Conditions for steady-state entanglement of quantum systems in a stationary environment under Markovian dissipation

Yuebing Zhou,[1,2] Jiawei Hu,[1,*] and Hongwei Yu[1,*]
[1]*Department of Physics and Synergetic Innovation Center for Quantum Effects and Applications, Hunan Normal University, Changsha, Hunan 410081, China*
[2]*Department of Physics, Huaihua University, Huaihua 418008, China*

(Received 24 September 2021; revised 19 December 2021; accepted 23 February 2022; published 14 March 2022)

We study the entanglement dynamics of an open quantum system composed of two identical two-level subsystems in a common stationary environment undergoing Markovian dissipation, with the help of a set of physical parameters defined with the collective transition coefficients of the system. We then systematically investigate the steady-state entanglement of such a system and obtain the necessary and sufficient condition for steady-state entanglement when it is initial-state dependent. We also show in particular conditions for steady-state entanglement in some circumstances and propose in general a conjecture for the necessary and sufficient condition when it is independent of the initial state.

DOI: 10.1103/PhysRevA.105.032426

I. INTRODUCTION

Quantum entanglement, which describes the nonlocal correlation between different quantum subsystems that has no classical counterpart [1], has become the core of quantum information science and technology [2,3], such as quantum cryptography [4], quantum teleportation [5], and quantum computation [6]. In recent decades, as one of the basic issues in quantum information science, the preparation of entangled states [7,8], has been extensively studied. However, a thorny problem is the inevitable coupling between a quantum system and the ubiquitous environmental fluctuations, which in general causes decoherence [9]. Moreover, unlike the asymptotic decoherence of a single quantum system, an initially entangled quantum system may get completely disentangled within a finite time due to spontaneous emission even in vacuum, which is called entanglement sudden death [10,11]. Consequently, the fragility of entanglement has become one of the main obstacles to the realization of quantum information technologies. To fight against the destructive environmental effects on entanglement, many active strategies have been proposed, such as entanglement distillation [12,13], dynamical decoupling [14], repeated projective measurements (i.e., using the quantum Zeno effect) [15], and weak measurement and quantum measurement reversal [16].

Nevertheless, on the other hand, the environmental noise does not always play a negative role. It can also serve as a medium to provide indirect interactions so that two otherwise separable quantum subsystems may get entangled [17–30]. Then a natural question is whether the entanglement generated for a quantum system undergoing dissipation can persist in the steady state. In fact, it has been shown that, in some specific scenarios, quantum systems undergoing purely dissipative dynamics can obtain steady-state entanglement, such as in a thermal bath in the limit of vanishing interatomic separation [31], in a plasmonic waveguide [32,33], and in a nonequilibrium environment [34–36]. Now a question arises naturally as to what the necessary and sufficient condition is for steady-state entanglement of an open quantum system undergoing purely dissipative dynamics.

In this paper we study the conditions for steady-state entanglement of a certain class of open quantum systems, i.e., an open quantum system composed of two identical two-level subsystems, each of which is weakly coupled to a common stationary environment, and the dynamics is Markovian. We define a set of parameters with the collective transition coefficients of the system. With the help of these parameters, we obtain the conditions for steady-state entanglement. Hereafter, & denotes the logical AND and $\|$ denotes the logical OR. Natural units with $\hbar = c = k_B = 1$ are used, where c is the speed of light, $\hbar$ the reduced Planck constant, and k_B the Boltzmann constant.

II. MASTER EQUATION

We consider a quantum system composed of a pair of identical two-level quantum subsystems weakly coupled to a common stationary dissipative environment. The Hamiltonian of the total system is

$$H = H_S + H_E + H_I. \tag{1}$$

We assume that the two subsystems do not interact directly with each other, so the Hamiltonian of the two subsystems H_S takes the form

$$H_S = \frac{\omega}{2}\sigma_3^{(1)} + \frac{\omega}{2}\sigma_3^{(2)}, \tag{2}$$

where ω is the energy-level spacing between the excited state $|1\rangle$ and the ground state $|0\rangle$, $\sigma_\mu^{(1)} = \sigma_\mu \otimes \sigma_0$, and

[*]Corresponding authors: jwhu@hunnu.edu.cn; hwyu@hunnu.edu.cn

$\sigma_\mu^{(2)} = \sigma_0 \otimes \sigma_\mu$. Here we define $\sigma_0 = |1\rangle\langle 1| + |0\rangle\langle 0|$, $\sigma_1 = |0\rangle\langle 1| + |1\rangle\langle 0|$, $\sigma_2 = i(|0\rangle\langle 1| - |1\rangle\langle 0|)$, and $\sigma_3 = |1\rangle\langle 1| - |0\rangle\langle 0|$. Further, H_E is the Hamiltonian of the environment. The interaction Hamiltonian describing the linear coupling between the two-level subsystems and the environment can be written as

$$H_I = \varepsilon \sum_{\alpha=1}^{2} \sum_m \mathcal{S}_m^{(\alpha)}(\tau) \otimes \Psi_m(x^{(\alpha)}(\tau)), \tag{3}$$

where ε is the coupling constant, which is assumed to be small, $\Psi_m(x^{(\alpha)}(\tau))$ is the operator of the environment, and $\mathcal{S}_m^{(1)}(\tau) = \mathcal{S}_m(\tau) \otimes \sigma_0$ and $\mathcal{S}_m^{(2)}(\tau) = \sigma_0 \otimes \mathcal{S}_m(\tau)$ are operators of the two subsystems, respectively, with $\mathcal{S}_m(\tau)$ the single-subsystem operator. In the interaction picture, the single-subsystem operator $\mathcal{S}_m(\tau)$ can be expressed as

$$\mathcal{S}_m(\tau) = d_m \sigma_+ e^{i\omega\tau} + d_m^* \sigma_- e^{-i\omega\tau}, \tag{4}$$

where $\sigma_+ = |1\rangle\langle 0|$ and $\sigma_- = |0\rangle\langle 1|$ are the transition operators of the two-level subsystems and $d_m = \langle 1|\mathcal{S}_m(\tau)|0\rangle$. Note that here the interaction Hamiltonian does not include terms leading to pure dephasing.

Under the Born-Markov approximation and the rotating-wave approximation, the evolution equation of the reduced density matrix $\rho(\tau) = \mathrm{Tr}_E[\rho_{\mathrm{tot}}(\tau)]$ can be described by the Gorini-Kossakowski-Lindblad-Sudarshan (GKLS) master equation [37,38]

$$\frac{d\rho(\tau)}{d\tau} = -i[H_{\mathrm{eff}}, \rho(\tau)] + \mathcal{D}[\rho(\tau)], \tag{5}$$

where the dissipator $\mathcal{D}[\rho(\tau)]$ takes the form

$$\mathcal{D}[\rho(\tau)] = \frac{1}{2} \sum_{\alpha,\varrho=1}^{2} [D_+^{(\alpha\varrho)}(2\sigma_-^{(\varrho)}\rho\sigma_+^{(\alpha)} - \{\sigma_+^{(\alpha)}\sigma_-^{(\varrho)}, \rho\}) + D_-^{(\alpha\varrho)}(2\sigma_+^{(\varrho)}\rho\sigma_-^{(\alpha)} - \{\sigma_-^{(\alpha)}\sigma_+^{(\varrho)}, \rho\})]. \tag{6}$$

Here $\sigma_\pm^{(1)} = \sigma_\pm \otimes \sigma_0$, $\sigma_\pm^{(2)} = \sigma_0 \otimes \sigma_\pm$, and the dissipative coefficients $D_\pm^{(\alpha\varrho)}$ can be written as

$$D_+^{(\alpha\varrho)} = \varepsilon^2 \sum_{m,n} d_m^{(\alpha)*} d_n^{(\varrho)} \int_{-\infty}^{\infty} G_{mn}^{(\alpha\varrho)}(\Delta\tau) e^{i\omega\Delta\tau} d\Delta\tau, \tag{7}$$

$$D_-^{(\alpha\varrho)} = \varepsilon^2 \sum_{m,n} d_m^{(\alpha)} d_n^{(\varrho)*} \int_{-\infty}^{\infty} G_{mn}^{(\alpha\varrho)}(\Delta\tau) e^{-i\omega\Delta\tau} d\Delta\tau. \tag{8}$$

In the derivation of the master equation, we assume that the environment perceived by the quantum system is stationary, i.e., the correlation functions of the environment

$$G_{mn}^{(\alpha\varrho)}(\tau, \tau') = \langle \Psi_m(x^{(\alpha)}(\tau))\Psi_n(x^{(\varrho)}(\tau'))\rangle \tag{9}$$

in Eqs. (7) and (8) are functions of $\Delta\tau = \tau - \tau'$, i.e., they are invariant under temporal translations. According to Eqs. (7), (8), (9), it can be proved that $D_\pm^{(\alpha\varrho)} = D_\pm^{(\varrho\alpha)*}$. Note that here we ignore the environment-induced energy shift of the subsystems and focus on the effects of dissipation, so the effective Hamiltonian H_{eff} in Eq. (5) is not shown.

On the other hand, for a quantum system composed of a pair of identical two-level subsystems in interaction with a stationary dissipative environment, the transition amplitude from the state $|k, \varphi_0\rangle$ to $|k', \varphi_f\rangle$ can be expressed as

$$\mathcal{A}_{|k,\varphi_0\rangle\to|k',\varphi_f\rangle}(\mathcal{T}) = i\langle k', \varphi_f| \int_{-\mathcal{T}/2}^{\mathcal{T}/2} d\tau\, H_I(\tau)|k, \varphi_0\rangle, \tag{10}$$

where $|\varphi_0\rangle$ and $|\varphi_f\rangle$ represent the initial and final states of the environment, respectively, $|k\rangle$ and $|k'\rangle$ are two arbitrary states of the quantum system, and $\mathcal{T}$ is the interaction time, which is treated as infinite here. Then we can define a set of transition coefficients as

$$\Gamma_{kl\to k'l'} = \lim_{\mathcal{T}\to+\infty} \frac{1}{\mathcal{T}} \sum_{\varphi_f} \mathcal{A}_{|k,\varphi_0\rangle\to|k',\varphi_f\rangle} \mathcal{A}^*_{|l,\varphi_0\rangle\to|l',\varphi_f\rangle}. \tag{11}$$

It is obvious that $\Gamma_{kl\to k'l'} = \Gamma^*_{lk\to l'k'}$. When $l = k$ and $l' = k'$, Eq. (11) becomes the transition rate of the quantum system from $|k\rangle$ to $|k'\rangle$. So $\Gamma_{kk\to k'k'} \geqslant 0$. If the state $|k\rangle$ is a superposition state which can be written as $|k\rangle = b|k_1\rangle + c|k_2\rangle$ with $|b|^2 + |c|^2 = 1$, then the transition rate $\Gamma_{kk\to k'k'}$ can be written according to Eq. (11) as

$$\Gamma_{kk\to k'k'} = |b|^2\Gamma_{k_1k_1\to k'k'} + |c|^2\Gamma_{k_2k_2\to k'k'} + (bc^*\Gamma_{k_1k_2\to k'k'} + b^*c\Gamma_{k_2k_1\to k'k'}). \tag{12}$$

The term in parentheses in Eq. (12) contributes an interference term. Therefore, when $l \neq k$ or $l' \neq k'$, $\Gamma_{kl\to k'l'}$ (which is in general a complex number) is referred to as a coherent transition coefficient, since it is related to the interference term in the transition rate between coherent superposition states.

In the coupled basis $\{|e\rangle, |s\rangle, |a\rangle, |g\rangle\}$, where

$$|e\rangle = |11\rangle, \quad |s\rangle = u|10\rangle + v|01\rangle,$$
$$|g\rangle = |00\rangle, \quad |a\rangle = v^*|10\rangle - u^*|01\rangle, \tag{13}$$

with $|u|^2 + |v|^2 = 1$, the master equation (5) (consider the dissipator only, i.e., $d\rho(\tau)/d\tau = \mathcal{D}[\rho(\tau)]$) can be written as a set of first-order linear differential equations using the transition coefficients defined in Eq. (11), i.e.,

$$\dot{\rho}_{kl} = \frac{1}{2} \sum_{k'l'} [2\rho_{k'l'}\Gamma_{k'l'\to kl} - \rho_{kl'}\Gamma_{ll'\to k'k'} - \rho_{l'l}\Gamma_{l'k\to k'k'}], \tag{14}$$

where $\rho_{kl} = \langle k|\rho|l\rangle$ and $\dot{\rho}_{kl} = d\rho_{kl}/d\tau$, with $k, l, k', l' \in \{e, s, a, g\}$. Here the transition coefficient $\Gamma_{kl\to k'l'}$ can be expressed as

$$\Gamma_{kl\to k'l'} = \sum_{\alpha,\varrho=1}^{2} \sum_{\kappa=\pm} \langle l|\sigma_\kappa^{(\alpha)}|l'\rangle\langle k'|\sigma_\kappa^{(\varrho)\dagger}|k\rangle D_\kappa^{(\alpha\varrho)}. \tag{15}$$

According to Eq. (15), for a quantum system described by the master equation (5), $\Gamma_{kl\to k'l'}$ is zero if $\Delta E_{kk'} = \Delta E_{ll'} = \pm\omega$ is not satisfied, where $\Delta E_{kk'} = E_k - E_{k'}$.

Substituting $|e\rangle = |11\rangle$, $|\phi_1\rangle = |10\rangle$, $|\phi_2\rangle = |01\rangle$, and $|g\rangle = |00\rangle$ into Eq. (15), we obtain

$$D_+^{(11)} = \Gamma_{\phi_1\phi_1\to gg} = \Gamma_{ee\to\phi_2\phi_2} = \Gamma_{\phi_1 e\to g\phi_2},$$
$$D_-^{(11)} = \Gamma_{gg\to\phi_1\phi_1} = \Gamma_{\phi_2\phi_2\to ee} = \Gamma_{g\phi_2\to\phi_1 e},$$
$$D_+^{(22)} = \Gamma_{\phi_2\phi_2\to gg} = \Gamma_{ee\to\phi_1\phi_1} = \Gamma_{\phi_2 e\to g\phi_1},$$
$$D_-^{(22)} = \Gamma_{gg\to\phi_2\phi_2} = \Gamma_{\phi_1\phi_1\to ee} = \Gamma_{g\phi_1\to\phi_2 e}. \tag{16}$$

Thus, the dissipation coefficients $D_+^{(\alpha\alpha)}$ and $D_-^{(\alpha\alpha)}$ are the individual downward and upward transition coefficients (rates) of the αth subsystem, respectively. Similarly,

$$\begin{aligned} D_+^{(12)} = D_+^{(21)*} &= \Gamma_{\phi_2\phi_1\to gg} = \Gamma_{ee\to\phi_1\phi_2} \\ &= \Gamma_{e\phi_1\to\phi_1 g} = \Gamma_{\phi_2 e\to g\phi_2}, \\ D_-^{(12)} = D_-^{(21)*} &= \Gamma_{gg\to\phi_2\phi_1} = \Gamma_{\phi_1\phi_2\to ee} \\ &= \Gamma_{\phi_1 g\to e\phi_1} = \Gamma_{g\phi_2\to\phi_2 e}, \end{aligned} \tag{17}$$

so the dissipation coefficients $D_+^{(\alpha\varrho)}|_{\alpha\neq\varrho}$ and $D_-^{(\alpha\varrho)}|_{\alpha\neq\varrho}$ are the collective downward and upward coherent transition coefficients of the two subsystems. Moreover, it can be found from Eq. (15) that all possible transition coefficients can be expressed with $D_\pm^{(\alpha\varrho)}$, since $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$ forms a set of the complete basis. The relationship between the dissipation coefficients in the master equation and the collective transitions coefficient of two subsystems shown in Eq. (17) can be regarded as a generalization of the result in Ref. [39], in which the relationship between the dissipation coefficients in the master equation of a single quantum system and its transition rates is shown.

III. ENTANGLEMENT DYNAMICS PARAMETERS

Now we define a set of eight parameters using $D_\pm^{(\alpha\varrho)}$ to describe the entanglement dynamics of an open quantum system composed of two two-level subsystems.

(i) The factor Γ is defined as

$$\Gamma \equiv \frac{\Gamma^{(11)} + \Gamma^{(22)}}{2}, \tag{18}$$

where

$$\Gamma^{(\alpha\varrho)} = D_+^{(\alpha\varrho)} - D_-^{(\varrho\alpha)} \tag{19}$$

is the spontaneous emission coefficient. That is, Γ is the average of the individual spontaneous emission rates of the two quantum subsystems, so $\Gamma \geqslant 0$.

(ii) The factor γ is defined as

$$\gamma \equiv \frac{\Gamma^{(22)} - \Gamma^{(11)}}{\Gamma^{(11)} + \Gamma^{(22)}}, \tag{20}$$

which describes the degree of difference between the individual spontaneous emission rates of the two quantum subsystems. Obviously, $|\gamma| \leqslant 1$. Here $\gamma = 0$ means $\Gamma^{(11)} = \Gamma^{(22)}$, i.e., the individual spontaneous emission rates of subsystems 1 and 2 are the same; $\gamma = 1$ (-1) means $\Gamma^{(11)}$ $(\Gamma^{(22)})$ is zero, i.e., the individual spontaneous emission rate of subsystem 1 (2) is zero. The larger the $|\gamma|$, the greater the difference between the spontaneous emission rates of the two subsystems.

(iii) For subsystem 1, we can define a parameter η_1 with the individual downward and upward transition rates of subsystem 1 as

$$\eta_1 \equiv \frac{D_+^{(11)} + D_-^{(11)}}{D_+^{(11)} - D_-^{(11)}} = 2N_1 + 1 = \coth\frac{\omega}{2T_1}, \tag{21}$$

where N_1 is the effective particle number perceived individually by subsystem 1. Then an equivalent temperature can be defined as $T_1 = \omega[\ln(1 + 1/N_1)]^{-1}$. It is obvious from Eq. (21) that $\eta_1 \geqslant 1$ since $N_1 \geqslant 0$.

(iv) Similarly, for subsystem 2, we can also define a parameter η_2,

$$\eta_2 \equiv \frac{D_+^{(22)} + D_-^{(22)}}{D_+^{(22)} - D_-^{(22)}} = 2N_2 + 1 = \coth\frac{\omega}{2T_2}, \tag{22}$$

and $\eta_2 \geqslant 1$.

(v) The angle θ_1 is defined as the argument of $\Gamma^{(12)}$, i.e.,

$$\theta_1 \equiv \mathrm{Arg}[\Gamma^{(12)}] = \mathrm{Arg}[D_+^{(12)} - D_-^{(21)}], \tag{23}$$

and the range is $\theta_1 \in (-\pi, \pi]$. Here $\theta_1 = 0$ means $\Gamma^{(12)} = \Gamma^{(21)} > 0$ and $\theta_1 = \pi$ means $\Gamma^{(12)} = \Gamma^{(21)} < 0$. The factor θ_1 characterizes the difference between the collective coherent emission coefficient $\Gamma^{(12)}$ and its complex conjugate $\Gamma^{(21)}$.

(vi) We define a parameter λ_1,

$$\lambda_1 \equiv \sqrt{\frac{\Gamma^{(12)}\Gamma^{(21)}}{\Gamma^{(11)}\Gamma^{(22)}}}, \tag{24}$$

to characterize the cooperative coherence of the two subsystems in collective transition processes. It is obvious from the definition that $\lambda_1 \geqslant 0$. In the following, we prove that $\lambda_1 \leqslant 1$, as required by the fact that the downward transition rate cannot be smaller than the upward one for two arbitrary energy eigenstates.

Proof. For a quantum system composed of two identical two-level quantum subsystems, the eigenvalues and the corresponding eigenstates of the Hamiltonian H_S can be written as

$$\begin{aligned} E_e &= \omega, \quad |e\rangle = |11\rangle, \\ E_\psi &= 0, \quad |\psi\rangle = u|10\rangle + v|01\rangle, \\ E_g &= -\omega, \quad |g\rangle = |00\rangle. \end{aligned} \tag{25}$$

Here $u = |u|e^{i\varphi_1}$, $v = |v|e^{i\varphi_2}$, and $|u|^2 + |v|^2 = 1$. The spontaneous emission rate from a higher level h to a lower level l can be expressed as $\Gamma_{hh\to ll} - \Gamma_{ll\to hh}$, since the stimulated absorption rate and the stimulated emission rate must be equal and are thus canceled. According to Eq. (12), we obtain

$$\begin{aligned} &\Gamma_{\psi\psi\to gg} - \Gamma_{gg\to\psi\psi} \\ &= |u|^2\Gamma^{(11)} + |v|^2\Gamma^{(22)} + (u^* v\Gamma^{(12)} + uv^*\Gamma^{(21)}) \qquad (26) \\ &= (|u|\sqrt{\Gamma^{(11)}} - |v|\sqrt{\Gamma^{(22)}})^2 + 2|u||v|\sqrt{\Gamma^{(11)}\Gamma^{(22)}} \\ &\quad + 2\,\mathrm{Re}[u^* v\, \Gamma^{(12)}] \\ &\geqslant 2|u||v|\sqrt{\Gamma^{(11)}\Gamma^{(22)}} + 2\,\mathrm{Re}[u^* v\Gamma^{(12)}] \qquad (27) \\ &= 2|u||v|(\sqrt{\Gamma^{(11)}\Gamma^{(22)}} \\ &\quad + \sqrt{\Gamma^{(12)}\Gamma^{(21)}}\mathrm{Re}[e^{i(\varphi_2 - \varphi_1 + \theta_1)}]) \\ &\geqslant 2|u||v|\sqrt{\Gamma^{(11)}\Gamma^{(22)}}\left(1 - \sqrt{\frac{\Gamma^{(12)}\Gamma^{(21)}}{\Gamma^{(11)}\Gamma^{(22)}}}\right), \qquad (28) \end{aligned}$$

where $\mathrm{Re}[x]$ means the real part of x. Note here that if and only if $|u| = \sqrt{\frac{\Gamma^{(22)}}{\Gamma^{(11)}+\Gamma^{(22)}}}$ and $|v| = \sqrt{\frac{\Gamma^{(11)}}{\Gamma^{(11)}+\Gamma^{(22)}}}$, the inequality (27) reduces to an equality. Moreover, the inequality (28) reduces to an equality if and only if $\varphi_1 - \varphi_2 = \theta_1 + \pi$, that is,

$$\Gamma_{\psi\psi\to gg} - \Gamma_{gg\to\psi\psi} = 2|u||v|\sqrt{\Gamma^{(11)}\Gamma^{(22)}}(1-\lambda_1) \quad (29)$$

if and only if $|\psi\rangle = \sqrt{\frac{\Gamma^{(22)}}{\Gamma^{(11)}+\Gamma^{(22)}}}e^{i\varphi_1}|10\rangle + \sqrt{\frac{\Gamma^{(11)}}{\Gamma^{(11)}+\Gamma^{(22)}}}e^{i(\varphi_1-\theta_1-\pi)}|01\rangle$. Similarly,

$$\Gamma_{ee\to\psi\psi} - \Gamma_{\psi\psi\to ee} \geqslant 2|u||v|\sqrt{\Gamma^{(11)}\Gamma^{(22)}}\left(1-\sqrt{\frac{\Gamma^{(12)}\Gamma^{(21)}}{\Gamma^{(11)}\Gamma^{(22)}}}\right) \quad (30)$$

and the equality holds if and only if $|\psi\rangle = \sqrt{\frac{\Gamma^{(11)}}{\Gamma^{(11)}+\Gamma^{(22)}}}e^{i\varphi_1}|10\rangle + \sqrt{\frac{\Gamma^{(22)}}{\Gamma^{(11)}+\Gamma^{(22)}}}e^{i(\varphi_1-\theta_1-\pi)}|01\rangle$. Therefore,

$$\{\Gamma_{\psi\psi\to gg} \geqslant \Gamma_{gg\to\psi\psi}\}\&\{\Gamma_{ee\to\psi\psi} \geqslant \Gamma_{\psi\psi\to ee}\} \Leftrightarrow \lambda_1 \leqslant 1. \quad (31)$$

. Q.E.D.

To conclude, $\lambda_1 \in [0, 1]$. When $\lambda_1 = 0$, $\Gamma^{(12)} = \Gamma^{(21)} = 0$, which indicates that there is no cooperative coherence between the subsystems in the process of collective spontaneous emission, e.g., when the spatial separation between the two subsystems $L \to \infty$. At this time, we find that the interference term in the collective spontaneous emission rate (26) vanishes. When $\lambda_1 = 1$, $\Gamma^{(12)}\Gamma^{(21)} = \Gamma^{(11)}\Gamma^{(22)}$, which indicates the strongest cooperative coherence between the two subsystems. As we have shown in the proof, when $\lambda_1 = 1$, there exists a collective state $|\psi\rangle$ such that the upward and downward collective transition rates are equal, i.e., the collective spontaneous emission rate is zero owing to the interference cancellation. Thus, the larger the λ_1, the stronger the cooperative coherence between the subsystems.

(vii) The angle θ_2 is defined as the argument of $D_-^{(21)}$, i.e.,

$$\theta_2 \equiv \text{Arg}[D_-^{(21)}]. \quad (32)$$

Then $\theta_2 \in (-\pi, \pi]$. Here $\theta_2 = 0$ means $D_-^{(12)}(\omega) = D_-^{(21)}(\omega) > 0$ and $\theta_2 = \pi$ means $D_-^{(12)}(\omega) = D_-^{(21)}(\omega) < 0$. The factor θ_2 characterizes the difference between the collective coherent emission coefficient $D_-^{(12)}$ and its complex conjugate $D_-^{(21)}$.

(viii) We define λ_2 as

$$\lambda_2 = \sqrt{\frac{D_-^{(12)}D_-^{(21)}}{D_-^{(11)}D_-^{(22)}}}, \quad (33)$$

whose range is $0 \leqslant \lambda_2 \leqslant 1$. The proof is shown as follows.

Proof. Similar to the proof of $0 \leqslant \lambda_1 \leqslant 1$, we can obtain that

$$\Gamma_{gg\to\psi\psi} = |u|^2D_-^{(11)} + |v|^2D_-^{(22)} + (uv^*D_-^{(12)} + u^*vD_-^{(21)}) \geqslant 2|u||v|\sqrt{D_-^{(11)}D_-^{(22)}}\left(1-\sqrt{\frac{D_-^{(12)}D_-^{(21)}}{D_-^{(11)}D_-^{(22)}}}\right). \quad (34)$$

The inequality (34) becomes an equality if and only if $|u| = \sqrt{\frac{D_-^{(22)}}{D_-^{(11)}+D_-^{(22)}}}$, $|v| = \sqrt{\frac{D_-^{(11)}}{D_-^{(11)}+D_-^{(22)}}}$, and $\varphi_1 - \varphi_2 = \theta_2 + \pi$. Similarly,

$$\Gamma_{\psi\psi\to ee} = |v|^2D_-^{(11)} + |u|^2D_-^{(22)} + (uv^*D_-^{(12)} + u^*vD_-^{(21)}) \geqslant 2|u||v|\sqrt{D_-^{(11)}D_-^{(22)}}\left(1-\sqrt{\frac{D_-^{(12)}D_-^{(21)}}{D_-^{(11)}D_-^{(22)}}}\right). \quad (35)$$

The inequality (35) becomes an equality if and only if $|u| = \sqrt{\frac{D_-^{(11)}}{D_-^{(11)}+D_-^{(22)}}}$, $|v| = \sqrt{\frac{D_-^{(22)}}{D_-^{(11)}+D_-^{(22)}}}$, and $\varphi_1 - \varphi_2 = \theta_2 + \pi$. Therefore,

$$\{\Gamma_{gg\to\psi\psi} \geqslant 0\}\ \&\ \{\Gamma_{\psi\psi\to ee} \geqslant 0\} \Leftrightarrow \lambda_2 \leqslant 1. \quad (36)$$

Q.E.D.

Thus, when $\lambda_2 = 0$, i.e., $D_-^{(12)} = D_-^{(21)} = 0$, the interference terms in Eqs. (34) and (35) vanish. As a result, there is no cooperative coherence between the two subsystems in the process of collective excitation. When $\lambda_2 = 1$, there exists a collective state $|\psi\rangle$ determined by $D_-^{(\alpha\varrho)}$ such that the upward collective transition rate is zero owing to the interference cancellation.

So far, we have defined eight parameters $\{\Gamma, \gamma, \eta_1, \eta_2, \lambda_1, \lambda_2, \theta_1, \theta_2\}$ and proved that $\Gamma \geqslant 0$, $|\gamma| \leqslant 1$, $\eta_1 \geqslant 1$, $\eta_2 \geqslant 1$, $0 \leqslant \lambda_1 \leqslant 1$, and $0 \leqslant \lambda_2 \leqslant 1$ by using that (a) the transition rate between any two collective states is non-negative and (b) the downward transition rate is greater than or equal to the upward one for two arbitrary energy eigenstates. With the parameters defined above, $D_\pm^{(\alpha\varrho)}$ can be reexpressed as

$$\begin{aligned}
D_\pm^{(11)} &= \tfrac{1}{2}(\eta_1 \pm 1)(1-\gamma)\Gamma,\\
D_\pm^{(22)} &= \tfrac{1}{2}(\eta_2 \pm 1)(1+\gamma)\Gamma,\\
D_-^{(12)} &= D_-^{(21)*} = \tfrac{1}{2}\sqrt{(\eta_1-1)(\eta_2-1)}\lambda_2 e^{-i\theta_2}\sqrt{1-\gamma^2}\Gamma,\\
D_+^{(12)} &= D_+^{(21)*} = (\sqrt{(\eta_1-1)(\eta_2-1)}\lambda_2 + 2\lambda_1 e^{i(\theta_1-\theta_2)})\\
&\quad\times \tfrac{1}{2}e^{i\theta_2}\sqrt{1-\gamma^2}\Gamma.
\end{aligned} \quad (37)$$

Then we can use this set of parameters $\{\Gamma, \gamma, \eta_1, \eta_2, \lambda_1, \lambda_2, \theta_1, \theta_2\}$ to describe the evolution of the quantum system.

IV. STEADY STATE

Under the coupled basis $\{|e\rangle, |s\rangle, |a\rangle, |g\rangle\}$, which is defined in Eq. (13), the first-order linear differential equations (14) describing the evolution of the system can be explicitly written as

$$\dot{\mathbf{X}} = -\mathbf{U}_1\mathbf{X}, \quad \dot{\rho}_{ge} = -\bar{\eta}\Gamma\rho_{ge}, \quad \dot{\mathbf{Y}} = -\mathbf{U}_2\mathbf{Y}. \quad (38)$$

Here $\mathbf{X} = (\rho_{ee}\ \ \rho_{ss}\ \ \rho_{aa}\ \ \rho_{gg}\ \ \rho_{as}\ \ \rho_{sa})^T$ and $\mathbf{Y} = (\rho_{se}\ \ \rho_{ae}\ \ \rho_{gs}\ \ \rho_{ga})^T$ are two column vectors and

$$\bar{\eta} = \tfrac{1}{2}[\eta_1 + \eta_2 + (\eta_2 - \eta_1)\gamma]. \quad (39)$$

The coefficient matrices $\mathbf{U}_1$ and $\mathbf{U}_2$ can be written explicitly as

$$\mathbf{U}_1 = \begin{pmatrix} \mathcal{P}_{ee} & -\Gamma_{ss\to ee} & -\Gamma_{aa\to ee} & 0 & -\Gamma_{as\to ee} & -\Gamma_{sa\to ee} \\ -\Gamma_{ee\to ss} & \mathcal{P}_{ss} & 0 & -\Gamma_{gg\to ss} & \mathcal{K} & \mathcal{K}^* \\ -\Gamma_{ee\to aa} & 0 & \mathcal{P}_{aa} & -\Gamma_{gg\to aa} & \mathcal{K} & \mathcal{K}^* \\ 0 & -\Gamma_{ss\to gg} & -\Gamma_{aa\to gg} & \mathcal{P}_{gg} & -\Gamma_{as\to gg} & -\Gamma_{sa\to gg} \\ -\Gamma_{ee\to as} & \mathcal{K}^* & \mathcal{K}^* & -\Gamma_{gg\to as} & \mathcal{P}_{as} & 0 \\ -\Gamma_{ee\to sa} & \mathcal{K} & \mathcal{K} & -\Gamma_{gg\to sa} & 0 & \mathcal{P}_{sa} \end{pmatrix} \tag{40}$$

and

$$\mathbf{U}_2 = \begin{pmatrix} \mathcal{P}_{se} & \mathcal{K} & -\Gamma_{gs\to se} & -\Gamma_{ga\to se} \\ \mathcal{K}^* & \mathcal{P}_{ae} & -\Gamma_{gs\to ae} & -\Gamma_{ga\to ae} \\ -\Gamma_{se\to gs} & -\Gamma_{ae\to gs} & \mathcal{P}_{gs} & \mathcal{K} \\ -\Gamma_{se\to ga} & -\Gamma_{ae\to ga} & \mathcal{K}^* & \mathcal{P}_{ga} \end{pmatrix}. \tag{41}$$

Moreover, here we define

$$\mathcal{P}_{kl} = \frac{1}{2}\sum_{k'}(\Gamma_{kk\to k'k'} + \Gamma_{ll\to k'k'}), \tag{42}$$

with $k, l, k' \in \{e, s, a, g\}$, and

$$\mathcal{K} = \tfrac{1}{2}(\Gamma_{as\to ee} + \Gamma_{as\to gg}). \tag{43}$$

Then the general solution of the differential equations (38) can be written in the form

$$\begin{aligned} \mathbf{X}(\tau) &= \mathbf{M}_0 + \sum_{i=1}^{5}\mathbf{M}_i e^{-\xi_i\bar{\eta}\Gamma\tau}, \\ \mathbf{Y}(\tau) &= \sum_{j=1}^{4}\mathbf{N}_j e^{-\zeta_j\bar{\eta}\Gamma\tau}, \\ \rho_{ge}(\tau) &= \rho_{ge}(0)e^{-\bar{\eta}\Gamma\tau}, \end{aligned} \tag{44}$$

where ξ_i and $\mathbf{M}_i$ $(i = 0, 1, 2, 3, 4, 5)$, and ζ_j and $\mathbf{N}_j$ $(j = 1, 2, 3, 4)$ are the eigenvalues and the corresponding eigenvectors of the dimensionless coefficient matrices $\mathbf{U}_1/\bar{\eta}\Gamma$ and $\mathbf{U}_2/\bar{\eta}\Gamma$, respectively. Note that $\xi_0 \equiv 0$, and $\mathbf{M}_0$ can be uniquely determined by the normalization condition $\sum_k \rho_{kk} = 1$, which is independent of the initial state. However, the rest components of $\mathbf{M}_i$ and $\mathbf{N}_j$ are related to the initial state.

The steady-state solution (when $\tau \to \infty$) of Eq. (38) must be finite, which requires that the real parts of ξ_i and ζ_j must be non-negative. This is naturally guaranteed since the dynamical map generated by the GKLS equation is completely positive and trace preserving [37,38,40]. Note that the two definite eigenvalues for $\mathbf{U}_1/\bar{\eta}\Gamma$ are labeled as $\xi_0 = 0$ and $\xi_5 = 1$ and the rest of the eigenvalues ξ_i and ζ_j are labeled as $\xi_1, \xi_2, \xi_3, \xi_4$ and $\zeta_1, \zeta_2, \zeta_3, \zeta_4$, respectively, in increasing order with respect to their real parts. Then, allowing for the ranges of parameters $|\gamma| \leqslant 1$, $\eta_1 \geqslant 1$, $\eta_2 \geqslant 1$, $0 \leqslant \lambda_1 \leqslant 1$, and $0 \leqslant \lambda_2 \leqslant 1$, it can be proved that there is at most one zero root in ξ_i (or ζ_j) $(i = 1, 2, 3, 4)$, i.e., only ξ_1 (or ζ_1) can be zero. See Appendix A for the proof.

First of all, if $\Gamma = 0$, then $\rho(\infty) = \rho(0)$ and the quantum system is locked up in the initial state. If $\Gamma \neq 0$, then the asymptotic state can be expressed as

$$\mathbf{X}(\infty) = \begin{cases} \mathbf{M}_0 + \mathbf{M}_1, & \xi_1 = 0 \\ \mathbf{M}_0, & \xi_1 \neq 0, \end{cases}$$

$$\mathbf{Y}(\infty) = \begin{cases} \mathbf{N}_1, & \zeta_1 = 0 \\ 0, & \zeta_1 \neq 0, \end{cases}$$

and $\rho_{ge}(\infty) = 0$. Moreover, there are three cases in which $\xi_1 = 0$, i.e.,

$|\gamma| = 1$ (case I),

$\lambda_1 = \lambda_2 = 1$ & $\gamma = 0$ & $\eta_1 = \eta_2 \neq 1$ & $\theta_1 = \theta_2$ (case II),

$\lambda_1 = 1$ & $|\gamma| \neq 1$ & $\eta_1 = \eta_2 = 1$ (case III),

and there are two cases in which $\zeta_1 = 0$, which are cases I and III shown above. The relevant proofs are given in Appendix A. Then, when none of Γ, ξ_1, or ζ_1 is zero, the steady state is independent of the initial state, while the steady state is dependent on the initial state if at least one of Γ, ξ_1, or ζ_1 is zero.

V. CONDITIONS FOR STEADY-STATE ENTANGLEMENT

We characterize the degree of entanglement by concurrence C [41], which is

$$C[\rho(\tau)] = \max\{0, K(\tau)\}, \tag{45}$$

where $K(\tau) = \kappa_1 - \kappa_2 - \kappa_3 - \kappa_4$, with κ_i $(i = 1, 2, 3, 4)$ the square roots of eigenvalues of the matrix $\rho(\sigma_2 \otimes \sigma_2)\rho^T(\sigma_2 \otimes \sigma_2)$ in decreasing order. Here ρ is the density matrix in the decoupled basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$ and ρ^T is its transpose. According to the discussion in Sec. IV, the steady states can be divided into two types, i.e., those related to the initial state and those independent of the initial state. In the following, we discuss the possible steady-state entanglement for the two types of steady states, respectively.

A. Steady-state entanglement depending on the initial state

First of all, if $\Gamma = 0$, the quantum system will be locked up in the initial state and the initial entanglement will be protected. As an example, in Ref. [42], a pair of two-level atoms placed in between two perfectly reflecting plate is shown to be locked up in its initial state when the distance between the plates is less than half of the transition wavelength of the atoms.

In the following, we focus on the case of $\Gamma \neq 0$. In this section, we work in the coupled basis $\{|e\rangle, |s\rangle, |a\rangle, |g\rangle\}$ [Eq. (13)] with

$$u = \sqrt{\frac{1-\gamma}{2}}e^{i\theta_1/2}, \quad v = \sqrt{\frac{1+\gamma}{2}}e^{-i\theta_1/2}. \tag{46}$$

Then the state $|a\rangle$ can be a subradiant state [43], i.e., the downward collective transition rate between the subradiant state and the ground state is zero.

Case I: $|\gamma| = 1$. In this case, $\xi_1 = 0$ and $\zeta_1 = 0$. The asymptotic state can be expressed as the block matrix

$$\rho(\infty) = \begin{pmatrix} \mathbf{P}^- & \mathbf{0} \\ \mathbf{0} & \mathbf{P}^+ \end{pmatrix}, \tag{47}$$

where

$$\mathbf{P}^\pm = \frac{\eta_i \pm 1}{2\eta_i} \begin{pmatrix} \rho_{aa}(0) + \rho_{ee}(0) & \rho_{ag}(0) + \gamma\rho_{es}(0) \\ \rho_{ga}(0) + \gamma\rho_{se}(0) & \rho_{ss}(0) + \rho_{gg}(0) \end{pmatrix}. \tag{48}$$

Here $\eta_i = \eta_2$ when $\gamma = 1$ and $\eta_i = \eta_1$ when $\gamma = -1$. Substituting $\rho(\infty)$ into Eq. (45), we can obtain

$$K(\infty) = -\big(\{[\rho_{ss}(0) + \rho_{gg}(0)][\rho_{aa}(0) + \rho_{ee}(0)] - |\rho_{ga}(0) + \gamma\rho_{se}(0)|^2\}\big(1 - \eta_i^{-2}\big)\big)^{1/2} \leqslant 0. \tag{49}$$

So there is no steady-state entanglement. Specific examples of this can been found in the literature. For example, in Refs. [42,44], it was shown that if one of the two atoms is placed extremely close to a reflecting plate ($|\gamma| = 1$), the entanglement will be completely degraded by dissipation.

Case II: $\lambda_1 = \lambda_2 = 1, \gamma = 0, \eta_1 = \eta_2 \neq 1$, and $\theta_1 = \theta_2$. In this case, $\xi_1 = 0$ and $\zeta_1 = \frac{1}{2}(1 - \eta^{-1}) > 0$, where $\eta \equiv \eta_1 = \eta_2$ and the nonzero density matrix elements of the asymptotic state can be obtained as

$$\begin{aligned} \rho_{ee}(\infty) &= \frac{(\eta - 1)^2[1 - \rho_{aa}(0)]}{1 + 3\eta^2}, \\ \rho_{ss}(\infty) &= \frac{(\eta^2 - 1)[1 - \rho_{aa}(0)]}{1 + 3\eta^2}, \\ \rho_{aa}(\infty) &= \rho_{aa}(0), \\ \rho_{gg}(\infty) &= \frac{(\eta + 1)^2[1 - \rho_{aa}(0)]}{1 + 3\eta^2}. \end{aligned} \tag{50}$$

For the expression of the asymptotic state in the uncoupled basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$, see Appendix B. Substituting Eq. (50) into Eq. (45), we find that

$$C(\infty) = \frac{2(3\eta^2 - 1)}{3\eta^2 + 1}\rho_{aa}(0) - \frac{3(\eta^2 - 1)}{3\eta^2 + 1} > 0 \tag{51}$$

if $\rho_{aa}(0) > \frac{3(\eta^2-1)}{2(3\eta^2-1)}$. Examples of this case include two atoms in certain initial states with a vanishing interatomic separation immersed in a thermal bath, which were shown (in Ref. [31]) to be able to obtain steady-state entanglement, and the two-qubit system prepared in $|10\rangle$ or $|01\rangle$ in a one-dimensional plasmonic waveguide, which was found (in Ref. [32]) to satisfy these conditions so that steady-state entanglement is obtained.

Case III: $\lambda_1 = 1$, $|\gamma| \neq 1$, and $\eta_1 = \eta_2 = 1$. In this case, $\xi_1 = 0$ and $\zeta_1 = 0$. Then the nonzero density matrix elements of the asymptotic state are

$$\begin{aligned} \rho_{aa}(\infty) &= \rho_{ee}(0)\gamma^2 + \rho_{aa}(0), \\ \rho_{gg}(\infty) &= 1 - \rho_{ee}(0)\gamma^2 - \rho_{aa}(0), \\ \rho_{ga}(\infty) &= \rho_{ag}^*(\infty) = \rho_{ga}(0) + \gamma\rho_{se}(0). \end{aligned} \tag{52}$$

For the expression of the asymptotic state in the uncoupled basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$, see Appendix B. Substituting Eq. (52) into Eq. (45), we obtain

$$C(\infty) = \sqrt{1 - \gamma^2}[\rho_{aa}(0) + \gamma^2\rho_{ee}(0)]. \tag{53}$$

This indicates that, as long as the initial state has components of $|a\rangle$ or $|e\rangle$, entanglement can be created and maintained in the asymptotic regime. An example of this case is when the environment can be modeled as a single-mode field (i.e., the Tavis-Cummings model) [45,46] in vacuum, which has been extensively studied [15,18,19,47]. For clarity, we summarize the necessary and sufficient condition for steady-state entanglement depending on the initial state in Table I.

B. Steady-state entanglement independent of the initial state

When $\Gamma \neq 0$, $\xi_1 \neq 0$, and $\zeta_1 \neq 0$, the asymptotic state $\rho(\infty)$ is independent of the initial state. See Appendix B for the explicit expressions in the uncoupled basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$. The concurrence coefficient $K(\infty)$ can be expressed as

$$K(\infty) = \frac{\sqrt{1 - \gamma^2}\sqrt{\bar{\eta}^2(\lambda_1 n_a - \lambda_2 n_g)^2 + \Theta} - \sqrt{\Phi_-\Phi_+}}{\mathcal{Z}}, \tag{54}$$

where $\bar{\eta}$ has been defined in Eq. (39), $n_a = (\eta_1 + \eta_2 - 2)/2$, $n_g = \sqrt{(\eta_1 - 1)(\eta_2 - 1)}$, and

$$\begin{aligned} \Phi_\pm = {} & \frac{1}{2}[\bar{\eta}^2 - (1 - \gamma^2)(\lambda_2 n_g + \lambda_1)^2](\eta_1 \pm 1)(\eta_2 \pm 1) + \frac{1}{2}[(\lambda_2 n_g + \lambda_1 - \bar{\eta}\lambda_1)^2 - \gamma^2(\lambda_2 n_g + \lambda_1 \pm \lambda_1)^2] \\ & + 2\lambda_1\lambda_2 n_g \sin^2\left(\frac{\Delta\theta}{2}\right)\left\{\bar{\eta} \pm 1 + (1 - \gamma^2)\left[\eta_1\eta_2 \pm (2n_a + 1) - \lambda_1\lambda_2 n_g \cos^2\left(\frac{\Delta\theta}{2}\right)\right]\right\}, \end{aligned} \tag{55}$$

$$\begin{aligned} \Theta = {} & 4\lambda_1\lambda_2 n_g \sin^2\left(\frac{\Delta\theta}{2}\right)\left\{\bar{\eta}^2 n_a + \lambda_1\lambda_2 n_g \cos^2\left(\frac{\Delta\theta}{2}\right)\left[[\lambda_1(n_a + 1 - n_a\gamma^2) - \lambda_2 n_g(n_a + 1)(1 - \gamma^2)]^2\right.\right. \\ & \left.\left. - 2\bar{\eta}^2\gamma^2 - 2\bar{\eta}^2(1 - \gamma^2)(n_a + 1)^2 + 4\lambda_1\lambda_2 n_g(n_a + 1)(1 - \gamma^2)(n_a + 1 - n_a\gamma^2)\cos^2\left(\frac{\Delta\theta}{2}\right)\right]\Big/\bar{\eta}^2\right\}, \end{aligned} \tag{56}$$

$$\mathcal{Z} = (1 - \gamma^2)\lambda_1[\lambda_1 n_a + \lambda_2 n_g(n_a + 1)\cos(\Delta\theta)] + \lambda_1^2 - \bar{\eta}^2 + \Phi_- + \Phi_+. \tag{57}$$

Here, $\Delta\theta = \theta_1 - \theta_2$. Thus, as long as $K(\infty) > 0$, the steady state is entangled, regardless of whether the initial state is entangled or separable. In the following, we discuss the conditions for steady-state entanglement independent of the initial state.

TABLE I. Necessary and sufficient conditions for steady-state entanglement related to the initial state. No entry denotes no constraints.

Case	Conditions Γ	λ_q	η_q	θ_q	γ	Initial state	Steady-state concurrence
1	$\Gamma = 0$					$C(0) > 0$	$C(0)$
2	$\Gamma \neq 0$	$\lambda_1 = \lambda_2 = 1$	$\eta_1 = \eta_2 \equiv \eta$	$\theta_1 = \theta_2$	$\gamma = 0$	$\rho_{aa}(0) > \frac{3(\eta^2-1)}{2(3\eta^2-1)}$	$\frac{2(3\eta^2-1)}{3\eta^2+1}\rho_{aa}(0) - \frac{3(\eta^2-1)}{3\eta^2+1}$
3	$\Gamma \neq 0$	$\lambda_1 = 1$	$\eta_1 = \eta_2 = 1$		$\lvert\gamma\rvert \neq 0, 1$	$\rho_{aa}(0) + \rho_{ee}(0) \neq 0$	$\sqrt{1-\gamma^2[\rho_{aa}(0)+\gamma^2\rho_{ee}(0)]}$

First of all, we can give a necessary condition for steady-state entanglement, that is,

$$\{\eta_1 \neq \eta_2 \parallel \lambda_1 \neq \lambda_2 \parallel \theta_1 \neq \theta_2\} \\ \& \{\eta_1 \neq 1 \parallel \eta_2 \neq 1\} \& \{\lambda_1 \neq 0\}. \tag{58}$$

See Appendix C for the proof. An example that satisfies the necessary condition (58) is when a two-qubit system in an arbitrary initial state is placed in a common stationary environment out of thermal equilibrium composed of a dielectric body whose temperature is different from its surroundings [35,36]. It is shown that the absence of equilibrium leads to the generation of steady-state entanglement, which is inaccessible when the environment is in thermal equilibrium.

Since the expression $K(\infty)$ [Eq. (54)] is very complicated, here we analyze a relatively simple case, i.e., $\{\eta_1 = \eta_2 \equiv \eta \neq 1\}$ & $\{\gamma = 0\}$, which means that the individual upward (nonzero) and downward transition rates of the two subsystems are the same. According to Eq. (54), we can obtain the necessary and sufficient condition for $K(\infty) > 0$ as

$$\lambda_1 > \lambda_c = \frac{\sqrt{\eta^2(\eta^2\lambda_2^2+4\eta+4)-Q}+(4-\eta^2-2\eta)\lambda_2\cos(\Delta\theta)}{4+(\eta-1)\lambda_2^2\sin^2(\Delta\theta)}, \tag{59}$$

where

$$Q = \lambda_2^2\{(\eta^2-4)[(\eta-1)^2\lambda_2^2+4\eta-3]+4\}\sin^2(\Delta\theta).$$

Moreover, some concise constraints on the parameters $\lambda_{1,2}$, η, and $\Delta\theta$ can be derived from the condition (59) together with $\eta \geqslant 1$ and $0 \leqslant \lambda_{1,2} \leqslant 1$. First, it is easy to prove that $\partial\lambda_c/\partial\eta \geqslant 0$. Then a necessary condition for steady-state entanglement is $\lambda_1 > \lambda_c|_{\eta=1}$, which can be equivalently written as

$$\lambda_1[2\lambda_1 - \lambda_2\cos(\Delta\theta)] > 1. \tag{60}$$

Second, it can be proved that λ_c is always larger than or equal to 1 when $\eta \geqslant \sqrt{2}$, so $\lambda_1 > \lambda_c$ cannot be satisfied since $0 \leqslant \lambda_1 \leqslant 1$. As a result,

$$\eta < \sqrt{2} \tag{61}$$

is also a necessary condition for steady-state entanglement. Third, it can be proved that λ_c reaches its minimum $\frac{1}{2}$ when $\eta = 1$, $\lambda_2 = 1$, and $\Delta\theta = \pm\pi$, which indicates that another necessary condition for steady-state entanglement is

$$\lambda_1 > \tfrac{1}{2}. \tag{62}$$

The conditions (61) and (62) suggest that a large enough λ_1 and a small enough η (but larger than 1) is the key to obtaining steady-state entanglement. The relevant proofs of the conditions (60)–(62) are given in Appendix D. These conditions (59)–(62) are helpful in judging whether steady-state entanglement exists in specific scenarios. For example, the entanglement dynamics of a pair of atoms rotating in coaxial orbits with the same radius and with their separation perpendicular to the rotating plane was studied in Refs. [48,49], in which numerical calculations were done systematically in the ultrarelativistic limit ($v/c \to 1$), and the phenomenon of steady-state entanglement was not found. However, it can actually be predicted directly with the necessary conditions (60)–(62) that steady-state entanglement is impossible in the ultrarelativistic limit and high angular velocity limit (i.e., the angular velocity Ω is much greater than the energy-level spacing of the atoms ω), as we will show elsewhere [50]. For more general cases, we can investigate numerically the steady-state concurrence coefficient $K(\infty)$ in Eq. (54). In Fig. 1 we plot the contour maps of the steady-state entanglement $C(\infty)$ as a function of λ_1 and an another parameter (γ, $\eta_{1,2}$, $\Delta\theta$, or λ_2) when the rest of the parameters are fixed. Numerical calculations, as shown in Fig. 1, suggest that $C(\infty)$ is always a nondecreasing function of λ_1. We conjecture that this proposition is true, although we do not have a rigorous proof for it at the moment. Then the necessary and sufficient condition for steady-state entanglement independent of the initial state can

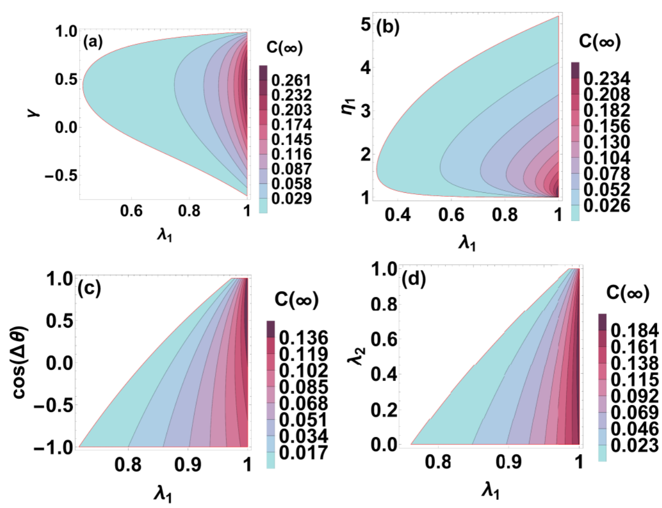

FIG. 1. Contour maps of the steady-state entanglement $C(\infty)$ as a function of (a) λ_1 and γ (with $\eta_1 = 1.1$, $\eta_2 = 1.005$, $\lambda_2 = 0.5$, and $\Delta\theta = \pi$), (b) λ_1 and η_1 (with $\gamma = 0.5$, $\eta_2 = 1.005$, $\lambda_2 = 0.5$, and $\Delta\theta = \pi$), (c) λ_1 and $\cos(\Delta\theta)$ (with $\gamma = -0.1$, $\eta_1 = 1.12$, $\eta_2 = 1.08$, and $\lambda_2 = 0.9$), and (d) λ_1 and λ_2 (with $\gamma = 0.1$, $\eta_1 = 1.04$, $\eta_2 = 1.08$, and $\Delta\theta = 0$).

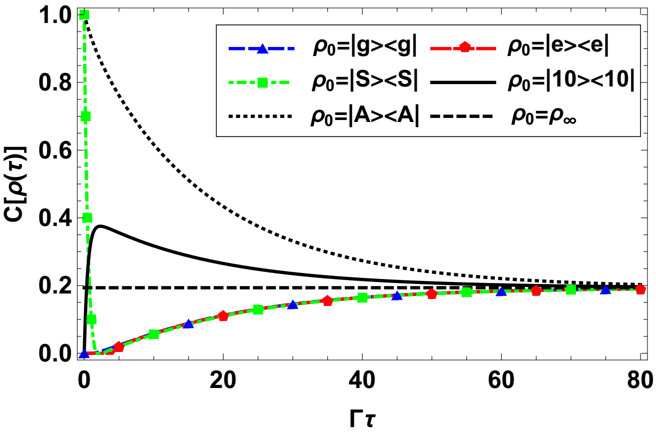

FIG. 2. Time evolution of concurrence C as a function of $\Gamma\tau$ when $\lambda_1 = 0.99$, $\lambda_2 = 0.98$, $\gamma = 0$, $\eta_1 = 1.01$, $\eta_2 = 1.02$, and $\Delta\theta = \pi$. Here $|S\rangle = (|10\rangle + |01\rangle)/\sqrt{2}$ and $|A\rangle = (|10\rangle - |01\rangle)/\sqrt{2}$.

be formally written as

$$\{\lambda_1 > \lambda_c\} \ \&\ \{\eta_1 \neq \eta_2 \parallel \lambda_1 \neq \lambda_2 \parallel \theta_1 \neq \theta_2\} \\ \&\ \{\eta_1 \neq 1 \parallel \eta_2 \neq 1\} \ \&\ \{\lambda_1 \neq 0\}, \tag{63}$$

where λ_c is a function of η_1, η_2, γ, λ_2, and $\Delta\theta$. Therefore, a large enough λ_1 is the key to obtaining steady-state entanglement.

VI. DISCUSSION

It should be pointed out that, in the present work, we have neglected the environment-induced energy shift, which is related to the effective Hamiltonian in the GKLS master equation (5). Here we would like to note that such a term will affect the entanglement dynamics but will not affect the asymptotic state [51]. Moreover, the interaction Hamiltonian does not include terms leading to pure dephasing. Nevertheless, in principle, all these factors can be included and our approach can be extended to more general cases if more physical parameters are introduced. This is left for future work.

An important question is whether the steady-state entanglement obtained from dissipative dynamics is useful in practice. As has been shown by Horodecki *et al.*, any two-qubit state is distillable if and only if it is entangled [1,13]. In fact, the efficiency of distillation from the steady entangled state obtained from dissipation into the maximal entangled states is an interesting question.

Another question of interest is how much time it takes to establish steady-state entanglement. This is related to the evolution rate Γ_{ev}. Actually, from the general solution (44), the evolution rate Γ_{ev} can be estimated as

$$\Gamma_{\text{ev}} = \chi\bar{\eta}\Gamma, \tag{64}$$

where χ is the minimum of the nonzero eigenvalues of ξ_i and ζ_i. Then the evolution time t_{ev}, which characterizes the time to establish steady-state entanglement, can be estimated as $t_{\text{ev}} = \Gamma_{\text{ev}}^{-1}$. In Fig. 2 we numerically study the evolution of concurrence C as a function of $\Gamma\tau$. It is shown that, in this specific case, the quantum system is driven to the steady entangled state at about $\tau \sim 80\Gamma^{-1}$.

VII. SUMMARY

We have studied the dynamics of an open quantum system composed of two identical two-level subsystems in a common stationary environment undergoing Markovian dissipation. With the help of a set of physical parameters defined with the collective transition coefficients of the system, we systematically investigated the steady-state entanglement that a quantum system composed of two identical two-level subsystems can obtain from purely dissipative dynamics, which can be classified into two categories, i.e., steady-state entanglement depending on and independent of the initial state of the quantum system. We demonstrated that a variety of works concerning entanglement dynamics reported in the literature could be viewed as specific examples considered here and we expect more in the future.

ACKNOWLEDGMENTS

This work was supported in part by the NSFC under Grants No. 11690034, No. 11805063, and No. 12075084 and the Hunan Provincial Natural Science Foundation of China under Grant No. 2020JJ3026.

APPENDIX A: PROOF OF THE NECESSARY AND SUFFICIENT CONDITIONS FOR $\xi_1 = 0$ AND $\zeta_1 = 0$

There are two definite eigenvalues for $\mathbf{U}_1/\bar{\eta}\Gamma$, which are labeled as $\xi_0 = 0$ and $\xi_5 = 1$. The remaining four eigenvalues for $\mathbf{U}_1/\bar{\eta}\Gamma$ and all four eigenvalues for $\mathbf{U}_2/\bar{\eta}\Gamma$ satisfy the characteristic root equations

$$\xi^4 - 5\xi^3 + G_2\xi^2 - G_1\xi + G_0 = 0, \tag{A1}$$

$$\zeta^4 - 4\zeta^3 + H_2\zeta^2 - H_1\zeta + H_0 = 0, \tag{A2}$$

respectively, and are labeled as $\xi_1, \xi_2, \xi_3, \xi_4$ and $\zeta_1, \zeta_2, \zeta_3, \zeta_4$, respectively, in increasing order with respect to their real parts. Here G_0, G_1, and G_2 (H_0, H_1, and H_2) are the coefficients of the zero, first, and second power terms of ξ (ζ), respectively. Here

G_0 and H_0 can be written as

$$G_0 = \frac{1-\gamma^2}{2\bar{\eta}^4}\{4(\eta_1\eta_2+\lambda_1^2)\bar{\eta}^2 + 4\eta_1\eta_2(1-\gamma^2)\lambda_1^2 - 4(\eta_1-1)(\eta_2-1)(1-\gamma^2)(\eta_1\eta_2+\lambda_1^2\sin^2\Delta\theta)\lambda_2^2 - 2[4\bar{\eta}+(1-\gamma^2)(4\eta_1\eta_2-\eta_1-\eta_2)][\lambda_1^2+\lambda_1\lambda_2\sqrt{(\eta_1-1)(\eta_2-1)}\cos\Delta\theta]\}, \tag{A3}$$

$$H_0 = \frac{1-\gamma^2}{16\bar{\eta}^4}\{8\eta_1\eta_2(\bar{\eta}^2-\lambda_1^2) + 8\bar{\eta}[(\eta_1+\eta_2-2)\lambda_1^2 - 2\sqrt{(\eta_1-1)(\eta_2-1)}\lambda_1\lambda_2\cos\Delta\theta] - (1-\gamma^2)[2(\eta_1^2+\eta_2^2)\lambda_1^2 - (\eta_1\eta_2+\lambda_1^2)^2 + 4(2\eta_1\eta_2-\eta_1-\eta_2)\sqrt{(\eta_1-1)(\eta_2-1)}\lambda_1\lambda_2\cos\Delta\theta + 4(\eta_1-1)(\eta_2-1)(\eta_1\eta_2\lambda_2^2+\lambda_1^2)] + 4i[(1+\gamma)^2\eta_2-(1-\gamma)^2\eta_1]\sqrt{(\eta_1-1)(\eta_2-1)}\lambda_1\lambda_2\sin\Delta\theta\}, \tag{A4}$$

with $\bar{\eta} = [\eta_1+\eta_2+(\eta_2-\eta_1)\gamma]/2$ and i the imaginary unit. Also, the coefficients G_1 and H_1 can be expressed as

$$G_1 = 4 + \frac{1-\gamma^2}{\bar{\eta}^3}[3\eta_1\eta_2\bar{\eta} - (\bar{\eta}+2)\lambda_1^2 - 4(\eta_1-1)(\eta_2-1)\bar{\eta}\lambda_2^2 - 2(4\bar{\eta}+1)\lambda_1\lambda_2\sqrt{(\eta_1-1)(\eta_2-1)}\cos\Delta\theta], \tag{A5}$$

$$H_1 = 2 + \frac{1-\gamma^2}{\bar{\eta}^3}\{\eta_1\eta_2\bar{\eta} - \lambda_1^2 - (\eta_1-1)(\eta_2-1)\bar{\eta}\lambda_2^2 - [(2\bar{\eta}+1)\cos\Delta\theta - i\gamma\sin\Delta\theta]\lambda_1\lambda_2\sqrt{(\eta_1-1)(\eta_2-1)}\}. \tag{A6}$$

According to the eigenvalue equations of the two coefficient matrices $\mathbf{U}_1/\bar{\eta}\Gamma$ and $\mathbf{U}_2/\bar{\eta}\Gamma$ (A1), we obtain the following two conclusions.

Conclusion 1. There is at most one zero root in ξ_i $(i = 1, 2, 3, 4)$, i.e., only ξ_1 can be equal to zero, and the necessary and sufficient condition for $\xi_1 = 0$ can be written as

$$\{|\gamma| = 1\} \parallel \{\lambda_1 = \lambda_2 = 1 \;\&\; \gamma = 0 \;\&\; \eta_1 = \eta_2 \neq 1 \;\&\; \theta_1 = \theta_2\} \parallel \{\lambda_1 = 1 \;\&\; |\gamma| \neq 1 \;\&\; \eta_1 = \eta_2 = 1\}. \tag{A7}$$

Conclusion 2. There is at most one zero root in ζ_j $(j = 1, 2, 3, 4)$, i.e., only ζ_1 can be equal to zero, and the necessary and sufficient condition for $\zeta_1 = 0$ can be written as

$$\{|\gamma| = 1\} \parallel \{\lambda_1 = 1 \;\&\; |\gamma| \neq 1 \;\&\; \eta_1 = \eta_2 = 1\}. \tag{A8}$$

1. Proof of conclusion 1

In Eq. (A1), according to Vieta's theorem, the necessary and sufficient condition that ξ_i $(i = 1, 2, 3, 4)$ has zero root is $G_0 = 0$. However, G_0 can be written in the non-negative form

$$G_0 = \frac{1-\gamma^2}{2\bar{\eta}^4}\left(A_1(1-\lambda_1) + A_2(1-\lambda_2) + \{A_3[1-\cos(\Delta\theta)] + A_4 + (1-\gamma^2)(\sqrt{A_5+A_6^2} - A_6)\}\lambda_1\lambda_2\right), \tag{A9}$$

where

$$A_1 = (1+\lambda_1)(1-\lambda_2)[(1-\gamma)^2\eta_1 + (1+\gamma)^2\eta_2 + 2(1-\gamma^2)\eta_1\eta_2] + \eta_1\eta_2\lambda_2(1-\lambda_1)\{4(1-\gamma^2)(\eta_1+\eta_2-1) + [(1+\gamma)\eta_2-(1-\gamma)\eta_1]^2\} + \lambda_1\lambda_2\{2(1-\gamma^2)(\eta_1+\eta_2-2) + (\eta_1\eta_2-1)[\eta_2-\eta_1+\gamma(\eta_1+\eta_2-2)]^2 + 4(\eta_1\eta_2+1) + 4\eta_1\eta_2[(1-\gamma)(\eta_1-1)+(1+\gamma)(\eta_2-1)]\} > 0, \tag{A10}$$

$$A_2 = (1-\gamma)^2\eta_1[\eta_1^2\eta_2 - 1 + (\eta_1-1)\lambda_1^2] + (1+\gamma)^2\eta_2[\eta_1\eta_2^2 - 1 + (\eta_2-1)\lambda_1^2] + 2(1-\gamma^2)\eta_1\eta_2[2(\eta_1-1)(\eta_2-1)\lambda_2 + \eta_1\eta_2 - 1] \geqslant 0, \tag{A11}$$

$$A_3 = 2\sqrt{(\eta_1-1)(\eta_2-1)}\left((1-\gamma)^2\eta_1 + (1+\gamma)^2\eta_2 + 2(1-\gamma^2)\left\{\eta_1\eta_2 + 1 + (\eta_1-1)(\eta_2-1) + (\sqrt{\eta_2-1}-\sqrt{\eta_1-1})^2 + 2\sqrt{(\eta_1-1)(\eta_2-1)}\left[1-\lambda_1\lambda_2\cos^2\left(\frac{\Delta\theta}{2}\right)\right]\right\}\right) \geqslant 0, \tag{A12}$$

$$A_4 = \eta_1\eta_2\left[(1+\gamma)\sqrt{\eta_2^2 - 1 + \frac{(\sqrt{\eta_2-1}-\sqrt{\eta_1-1})^2}{\eta_1}} - (1-\gamma)\sqrt{\eta_1^2 - 1 + \frac{(\sqrt{\eta_2-1}-\sqrt{\eta_1-1})^2}{\eta_2}}\right]^2 \geqslant 0, \tag{A13}$$

$$A_5 = 8\eta_1\eta_2(\sqrt{\eta_2-1}-\sqrt{\eta_1-1})^2(1+\sqrt{\eta_1-1}\sqrt{\eta_2-1})(\eta_1\eta_2 - 1 + 2\eta_1\eta_2\sqrt{\eta_1-1}\sqrt{\eta_2-1}) \geqslant 0, \tag{A14}$$

$$A_6 = 2\eta_1\eta_2[(\sqrt{\eta_1-1}\sqrt{\eta_2-1}+2)^2 - (\eta_1+\eta_2+2)]. \tag{A15}$$

Since $\eta_{1,2} \geqslant 1$, $0 \leqslant \lambda_{1,2} \leqslant 1$, and $|\gamma| \leqslant 1$, it is obvious that $A_1 > 0$, $A_{2,3,4,5} \geqslant 0$, and

$$A_2 = 0 \Leftrightarrow \eta_1 = \eta_2 = 1, \tag{A16}$$

$$A_3 = 0 \Leftrightarrow \{\eta_1 = 1\} \parallel \{\eta_2 = 1\}, \tag{A17}$$

$$A_4 + (1-\gamma^2)\big(\sqrt{A_5 + A_6^2} - A_6\big) = 0 \Leftrightarrow \{\eta_1 = \eta_2 = 1\} \parallel \{\eta_1 = \eta_2 \neq 1 \;\&\; \gamma = 0\}. \tag{A18}$$

Furthermore, applying the conclusions (A16)–(A18) to Eq. (A9), it is easy to obtain that $G_0 \geqslant 0$, and the necessary and sufficient condition of $G_0 = 0$ can be written as

$$\{|\gamma| = 1\} \parallel \{\lambda_1 = \lambda_2 = 1 \;\&\; \gamma = 0 \;\&\; \eta_1 = \eta_2 \neq 1 \;\&\; \theta_1 = \theta_2\} \parallel \{\lambda_1 = 1 \;\&\; |\gamma| \neq 1 \;\&\; \eta_1 = \eta_2 = 1\}. \tag{A19}$$

Now we prove that there is at most one zero root in ξ_i $(i = 1, 2, 3, 4)$, that is, that $G_1 \neq 0$ when $G_0 = 0$. According to Eq. (A19), there are three cases when $G_0 = 0$.

Case 1. If $|\gamma| = 1$, a direct calculation of Eq. (A5) shows that $G_1 = 4 > 0$.

Case 2. If $\lambda_1 = \lambda_2 = 1$, $\gamma = 0$, $\eta_1 = \eta_2 \equiv \eta$, and $\theta_1 = \theta_2$, then it is found that $G_1 = 3 + \eta^{-2} > 0$.

Case 3. If $\lambda_1 = 1$, $|\gamma| \neq 1$, and $\eta_1 = \eta_2 = 1$, then $G_1 = 4 > 0$.

Thus, $G_1 > 0$ when $G_0 = 0$. Q.E.D.

2. Proof of conclusion 2

Similarly, according to Vieta's theorem, the necessary and sufficient condition for ζ_j to have a zero root is $H_0 = 0$. Also, H_0 can be written in the form

$$H_0 = \frac{1-\gamma^2}{16\bar{\eta}^4}[B_1(1-\gamma)^2 + B_2(1+\gamma)^2 + B_3(1-\gamma^2) + iB_4], \tag{A20}$$

where

$$\begin{aligned} B_1 = 2\eta_1\big\{&2\sqrt{(\eta_1-1)(\eta_2-1)}[1-\lambda_1\lambda_2\cos(\Delta\theta)] + (\sqrt{\eta_2-1} - \sqrt{\eta_1-1})^2 \\ &+ (\eta_1-1)(\eta_1\eta_2+\eta_2-1) + (\eta_1-1)\lambda_1^2 + \big(1-\lambda_1^2\big)\big\} \geqslant 0, \end{aligned} \tag{A21}$$

$$\begin{aligned} B_2 = 2\eta_2\big\{&2\sqrt{(\eta_1-1)(\eta_2-1)}[1-\lambda_1\lambda_2\cos(\Delta\theta)] + (\sqrt{\eta_2-1} - \sqrt{\eta_1-1})^2 \\ &+ (\eta_2-1)(\eta_1\eta_2+\eta_1-1) + (\eta_2-1)\lambda_1^2 + \big(1-\lambda_1^2\big)\big\} \geqslant 0, \end{aligned} \tag{A22}$$

$$\begin{aligned} B_3 = 4\eta_1\eta_2\big\{&2\sqrt{(\eta_1-1)(\eta_2-1)}[1-\lambda_1\lambda_2\cos(\Delta\theta)] + (\sqrt{\eta_2-1} - \sqrt{\eta_1-1})^2 + (\eta_1-1)(\eta_2-1)\big(1-\lambda_2^2\big)\big\} \\ &+ 2\big(1-\lambda_1^2\big)(\eta_1\eta_2+1) + (\eta_1\eta_2-1)(\eta_1\eta_2+3) + \big(1-\lambda_1^2\big)^2 \geqslant 0, \end{aligned} \tag{A23}$$

$$B_4 = 4\lambda_1\lambda_2\sqrt{(\eta_1-1)(\eta_2-1)}[(1+\gamma)^2\eta_2 - (1-\gamma)^2\eta_1]\sin(\Delta\theta). \tag{A24}$$

With the help of $\eta_{1,2} \geqslant 1$, $0 \leqslant \lambda_{1,2} \leqslant 1$, and $|\gamma| \leqslant 1$, it is obvious that $B_{1,2,3} \geqslant 0$ and

$$B_{1,2,3} = 0 \Leftrightarrow \lambda_1 = 1 \;\&\; \eta_1 = \eta_2 = 1. \tag{A25}$$

Moreover, it is found that $B_4 = 0$ when $\lambda_1 = 1$ and $\eta_1 = \eta_2 = 1$. Therefore, according to the conclusions above, it is easy to obtain that the necessary and sufficient condition for $H_0 = 0$ can be written as

$$\{|\gamma| = 1\} \parallel \{\lambda_1 = 1 \;\&\; |\gamma| \neq 1 \;\&\; \eta_1 = \eta_2 = 1\}, \tag{A26}$$

which is also the necessary and sufficient condition for ζ_j to have a zero root.

Furthermore, from Eq. (A6), we find that $H_1 = 2 > 0$ when $H_0 = 0$. Thus, there is at most one zero root in ζ_i $(i = 1, 2, 3, 4)$. Q.E.D.

APPENDIX B: EXPRESSIONS OF THE ENTANGLED STEADY STATE IN THE UNCOUPLED BASIS $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$

For convenience, we give the expressions of the entangled steady state obtained in this paper in the uncoupled basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$.

1. Entangled steady state corresponding to Eq. (50)

First, when $\lambda_1 = \lambda_2 = 1$, $\eta_1 = \eta_2 \equiv \eta \neq 1$, $\theta_1 = \theta_2$, $\gamma = 0$, and the corresponding initial condition is satisfied, we obtain an entangled steady state related to the initial state, which has been shown in Eq. (50) in the coupled basis. In the uncoupled basis

$\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$ it takes the form

$$\rho(\infty) = \begin{pmatrix} \rho_{11}(\infty) & 0 & 0 & 0 \\ 0 & \rho_{22}(\infty) & \rho_{23}(\infty) & 0 \\ 0 & \rho_{32}(\infty) & \rho_{33}(\infty) & 0 \\ 0 & 0 & 0 & \rho_{44}(\infty) \end{pmatrix}. \tag{B1}$$

Then the nonzero density matrix elements in Eq. (B1) can be written as

$$\begin{aligned}
\rho_{11}(\infty) &= \frac{(\eta-1)^2}{3\eta^2+1}\left\{1 - \frac{\rho_{22}(0)+\rho_{33}(0)-[e^{-i\theta_1}\rho_{23}(0)+e^{i\theta_1}\rho_{32}(0)]}{2}\right\},\\
\rho_{22}(\infty) &= \rho_{33}(\infty) = \frac{\eta^2+1}{3\eta^2+1}\left\{1 + \frac{\rho_{22}(0)+\rho_{33}(0)-[e^{-i\theta_1}\rho_{23}(0)+e^{i\theta_1}\rho_{32}(0)]}{2}\right\} - \frac{\eta^2+3}{2(3\eta^2+1)},\\
\rho_{44}(\infty) &= \frac{(\eta+1)^2}{3\eta^2+1}\left\{1 - \frac{\rho_{22}(0)+\rho_{33}(0)-[e^{-i\theta_1}\rho_{23}(0)+e^{i\theta_1}\rho_{32}(0)]}{2}\right\},\\
\rho_{23}(\infty) &= \rho_{32}(\infty)^* = \frac{\eta^2[\rho_{23}(0)+e^{2i\theta_1}\rho_{32}(0)]}{3\eta^2+1} + e^{i\theta_1}\left\{\frac{\eta^2-1}{2(3\eta^2+1)} - \frac{\eta^2[\rho_{22}(0)+\rho_{33}(0)]}{3\eta^2+1}\right\}.
\end{aligned} \tag{B2}$$

2. Entangled steady state corresponding to Eq. (52)

Second, when $\lambda_1 = 1$, $\eta_1 = \eta_2 = 1$, $|\gamma| \neq 0, 1$, and the corresponding initial condition is satisfied, we also obtain an entangled steady state which has been shown in Eq. (52) in the coupled basis. In the uncoupled basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$, this state takes the form

$$\rho(\infty) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22}(\infty) & \rho_{23}(\infty) & \rho_{24}(\infty) \\ 0 & \rho_{32}(\infty) & \rho_{33}(\infty) & \rho_{34}(\infty) \\ 0 & \rho_{42}(\infty) & \rho_{43}(\infty) & \rho_{44}(\infty) \end{pmatrix}, \tag{B3}$$

where

$$\begin{aligned}
\rho_{22}(\infty) &= \frac{1+\gamma}{2}\left\{\gamma^2\rho_{11}(0) + \frac{(1+\gamma)\rho_{22}(0)+(1-\gamma)\rho_{33}(0)-\sqrt{1-\gamma^2}[e^{-i\theta_1}\rho_{23}(0)+e^{i\theta_1}\rho_{32}(0)]}{2}\right\},\\
\rho_{33}(\infty) &= \frac{1-\gamma}{2}\left\{\gamma^2\rho_{11}(0) + \frac{(1+\gamma)\rho_{22}(0)+(1-\gamma)\rho_{33}(0)-\sqrt{1-\gamma^2}[e^{-i\theta_1}\rho_{23}(0)+e^{i\theta_1}\rho_{32}(0)]}{2}\right\},\\
\rho_{44}(\infty) &= 1-[\rho_{22}(\infty)+\rho_{33}(\infty)], \quad \rho_{23}(\infty) = \rho_{32}(\infty)^* = -\frac{\sqrt{1-\gamma^2}}{2}e^{i\theta_1}[\rho_{22}(\infty)+\rho_{33}(\infty)],\\
\rho_{24}(\infty) &= \rho_{42}(\infty)^* = \frac{(1+\gamma)[\rho_{24}(0)+\gamma\rho_{12}(0)]+\sqrt{1-\gamma^2}[e^{i\theta_1}\rho_{34}(0)-\gamma\rho_{13}(0)]}{2},\\
\rho_{34}(\infty) &= \rho_{43}(\infty)^* = \frac{(1-\gamma)[e^{i\theta_1}\rho_{34}(0)+\gamma\rho_{13}(0)]-\sqrt{1-\gamma^2}[\rho_{24}(0)+\gamma\rho_{12}(0)]}{2}e^{-i\theta_1}.
\end{aligned} \tag{B4}$$

3. Entangled steady state independent of the initial state

Third, when $\Gamma \neq 0$, $\xi_1 \neq 0$, $\zeta_1 \neq 0$, and the corresponding condition (63) is satisfied, we obtain an entangled steady state independent of the initial state, which takes the same form as Eq. (B1) in the uncoupled basis $\{|11\rangle, |10\rangle, |01\rangle, |00\rangle\}$, and the nonzero density matrix elements are

$$\begin{aligned}
&\rho_{11}(\infty) = \frac{\Phi_-}{2\mathcal{Z}}, \quad \rho_{44}(\infty) = \frac{\Phi_+}{2\mathcal{Z}}, \quad \rho_{22}(\infty) = \frac{\bar{\eta}(2\mathcal{Z}-\Phi_+-\Phi_-)+W}{4\bar{\eta}\mathcal{Z}},\\
&\rho_{23}(\infty) = [\rho_{32}(\infty)]^* = -\frac{\sqrt{1-\gamma^2}(P_1+iP_2)}{4\bar{\eta}\mathcal{Z}}, \quad \rho_{33}(\infty) = \frac{\bar{\eta}(2\mathcal{Z}-\Phi_+-\Phi_-)-W}{4\bar{\eta}\mathcal{Z}},
\end{aligned} \tag{B5}$$

where $\Phi_\pm$ and $\mathcal{Z}$ have been defined in Eqs. (55) and (57) respectively, and $P_{1,2}$ and W can be expressed as

$$P_1 = 2\bar{\eta}^2(n_a\lambda_1\cos\theta_1 - n_g\lambda_2\cos\theta_2) + 2\lambda_1\lambda_2 n_g\sin(\Delta\theta)\{[1+n_a(1-\gamma^2)]\lambda_1\sin\theta_1 + n_g(n_a+1)(1-\gamma^2)\lambda_2\sin\theta_2\}, \tag{B6}$$

$$\begin{aligned} P_2 &= 2\bar{\eta}^2(n_a\lambda_1\sin\theta_1 - n_g\lambda_2\sin\theta_2) - 2\lambda_1\lambda_2 n_g\sin(\Delta\theta) \\ &\quad\times\{[1+n_a(1-\gamma^2)]\lambda_1\cos\theta_1 + n_g(n_a+1)(1-\gamma^2)\lambda_2\cos\theta_2\}, \end{aligned} \tag{B7}$$

$$\begin{aligned} W &= 2\gamma\lambda_1\bar{\eta}(\lambda_1 n_a - \lambda_2 n_g) + (\eta_1-\eta_2)\bar{\eta}\big[\bar{\eta}^2 - \lambda_1^2 - (1-\gamma^2)\big(2\lambda_1\lambda_2 n_g + \lambda_2^2 n_g^2\big)\big] \\ &\quad + 4n_g\lambda_1\lambda_2\sin^2\left(\frac{\Delta\theta}{2}\right)\left\{\gamma\bar{\eta} - n_g(1-\gamma^2)\left[(\eta_2-\eta_1)\bar{\eta} + 2\gamma\lambda_1\lambda_2\cos^2\left(\frac{\Delta\theta}{2}\right)\right]\right\}, \end{aligned} \tag{B8}$$

with $\Delta\theta = \theta_1 - \theta_2$, $n_a = \frac{1}{2}(\eta_1+\eta_2-2)$, $n_g = \sqrt{(\eta_1-1)(\eta_2-1)}$, and $\bar{\eta} = \frac{1}{2}[\eta_1+\eta_2+(\eta_2-\eta_1)\gamma]$.

APPENDIX C: PROOF OF A NECESSARY CONDITION OF THE STEADY-STATE ENTANGLEMENT INDEPENDENT OF THE INITIAL STATE

To prove that a necessary condition for steady-state entanglement independent of the initial state is

$$\{\eta_1 \neq \eta_2 \parallel \lambda_1 \neq \lambda_2 \parallel \theta_1 \neq \theta_2\} \,\&\, \{\eta_1 \neq 1 \parallel \eta_2 \neq 1\} \,\&\, \{\lambda_1 \neq 0\}, \tag{C1}$$

that is, to prove that $K(\infty) \leqslant 0$ [Eq. (54)] when $\{\eta_1 = \eta_2 \,\&\, \lambda_1 = \lambda_2 \,\&\, \theta_1 = \theta_2\} \parallel \{\eta_1 = 1 \,\&\, \eta_2 = 1\} \parallel \{\lambda_1 = 0\}$, we have the following.

(i) When $\eta_1 = \eta_2 \,\&\, \lambda_1 = \lambda_2 \,\&\, \theta_1 = \theta_2$, if the conditions shown in Table I are not satisfied, then direct calculations show that $K(\infty) = \frac{1}{2}(\eta_1^{-2} - 1) \leqslant 0$.

(ii) When $\eta_1 = 1 \,\&\, \eta_2 = 1$, we can directly obtain that $K(\infty) = 0$.

(iii) When $\lambda_1 = 0$, $K(\infty)$ can be written as

$$K(\infty) = \frac{\mathcal{U}_1 - \mathcal{U}_2}{2\eta_1\eta_2\{[\eta_2-\eta_1+\gamma(\eta_1+\eta_2)]^2 + 4(1-\gamma^2)\big[\eta_1\eta_2\big(1-\lambda_2^2\big) + (\eta_1+\eta_2-1)\lambda_2^2\big]\}}, \tag{C2}$$

where

$$\begin{aligned} \mathcal{U}_1 &= 4\bar{\eta}\lambda_2\sqrt{(1-\gamma^2)(\eta_1-1)(\eta_2-1)}, \\ \mathcal{U}_2 &= 4\sqrt{\big(\eta_1^2-1\big)\big(\eta_2^2-1\big)\big[\bar{\eta}^2 - (1-\gamma^2)\mathcal{Q}(\eta_1\eta_2+\eta_1+\eta_2)\lambda_2^2\big]\big[\bar{\eta}^2 - (1-\gamma^2)(\eta_1\eta_2-\eta_1-\eta_2)\lambda_2^2\big]}, \end{aligned} \tag{C3}$$

with $\mathcal{Q} = (\eta_1-1)(\eta_2-1)/(\eta_1+1)(\eta_2+1)$. It is obvious that the denominator in Eq. (C2) is positive. Thus, to prove $K(\infty) \leqslant 0$, we just need to prove $\mathcal{U}_1^2 \leqslant \mathcal{U}_2^2$, which can be written as

$$\mathcal{U}_2^2 - \mathcal{U}_1^2 = \big(\eta_1^2-1\big)\big(\eta_2^2-1\big)\mathcal{V}_+\mathcal{V}_-, \tag{C4}$$

where

$$\mathcal{V}_\pm = 4\bar{\eta}^2 + \frac{4(1-\gamma^2)\lambda_2^2\Big[(\eta_1\eta_2-1) - \big(\eta_1^2-1\big)\big(\eta_2^2-1\big) \pm \sqrt{3\big(\eta_1^2-1\big)\big(\eta_2^2-1\big) + (\eta_1\eta_2-1)^2}\Big]}{(\eta_1+1)(\eta_2+1)}. \tag{C5}$$

It is obvious that $\mathcal{V}_+ > 0$ and $\mathcal{V}_- = 1 > 0$ if $\eta_1 = \eta_2 = 1$. When $\eta_1 \neq 1$ and $\eta_2 \neq 1$, $\mathcal{V}_-$ can be written as

$$\mathcal{V}_- = [(\eta_1-\eta_2)^2 + \mathcal{W}]\left[\gamma + \frac{\eta_2^2-\eta_1^2}{(\eta_1-\eta_2)^2+\mathcal{W}}\right]^2 + \frac{\mathcal{W}(4\eta_1\eta_2 - \mathcal{W})}{(\eta_1-\eta_2)^2+\mathcal{W}}, \tag{C6}$$

where

$$\mathcal{W} = \frac{4\lambda_2^2\Big[\big(\eta_1^2-1\big)\big(\eta_2^2-1\big) + \sqrt{3\big(\eta_1^2-1\big)\big(\eta_2^2-1\big) + (\eta_1\eta_2-1)^2} - (\eta_1\eta_2-1)\Big]}{(\eta_1+1)(\eta_2+1)} \geqslant 0 \tag{C7}$$

and $\mathcal{W} = 0$ if and only if $\eta_1 = \eta_2 = 1$. Now we prove $4\eta_1\eta_2 > \mathcal{W}$, as long as $4\eta_1\eta_2 > \mathcal{W}|_{\lambda_2=1}$. It can be obtained that

$$4\eta_1\eta_2 - \mathcal{W}|_{\lambda_2=1} = \frac{\sqrt{\mathcal{X}^2 + 16(\eta_1+1)(\eta_2+1)[(\eta_1-\eta_2)^2 + (\eta_1+\eta_2)^2(\eta_1\eta_2+\eta_1+\eta_2-2)]} - \mathcal{X}}{(\eta_1+1)(\eta_2+1)}, \tag{C8}$$

where $\mathcal{X} = 4\sqrt{3(\eta_1^2-1)(\eta_2^2-1) + (\eta_1\eta_2-1)^2}$. It is obvious from Eq. (C8) that $4\eta_1\eta_2 - \mathcal{W}|_{\lambda_2=1} > 0$, so $4\eta_1\eta_2 > \mathcal{W}$. Therefore, $\mathcal{V}_- > 0$, $\mathcal{U}_1 \leqslant \mathcal{U}_2$, and $K(\infty) \leqslant 0$. Moreover, $K(\infty) = 0$ if and only if $\eta_1 = 1$ or $\eta_2 = 1$. Q.E.D.

APPENDIX D: PROOF OF SEVERAL NECESSARY CONDITIONS FOR STEADY-STATE ENTANGLEMENT WHEN $\eta_1 = \eta_2 \equiv \eta \neq 1$ AND $\gamma = 0$

When $\eta_1 = \eta_2 \equiv \eta \neq 1$ and $\gamma = 0$, some concise necessary conditions for steady-state entanglement can be derived from the condition (59) together with $\eta \geqslant 1$ and $0 \leqslant \lambda_{1,2} \leqslant 1$, i.e.,

$$\lambda_1[2\lambda_1 - \lambda_2\cos(\Delta\theta)] > 1, \quad \eta < \sqrt{2}, \quad \lambda_1 > \tfrac{1}{2}. \tag{D1}$$

Proof. First, we prove $\partial\lambda_c/\partial\eta \geqslant 0$. When $\eta_1 = \eta_2 \equiv \eta \neq 1$ and $\gamma = 0$, λ_c can be written as

$$\lambda_c = \frac{\sqrt{\mathcal{J}} + (4 - \eta^2 - 2\eta)\lambda_2\cos(\Delta\theta)}{4 + (\eta - 1)\lambda_2^2\sin^2(\Delta\theta)}, \tag{D2}$$

where $\mathcal{J}$ is a positive parameter defined as

$$\mathcal{J} = \eta^2\big(4 + 4\eta + \eta^2\lambda_2^2\big)\big[1 - \lambda_2^2\sin^2(\Delta\theta)\big] + \big\{7 + \lambda_2^2 + (\eta - 1)[(2\eta - 1)(\eta + 3)\lambda_2^2 + 7(\eta - 1) + 30]\big\}\lambda_2^2\sin^2(\Delta\theta). \tag{D3}$$

Then the first derivative of λ_c with respect to η can be expressed as

$$\frac{\partial\lambda_c}{\partial\eta} = \frac{\sqrt{\mathcal{N} + \mathcal{M}^2} - \mathcal{M}}{\sqrt{\mathcal{J}}\big[4 + (\eta - 1)\lambda_2^2\sin^2(\Delta\theta)\big]^2}, \tag{D4}$$

where

$$\mathcal{M} = \cos(\Delta\theta)\sqrt{\mathcal{J}}\lambda_2\big\{8(\eta + 1) + \sin^2(\Delta\theta)\lambda_2^2[(\eta - 1)^2 + 1]\big\}, \tag{D5}$$

$$\begin{aligned}\mathcal{N} = {}& \big[4 + (\eta - 1)\lambda_2^2\sin^2(\Delta\theta)\big]^2\big\{\lambda_2^8\sin^4(\Delta\theta)(\eta^2 + 6\eta + 8)^2\\ &+ 8\lambda_2^6\sin^2(\Delta\theta)\big[1 - \lambda_2^2\sin^2(\Delta\theta)\big](\eta + 1)(\eta^2 + 6\eta + 6)\\ &+ \big(1 - \lambda_2^2\big)\lambda_2^6\sin^4(\Delta\theta)(2\eta^5 + 15\eta^4 + 32\eta^3 + 15\eta^2 + 96\eta + 80)\\ &+ \big(1 - \lambda_2^2\big)\lambda_2^4\sin^4(\Delta\theta)\eta(7\eta^3 + 58\eta^2 + 157\eta + 80)\\ &+ 4\big(1 - \lambda_2^2\big)\big[1 - \lambda_2^2\sin^2(\Delta\theta)\big]\eta^2\big[\eta^2(2\eta + 1)\lambda_2^2 + (3\eta + 2)^2\big]\\ &+ 4\big(1 - \lambda_2^2\big)\lambda_2^2\sin^2(\Delta\theta)\big[1 - \lambda_2^2\sin^2(\Delta\theta)\big]\eta(\eta + 1)[(\eta - 1)^3 + 19\eta + 17]\\ &+ \big(1 - \lambda_2^2\big)\lambda_2^4\sin^2(\Delta\theta)\big[1 - \lambda_2^2\sin^2(\Delta\theta)\big][(\eta - 1)^2\eta^4 + 19\eta^4 + 24\eta^3 + 20\eta^2 + 96\eta + 64]\big\}.\end{aligned} \tag{D6}$$

Since $0 \leqslant \lambda_2 \leqslant 1$ and $\eta \geqslant 1$, it is obvious that $\mathcal{N} \geqslant 0$. Thus $\partial\lambda_c/\partial\eta \geqslant 0$.

Second, we prove $\partial\lambda_c(1, \lambda_2, \Delta\theta)/\partial[\lambda_2\cos(\Delta\theta)] > 0$. From Eq. (D2) we easily obtain that

$$\lambda_c(1, \lambda_2, \Delta\theta) = \tfrac{1}{4}\Big[\lambda_2\cos(\Delta\theta) + \sqrt{8 + \lambda_2^2\cos^2(\Delta\theta)}\Big]. \tag{D7}$$

Then

$$\frac{\partial\lambda_c(1, \lambda_2, \Delta\theta)}{\partial[\lambda_2\cos(\Delta\theta)]} = \frac{\sqrt{\lambda_2^2\cos^2(\Delta\theta) + 8} - \lambda_2\cos(\Delta\theta)}{4\sqrt{\lambda_2^2\cos^2(\Delta\theta) + 8}} > 0. \tag{D8}$$

Now, according to $0 \leqslant \lambda_{1,2} \leqslant 1$, $\eta \geqslant 1$, $\partial\lambda_c/\partial\eta \geqslant 0$, $\partial\lambda_c(1, \lambda_2, \Delta\theta)/\partial[\lambda_2\cos(\Delta\theta)] > 0$, and the condition (59) (i.e., $\lambda_1 > \lambda_c$), we obtain

$$1 \geqslant \lambda_1 > \lambda_c(\eta, \lambda_2, \Delta\theta) \geqslant \lambda_c(1, \lambda_2, \Delta\theta) \geqslant \lambda_c(1, 1, \pm\pi) = \tfrac{1}{2}, \tag{D9}$$

in which $\lambda_1 > \lambda_c(1, \lambda_2, \Delta\theta)$ can be equivalently expressed as

$$\lambda_1[2\lambda_1 - \lambda_2\cos(\Delta\theta)] > 1. \tag{D10}$$

From the inequality (D9), $1 > \lambda_c(\eta, \lambda_2, \Delta\theta)$ can be equivalently expressed as

$$\cos(\Delta\theta) > \frac{\eta^2 + 2\eta - 4 - \sqrt{4(\eta - 1)^3 + (3\eta - 4)^2 + (\eta^2 - 3)(\eta - 1)^2\lambda_2^2}}{(\eta - 1)\lambda_2}. \tag{D11}$$

Combining the inequality (D11) with $\cos(\Delta\theta) \leqslant 1$, we can further obtain that

$$1 > \lambda_2 > \frac{4 - \eta^3 - \eta^2}{(\eta - 1)(\eta^2 - 4)} \;\&\; \eta < 2. \tag{D12}$$

When $\lambda_2 \to 1$, we can obtain from the inequality (D12) that the maximal upper limit of η to guarantee $\lambda_1 > \lambda_c$ is $\sqrt{2}$, i.e.,

$$\eta < \sqrt{2} \tag{D13}$$

is a necessary condition for steady-state entanglement. Q.E.D.

[1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, Rev. Mod. Phys. **81**, 865 (2009).

[2] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, 2000).

[3] D. Bouwmeester, A. Ekert, and A. Zeilinger, *The Physics of Quantum Information* (Springer, Berlin, 2000).

[4] A. K. Ekert, Phys. Rev. Lett. **67**, 661 (1991).

[5] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[6] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).

[7] E. Hagley, X. Maître, G. Nogues, C. Wunderlich, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **79**, 1 (1997).

[8] S. B. Zheng and G. C. Guo, Phys. Rev. Lett. **85**, 2392 (2000).

[9] D. Guilini, E. Joos, C. Kiefer, J. Kupsch, I.-O. Stamatescu, and H. D. Zeh, *Decoherence and the Appearance of a Classical World in Quantum Theory* (Springer, Berlin, 1996).

[10] T. Yu and J. H. Eberly, Science **323**, 598 (2009).

[11] T. Yu and J. H. Eberly, Phys. Rev. Lett. **93**, 140404 (2004).

[12] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).

[13] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **78**, 574 (1997).

[14] L. Viola, E. Knill, and S. Lloyd, Phys. Rev. Lett. **82**, 2417 (1999).

[15] S. Maniscalco, F. Francica, R. L. Zaffino, N. Lo Gullo, and F. Plastina, Phys. Rev. Lett. **100**, 090503 (2008).

[16] Y.-S. Kim, J.-C. Lee, O. Kwon, and Y.-H. Kim, Nat. Phys. **8**, 117 (2012).

[17] D. Braun, Phys. Rev. Lett. **89**, 277901 (2002).

[18] M. S. Kim, J. Lee, D. Ahn, and P. L. Knight, Phys. Rev. A **65**, 040101(R) (2002).

[19] S. Schneider and G. J. Milburn, Phys. Rev. A **65**, 042107 (2002).

[20] A. M. Basharov, J. Exp. Theor. Phys. **94**, 1070 (2002).

[21] L. Jakóbczyk, J. Phys. A: Math. Gen. **35**, 6383 (2002).

[22] B. Reznik, Found. Phys. **33**, 167 (2003).

[23] F. Benatti, R. Floreanini, and M. Piani, Phys. Rev. Lett. **91**, 070402 (2003).

[24] Z. Ficek and R. Tanaś, J. Mod. Opt. **50**, 2765 (2003).

[25] R. Tanaś and Z. Ficek, J. Opt. B **6**, S90 (2004).

[26] Z. Ficek and R. Tanaś, Phys. Rev. A **77**, 054301 (2008).

[27] L. Mazzola, S. Maniscalco, J. Piilo, K.-A. Suominen, and B. M. Garraway, Phys. Rev. A **79**, 042302 (2009).

[28] S. Das and G. S. Agarwal, J. Phys. B **42**, 141003 (2009).

[29] Z. Ficek and R. Tanaś, Phys. Rev. A **74**, 024304 (2006).

[30] J. Hu and H. Yu, Phys. Rev. A **91**, 012327 (2015).

[31] F. Benatti and R. Floreanini, J. Opt. B **7**, S429 (2005).

[32] A. Gonzalez-Tudela, D. Martin-Cano, E. Moreno, L. Martin-Moreno, C. Tejedor, and F. J. Garcia-Vidal, Phys. Rev. Lett. **106**, 020501 (2011).

[33] W. Tang, F. Lin, X. Zhu, and Z. Fang, Phys. Rev. B **100**, 165415 (2019).

[34] L. Hartmann, W. Dür, and H.-J. Briegel, Phys. Rev. A **74**, 052304 (2006).

[35] B. Bellomo and M. Antezza, Europhys. Lett. **104**, 10006 (2013).

[36] B. Bellomo and M. Antezza, New J. Phys. **15**, 113052 (2013).

[37] V. Gorini, A. Kossakowski, and E. C. G. Surdarshan, J. Math. Phys. **17**, 821 (1976).

[38] G. Lindblad, Commun. Math. Phys. **48**, 119 (1976).

[39] R. Alicki, Int. J. Theor. Phys. **16**, 351 (1977).

[40] L. Aolita, F. de Melo, and L. Davidovich, Rep. Prog. Phys. **78**, 042001 (2015).

[41] W. K. Wootters, Phys. Rev. Lett. **80**, 2245 (1998).

[42] Y. Ye, H. Yu, and J. Hu, Commun. Theor. Phys. **73**, 065104 (2021).

[43] R. H. Dicke, Phys. Rev. **93**, 99 (1954).

[44] S. Cheng, H. Yu, and J. Hu, Phys. Rev. D **98**, 025001 (2018).

[45] M. Tavis and F. W. Cummings, Phys. Rev. **170**, 379 (1968).

[46] M. Tavis and F. W. Cummings, Phys. Rev. **188**, 692 (1969).

[47] K. Härkönen, F. Plastina, and S. Maniscalco, Phys. Rev. A **80**, 033841 (2009).

[48] J. Hu and H. Yu, Ann. Phys. (NY) **363**, 243 (2015).

[49] J. She, J. Hu, and H. Yu, Phys. Rev. D **99**, 105009 (2019).

[50] Y. Zhou, J. Hu, and H. Yu (unpublished).

[51] Y. Chen, J. Hu, and H. Yu, Phys. Rev. D **105**, 045013 (2022).