General lower and upper bounds under minimum-error quantum state discrimination

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(Received 13 May 2021; accepted 11 January 2022; published 4 March 2022)

For the optimal success probability under minimum-error discrimination between $r \geqslant 2$ arbitrary quantum states prepared with any *a priori* probabilities, we find new general analytical lower and upper bounds and specify the relations between these new general bounds and the known general bounds, lower and upper. We also present the example where the values of the new general lower and upper bounds on the optimal success probability are tighter than the values of most of the general analytical bounds known in the literature. The new upper bound on the optimal success probability explicitly generalizes to $r > 2$ the form of the Helstrom bound. For $r = 2$, each of our new bounds, lower and upper, reduces to the Helstrom bound.

DOI: 10.1103/PhysRevA.105.032410

I. INTRODUCTION

Different aspects of quantum state discrimination have been discussed in the literature ever since the seminal papers of Helstrom and Holevo [1–5] and are now presented in many textbooks and reviews; see, for example, Refs. [6–8] and references therein.

Let a sender prepare a quantum system described in terms of a complex Hilbert space $\mathcal{H}$ in one of $r \geqslant 2$ quantum states $\rho_1, \ldots, \rho_r$, pure or mixed, with probabilities $q_1, \ldots, q_r$, $\sum_i q_i = 1$, $q_i > 0$, and send this quantum system in an initial state

$$\rho = \sum_{i=1,\ldots,r} q_i \rho_i, \qquad \sum_{i=1,\ldots,r} q_i = 1, \quad q_i > 0, \tag{1}$$

to a receiver. For discriminating between states $\rho_1, \ldots, \rho_r$, a receiver performs a measurement described by a positive operator-valued (POV) measure

$$\mathrm{M}_r = \left\{ \mathrm{M}_r(i),\ i = 1, \ldots, r \sum_{i=1,\ldots,r} \mathrm{M}_r(i) = \mathbb{I}_{\mathcal{H}} \right\} \tag{2}$$

and the success probability to take under this measurement the proper decision equals to

$$\mathrm{P}^{\text{success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}(\mathrm{M}_r) = \sum_{i=1,\ldots,r} q_i \mathrm{tr}[\rho_i \mathrm{M}_r(i)]; \tag{3}$$

correspondingly, the error probability

$$\mathrm{P}^{\text{error}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}(\mathrm{M}_r) = \sum_{i=1,\ldots,r} q_i \mathrm{tr}[\rho_i(\mathbb{I} - \mathrm{M}_r(i))] = 1 - \mathrm{P}^{\text{success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}(\mathrm{M}_r). \tag{4}$$

Denote by $\mathfrak{M}_r = \{\mathrm{M}_r\}$, $r \geqslant 2$, the set of all possible POV measures (2). Under the maximum likelihood (the minimum error) state discrimination strategy, the optimal success probability and the optimal error probability are given by

$$\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} := \max_{\mathrm{M}_r \in \mathfrak{M}_r} \mathrm{P}^{\text{success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}(\mathrm{M}_r), \tag{5}$$

$$\mathrm{P}^{\text{opt.error}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} := \min_{\mathrm{M}_r \in \mathfrak{M}_r} \mathrm{P}^{\text{error}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}(\mathrm{M}_r) = 1 - \mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} \tag{6}$$

and are attained at some extreme point of the convex set $\mathfrak{M}_r$.

The alternative expressions for the optimal error probability (6) are presented in Theorem 1 and Corollary 1 of Ref. [9].

The following general statement was first formulated and proved by Holevo in [3,4].

Theorem 1. Under the maximum likelihood (the minimum error) state discrimination strategy, a POV measure $\mathrm{M}_r^{(\text{opt})} \in \mathfrak{M}_r$ is optimal if and only if there exists a self-adjoint trace class operator Λ_0 such that (i) $(\Lambda_0 - q_i\rho_i)\mathrm{M}_r^{(\text{opt})}(i) = 0$; and (ii) $\Lambda_0 \geqslant q_i\rho_i$, for all $i = 1, \ldots, r$. Herewith,

$$\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} = \mathrm{tr}[\Lambda_0], \quad \Lambda_0 = \sum_{i=1,\ldots,r} q_i\rho_i \mathrm{M}_r^{(\text{opt})}(i). \tag{7}$$

For $r = 2$, the success probability (3) admits the Helstrom upper bound [1,2,5]

$$\mathrm{P}^{\text{success}}_{\rho_1,\rho_2|q_1,q_2}(\mathrm{M}_2) \leqslant \tfrac{1}{2}(1 + \|q_1\rho_1 - q_2\rho_2\|_1), \tag{8}$$

which is attained, so that, for $r = 2$, the optimal success probability is equal to [1,2,5]

$$\mathrm{P}^{\text{opt.success}}_{\rho_1,\rho_2|q_1,q_2} = \tfrac{1}{2}(1 + \|q_1\rho_1 - q_2\rho_2\|_1), \tag{9}$$

where $\|\cdot\|_1$ is the trace norm.

For an arbitrary $r > 2$, a precise general expression for the optimal success probability (5) in terms of only states $\rho_1, \ldots, \rho_r$ and *a priori* probabilities $q_1, \ldots, q_r$ is not known.

However, there were introduced and studied several general upper and lower bounds [10–21] on the optimal success probability $\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}$, expressed via different characteristics of states $\rho_1, \ldots, \rho_r$ with *a priori* probabilities $q_1, \ldots, q_r$. As proved by Qiu and Li [17], *in some cases*, the general lower bound on the optimal error probability $P^{\text{opt.error}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}$ (correspondingly, the upper bound on the optimal success probability $P^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}$), introduced by them in Ref. [17], is tighter than the other general lower bounds known in the literature. Computation of bounds on $P^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}$ within semidefinite programming was considered recently in Ref. [21].

In the present article, we find (Theorems 2 and 3) the new general lower and upper bounds on the optimal success probability (5) valid for all $r \geqslant 2$ and specify (Propositions 1 and 2) the relation of these new general bounds to the general lower and upper bounds known in literature. For $r = 2$, each of the new general bounds, lower and upper, reduces to the Helstrom bound in (8), and this proves in the other way the Helstrom result (9).

II. GENERAL LOWER BOUNDS

Taking into account that $M_r(j) = \mathbb{I}_{\mathcal{H}} - \sum_{i\neq j} M_r(i)$, we rewrite the right-hand side of expression (3) in either of jth representations:

$$\begin{aligned} P^{\text{success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}(M_r) &= \sum_{i=1,\ldots,r} q_i \,\text{tr}[\rho_i M_r(i)] \\ &= q_j + \sum_{i=1,\ldots,r} \text{tr}[(q_i\rho_i - q_j\rho_j)M_r(i)], \\ & j = 1,\ldots,r, \end{aligned} \tag{10}$$

for every POV measure M_r. Summing up the left-hand and the right-hand sides of (10) over all $j = 1,\ldots,r$, for any POV measure $M_r \in \mathfrak{M}_r$, we also come to the following representation for the success probability

$$P^{\text{success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}(M_r) = \frac{1}{r}\left(1 + \sum_{i,j=1,\ldots,r} \text{tr}[(q_i\rho_i - q_j\rho_j)M_r(i)]\right). \tag{11}$$

Recall that a self-adjoint (Hermitian) bounded operator X on $\mathcal{H}$ admits the decomposition:

$$\begin{aligned} &X = X^{(+)} - X^{(-)}, \quad X^{(\pm)} \geqslant 0, \\ &X^{(+)} = \sum_{\lambda_k>0,} \lambda_k E_X(\lambda_k), \quad X^{(-)} = \sum_{\lambda_k\leqslant 0,} |\lambda_k| E_X(\lambda_k), \end{aligned} \tag{12}$$

where $E_X(\lambda_k)$ the spectral projections of a Hermitian operator X. If a bounded operator X is trace class, then operators $X^{(\pm)} \geqslant 0$ are also trace class and

$$\|X\|_1 := \text{tr}|X|, \quad |X| = X^{(+)} + X^{(-)}, \quad \|X^{(\pm)}\|_1 = \text{tr}[X^{(\pm)}]. \tag{13}$$

From relations [7] $|\text{tr}[W]| \leqslant \|W\|_1$ and $\|AB\|_1 \leqslant \|A\|_1\|B\|_0$, valid for all trace-class operators W, A and all bounded operators B, it follows that if $X, Y \geqslant 0$ (hence, $\text{tr}[XY] \geqslant 0$), then

$$0 \leqslant \text{tr}[XY] \leqslant \|X\|_1\|Y\|_0, \tag{14}$$

where notation $\|\cdot\|_0$ means the operator norm.

Definition (5) and relations (10)–(14) imply Theorem 2.

Theorem 2 (New lower bounds). For any number $r \geqslant 2$ of arbitrary quantum states $\rho_1,\ldots,\rho_r$ prepared with probabilities $q_1,\ldots,q_r$, the optimal success probability (5) admits the lower bounds

$$P^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} \geqslant \mathfrak{L}^{(r)}_{1,\text{new}} := \max_{j=1,\ldots,r}\left\{q_j + \frac{1}{r-1}\sum_{i=1,\ldots,r} \|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1\right\} \tag{15}$$

$$= \frac{1}{2(r-1)} + \frac{1}{2(r-1)}\max_{j=1,\ldots,r}\left\{\sum_{i=1,\ldots,r} \|q_i\rho_i - q_j\rho_j\|_1 + q_j(r-2)\right\} \tag{16}$$

$$\geqslant \mathfrak{L}^{(r)}_{2,\text{new}} := \frac{1}{r}\left(1 + \frac{1}{r-1}\sum_{1\leqslant i<j\leqslant r} \|q_i\rho_i - q_j\rho_j\|_1\right). \tag{17}$$

For $r = 2$, each of these new lower bounds reduces to the Helstrom bound in (8).

Proof. Let $E_{(q_i\rho_i - q_j\rho_j)}(\lambda_k)$ be the spectral projections of the Hermitian operator $(q_i\rho_i - q_j\rho_j)$ on $\mathcal{H}$ and

$$P^{(+)}_{ij} := \sum_{\lambda_k>0} E_{(q_i\rho_i - q_j\rho_j)}(\lambda_k), \quad i \neq j, \tag{18}$$

denote the orthogonal projection on the proper subspace of operator $(q_i\rho_i - q_j\rho_j)$, corresponding to its positive eigenvalues. Note that by (13)

$$\begin{aligned} q_i - q_j &= \|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1 - \|(q_i\rho_i - q_j\rho_j)^{(-)}\|_1, \\ \|q_i\rho_i - q_j\rho_j\|_1 &= \|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1 + \left\|(q_i\rho_i - q_j\rho_j)^{(-)}\right\|_1. \end{aligned} \tag{19}$$

Introduce the POV measures $M^{(j)}_r$, $j = 1,\ldots,r$, each with the elements

$$M^{(j)}_r(i) = \frac{1}{r-1}P^{(+)}_{ij}, \quad i \neq j, \quad M^{(j)}_r(j) = \mathbb{I}_{\mathcal{H}} - \frac{1}{r-1}\sum_{i=1,\ldots,r,\, i\neq j} P^{(+)}_{ij} = \frac{1}{r-1}\sum_{i=1,\ldots,r,\, i\neq j} (\mathbb{I}_{\mathcal{H}} - P^{(+)}_{ij}) \geqslant 0. \tag{20}$$

From the jth representation in (10) and relations (19) it follows that, for the jth POV measure (20), we have

$$\mathrm{P}^{\text{success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}(\mathrm{M}_r^{(j)}) = q_j + \sum_{i=1,\ldots,N} \mathrm{tr}[(q_i\rho_i - q_j\rho_j)\mathrm{M}_r^{(j)}(i)]$$
$$= q_j + \frac{1}{r-1}\sum_{i=1,\ldots,r} \|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1,$$
$$j = 1,\ldots,r. \tag{21}$$

For the optimal success probability (5), equalities (21) imply

$$\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} \geqslant q_j + \frac{1}{r-1}\sum_{i=1,\ldots,r} \|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1,$$
$$\forall j = 1,\ldots,r, \tag{22}$$

and hence,

$$\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} \geqslant \max_{j=1,\ldots,r}\left\{q_j + \frac{1}{r-1}\sum_{i=1,\ldots,r} \|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1\right\}. \tag{23}$$

Since by (19)

$$\|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1 = \tfrac{1}{2}(q_i - q_j) + \tfrac{1}{2}\|q_i\rho_i - q_j\rho_j\|_1, \tag{24}$$

the expression in the right-hand side of (22) is otherwise equal to

$$q_j + \frac{1}{r-1}\sum_{i=1,\ldots,r} \|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1$$
$$= \frac{1}{2(r-1)} + \frac{1}{2(r-1)}\left\{\sum_{i=1,\ldots,r} \|q_i\rho_i - q_j\rho_j\|_1 + q_j(r-2)\right\}. \tag{25}$$

This and relation (23) imply the lower bounds (15) and (16). Summing up the left-hand and the right-hand sides of (22) over all $j = 1,\ldots,r$ and taking into account $\sum_{j=1,\ldots,r} q_j = 1$, relations (19) and

$$(q_i\rho_i - q_j\rho_j)^{+} = (q_j\rho_j - q_i\rho_i)^{(-)}, \tag{26}$$

we derive

$$\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} \geqslant \frac{1}{r}\left(1 + \frac{1}{r-1}\sum_{1\leqslant i<j\leqslant r} \|q_i\rho_i - q_j\rho_j\|_1\right), \tag{27}$$

that is, the lower bound (17). Furthermore, since, for any positive numbers α_j, $j = 1,\ldots,r$, their sum

$$\sum_{j=1,\ldots,r} \alpha_j \leqslant r \max_{j=1,\ldots,r} \alpha_j, \tag{28}$$

we have

$$\max_{j=1,\ldots,r}\left\{q_j + \frac{1}{r-1}\sum_{i=1,\ldots,N} \|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1\right\}$$
$$\geqslant \frac{1}{r}\left(1 + \frac{1}{r-1}\sum_{1\leqslant i<j\leqslant r} \|q_i\rho_i - q_j\rho_j\|_1\right). \tag{29}$$

Relations (23), (25), and (29) prove the statement of Theorem 2. ■

Consider now the relation of the new lower bounds (15)–(17) to the known [10,19,20] general lower bounds on $\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}$:

$$\mathfrak{L}_1^{(r)} := \max_{j=1,\ldots,r} q_j \geqslant \frac{1}{r}, \tag{30}$$

$$\mathfrak{L}_2^{(r)} := 1 - \sum_{1\leqslant i<j\leqslant r} \sqrt{q_i q_j} F_{ij}, \tag{31}$$

$$\mathfrak{L}_3^{(r)} := \left(\mathrm{tr}\left[\sqrt{\sum_{i=1,\ldots,r} q_i^2\rho_i^2}\right]\right)^2, \tag{32}$$

where $F_{ij} := \|\sqrt{\rho_i}\sqrt{\rho_j}\|_1$ is the pairwise fidelity. Here (i) bound (30) follows from item (ii) and relation (7) in Theorem 1; (ii) bound (31) was introduced by Barnum and Knill in Ref. [10] and further studied by Audenaert and Mosonyi in Ref. [20]; and (iii) bound (32) was introduced by Tyson in Ref. [19].

Note that if trace class operators $B \geqslant A$, then $\mathrm{tr}[B] \geqslant \mathrm{tr}[A]$. Therefore, in bound (32), parameter $\mathrm{tr}[\sqrt{\sum_{i=1,\ldots,r} q_i^2\rho_i^2}] \geqslant q_i$, for all $i = 1,\ldots,r$. This implies

$$\mathrm{tr}\left[\sqrt{\sum_{i=1,\ldots,r} q_i^2\rho_i^2}\right] \geqslant \max_{i=1,\ldots,r} q_i \geqslant \frac{1}{r}, \tag{33}$$

so that

$$\mathfrak{L}_3^{(r)} \geqslant \left(\mathfrak{L}_1^{(r)}\right)^2. \tag{34}$$

Proposition 1. (a) The new lower bound (15) is tighter than the known lower bound (30)

$$\mathfrak{L}_{1,\text{new}}^{(r)} \geqslant \mathfrak{L}_1^{(r)} \tag{35}$$

for any number $r \geqslant 2$ of arbitrary quantum states $\rho_1,\ldots,\rho_r$ and probabilities $q_1,\ldots,q_r$. (b) For $r = 2$, the new lower bounds (15) and (17) are equal to the Helstrom bound and $\mathfrak{L}_{1,\text{new}}^{(2)} = \mathfrak{L}_{2,\text{new}}^{(2)} \geqslant \mathfrak{L}_2^{(2)}$ for all states ρ_1, ρ_2 and probabilities q_1, q_2. (c) For any $r > 2$, the new lower bound (17) is tighter than the lower bound (31)

$$\mathfrak{L}_{2,\text{new}}^{(r)} \geqslant \mathfrak{L}_2^{(r)} \tag{36}$$

if

$$\sum_{1\leqslant i<j\leqslant r} \|q_i\rho_i - q_j\rho_j\|_1 \leqslant \frac{(r-1)^2}{r+1}. \tag{37}$$

(d) The new lower bound (15) is tighter than the lower bound (32)

$$\mathfrak{L}_{1,\text{new}}^{(r)} \geqslant \mathfrak{L}_3^{(r)} \tag{38}$$

if

$$\mathrm{tr}\left[\sqrt{\sum_{i=1,\ldots,r} q_i^2\rho_i^2}\right] \leqslant \sqrt{\max_{i=1,\ldots,r} q_i}. \tag{39}$$

(e) The new lower bound (17) is tighter than the lower bound (32)

$$\mathfrak{L}_{2,\text{new}}^{(r)} \geqslant \mathfrak{L}_3^{(r)} \tag{40}$$

if

$$\mathrm{tr}\left[\sqrt{\sum_{i=1,\ldots,r} q_i^2\rho_i^2}\right] \leqslant \frac{1}{\sqrt{r}} \leqslant \sqrt{\max_{i=1,\ldots,r} q_i}. \tag{41}$$

Proof. Due to the structure of the new lower bound (15), relation (35) is obvious. By the Helstrom result (9) and Theorem 2 item (b) is also obvious. In order to prove item (c), we consider the difference

$$\mathfrak{L}_{2,\text{new}}^{(r)} - \mathfrak{L}_2^{(r)} = \frac{1}{r} + \frac{1}{r(r-1)} \sum_{1\leqslant i<j\leqslant r} \|q_i\rho_i - q_j\rho_j\|_1 - 1 + \sum_{1\leqslant i<j\leqslant r} \sqrt{q_iq_j}F(\rho_i,\rho_j). \tag{42}$$

By Lemma 5 in Ref. [17]

$$\sqrt{q_iq_j}F(\rho_i,\rho_j) \geqslant \tfrac{1}{2}(q_i+q_j) - \tfrac{1}{2}\|q_i\rho_i - q_j\rho_j\|_1. \tag{43}$$

Substituting this relation into (42) and taking into account

$$\sum_{1\leqslant i<j\leqslant r} (q_i+q_j) = r-1, \tag{44}$$

we derive

$$\begin{aligned}\mathfrak{L}_{2,\text{new}}^{(r)} - \mathfrak{L}_2^{(r)} &\geqslant \frac{(r-2)(r-1)}{2r} - \left(\frac{1}{2} - \frac{1}{r(r-1)}\right) \times \sum_{1\leqslant i<j\leqslant r} \|q_i\rho_i - q_j\rho_j\|_1 \\ &= \frac{(r-2)(r-1)}{2r} \times \left(1 - \frac{r+1}{(r-1)^2} \sum_{1\leqslant i<j\leqslant r} \|q_i\rho_i - q_j\rho_j\|_1\right).\end{aligned} \tag{45}$$

Under condition (37), this proves relation (36). Note that

$$\sum_{1\leqslant i<j\leqslant r} \|q_i\rho_i - q_j\rho_j\|_1 \leqslant \sum_{1\leqslant i<j\leqslant r} (q_i+q_j) = r-1, \tag{46}$$

where we took into account (44). In order to prove (d), we consider the value of difference $(\mathfrak{L}_{1,\text{new}}^{(r)} - \mathfrak{L}_3^{(r)})$ under condition (39):

$$\begin{aligned}&\mathfrak{L}_{1,\text{new}}^{(r)} - \mathfrak{L}_3^{(r)} \\ &= \max_{j=1,\ldots,r}\left\{q_j + \frac{1}{r-1}\sum_{i=1,\ldots,r} \|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1\right\} - \left(\operatorname{tr}\left[\sqrt{\sum_{i=1,\ldots,r} q_i^2\rho_i^2}\right]\right)^2 \\ &\geqslant \max_{j=1,\ldots,r} q_j - \max_{j=1,\ldots,r} q_j = 0.\end{aligned} \tag{47}$$

This proves relation (38). In item (e), under condition (41),

$$\mathfrak{L}_{2,\text{new}}^{(r)} - \mathfrak{L}_3^{(r)} = \frac{1}{r} + \frac{1}{r(r-1)}\sum_{1\leqslant i<j\leqslant r} \|q_i\rho_i - q_j\rho_j\|_1 - \left(\operatorname{tr}\left[\sqrt{\sum_{i=1,\ldots,r} q_i^2\rho_i^2}\right]\right)^2 \geqslant 0. \tag{48}$$

This proves (40). ■

III. GENERAL UPPER BOUNDS

From relations (12)–(14) and inequality $\|\mathrm{M}_r(i)\|_0 \leqslant 1$ it follows that, for every POV measure M_r, in representation (10)

$$\begin{aligned}&\mathrm{P}^{\text{success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}(\mathrm{M}_r) \\ &= q_k + \sum_{i=1,\ldots,r} \operatorname{tr}[(q_i\rho_i - q_k\rho_k)\mathrm{M}_r(i)] \\ &\leqslant q_m + \sum_{i=1,\ldots,r} \|(q_i\rho_i - q_m\rho_m)^{(+)}\|_1, \quad k,m = 1,\ldots,r,\end{aligned} \tag{49}$$

and in representation (11)

$$\begin{aligned}&\mathrm{P}^{\text{success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}(\mathrm{M}_r) \\ &= \frac{1}{r}\left(1 + \sum_{i,j=1,\ldots,r} \operatorname{tr}\{(q_i\rho_i - q_j\rho_j)\mathrm{M}_r(i)\}\right) \\ &\leqslant \frac{1}{r}\left(1 + \sum_{i,j=1,\ldots,r} \|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1\right).\end{aligned} \tag{50}$$

Relation (49) immediately implies the upper bound

$$\begin{aligned}&\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} \\ &\leqslant Q_4^{(r)} := \min_{j=1,\ldots,r}\left\{q_j + \sum_{i=1,\ldots,r} \left\|(q_i\rho_i - q_j\rho_j)^{(+)}\right\|_1\right\} \\ &= \frac{1}{2} + \frac{1}{2}\min_{j=1,\ldots,r}\left\{\sum_{i=1,\ldots,r} \|q_i\rho_i - q_j\rho_j\|_1 - q_j(r-2)\right\},\end{aligned} \tag{51}$$

which agrees due to the relation $\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} = 1 - \mathrm{P}^{\text{opt.error}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}$ with the lower bound L_4 by Qiu&Li on $\mathrm{P}^{\text{opt.error}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}$ introduced in Ref. [17]. Here, in order to derive the expression in the last expression of (51) we took into account relation (24).

For convenience in comparison, we take for the upper bound (51) and the below upper bounds (52), (63), and (64) on the optimal success probability (5) the numeration similar to that for the lower bounds $L_n^{(r)}$ in Ref. [17] on the optimal error probability (6) with the obvious correspondence $Q_n^{(r)} = 1 - L_n^{(r)}$.

The following theorem introduces a new upper bound on the optimal success probability $\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}$ and establishes its relation to the upper bound (51) by Qiu and Li in Ref. [17] and the upper bound

$$Q_2^{(r)} := \frac{1}{2}\left(1 + \frac{1}{r-1}\sum_{1\leqslant i<j\leqslant r} \|q_i\rho_i - q_j\rho_j\|_1\right), \tag{52}$$

introduced by Qiu [15] earlier than bound (51) in Ref. [17].

Theorem 3 (New upper bound). For any number $r \geqslant 2$ of arbitrary quantum states $\rho_1,\ldots,\rho_r$ prepared with probabilities $q_1,\ldots,q_r$, the optimal success probability (5) admits the upper bound

$$\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} \leqslant Q^{(r)}_{\text{new}} := \frac{1}{r}\left(1+\sum_{1\leqslant i<j\leqslant r}\|q_i\rho_i - q_j\rho_j\|_1\right), \tag{53}$$

explicitly generalizing to $r > 2$ the form of the Helstrom upper bound in (8) and relating to the upper bounds (51) and (52) as

$$Q^{(r)}_4 \leqslant Q^{(r)}_{\text{new}} \leqslant Q^{(r)}_2. \tag{54}$$

Proof. In view of (5), relations (50) and (26) immediately imply the upper bound (53). Also, by (49)

$$\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} \leqslant q_j + \sum_{i=1,\ldots,r}\|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1, \quad j = 1,\ldots,r. \tag{55}$$

Summing up the left-hand and the right-hand sides of this relation over $j = 1,\ldots,r$, and taking into account (26), we derive

$$r\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} \leqslant \sum_{j=1,\ldots,r} q_j + \sum_{i=1,\ldots,r}\|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1 = 1 + \sum_{1\leqslant i<j\leqslant r}\|q_i\rho_i - q_j\rho_j\|_1. \tag{56}$$

Since, for any positive numbers α_j, $j = 1,\ldots,r$, their sum

$$\sum_{j=1,\ldots,r}\alpha_j \geqslant r\min_{j=1,\ldots,r}\alpha_j, \tag{57}$$

we have

$$\min_{j=1,\ldots,r}\left\{q_j + \sum_i \|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1\right\} \leqslant \frac{1}{r}\left(1+\sum_{1\leqslant i<j\leqslant r}\|q_i\rho_i - q_j\rho_j\|_1\right), \tag{58}$$

that is, the first inequality in relation (54). In order to prove the second inequality in (54), we recall (46) which implies

$$Q^{(r)}_2 - Q^{(r)}_{\text{new}} = \frac{r-2}{2r}\left(1 - \frac{1}{r-1}\sum_{1\leqslant i<j\leqslant r}\|q_i\rho_i - q_j\rho_j\|_1\right) \geqslant 0. \tag{59}$$

This proves the statement of Theorem 3. ■

We stress that, though the upper bound (51) in Ref. [17] is tighter than the new upper bound (53), for a large number $r \geqslant 2$ of quantum states to be discriminated, the straightforward calculation of the expression in bound (53) is much easier than finding minimum in bound (51).

Remark 1. From relations (24) and $\|q_i\rho_i - q_j\rho_j\|_1 \leqslant q_i + q_j$, it follows $\|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1 \leqslant q_i$, and therefore

$$\sum_{i=1,\ldots,r}\|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1 \leqslant \sum_{i=1,\ldots,r,\ i\neq j} q_i = 1 - q_j. \tag{60}$$

Relations (26), (44), and (60) imply

$$\sum_{1\leqslant i<j\leqslant r}\|q_i\rho_i - q_j\rho_j\|_1 = \sum_{i,j=1,\ldots,r,}\|(q_i\rho_i - q_j\rho_j)^{(+)}\|_1 \leqslant r-1, \tag{61}$$

which agrees with (46). Therefore, the new upper bound in Theorem 3 is nontrivial (i.e., not more than one). Also, by relation (49) specified for the POV measure (20), the bounds in Theorem 2 and the upper bound (51) are consistent in the sense that

$$\max_{k=1,\ldots,r}\left\{q_k + \frac{1}{r-1}\sum_{i=1,\ldots,r}\|(q_i\rho_i - q_k\rho_k)^{(+)}\|_1\right\} \leqslant \min_{m=1,\ldots,r}\left\{q_m + \sum_{i=1,\ldots,r}\|(q_i\rho_i - q_m\rho_m)^{(+)}\|_1\right\} \tag{62}$$

for all $r \geqslant 2$.

Besides relation (54) of the new upper bound (53) to the known upper bounds (51) and (52) of Qiu and Li in Ref. [17] and Qui in Ref. [15], let us also consider its relation to the following known [16,18,19] general upper bounds on $\mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r}$:

$$Q^{(r)}_3 := 1 - \sum_{1\leqslant i<j\leqslant r} q_iq_jF^2_{ij}, \tag{63}$$

$$Q^{(r)}_5 := \operatorname{tr}\left[\sqrt{\sum_{i=1,\ldots,r} q_i^2\rho_i^2}\right], \tag{64}$$

which follow from the lower bounds $L^{(r)}_n$ on the optimal error probability introduced, correspondingly, (i) by Montanaro in Ref. [16] and (ii) by Ogawa and Nagaoka in Ref. [18] for the equiprobable case and by Tyson [19] for a general case.

The detailed study of these general bounds and some other bounds for specific families of quantum states is presented in Ref. [20].

Proposition 2. (i) For any number $r \geqslant 2$, of arbitrary quantum states $\rho_1,\ldots,\rho_r$ and *a priori* probabilities $q_1,\ldots,q_r$, the new upper bound (53) and the upper bound (63) satisfy the relation

$$Q^{(r)}_{\text{new}} \leqslant \frac{r-2}{r-1} + \frac{1}{r-1}Q^{(r)}_3. \tag{65}$$

(ii) In the equiprobable case, the new upper bound (53) is tighter than the known upper bound (63):

$$Q^{(r)}_{\text{new}} \leqslant Q^{(r)}_3. \tag{66}$$

(iii) The new upper bound (53) is tighter than the upper bound (64)

$$Q^{(r)}_{\text{new}} \leqslant Q^{(r)}_5 \tag{67}$$

if

$$\operatorname{tr}\left[\sqrt{\sum_{i=1,\ldots,r} q_i^2\rho_i^2}\right] \geqslant \frac{1}{\sqrt{r}}, \qquad \sum_{1\leqslant i<j\leqslant r}\|q_i\rho_i - q_j\rho_j\|_1 \leqslant \sqrt{r} - 1. \tag{68}$$

Proof. (i) Inequality in (65) follows from the second inequality $Q^{(r)}_{\text{new}} \leqslant Q^{(r)}_2$ in (54) and estimate (55) in Ref. [16]:

$$1 - Q^{(r)}_2 \geqslant \frac{1 - Q^{(r)}_3}{r-1}. \tag{69}$$

(ii) For the equiprobable case $q_1 = \cdots = q_r = \frac{1}{r}$, we consider the difference

$$Q_{\text{new}}^{(r)} - Q_3^{(r)} = \frac{1}{r} + \frac{1}{r^2}\sum_{1\leqslant i<j\leqslant r}\|\rho_i - \rho_j\|_1 - 1 + \frac{1}{r^2} \times \sum_{1\leqslant i<j\leqslant r} F^2(\rho_i,\rho_j)$$
$$= \frac{1}{r^2}\sum_{1\leqslant i<j\leqslant r}(F^2(\rho_i,\rho_j) + \|\rho_i-\rho_j\|_1) - \frac{r-1}{r}. \quad (70)$$

Taking into account relation (20) in Ref. [17], which implies

$$\sum_{1\leqslant i<j\leqslant r}(F^2(\rho_i,\rho_j)+\|\rho_i-\rho_j\|_1) \leqslant \sum_{1\leqslant i<j\leqslant r} 2 = r(r-1), \quad (71)$$

and substituting this into (70), we have

$$Q_{\text{new}}^{(r)} - Q_3^{(r)} \leqslant \frac{r-1}{r} - \frac{r-1}{r} = 0. \quad (72)$$

This implies (66). (iii) Taking into the account (33), for the difference $(Q_{\text{new}}^{(r)} - Q_5^{(r)})$ under conditions (68) we derive

$$Q_{\text{new}}^{(r)} - Q_5^{(r)} = \frac{1}{r} + \frac{1}{r}\sum_{1\leqslant i<j\leqslant r}\|q_i\rho_i - q_j\rho_j\|_1 - \text{tr}\left[\sqrt{\sum_{i=1,\ldots,r} q_i^2\rho_i^2}\right]$$
$$\leqslant \frac{1}{r} + \frac{\sqrt{r}-1}{r} - \frac{1}{\sqrt{r}} = 0. \quad (73)$$

This proves (67). ■

Proposition 2 implies Corollary 1.

Corollary 1. If conditions (68) are fulfilled, then the upper bound (51) by Qiu and Li [17] is tighter than the upper bound (64) by Tyson [19]:

$$Q_4^{(r)} \leqslant Q_5^{(r)} \quad (74)$$

To our knowledge, except for the specific examples considered by Qiu and Li [17], in general, the comparison of the upper bound (51) in Ref. [17] with the upper bound (64) in Ref. [19] has not been reported in the literature.

IV. GENERAL RELATIONS

Theorems 2 and 3 imply the following general relations.

Corollary 2. For any number $r \geqslant 2$ of arbitrary quantum states $\rho_1, \ldots, \rho$ prepared with any probabilities $q_1, \ldots, q_r$, the optimal success probability (5) admits the following general bounds:

$$\frac{1}{r}\left(1 + \frac{1}{(r-1)}\sum_{1\leqslant i<j\leqslant r}\|q_i\rho_i - q_j\rho_j\|_1\right)$$
$$\leqslant \frac{1}{2(r-1)} + \frac{1}{2(r-1)}\max_{j=1,\ldots,r}\left\{\sum_{i=1,\ldots,r}\|q_i\rho_i - q_j\rho_j\|_1 + q_j(r-2)\right\}$$
$$\leqslant \mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|q_1,\ldots,q_r} \leqslant \frac{1}{2} + \frac{1}{2}\min_{j=1,\ldots,r}\left\{\sum_{i=1,\ldots,r}\|q_i\rho_i - q_j\rho_j\|_1 - q_j(r-2)\right\}$$
$$\leqslant \frac{1}{r}\left(1 + \sum_{1\leqslant i<j\leqslant r}\|q_i\rho_i - q_j\rho_j\|_1\right). \quad (75)$$

In the first and the last lines of (75), the bounds are quite similar by its form to the Helstrom upper bound (8). For $r = 2$, each of the lower and all bounds in (75) reduces to the Helstrom bound in (8) and this proves in the other way the Helstrom result (9).

For the equiprobable case, bounds (75) take the following forms.

Corollary 3. For any number $r \geqslant 2$, of arbitrary quantum states $\rho_1, \ldots, \rho_r$, prepared with equal probabilities $q_1 = \cdots = q_1 = \frac{1}{r}$, the optimal success probability (5) admits the bounds

$$\frac{1}{r}\left(1 + \frac{1}{r(r-1)}\sum_{1\leqslant i<j\leqslant r}\|\rho_i - \rho_j\|_1\right) \leqslant \frac{1}{r} + \frac{1}{2r(r-1)}\max_{j=1,\ldots,r}\left\{\sum_{i=1,\ldots,r}\|\rho_i - \rho_j\|_1\right\}$$
$$\leqslant \mathrm{P}^{\text{opt.success}}_{\rho_1,\ldots,\rho_r|\frac{1}{r},\ldots,\frac{1}{r}} \leqslant \frac{1}{r} + \frac{1}{2r}\min_{j=1,\ldots,r}\left\{\sum_{i=1,\ldots,r}\|\rho_i - \rho_j\|_1\right\}$$
$$\leqslant \frac{1}{r}\left(1 + \frac{1}{r}\sum_{1\leqslant i<j\leqslant r}\|\rho_i - \rho_j\|_1\right). \quad (76)$$

Example

For the numerical comparison of the new lower bounds (15)–(17) and the new upper bound (53) with the known lower bounds (30)–(32) and the known upper bounds (51), (52), (63), and (64), we analyze the discrimination between the following three equiprobable qubit states:

$$\rho_1 = \tfrac{7}{8}|0\rangle\langle 0| + \tfrac{1}{8}|1\rangle\langle 1|, \quad \rho_2 = \tfrac{5}{8}|0\rangle\langle 0| + \tfrac{3}{8}|1\rangle\langle 1|,$$
$$\rho_3 = \tfrac{3}{4}|0\rangle\langle 0| + \tfrac{1}{4}|1\rangle\langle 1|. \qquad (77)$$

In this case, $\|\rho_1 - \rho_2\|_1 = \frac{1}{2}$, $\|\rho_1 - \rho_3\|_1 = \frac{1}{4}$, $\|\rho_2 - \rho_3\|_1 = \frac{1}{4}$, and

$$\sum_{1\leqslant i<j\leqslant 3} \|\rho_i - \rho_j\|_1 = 1, \quad \min_{j=1,2,3}\left\{\sum_{i=1,2,3}\|\rho_i - \rho_j\|_1\right\} = \frac{1}{2}, \quad \max_{j=1,2,3}\left\{\sum_{i=1,2,3}\|\rho_i - \rho_j\|_1\right\} = \frac{3}{4}. \qquad (78)$$

Therefore, in the case considered, the values of the new bounds (53), (15), and (17) are equal to

$$\mathfrak{L}_{1,\mathrm{new}} = \tfrac{19}{48} = 0.3958, \quad \mathfrak{L}_{2,\mathrm{new}} = \tfrac{7}{18} \simeq 0.3889,$$
$$Q_{\mathrm{new}} = \tfrac{4}{9} \simeq 0.4444, \qquad (79)$$

while the values of the known bounds (51), (52), and (30) are

$$Q_4 = \tfrac{5}{12} \simeq 0.4166, \quad Q_2 = \tfrac{7}{12} \simeq 0.5833, \quad \mathfrak{L}_1 = \tfrac{1}{3}. \qquad (80)$$

Note that since states (77) mutually commute and are of the form $\rho_i = \sum_{n=0,1} \lambda_n^{(i)} |n\rangle\langle n|$, $i = 1, 2, 3$, the optimal success probability [4]

$$\mathrm{P}^{\mathrm{opt.success}}_{\rho_1,\rho_2,\rho_3|\frac{1}{3},\frac{1}{3},\frac{1}{3}} = \frac{1}{3}\sum_{n=0,1}\left(\max_{i=1,2,3}\lambda_n^{(i)}\right) = \frac{5}{12} = Q_4 \simeq 0.4166. \qquad (81)$$

For the calculation of the lower bounds (31) and (32) and the upper bounds (63) and (64) for equiprobable states (77), we find fidelities $F_{ij} := \|\sqrt{\rho_i}\sqrt{\rho_j}\|_1$ for states (77):

$$F_{12} = \frac{\sqrt{35}+\sqrt{3}}{8} \simeq 0.9560, \quad F_{13} = \frac{\sqrt{42}+\sqrt{2}}{8} \simeq 0,9868,$$
$$F_{23} = \frac{\sqrt{30}+\sqrt{6}}{8} \simeq 0,9909, \qquad (82)$$

and also the trace

$$\mathrm{tr}\left[\sqrt{\sum_{i=1,2,3}\rho_i^2}\right] = \frac{\sqrt{110}+\sqrt{14}}{8} \simeq 1,7787. \qquad (83)$$

Therefore,

$$Q_3 = 1 - \frac{1}{9}\sum_{1\leqslant i<j\leqslant 3} F_{ij}^2 \simeq 0.6812, \quad Q_5 = \frac{1}{3}\mathrm{tr}\left[\sqrt{\sum_{i=1,2,3}\rho_i^2}\right] \simeq 0,5929,$$
$$\mathfrak{L}_2 = 1 - \frac{1}{3}\sum_{1\leqslant i<j\leqslant 3} F_{ij} \simeq 0,0221, \quad \mathfrak{L}_3 = (Q_5)^2 \simeq 0.3515. \qquad (84)$$

From (79)–(84) it follows that, in the case considered,

$$\mathrm{P}^{\mathrm{opt.success}}_{\rho_1,\rho_2,\rho_3|\frac{1}{3},\frac{1}{3},\frac{1}{3}} = Q_4 < Q_{\mathrm{new}} < Q_2 < Q_5 < Q_3 \qquad (85)$$

and

$$\mathrm{P}^{\mathrm{opt.success}}_{\rho_1,\rho_2,\rho_3|\frac{1}{3},\frac{1}{3},\frac{1}{3}} > \mathfrak{L}_{1,\mathrm{new}} > \mathfrak{L}_{2,\mathrm{new}} > \mathfrak{L}_3 > \mathfrak{L}_1 > \mathfrak{L}_2, \qquad (86)$$

that is, the values of the new lower bounds $\mathfrak{L}_{\mathrm{new},1}$, $\mathfrak{L}_{\mathrm{new},2}$ and the new upper bound Q_{new} are tighter than the values of the known lower bounds (30)–(32) and the known upper bounds (52), (63), and (64), respectively.

V. CONCLUSIONS

In the present article, for the optimal success probability (5), we find for all $r \geqslant 2$: (i) the new general lower bounds (Theorem 2) and specify their relation (Proposition 1) to the general lower bounds (30)–(32) in Refs. [10,19,20]; and (ii) the new general upper bound (Theorem 3) and specify its relation (Proposition 2) to the general upper bounds (51), (52), (63), and (64) in Refs. [15–19].

We stress that, though the general upper bound (51) in Ref. [17] is tighter than our new general upper bound (53), for a large number $r \geqslant 2$ of quantum states to be discriminated, the straightforward calculation of the expression in bound (53) is much easier than finding the minimum in bound (51).

We also present the example where the values of the new general lower bounds (15)-(17) and the new general upper bound (53) on the optimal success probability are tighter than the values of the known general lower bounds (30)–(32) and the known general upper bounds (52), (63), and (64), respectively.

The new upper bound (53) on the optimal success probability has the form explicitly generalizing to $r > 2$ the Helstrom bound in (8) and is easily calculated. For $r = 2$, each of our new bounds, lower and upper, reduces to the Helstrom bound in (8), and this proves in the other way the Helstrom result (9).

ACKNOWLEDGMENTS

The author is grateful to A. S. Holevo and Min Namkung for useful discussions and acknowledge the valuable comments of the anonymous reviewers. The study presented in this article is supported within the framework of State Assignment of Russian Ministry of Science and Higher Education.

[1] C. W. Helstrom, Inf. Control **10**, 254 (1967).

[2] C. W. Helstrom, J. Liu, and J. Gordon, Proc. IEEE **58**, 1578 (1970).

[3] A. S. Holevo, Teor. Veroyatnost. Primenen. (in Russian) **18**, 442 (1973).

[4] A. S. Holevo, J. Multiv. Anal. **3**, 337 (1973).

[5] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic Press, San Diego, 1976).
[6] H. P. Yuen, R. S. Kennedy, and M. Lax, IEEE Trans. Inf. Theory **21**, 125 (1975).
[7] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1979).
[8] S. M. Barnett and S. Croke, Adv. Opt. Photon. **1**, 238 (2009).
[9] A. B. Coll, G. Vazquez-Vilar, and J. R. Fonollosa, arXiv:2007.07958.
[10] H. Barnum and E. Knill, J. Math. Phys. **43**, 2097 (2002).
[11] P. Hayden, D. Leung, and G. Smith, Phys. Rev. A **71**, 062339 (2005).
[12] A. Nayak and J. Salzman, J. ACM **53**, 184 (2006).
[13] M. Hayashi, A. Kawachi, and H. Kobayashi, Quantum Inf. Comput. **8**, 0345 (2008).
[14] W. Hwang and J. Bae, J. Math. Phys. **51**, 022202 (2010).
[15] D. Qiu, Phys. Rev. A **77**, 012328 (2008).
[16] A. Montanaro, in *IEEE Information Theory Workshop, ITW'08* (IEEE, Piscataway, NJ, 2008), p. 378.
[17] D. Qiu and L. Li, Phys. Rev. A **81**, 042329 (2010).
[18] T. Ogawa and H. Nagaoka, IEEE Trans. Inf. Theory **45**, 2486 (1999).
[19] J. Tyson, J. Math. Phys. **50**, 032106 (2009).
[20] K. M. R. Audenaert and M. Mosonyi, J. Math. Phys. **55**, 102201 (2014).
[21] K. Nakahira, T. S. Usuda, and K. Kato, Phys. Rev. A **97**, 012103 (2018).