E^\dagger E)^{-1} = (I - EE^\dagger)^{-1}E \quad \text{and} \quad (I - E^\dagger E)^{-1}E^\dagger = E^\dagger(I - EE^\dagger)^{-1}.$$

It is easily checked that

$$\begin{pmatrix} I & E \\ E^\dagger & I \end{pmatrix}^{-1} = \begin{pmatrix} (I - EE^\dagger)^{-1} & -E(I - E^\dagger E)^{-1} \\ -(I - E^\dagger E)^{-1}E^\dagger & (I - E^\dagger E)^{-1} \end{pmatrix}.$$

Then,

$$\begin{aligned} (D \ F) \begin{pmatrix} I & E \\ E^\dagger & I \end{pmatrix}^{-1} \begin{pmatrix} D^\dagger \\ F^\dagger \end{pmatrix} &= D(I - EE^\dagger)^{-1}D^\dagger - F(I - E^\dagger E)^{-1}E^\dagger D^\dagger - DE(I - E^\dagger E)^{-1}F^\dagger + F(I - E^\dagger E)^{-1}F^\dagger \\ &= DD^\dagger + DE(I - E^\dagger E)^{-1}E^\dagger D^\dagger - F(I - E^\dagger E)^{-1}E^\dagger D^\dagger - DE(I - E^\dagger E)^{-1}F^\dagger + F(I - E^\dagger E)^{-1}F^\dagger \\ &= DD^\dagger + (D \ F) \begin{pmatrix} E(I - E^\dagger E)^{-1}E^\dagger & -E(I - E^\dagger E)^{-1} \\ -(I - E^\dagger E)^{-1}E^\dagger & (I - E^\dagger E)^{-1} \end{pmatrix} \begin{pmatrix} D^\dagger \\ F^\dagger \end{pmatrix}. \end{aligned}$$

Obviously,

$$\begin{pmatrix} E(I - E^\dagger E)^{-1}E^\dagger & -E(I - E^\dagger E)^{-1} \\ -(I - E^\dagger E)^{-1}E^\dagger & (I - E^\dagger E)^{-1} \end{pmatrix} \geq 0$$

since  $[-E(I - E^\dagger E)^{-1}][(I - E^\dagger E)^{-1}]^{-1}[-(I - E^\dagger E)^{-1}E^\dagger] = E(I - E^\dagger E)^{-1}E^\dagger$ . Hence we have

$$(D \ F) \begin{pmatrix} I & E \\ E^\dagger & I \end{pmatrix}^{-1} \begin{pmatrix} D^\dagger \\ F^\dagger \end{pmatrix} = DD^\dagger + (D \ F) \begin{pmatrix} E(I - E^\dagger E)^{-1}E^\dagger & -E(I - E^\dagger E)^{-1} \\ -(I - E^\dagger E)^{-1}E^\dagger & (I - E^\dagger E)^{-1} \end{pmatrix} \begin{pmatrix} D^\dagger \\ F^\dagger \end{pmatrix} \geq DD^\dagger, \tag{B1}$$

as desired.

It is clear from Inequality (B1) that the equality holds if and only if

$$(D \ F) \begin{pmatrix} E(I - E^\dagger E)^{-1}E^\dagger & -E(I - E^\dagger E)^{-1} \\ -(I - E^\dagger E)^{-1}E^\dagger & (I - E^\dagger E)^{-1} \end{pmatrix} \begin{pmatrix} D^\dagger \\ F^\dagger \end{pmatrix} = 0.$$

As for operators  $A, C$  with  $A \geq 0, CAC^\dagger = 0 \Leftrightarrow CA = 0$ , we see that the above equation holds if and only if

$$DE(I - E^\dagger E)^{-1}E^\dagger - F(I - E^\dagger E)^{-1}E^\dagger = 0, \quad -DE(I - E^\dagger E)^{-1} + F(I - E^\dagger E)^{-1} = 0;$$

and in turn, if and only if  $F = DE$ .

It is similar to show that the equality for any one of the other three inequalities holds if and only if the same condition  $F = DE$  is satisfied, completing the proof. ■

The following lemma is a generalization of Lemma B1, which is also useful.

*Lemma B2.* Let  $\begin{pmatrix} A & X & Z \\ X^\dagger & B & Y \\ Z^\dagger & Y^\dagger & C \end{pmatrix}$  be a positive-definite block matrix over the complex field  $\mathbb{C}$ . Then

$$\begin{aligned} (X \ Z) \begin{pmatrix} B & Y \\ Y^\dagger & C \end{pmatrix}^{-1} \begin{pmatrix} X^\dagger \\ Z^\dagger \end{pmatrix} &\geq XB^{-1}X^\dagger, & (X \ Z) \begin{pmatrix} B & Y \\ Y^\dagger & C \end{pmatrix}^{-1} \begin{pmatrix} X^\dagger \\ Z^\dagger \end{pmatrix} &\geq ZC^{-1}Z^\dagger, \\ (Z^\dagger \ Y^\dagger) \begin{pmatrix} A & X \\ X^\dagger & B \end{pmatrix}^{-1} \begin{pmatrix} Z \\ Y \end{pmatrix} &\geq Z^\dagger A^{-1}Z \quad \text{and} \quad (Z^\dagger \ Y^\dagger) \begin{pmatrix} A & X \\ X^\dagger & B \end{pmatrix}^{-1} \begin{pmatrix} Z \\ Y \end{pmatrix} &\geq Y^\dagger B^{-1}Y. \end{aligned}$$

Furthermore, the equality holds for any one of the above four inequalities if and only if the equality holds for all of the above four inequalities, and in turn, if and only if  $Z = XB^{-1}Y$ .

*Proof.* Clearly,

$$\Gamma = \begin{pmatrix} A & X & Z \\ X^\dagger & B & Y \\ Z^\dagger & Y^\dagger & C \end{pmatrix} = \begin{pmatrix} A^{\frac{1}{2}} & 0 & 0 \\ 0 & B^{\frac{1}{2}} & 0 \\ 0 & 0 & C^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I & D & F \\ D^\dagger & I & E \\ F^\dagger & E^\dagger & I \end{pmatrix} \begin{pmatrix} A^{\frac{1}{2}} & 0 & 0 \\ 0 & B^{\frac{1}{2}} & 0 \\ 0 & 0 & C^{\frac{1}{2}} \end{pmatrix},$$

where  $D = A^{-\frac{1}{2}}XB^{-\frac{1}{2}}$ ,  $E = B^{-\frac{1}{2}}YC^{-\frac{1}{2}}$ , and  $F = A^{-\frac{1}{2}}ZC^{-\frac{1}{2}}$ . As  $\Gamma$  is positive and invertible, we have  $\{\|D\|, \|E\|, \|F\|\} \subset [0, 1)$ . Then Lemma B1 is applicable. Let us give a proof of the second inequality in detail. By Lemma 4 we have

$$(D \ F) \begin{pmatrix} I & E \\ E^\dagger & I \end{pmatrix}^{-1} \begin{pmatrix} D^\dagger \\ F^\dagger \end{pmatrix} \geq FF^\dagger.$$

Substituting  $D = A^{-\frac{1}{2}}XB^{-\frac{1}{2}}$ ,  $E = B^{-\frac{1}{2}}YC^{-\frac{1}{2}}$ , and  $F = A^{-\frac{1}{2}}ZC^{-\frac{1}{2}}$  into the above inequality leads to

$$A^{-\frac{1}{2}}(X \ Z) \begin{pmatrix} B & Y \\ Y^\dagger & C \end{pmatrix}^{-1} \begin{pmatrix} X^\dagger \\ Z^\dagger \end{pmatrix} A^{-\frac{1}{2}} = (D \ F) \begin{pmatrix} I & E \\ E^\dagger & I \end{pmatrix}^{-1} \begin{pmatrix} D^\dagger \\ F^\dagger \end{pmatrix} \geq FF^\dagger = A^{-\frac{1}{2}}ZC^{-1}Z^\dagger A^{-\frac{1}{2}},$$

which entails that

$$(X \ Z) \begin{pmatrix} B & Y \\ Y^\dagger & C \end{pmatrix}^{-1} \begin{pmatrix} X^\dagger \\ Z^\dagger \end{pmatrix} \geq ZC^{-1}Z^\dagger.$$

By Lemma B1, the equality holds if and only if  $F = DE$ , which holds if and only if  $Z = XB^{-1}Y$ , completing the proof. ■

The next result reveals that the bipartite Gaussian quantum correlation measure  $\mathcal{M} = \mathcal{M}^{(2)}$  satisfies the hierarchy condition.

*Theorem B3.* For any  $(m + n + l)$ -mode tripartite state  $\rho_{ABC} \in \mathcal{FS}(H_A \otimes H_B \otimes H_C)$ , we have  $\mathcal{M}(\rho_{A|BC}) \geq \mathcal{M}(\rho_{AB})$ . Furthermore,  $\mathcal{M}(\rho_{A|BC}) = \mathcal{M}(\rho_{AB})$  if and only if  $Z = XB^{-1}Y$ , where  $\Gamma_{ABC} = \begin{pmatrix} A & X & Z \\ X^\dagger & B & Y \\ Z^\dagger & Y^\dagger & C \end{pmatrix}$  is the covariance matrix of  $\rho_{ABC}$ .

*Proof.* Let  $\rho_{ABC}$  be an  $(m + n + l)$ -mode tripartite state with CM  $\Gamma_{ABC} = \begin{pmatrix} A & X & Z \\ X^\dagger & B & Y \\ Z^\dagger & Y^\dagger & C \end{pmatrix}$ . Then the CM of  $\rho_A$  is  $A$  and the CM of  $\rho_{BC}$  is  $\Gamma_{BC} = \begin{pmatrix} B & Y \\ Y^\dagger & C \end{pmatrix}$ . Clearly,

$$\Gamma_{ABC} = \begin{pmatrix} A^{\frac{1}{2}} & 0 & 0 \\ 0 & B^{\frac{1}{2}} & 0 \\ 0 & 0 & C^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I & D & F \\ D^T & I & E \\ F^T & E^T & I \end{pmatrix} \begin{pmatrix} A^{\frac{1}{2}} & 0 & 0 \\ 0 & B^{\frac{1}{2}} & 0 \\ 0 & 0 & C^{\frac{1}{2}} \end{pmatrix},$$

where  $D = A^{-\frac{1}{2}}XB^{-\frac{1}{2}}$ ,  $E = B^{-\frac{1}{2}}YC^{-\frac{1}{2}}$ , and  $F = A^{-\frac{1}{2}}ZC^{-\frac{1}{2}}$ . As  $\Gamma_{ABC}$  is invertible, we have  $\{\|D\|, \|E\|, \|F\|\} \subset [0, 1)$ . It follows that

$$\mathcal{M}(\rho_{A|BC}) = 1 - \frac{\det(\Gamma_{ABC})}{\det(A)\det(\Gamma_{BC})} = 1 - \frac{\det \left( \begin{pmatrix} I & D & F \\ D^T & I & E \\ F^T & E^T & I \end{pmatrix} \right)}{\det \left( \begin{pmatrix} I & E \\ E^T & I \end{pmatrix} \right)} = 1 - \det \left[ I - (DF) \begin{pmatrix} I & E \\ E^T & I \end{pmatrix}^{-1} \begin{pmatrix} D^T \\ F^T \end{pmatrix} \right]$$

and

$$\mathcal{M}(\rho_{AB}) = 1 - \frac{\det(\Gamma_{AB})}{\det(A)\det(B)} = 1 - \det(I - D^T D) = 1 - \det(I - DD^T).$$

Thus, by Lemma B1, we have

$$(D \ F) \begin{pmatrix} I & E \\ E^T & I \end{pmatrix}^{-1} \begin{pmatrix} D^T \\ F^T \end{pmatrix} \geq DD^T.$$

Hence

$$\det \left( I - (DF) \begin{pmatrix} I & E \\ E^T & I \end{pmatrix}^{-1} \begin{pmatrix} D^T \\ F^T \end{pmatrix} \right) \leq \det(I - DD^T),$$

and consequently,  $\mathcal{M}(\rho_{A|BC}) \geq \mathcal{M}(\rho_{AB})$ .

Now, by Lemma A2 it is easily seen that  $\mathcal{M}(\rho_{A|BC}) = \mathcal{M}(\rho_{AB})$  if and only if

$$(D \ F) \begin{pmatrix} I & E \\ E^T & I \end{pmatrix}^{-1} \begin{pmatrix} D^T \\ F^T \end{pmatrix} = DD^T.$$

Therefore, by Lemma B1 we conclude that  $\mathcal{M}(\rho_{A|BC}) = \mathcal{M}(\rho_{AB})$  if and only if  $F = DE$ , which is equivalent to say that  $Z = XB^{-1}Y$ . This completes the proof. ■

Now let us consider the general case.

*Theorem B4.* Let  $\rho_{A_1, A_2, \dots, A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$  with the CM as in Eq. (1). For any  $l$ -subpartition ( $2 \leq l < k$ ) of a  $k$ -partite system ( $k \geq 3$ )  $A_1 A_2 \dots A_k$  as in Eq. (6), the following statements are true.

(i) (Nonincreasing under subpartition)

$$\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) \geq \mathcal{M}^{(l)}(\rho_{A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} \dots A_{\pi(i_1+i_2)} | \dots | A_{\pi(i_1+\dots+i_{l-1}+1)} \dots A_{\pi(i_1+\dots+i_l)}}).$$

(ii) (Nonincreasing under taking subgroup) With  $i_0 = 0$ , for each  $h \in \{1, 2, \dots, l\}$ , we have

$$\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) \geq \mathcal{M}^{(i_h)}(\rho_{A_{\pi(i_0+i_1+\dots+i_{h-1}+1)} \dots A_{\pi(i_0+i_1+\dots+i_h)}}).$$

*Proof.* (i) Let  $\rho_{A_1, A_2, \dots, A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$  with the CM as in Eq. (1) and  $\mathcal{P}_l(A_1 A_2 \dots A_k)$  be an  $l$  partition of  $A_1 A_2 \dots A_k$  as in Eq. (6). Denote by  $B_h = A_{\pi(i_0+i_1+\dots+i_{h-1}+1)} \dots A_{\pi(i_0+i_1+\dots+i_h)}$  and  $\Gamma_{B_h}$  the CM of  $\rho_{B_h}$ ,  $h = 1, 2, \dots, k$ . Then, by Definition 1 and Lemma A1, we have, with  $i_0 = 0$ ,

$$\begin{aligned} \mathcal{M}^{(l)}(\rho_{A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} \dots A_{\pi(i_1+i_2)} | \dots | A_{\pi(i_1+\dots+i_{l-1}+1)} \dots A_{\pi(k)}}) &= 1 - \frac{\det(\Gamma_{\rho_{A_1 A_2 \dots A_k}})}{\prod_{h=1}^l \det(\Gamma_{B_h})} \leq 1 - \frac{\det(\Gamma_{\rho_{A_1 A_2 \dots A_k}})}{\prod_{h=1}^l [\prod_{j=i_0+i_1+\dots+i_{h-1}+1}^{i_1+\dots+i_{h-1}+i_h} \det(A_{\pi(j), \pi(j)})]} \\ &= \mathcal{M}^{(k)}(\rho_{A_1 A_2 \dots A_k}). \end{aligned} \tag{B2}$$

This completes the proof of (i).

(ii) For any  $h \in \{1, 2, \dots, l\}$ , we also denote by  $B_h = \{\pi(i_0 + i_1 + \dots + i_{h-1} + 1), \dots, \pi(i_0 + i_1 + \dots + i_{h-1} + i_h)\}$  and  $\Gamma_{B_h^c} = (A_{ij})$  with  $i, j \in B_h^c = \{1, 2, \dots, k\} \setminus B_h$ , which is the CM of  $\rho_{B_h^c}$ . Then

$$\Gamma_{\rho} \cong \Gamma_{\rho_{A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} \dots A_{\pi(i_1+i_2)} | \dots | A_{\pi(i_1+\dots+i_{l-1}+1)} \dots A_{\pi(i_1+\dots+i_l)}}} \cong \Gamma_{\rho_{B_h, B_h^c}} = \begin{pmatrix} \Gamma_{B_h} & C_h \\ C_h^T & \Gamma_{B_h^c} \end{pmatrix}.$$

Thus for any  $h = 1, 2, \dots, l$ , we have

$$\begin{aligned} \mathcal{M}^{(i_h)}(\rho_{A_{\pi(i_0+i_1+\dots+i_{h-1}+1)} \dots A_{\pi(i_0+i_1+\dots+i_h)}}) &= 1 - \frac{\det(\Gamma_{B_h})}{\prod_{j=i_0+i_1+\dots+i_{h-1}+1}^{i_0+i_1+\dots+i_{h-1}+i_h} \det(A_{\pi(j), \pi(j)})} \\ &= 1 - \frac{\det(\Gamma_{B_h}) \det(\Gamma_{B_h^c} - C_h^T \Gamma_{B_h}^{-1} C_h)}{\det(\Gamma_{B_h^c} - C_h^T \Gamma_{B_h}^{-1} C_h) \prod_{j=i_0+i_1+\dots+i_{h-1}+1}^{i_0+i_1+\dots+i_{h-1}+i_h} \det(A_{\pi(j), \pi(j)})} \\ &\leq 1 - \frac{\det(\Gamma_{\rho_{A_1 A_2 \dots A_k}})}{\det(\Gamma_{B_h^c}) \prod_{j=i_0+i_1+\dots+i_{h-1}+1}^{i_0+i_1+\dots+i_{h-1}+i_h} \det(A_{\pi(j), \pi(j)})} \leq 1 - \frac{\det(\Gamma_{\rho_{A_1 A_2 \dots A_k}})}{\prod_{j=1}^k \det(A_{jj})} \\ &= \mathcal{M}^{(k)}(\rho_{A_1 A_2 \dots A_k}). \end{aligned} \tag{B3}$$

This completes the proof of statement (ii). ■

To prove that  $\mathcal{M}^{(k)}$  satisfies the hierarchy condition, we still need a multipartite version of Theorem B3.

*Theorem B5.* (Nonincreasing under kickout) For any  $l$  subpartition ( $2 \leq l < k$ ) of a  $k$ -partite system ( $k \geq 3$ )  $A_1 A_2 \dots A_k$  as in Eq. (6) with  $i_l \geq 2$  and any  $\rho_{A_1 A_2 \dots A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$ , we have

$$\mathcal{M}^{(l)}(\rho_{A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} \dots A_{\pi(i_1+i_2)} | \dots | A_{\pi(i_1+\dots+i_{l-1}+1)} \dots A_{\pi(k-1)} A_{\pi(k)}}) \geq \mathcal{M}^{(l)}(\rho_{A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} \dots A_{\pi(i_1+i_2)} | \dots | A_{\pi(i_1+\dots+i_{l-1}+1)} \dots A_{\pi(k-1)}}).$$

*Corollary B6.* For  $k \geq 2$  and  $\rho = \rho_{A_1 A_2 \dots A_k A_{k+1}} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k} \otimes H_{A_{k+1}})$  with CM  $\Gamma_{\rho} = (A_{ij})_{(k+1) \times (k+1)}$ , we always have  $\mathcal{M}^{(k)}(\rho_{A_1 A_2 \dots A_{k-1} | A_k A_{k+1}}) \geq \mathcal{M}^{(k)}(\rho_{A_1 A_2 \dots A_k})$ .

Note that  $\mathcal{M}^{(k)}$  is symmetric about subsystems, so the following corollary is true, which is a generalization of Theorem B5 and reveals exactly the meaning of “nonincreasing under kickout.”

*Corollary B7.* (Nonincreasing under kickout) Assume  $k \geq 3$  and consider any  $l$  partition ( $2 \leq l < k$ ) of a  $k$ -partite system  $A_1 A_2 \dots A_k$  as in Eq. (5). For each  $h = 1, 2, \dots, l$ , let  $C_h$  be a nonempty subset of  $B_h = \{A_{\pi([\sum_{j=0}^{h-1} i_j]+1)}, A_{\pi([\sum_{j=0}^{h-1} i_j]+2)}, \dots, A_{\pi([\sum_{j=0}^h i_j])}\}$ , where  $i_0 = 0$ . Then, for any  $\rho_{A_1 A_2 \dots A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$ , we have

$$\mathcal{M}^{(l)}(\rho_{A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} \dots A_{\pi(i_1+i_2)} | \dots | A_{\pi(i_1+\dots+i_{l-1}+1)} \dots A_{\pi(k)}}) \geq \mathcal{M}^{(l)}(\rho_{C_1 | C_2 | \dots | C_{l-1} | C_l}).$$

*Proof of Theorem B5.* By the invariance of  $\mathcal{M}^{(k)}$  under permutations, with no loss of generality, we may assume that the  $l$  subpartition of  $A_1 A_2 \dots A_{k-1} A_k$  is

$$A_1 \dots A_{i_1} | A_{i_1+1} \dots A_{i_1+i_2} | A_{i_1+i_2+1} \dots \dots A_{i_1+\dots+i_{l-1}} | A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1} A_{kk}. \tag{B4}$$

As  $i_l \geq 2$ , we see that  $i_1 + i_2 + \dots + i_{l-1} < k - 1$ .

For any  $k$ -partite state  $\rho = \rho_{A_1 A_2 \dots A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$  with CM as represented as in Eq. (1), write  $B_h$  the  $h$ th party of the  $l$  subpartition in Eq. (B4), that is,  $B_h = A_{i_0+i_1+\dots+i_{h-1}+1} \dots A_{i_0+i_1+\dots+i_h}$ , and  $\rho_{B_h}$  the corresponding reduced state of

$\rho_{A_1 A_2 \dots A_k}$ ,  $h = 1, 2, \dots, l$ . Let  $\rho_{k^c}$  be the reduced state  $\rho_{k^c} = \rho_{A_1 A_2 \dots A_{k-1, k-1}}$  of  $\rho_{A_1 A_2 \dots A_k}$ . Then

$$\mathcal{M}^{(l)}(\rho_{A_1 \dots A_{i_1} | A_{i_1+1} \dots A_{i_1+i_2} | \dots | A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1} A_{kk})} = 1 - \frac{\det(\Gamma_\rho)}{\prod_{h=1}^l \det(\Gamma_{\rho_{B_h}})}$$

and

$$\mathcal{M}^{(l)}(\rho_{A_1 \dots A_{i_1} | A_{i_1+1} \dots A_{i_1+i_2} | \dots | A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1}}) = 1 - \frac{\det(\Gamma_{\rho_{k^c}})}{\det(\Gamma_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1}}}) \prod_{h=1}^{l-1} \det(\Gamma_{\rho_{B_h}})}.$$

So the theorem is true if and only if

$$\frac{\det(\Gamma_\rho)}{\det(\Gamma_{\rho_{B_l}})} \leq \frac{\det(\Gamma_{\rho_{k^c}})}{\det(\Gamma_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1}}})}. \quad (\text{B5})$$

Decompose  $\Gamma_\rho$  into

$$\Gamma_\rho = (A_{ij}) = \begin{pmatrix} A_{11}^{\frac{1}{2}} & 0 & \dots & 0 \\ 0 & A_{22}^{\frac{1}{2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{kk}^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} I & E_{12} & \dots & E_{1k} \\ E_{12}^T & I & \dots & E_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{1k}^T & E_{2k}^T & \dots & I \end{pmatrix} \begin{pmatrix} A_{11}^{\frac{1}{2}} & 0 & \dots & 0 \\ 0 & A_{22}^{\frac{1}{2}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{kk}^{\frac{1}{2}} \end{pmatrix},$$

where  $E_{ij} = A_{ii}^{-\frac{1}{2}} A_{ij} A_{jj}^{-\frac{1}{2}}$  ( $i \neq j$ ), and denote

$$\Lambda_\rho = \begin{pmatrix} I & E_{12} & \dots & E_{1k} \\ E_{12}^T & I & \dots & E_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{1k}^T & E_{2k}^T & \dots & I \end{pmatrix}.$$

Then the inequality in Eq. (B5) holds if and only if

$$\frac{\det(\Lambda_\rho)}{\det(\Lambda_{\rho_{B_l}})} \leq \frac{\det(\Lambda_{\rho_{k^c}})}{\det(\Lambda_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1}}})}. \quad (\text{B6})$$

Rewrite

$$\Lambda_\rho = \begin{pmatrix} I & E_{12} & \dots & E_{1k} \\ E_{12}^T & I & \dots & E_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{1k}^T & E_{2k}^T & \dots & I \end{pmatrix} = \begin{pmatrix} \Lambda_{\rho_{A_1 \dots A_{i_1+\dots+i_{l-1}}}} & X & Z \\ X^T & \Lambda_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1}}} & Y \\ Z^T & Y^T & I \end{pmatrix},$$

where

$$X = \begin{pmatrix} E_{1, i_1+\dots+i_{l-1}+1} & E_{1, i_1+\dots+i_{l-1}+2} & \dots & E_{1, k-1} \\ E_{2, i_1+\dots+i_{l-1}+1} & E_{2, i_1+\dots+i_{l-1}+2} & \dots & E_{2, k-1} \\ \vdots & \vdots & \ddots & \vdots \\ E_{i_1+\dots+i_{l-1}, i_1+\dots+i_{l-1}+1} & E_{i_1+\dots+i_{l-1}, i_1+\dots+i_{l-1}+2} & \dots & E_{i_1+\dots+i_{l-1}, k-1} \end{pmatrix},$$

$$Z = \begin{pmatrix} E_{1k} \\ E_{2k} \\ \vdots \\ E_{i_1+\dots+i_{l-1}, k} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} E_{i_1+\dots+i_{l-1}+1, k} \\ E_{i_1+\dots+i_{l-1}+2, k} \\ \vdots \\ E_{k-1, k} \end{pmatrix}.$$

Then  $\Lambda_{\rho_{k^c}} = \begin{pmatrix} \Lambda_{\rho_{A_1 \dots A_{i_1+\dots+i_{l-1}}}} & X \\ X^T & \Lambda_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1}}} \end{pmatrix}$ ,

$$\det(\Lambda_\rho) = \det(\Lambda_{\rho_{k^c}}) \det \left( I - \begin{pmatrix} Z^T & Y^T \end{pmatrix} \Lambda_{\rho_{k^c}}^{-1} \begin{pmatrix} Z \\ Y \end{pmatrix} \right),$$

and

$$\det(\Lambda_{\rho_{B_l}}) = \det(\Lambda_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1}}}) \det \left( I - Y^T \Lambda_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1}}}^{-1} Y \right).$$

So the inequality in Eq. (B6) holds if and only if

$$\det \left[ I - \begin{pmatrix} Z^T & Y^T \\ \Lambda_{\rho_k}^{-1} & \end{pmatrix} \begin{pmatrix} Z \\ Y \end{pmatrix} \right] \leq \det \left( I - Y^T \Lambda_{\rho_{A_{i_1} + \dots + i_{l-1} + 1} \dots A_{k-1, k-1}}^{-1} Y \right). \tag{B7}$$

Now, by the fourth inequality in Lemma B2, we have

$$\begin{pmatrix} Z^T & Y^T \end{pmatrix} \Lambda_{\rho_k}^{-1} \begin{pmatrix} Z \\ Y \end{pmatrix} \geq Y^T \Lambda_{\rho_{A_{i_1} + \dots + i_{l-1} + 1} \dots A_{k-1, k-1}}^{-1} Y, \tag{B8}$$

which implies that the inequality in Eq. (B7) is true. Consequently, the inequality in Eq. (B6) is true, and hence the inequality in Eq. (B5) holds, which completes the proof.  $\blacksquare$

It is clear from Theorem B4 and Corollary B7 that  $\mathcal{M}^{(k)}$  satisfies the conditions (1)–(3) in Definition 2 and thus meets the hierarchy condition. This completes the proof of Theorem 1.

### APPENDIX C: PROOF OF THEOREM 3

We first complete the proof of Example 1, that is, to show that, for any tripartite fully symmetric Gaussian state  $\sigma_{ABC}$ ,  $\mathcal{M}(\sigma_{A|BC}) = \mathcal{M}(\sigma_{AB})$  if and only if  $\sigma_{ABC} = \sigma_A \otimes \sigma_B \otimes \sigma_C$ .

*Proof of Example 1.* Recall that  $\sigma_{ABC}$  is fully symmetric if it is an  $(n + n + n)$ -mode Gaussian state with CM having the form

$$\Gamma_{\sigma_{ABC}} = \begin{pmatrix} A & X & X \\ X^T & A & X \\ X^T & X^T & A \end{pmatrix},$$

where  $A, X \in M_{2n}(\mathbb{R})$ . We have to show that  $\mathcal{M}(\sigma_{A|BC}) = \mathcal{M}(\sigma_{AB})$  if and only if  $X = 0$ .

The ‘‘if’’ part is obvious. For the ‘‘only if’’ part, assume  $\mathcal{M}(\sigma_{A|BC}) = \mathcal{M}(\sigma_{AB})$ . By Theorem B3,  $X = XA^{-1}X$ .

Assume that  $X \neq 0$ . If  $X$  is invertible, then  $X = A$ , which is impossible as  $\Gamma_{\sigma_{ABC}}$  is invertible. So  $\ker X \neq \{0\}$ . Under the space decomposition  $\mathbb{R}^{2n} = \ker X \oplus (\ker X)^\perp$ ,  $A$  and  $X$  can be represented as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} 0 & X_{12} \\ 0 & X_{22} \end{pmatrix}.$$

Notice that

$$A^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{12}^T)^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{12}^T A_{11}^{-1}A_{12})^{-1} \\ -(A_{22} - A_{12}^T A_{11}^{-1}A_{12})^{-1}A_{12}^T A_{11}^{-1} & (A_{22} - A_{12}^T A_{11}^{-1}A_{12})^{-1} \end{pmatrix}.$$

Then  $X = XA^{-1}X$  gives

$$X_{22} = 0 \quad \text{and} \quad X_{12} = -X_{12}(A_{22} - A_{12}^T A_{11}^{-1}A_{12})^{-1}A_{12}^T A_{11}^{-1}X_{12}$$

with  $\ker X_{12} = \{0\}$ . It follows that  $A_{ij}, X_{ij} \in M_n(\mathbb{R})$  and  $A_{12}, X_{12}$  are invertible. Therefore a tripartite fully symmetric Gaussian state  $\sigma_{ABC}$  satisfies  $\mathcal{M}(\sigma_{A|BC}) = \mathcal{M}(\sigma_{AB})$  if and only if its CM has the form

$$\Gamma_{\sigma_{ABC}} = \left( \begin{array}{cc|cc|cc} A_{11} & A_{12} & 0 & X_{12} & 0 & X_{12} \\ A_{12}^T & A_{22} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & A_{11} & A_{12} & 0 & X_{12} \\ X_{12}^T & 0 & A_{12}^T & A_{22} & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & A_{11} & A_{12} \\ X_{12}^T & 0 & X_{12}^T & 0 & A_{12}^T & A_{22} \end{array} \right),$$

where  $A_{ij}, X_{12} \in M_n(\mathbb{R})$  are invertible and

$$X_{12} = -A_{11}(A_{12}^{-1})^T (A_{22} - A_{12}^T A_{11}^{-1}A_{12}).$$

As  $\Gamma_{\sigma_{ABC}} > 0$ , we have

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} \geq \begin{pmatrix} 0 & X_{12} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix}^{-1} \begin{pmatrix} 0 & 0 \\ X_{12}^T & 0 \end{pmatrix} = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} D &= X_{12}(A_{22} - A_{12}^T A_{11}^{-1}A_{12})^{-1}X_{12}^T = A_{11}(A_{12}^{-1})^T (A_{22} - A_{12}^T A_{11}^{-1}A_{12})A_{12}^{-1}A_{11} \\ &= A_{11}(A_{12}^{-1})^T A_{22}A_{12}^{-1}A_{11} - A_{11}. \end{aligned}$$

Thus we have

$$2A_{11} - A_{11}(A_{12}^{-1})^T A_{22}A_{12}^{-1}A_{11} \geq A_{12}A_{22}^{-1}A_{12}^T.$$

Note that  $\begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} > 0$  implies that there is a contractive matrix  $E$  with  $\|E\| < 1$  such that  $A_{12} = A_{11}^{\frac{1}{2}}EA_{22}^{\frac{1}{2}}$ .  $E$  is invertible as  $A_{12}$  is. So the above inequality becomes to

$$2A_{11} \geq A_{11}^{\frac{1}{2}}EE^T A_{11}^{\frac{1}{2}} + A_{11}^{\frac{1}{2}}(EE^T)^{-1}A_{11}^{\frac{1}{2}},$$

and consequently,

$$2I \geq EE^T + (EE^T)^{-1}.$$

This is impossible because it leads to a contraction  $0 \geq (I - EE^T)^2 > 0$ . Therefore we must have  $X = 0$  and  $\sigma_{ABC} = \sigma_A \otimes \sigma_B \otimes \sigma_C$ .

As an illustration, let us consider the  $(1 + 1 + 1)$ -mode case. If  $\Gamma$  is a CM of an  $(1 + 1 + 1)$ -mode symmetric Gaussian state  $\sigma_{ABC}$  satisfying  $\mathcal{M}(\sigma_{A|BC}) = \mathcal{M}(\sigma_{AB})$  and  $\sigma_{ABC}$  not a product state, then, by what discussed above,  $\Gamma$  has the form

$$\Gamma = \left( \begin{array}{cc|cc|cc} a & c & 0 & -\frac{ab-c^2}{c} & 0 & -\frac{ab-c^2}{c} \\ c & b & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a & c & 0 & -\frac{ab-c^2}{c} \\ -\frac{ab-c^2}{c} & 0 & c & b & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & a & c \\ -\frac{ab-c^2}{c} & 0 & -\frac{ab-c^2}{c} & 0 & c & b \end{array} \right),$$

where  $a, b, c \in \mathbb{R}$  with  $a > 0, b > 0$  and  $ab > c^2 > 0$ . Since

$$\Gamma + i\Delta = \left( \begin{array}{cc|cc|cc} a & c+i & 0 & -\frac{ab-c^2}{c} & 0 & -\frac{ab-c^2}{c} \\ c-i & b & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & a & c+i & 0 & -\frac{ab-c^2}{c} \\ -\frac{ab-c^2}{c} & 0 & c-i & b & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & a & c+i \\ -\frac{ab-c^2}{c} & 0 & -\frac{ab-c^2}{c} & 0 & c-i & b \end{array} \right) \geq 0,$$

we have

$$ab - c^2 - 1 \geq 0$$

and

$$(ab - c^2 - 1)^2 - \frac{ab(ab - c^2)^2}{c^2} \geq 0.$$

However, the last inequality is not true, as it will lead to a contradiction  $ab(ab - c^2)^2 \leq c^2(ab - c^2 - 1)^2 < ab(ab - c^2)^2$ . ■

The ‘‘only if’’ parts of statements (i) and (ii) of the next general result imply respectively that  $\mathcal{M}^{(k)}$  is tightly monogamous and completely monogamous.

*Theorem C1.* Assume  $k \geq 3$ . Let  $\rho = \rho_{A_1, A_2, \dots, A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$  with the CM  $\Gamma_\rho = (A_{ij})_{k \times k}$  as in Eq. (1). For any  $l$  partition ( $2 \leq l < k$ ) of a  $k$ -partite system  $A_1 A_2 \dots A_k$  determined by a permutation  $\pi$  of  $k$  as in Eq. (6), the following statements are true.

(i) With  $i_0 = 0$ , we have

$$\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) = \mathcal{M}^{(l)}(\rho_{A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} \dots A_{\pi(i_1+i_2)} | \dots | A_{\pi(i_1+\dots+i_{l-1}+1)} \dots A_{\pi(i_1+\dots+i_l)}})$$

if and only if

$$\mathcal{M}^{(i_h)}(\rho_{A_{\pi(i_0+i_1+\dots+i_{h-1}+1)} \dots A_{\pi(i_0+i_1+\dots+i_h)}}) = 0 \text{ for all } h \in \{1, 2, \dots, l\} \text{ with } i_h \geq 2.$$

(ii) For each  $h \in \{1, 2, \dots, l\}$ , we have

$$\mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) = \mathcal{M}^{(i_h)}(\rho_{A_{\pi(i_0+i_1+\dots+i_{h-1}+1)} \dots A_{\pi(i_0+i_1+\dots+i_h)}})$$

if and only if, when  $h > 1$ ,

$$\mathcal{M}^{(2)}(\rho_{A_{\pi(i_1+\dots+i_{h-1}+1)} \dots A_{\pi(i_1+\dots+i_h)} | A_{\pi(1)} \dots A_{\pi(i_1+\dots+i_{h-1})} A_{\pi(i_1+\dots+i_h+1)} \dots A_{\pi(k)}}) = 0$$

and

$$\mathcal{M}^{(k-i_h)}(\rho_{A_{\pi(1)} \dots A_{\pi(i_1+\dots+i_{h-1})} A_{\pi(i_1+\dots+i_h+1)} \dots A_{\pi(k)}}) = 0;$$

when  $h = 1$ ,

$$\mathcal{M}^{(2)}(\rho_{A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} A_{\pi(i_1+2)} \dots A_{\pi(k)}}) = 0 \quad \text{and} \quad \mathcal{M}^{(k-i_1)}(\rho_{A_{\pi(i_1+1)} A_{\pi(i_1+2)} \dots A_{\pi(k)}}) = 0.$$

*Proof.* (i) Denote by  $B_h = A_{\pi(i_0+i_1+\dots+i_{h-1}+1)} \dots A_{\pi(i_0+i_1+\dots+i_{h-1}+i_h)}$  and  $\Gamma_{B_h}$  the CM of  $\rho_{B_h}$ ,  $h = 1, 2, \dots, l$ . Then, by Eq. (B2),

$$\mathcal{M}^{(l)}(\rho_{A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} \dots A_{\pi(i_1+i_2)} | \dots | A_{\pi(i_1+\dots+i_{l-1}+1)} \dots A_{\pi(k)}}) = \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k})$$

if and only if

$$\prod_{h=1}^l \det(\Gamma_{B_h}) = \prod_{h=1}^l \left[ \prod_{j=i_0+i_1+\dots+i_{h-1}+1}^{i_1+\dots+i_{h-1}+i_h} \det(A_{\pi(j), \pi(j)}) \right].$$

By Lemma A1, the above equation holds if and only if  $\det(\Gamma_{B_h}) = \prod_{j=i_0+i_1+\dots+i_{h-1}+1}^{i_1+\dots+i_{h-1}+i_h} \det(A_{\pi(j), \pi(j)})$  for each  $h = 1, 2, \dots, l$ , and in turn, if and only if  $\mathcal{M}^{(i_h)}(\rho_{A_{\pi(i_0+i_1+\dots+i_{h-1}+1)} \dots A_{\pi(i_1+\dots+i_h)}}) = 0$  for each  $h = 1, 2, \dots, l$  whenever  $i_h \geq 2$ .

(ii) For any  $h \in \{1, 2, \dots, l\}$  with  $i_h \geq 2$ , denote by  $\Gamma_{B_h^c} = (A_{ij})$  with  $i, j \in \{1, 2, \dots, k\} \setminus B_h = B_h^c$ , then

$$\Gamma_{\rho} \cong \Gamma_{\rho_{B_h, B_h^c}} = \begin{pmatrix} \Gamma_{B_h} & C_h \\ C_h^T & \Gamma_{B_h^c} \end{pmatrix}.$$

By Eq. (B3),

$$\mathcal{M}^{(i_h)}(\rho_{A_{\pi(i_0+i_1+\dots+i_{h-1}+1)} \dots A_{\pi(i_0+i_1+\dots+i_h)}}) = \mathcal{M}^{(i_h)}(\rho_{B_h}) = \mathcal{M}^{(k)}(\rho_{A_1, A_2, \dots, A_k}) \tag{C1}$$

if and only if

$$\det(\Gamma_{B_h^c} - C_h^T \Gamma_{B_h}^{-1} C_h) = \prod_{j \in B_h^c} \det(A_{jj}).$$

By Lemma A1 and Lemma A2, the above equation holds if and only if  $C_h = 0$  and  $\det(\Gamma_{B_h^c}) = \prod_{j \in B_h^c} \det(A_{jj})$ . Therefore Eq. (C1) is true if and only if

$$\mathcal{M}^{(2)}(\rho_{B_h, B_h^c}) = 0 \quad \text{and} \quad \mathcal{M}^{(k-i_h)}(\rho_{B_h^c}) = 0.$$

This completes the proof of statement (ii). ■

To prove that  $\mathcal{M}^{(k)}$  is not strongly monogamous, the following general result is useful.

*Theorem C2.* For any  $\rho = \rho_{A_1 A_2 \dots A_k} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k})$  with CM  $\Gamma_{\rho} = (A_{ij})_{k \times k}$  as in Eq. (1) and any  $l$  partition ( $2 \leq l < k$ ) of  $A_1 A_2 \dots A_k$  determined by a permutation  $\pi$  as in Eq. (6) with  $i_l \geq 2$ , we have

$$\begin{aligned} &\mathcal{M}^{(l)}(\rho_{A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} \dots A_{\pi(i_1+i_2)} | \dots | A_{\pi(i_1+\dots+i_{l-1}+1)} \dots A_{\pi(k)}}) \\ &= \mathcal{M}^{(l)}(\rho_{A_{\pi(1)} \dots A_{\pi(i_1)} | A_{\pi(i_1+1)} \dots A_{\pi(i_1+i_2)} | \dots | A_{\pi(i_1+\dots+i_{l-1}+1)} \dots A_{\pi(k-1)}}) \end{aligned}$$

if and only if  $F = D(\Gamma_{\rho_{A_{\pi(i_1+\dots+i_{l-1}+1)} \dots A_{\pi(k-1)}}})^{-1} E$ , where

$$D = \begin{pmatrix} A_{\pi(1), \pi(i_1+\dots+i_{l-1}+1)} & A_{\pi(1), \pi(i_1+\dots+i_{l-1}+2)} & \dots & E_{\pi(1), \pi(k-1)} \\ A_{\pi(2), \pi(i_1+\dots+i_{l-1}+1)} & A_{\pi(2), \pi(i_1+\dots+i_{l-1}+2)} & \dots & E_{\pi(2), \pi(k-1)} \\ \vdots & \vdots & \ddots & \vdots \\ A_{\pi(i_1+\dots+i_{l-1}), \pi(i_1+\dots+i_{l-1}+1)} & A_{\pi(i_1+\dots+i_{l-1}), \pi(i_1+\dots+i_{l-1}+2)} & \dots & A_{\pi(i_1+\dots+i_{l-1}), \pi(k-1)} \end{pmatrix}$$

is an  $(k - i_l) \times (i_l - 1)$  block matrix,

$$F = \begin{pmatrix} A_{\pi(1), \pi(k)} \\ A_{\pi(2), \pi(k)} \\ \vdots \\ A_{\pi(i_1+\dots+i_{l-1}), \pi(k)} \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} A_{\pi(i_1+\dots+i_{l-1}+1), \pi(k)} \\ A_{\pi(i_1+\dots+i_{l-1}+2), \pi(k)} \\ \vdots \\ A_{\pi(k-1), \pi(k)} \end{pmatrix}.$$

The following result is a special case of Theorem C1, which illustrates the exact meaning of Theorem C2 plainly.

*Theorem C3.* For  $k \geq 2$  and  $\rho = \rho_{A_1 A_2 \dots A_k A_{k+1}} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k} \otimes H_{A_{k+1}})$  with CM  $\Gamma_{\rho} = (A_{ij})_{(k+1) \times (k+1)}$ ,

$$\mathcal{M}^{(k)}(\rho_{A_1 A_2 \dots A_{k-1} | A_k A_{k+1}}) = \mathcal{M}^{(k)}(\rho_{A_1 A_2 \dots A_k}) \quad \text{if and only if} \quad \begin{pmatrix} A_{1, k+1} \\ A_{2, k+1} \\ \vdots \\ A_{k-1, k+1} \end{pmatrix} = \begin{pmatrix} A_{1, k} \\ A_{2, k} \\ \vdots \\ A_{k-1, k} \end{pmatrix} A_{kk}^{-1} A_{k, k+1}.$$

*Proof of Theorem C2.* Without loss of generality, we may assume that the  $l$  partition of  $A_1 A_2 \dots A_{k-1} A_k$  is

$$A_1 \dots A_{i_l} | A_{i_l+1} \dots A_{i_l+i_2} | A_{i_l+i_2+1} \dots A_{i_l+\dots+i_{l-1}} | A_{i_l+\dots+i_{l-1}+1} \dots A_{k-1, k-1} A_{kk},$$



since  $\mathcal{M}^{(k)}$  is invariant under any permutation of subsystems. It is clear by the assumption  $i_l \geq 2$  that  $i_1 + i_2 + \dots + i_{l-1} < k - 1$ . With the same symbols used in the proof of Theorem B5, and by Eqs. (B5)–(B8),

$$\begin{aligned} & \mathcal{M}^{(l)}(\rho_{A_1 \dots A_{i_1} | A_{i_1+1} \dots A_{i_1+i_2} | A_{i_1+i_2+1} \dots A_{i_1+\dots+i_{l-1}} | A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1} A_{kk}) \\ &= \mathcal{M}^{(l)}(\rho_{A_1 \dots A_{i_1} | A_{i_1+1} \dots A_{i_1+i_2} | A_{i_1+i_2+1} \dots A_{i_1+\dots+i_{l-1}} | A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1}), \end{aligned}$$

if and only if

$$\frac{\det(\Lambda_\rho)}{\det(\Lambda_{\rho_{B_l}})} = \frac{\det(\Gamma_\rho)}{\det(\Gamma_{B_l})} = \frac{\det(\Gamma_{\rho_{k^c}})}{\det(\Gamma_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1}}})} = \frac{\det(\Lambda_{\rho_{k^c}})}{\det(\Lambda_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1}}})}, \tag{C2}$$

where

$$\Lambda_\rho = \begin{pmatrix} I & E_{12} & \dots & E_{1k} \\ E_{12}^T & I & \dots & E_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ E_{1k}^T & E_{2k}^T & \dots & I \end{pmatrix} = \begin{pmatrix} \Lambda_{\rho_{A_1 \dots A_{i_1+\dots+i_{l-1}}}} & X & Z \\ X^T & \Lambda_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1}}} & Y \\ Z^T & Y^T & I \end{pmatrix}$$

with  $E_{ij} = A_{ii}^{-\frac{1}{2}} A_{ij} A_{jj}^{-\frac{1}{2}}$ ,  $i \neq j$ ,

$$\begin{aligned} X &= \begin{pmatrix} E_{1, i_1+\dots+i_{l-1}+1} & E_{1, i_1+\dots+i_{l-1}+2} & \dots & E_{1, k-1} \\ E_{2, i_1+\dots+i_{l-1}+1} & E_{2, i_1+\dots+i_{l-1}+2} & \dots & E_{2, k-1} \\ \vdots & \vdots & \ddots & \vdots \\ E_{i_1+\dots+i_{l-1}, i_1+\dots+i_{l-1}+1} & E_{i_1+\dots+i_{l-1}, i_1+\dots+i_{l-1}+2} & \dots & E_{i_1+\dots+i_{l-1}, k-1} \end{pmatrix}, \\ Z &= \begin{pmatrix} E_{1k} \\ E_{2k} \\ \vdots \\ E_{i_1+\dots+i_{l-1}, k} \end{pmatrix} \quad \text{and} \quad Y = \begin{pmatrix} E_{i_1+\dots+i_{l-1}+1, k} \\ E_{i_1+\dots+i_{l-1}+2, k} \\ \vdots \\ E_{k-1, k} \end{pmatrix}. \end{aligned}$$

It is clear that Eq. (C2) holds if and only if

$$(Z^T \quad Y^T) \Lambda_{\rho_{k^c}}^{-1} \begin{pmatrix} Z \\ Y \end{pmatrix} = Y^T \Lambda_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1}}}^{-1} Y. \tag{C3}$$

By Lemma B2, Eq. (C3) is true if and only if  $Z = X \Lambda_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1}}}^{-1} Y$ , which is equivalent to

$$F = D(\Gamma_{\rho_{A_{i_1+\dots+i_{l-1}+1} \dots A_{k-1, k-1}}})^{-1} E, \tag{C4}$$

where

$$\begin{aligned} D &= \begin{pmatrix} A_{1, i_1+\dots+i_{l-1}+1} & A_{1, i_1+\dots+i_{l-1}+2} & \dots & A_{1, k-1} \\ A_{2, i_1+\dots+i_{l-1}+1} & A_{2, i_1+\dots+i_{l-1}+2} & \dots & A_{2, k-1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{i_1+\dots+i_{l-1}, i_1+\dots+i_{l-1}+1} & A_{i_1+\dots+i_{l-1}, i_1+\dots+i_{l-1}+2} & \dots & A_{i_1+\dots+i_{l-1}, k-1} \end{pmatrix}, \\ F &= \begin{pmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{i_1+\dots+i_{l-1}, k} \end{pmatrix} \quad \text{and} \quad E = \begin{pmatrix} A_{i_1+\dots+i_{l-1}+1, k} \\ A_{i_1+\dots+i_{l-1}+2, k} \\ \vdots \\ A_{k-1, k} \end{pmatrix}. \end{aligned}$$

Hence the statement of the theorem is true and the proof is completed.  $\blacksquare$

Now we are in a position to prove Theorem 3.

*Proof of Theorem 3.* Assume  $k \geq 3$ . It is obvious that  $\mathcal{M}^{(k)}$  is tightly monogamous by (i) and is completely monogamous by (ii) in Theorem C1. Hence the statements (1) and (2) of Theorem 3 are true.

To prove the statement (3), assume  $k \geq 2$ . Let  $\rho_{A_1 A_2 \dots A_k A_{k+1}} \in \mathcal{FS}(H_{A_1} \otimes H_{A_2} \otimes \dots \otimes H_{A_k} \otimes H_{A_{k+1}})$  with

CM  $\Gamma_\rho = (A_{ij})_{(k+1) \times (k+1)}$ . It is clear that  $\mathcal{M}^{(2)}(\rho_{A_1 A_{k+1}}) = \mathcal{M}^{(2)}(\rho_{A_2 A_{k+1}}) = \dots = \mathcal{M}^{(2)}(\rho_{A_k A_{k+1}}) = 0$  if and only if

$$\begin{pmatrix} A_{1, k+1} \\ A_{2, k+1} \\ \vdots \\ A_{k-1, k+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = 0.$$

However, by Theorem C3,  $\mathcal{M}^{(k)}(\rho_{A_1 A_2 \dots A_{k-1} | A_k A_{k+1}}) = \mathcal{M}^{(k)}(\rho_{A_1 A_2 \dots A_k})$  if and only if

$$\begin{pmatrix} A_{1,k+1} \\ A_{2,k+1} \\ \vdots \\ A_{k-1,k+1} \end{pmatrix} = \begin{pmatrix} A_{1,k} A_{kk}^{-1} A_{k,k+1} \\ A_{2,k} A_{kk}^{-1} A_{k,k+1} \\ \vdots \\ A_{k-1,k} A_{kk}^{-1} A_{k,k+1} \end{pmatrix},$$

which may not be zero. To make this sure, let us give an example here. Consider the  $(k+1)$ -partite  $(1+1+\dots+1)$ -mode case. Let  $\Gamma = (A_{ij})$  be a real symmetric matrix, where  $A_{jj} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$  for  $j = 1, 2, \dots, k-1, k+1$ ,  $A_{kk} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ ,  $A_{k-1,k} = A_{k,k+1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ,  $A_{k-1,k+1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ , otherwise,  $A_{ij} = 0$  for

$i < j$ . It is easily checked that  $\Gamma = \Gamma_\rho$  is a CM for a Gaussian state  $\rho = \rho_{A_1 A_2 \dots A_k A_{k+1}}$  since  $\Gamma + i\Delta \geq 0$ , where  $\Delta = \bigoplus_{j=1}^{k+1} \Delta_j$  with  $\Delta_j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ . Obviously, we have

$$\begin{pmatrix} A_{1,k+1} \\ A_{2,k+1} \\ \vdots \\ A_{k-1,k+1} \end{pmatrix} = \begin{pmatrix} A_{1,k} A_{kk}^{-1} A_{k,k+1} \\ A_{2,k} A_{kk}^{-1} A_{k,k+1} \\ \vdots \\ A_{k-1,k} A_{kk}^{-1} A_{k,k+1} \end{pmatrix} \neq 0$$

as  $A_{k-1,k+1} = A_{k-1,k} A_{kk}^{-1} A_{k,k+1} \neq 0$ . In fact, for this state  $\rho$ , we have  $\mathcal{M}^{(k)}(\rho_{A_1 A_2 \dots A_{k-1} | A_k A_{k+1}}) = \mathcal{M}^{(k)}(\rho_{A_1 A_2 \dots A_{k-1} A_k}) \approx 0.3056$ , but  $\mathcal{M}^{(2)}(\rho_{A_{k-1} A_{k+1}}) \approx 0.0548 \neq 0$ .

Hence  $\mathcal{M}^{(k)}$  is not strongly monogamous, which completes the proof of statement (3) in Theorem 3.  $\blacksquare$

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