Einstein-Podolsky-Rosen steering in two-sided sequential measurements with one entangled pair

Jie Zhu,[1,2] Meng-Jun Hu,[3,1,2,*] Chuan-Feng Li,[1,2] Guang-Can Guo,[1,2] and Yong-Sheng Zhang[1,2,†]
[1]*Key Laboratory of Quantum Information, University of Science and Technology of China, CAS, Hefei 230026, China*
[2]*CAS Center for Excellence in Quantum Information and Quantum Physics, Hefei 230026, China*
[3]*Beijing Academy of Quantum Information Sciences, Beijing 100089, China*

(Received 7 January 2022; accepted 28 February 2022; published 25 March 2022)

Nonlocality and quantum measurement are two fundamental topics in quantum theory and their interplay attracts intensive focuses since the discovery of Bell theorem. Recently, nonlocality sharing among multiple observers with one entangled pair has been predicted and experimentally observed by generalized quantum measurement—weak measurement. However, only the one-sided sequential case, i.e., one Alice and multiple Bobs, is widely discussed and little is known about the two-sided case. Here, we theoretically and experimentally explore the nonlocality sharing in the two-sided sequential measurements case in which one entangled pair is distributed to multiple Alices and Bobs. We experimentally observed double Einstein-Podolsky-Rosen (EPR) steering among four observers in a photonic system. In the case that all observers adopt the same measurement strength of the weak measurement, it is observed that double EPR steering can be demonstrated simultaneously. The results not only deepen our understanding of relation between sequential measurements and nonlocality but also may find important applications in many quantum information tasks, such as randomness certification.

DOI: 10.1103/PhysRevA.105.032211

I. INTRODUCTION

Nonlocality, which is the core characteristic of quantum theory [1], plays a fundamental role in many quantum information tasks. Bell nonlocality [2,3] and Einstein-Podolsky-Rosen (EPR) steering [4,5] are two extensively investigated notions that capture the quantum nonlocality in which EPR steering is proved to be a more general form than Bell nonlocality [6]. From the perspective of quantum information, both Bell nonlocality and EPR steering can be demonstrated by considering two separated observers, Alice and Bob, that perform local measurements on a shared quantum state ρ_{AB} and quantum nonlocality is witnessed via violation of corresponding inequalities. Recently, Silva *et al.* extended the Bell test to include one Alice and many Bobs with intermediate Bobs performing sequential weak measurements and showed that Bell nonlocality can be shared among multiple observers with one entangled pair [7]. Double Bell-Clauser-Horne-Shimony-Holt (Bell-CHSH) [8] inequality violations among three observers were then experimentally observed with one entangled photon pair by two independent groups including ours [9,10]. Based on this sequential scenario, lots of works have been reported [11–22] and EPR steering among multiple observers is experimentally demonstrated very recently [23].

To date, however, almost all discussions are limited to the one-sided sequential case, i.e., one entangled pair is distributed to one Alice and multiple Bobs. As emphasized bySilva *et al.* in their last sentence in Ref. [7], it would be interesting to investigate the two-sided sequential case, i.e., including multiple Alices in the setup. In this article, we theoretically and experimentally explore the two-sided sequential case that one entangled pair is distributed to multiple Alices and Bobs, in which middle Alices and Bobs perform optimal weak measurements and the last Alice and Bob perform projective measurement. The relation between sequential weak measurements and Bell nonlocality is explicitly derived under the unbiased input condition for the case of a two-sided sequential. It is shown that no more than two Bell-CHSH inequality violations can be obtained in the same method [24]. However, here the analytical forms of EPR steering are obtained for the case of two Alices and two Bobs, showing that Alice1-Bob1 and Alice2-Bob2 can demonstrate EPR steering simultaneously. Using an entangled photon pair, we experimentally observed double EPR steering simultaneously with $n = 6$ and $n = 10$ measurement settings in the case of two pairs of observers [4].

II. THEORETICAL FRAMEWORK

A. Weak measurement

Consider a two-party state ρ_{AB} distributed to Alices and Bobs who perform sequential weak measurements as shown in Fig. 1. For convenience of calculations and experimental realization, we choose $\rho_{AB} = |\Psi^-\rangle_{AB}\langle\Psi^-|$ with the singlet state $|\Psi^-\rangle_{AB} = (|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B)/\sqrt{2}$ and require that optimal weak measurements are performed.

For a projective measurement $\{|k^+\rangle, |k^-\rangle\}$ in a two-level system with $\langle k^+|k^-\rangle = 0$, the projective measurement operators are $P_+ = |k^+\rangle\langle k^+|$, $P_- = |k^-\rangle\langle k^-|$, and $\hat{\sigma}_k = P_+ - P_- = |k^+\rangle\langle k^+| - |k^-\rangle\langle k^-|$.

*humj@baqis.ac.cn
†yshzhang@ustc.edu.cn

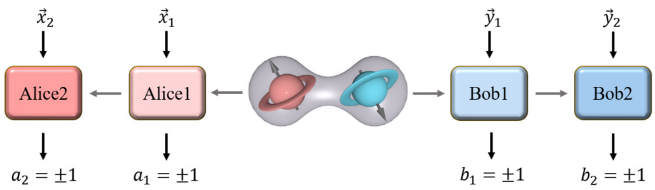

FIG. 1. Theoretical sketch. The two-sided sequential scenario in which one entangled pair is distributed to multiple Alices and Bobs. Here $\vec{x}_i, \vec{y}_j$ represent measurement inputs and a_i, b_j are corresponding dichotomic measurement outcomes, respectively.

For a weak measurement, the positive-operator valued measurement [25] can be written as Kraus operators

$$\hat{M}_{\pm} = \cos(\theta)|k^{\pm}\rangle\langle k^{\pm}| + \sin(\theta)|k^{\mp}\rangle\langle k^{\mp}|, \tag{1}$$

where parameter $\theta \in [0, \pi/4]$ determines the strength of measurement. When $\theta = 0$, $\hat{M}_{\pm 1|\vec{k}}$ reduces to the projector $|k^{\pm}\rangle\langle k^{\pm}|$ corresponding to project measurement, while $\theta = \pi/4$ gives $\hat{M}_{\pm 1|\vec{k}} = \hat{I}/\sqrt{2}$ representing no measurement at all. Two quantities are of particular interest in weak measurement, which are quality factor F measuring the disturbance of measurement and information gain G and they satisfy the trade-off relation $F^2 + G^2 \leqslant 1$ [7]. For an initial state $|\psi\rangle$, the result of the weak measurement is

$$\begin{aligned}\rho_{\pm} &= \hat{M}_{\pm}|\psi\rangle\langle\psi|\hat{M}_{\pm}^{\dagger}/\mathrm{Tr}(\hat{M}_{\pm}|\psi\rangle\langle\psi|\hat{M}_{\pm}^{\dagger}) \\ &= F|\psi\rangle\langle\psi| + (1-F)(P_{+}|\psi\rangle\langle\psi|P_{+} + P_{-}|\psi\rangle\langle\psi|P_{-}),\end{aligned} \tag{2}$$

with probability $P(\pm) = \mathrm{Tr}(\hat{M}_{\pm}|\psi\rangle\langle\psi|\hat{M}_{\pm}^{\dagger}) = G\langle\psi|P_{\pm}|\psi\rangle + \frac{1}{2}(1-G)$. The meanings of factors F and G are described in Ref. [7] and will be illustrated in the following.

For an initial two-level state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$, a projective measurement can be realized by coupling a pointer state that can be written as

$$(\alpha|0\rangle + \beta|1\rangle)|0\rangle_{\text{pointer}} \rightarrow \alpha|0\rangle|+\rangle_{\text{pointer}} + \beta|1\rangle|-\rangle_{\text{pointer}} \tag{3}$$

with $\langle +|-\rangle = 0$. However, for a weak measurement, the evolution of the pointer state is

$$\begin{aligned}|\psi\rangle &= (\alpha|0\rangle + \beta|1\rangle)|0\rangle_{\text{pointer}} \\ \rightarrow |\psi'\rangle &= \alpha|0\rangle|+'\rangle_{\text{pointer}} + \beta|1\rangle|-'\rangle_{\text{pointer}},\end{aligned} \tag{4}$$

where $|+'\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle$ and $|-'\rangle = \sin(\theta)|0\rangle + \cos(\theta)|1\rangle$, with $\langle +'|-'\rangle = \sin(2\theta)$ indicating the strength of the weak measurement. Then a projective measurement with the basis $\{|0\rangle, |1\rangle\}$ on the pointer state is performed. Hereto a completed weak measurement of $\hat{M}_{+} = \cos(\theta)|0\rangle\langle 0| + \sin(\theta)|1\rangle\langle 1|$ and $\hat{M}_{-} = \cos(\theta)|1\rangle\langle 1| + \sin(\theta)|0\rangle\langle 0|$ is fulfilled. The probabilities of two outcomes are $P(+) = |\langle 0|\psi'\rangle|^2$ and $P(-) = |\langle 1|\psi'\rangle|^2$.

In our experiment, the factors F and G are defined the same as in Refs. [7,9]. F denotes the disturbance of the measurement, which can be written as $F = \langle +'|-'\rangle$. G denotes the information gain of the measurement, which can be written as $G = 1 - |\langle 0|-'\rangle|^2 - |\langle 1|+'\rangle|^2$, where $|\langle 0|-'\rangle|^2$ and $|\langle 1|+'\rangle|^2$ are error rates of the weak measurement. Here $F = \sin 2\theta$, $G = \cos 2\theta$ and the optimal condition, $F^2 + G^2 = 1$, is satisfied.

B. EPR steering in two-sided sequential measurement

The quantum nonlocality can be witnessed via violations of corresponding inequalities. The quantitative measurement of quantum correlation needs to be calculated to see whether or not they can surpass the threshold supported by local hidden variables and local hidden states theory [8,26]. It is shown in the following that these quantities are deeply connected to the quality factor F and information gain G of weak measurements. Bell quantity I and EPR steering quantity S both are determined by the two-party correlation $C_{(\vec{x},\vec{y})} = \sum_{a,b} abP(a, b|\vec{x}, \vec{y})$. In order to obtain a general process of calculations, we first consider the joint conditional probability distribution of four observers in the two-sided sequential case, which is given as

$$\begin{aligned}&P(a_1, a_2, b_1, b_2|\vec{x}_1, \vec{x}_2, \vec{y}_1, \vec{y}_2) \\ &\quad = \mathrm{Tr}\big[\big(\hat{H}_{a_1,a_2|\vec{x}_1,\vec{x}_2} \otimes \hat{H}_{b_1,b_2|\vec{y}_1,\vec{y}_2}\big)\rho_{AB}\big],\end{aligned} \tag{5}$$

where $\hat{H}_{a_1,a_2|\vec{x}_1,\vec{x}_2} \equiv \hat{M}_{a_1|\vec{x}_1}^{\dagger}\hat{\Pi}_{a_2|\vec{x}_2}\hat{M}_{a_1|\vec{x}_1}$ where $\hat{\Pi}$ represents the projection operator and $\hat{H}_{b_1,b_2|\vec{y}_1,\vec{y}_2}$ is defined in the same way. The joint conditional probability distribution of any two observers is

$$\begin{aligned}&P(a_i, b_j|\vec{x}_i, \vec{y}_j) \\ &\quad = \sum_{a_{i'}b_{j'}\vec{x}_{i'}\vec{y}_{j'}} P(\vec{x}_{i'}, \vec{y}_{j'})P(a_i, a_{i'}, b_j, b_{j'}|\vec{x}_i, \vec{x}_{i'}, \vec{y}_j, \vec{y}_{j'})\end{aligned} \tag{6}$$

with $i, i', j, j' \in \{1, 2\}$ and $i \neq i', j \neq j'$. Since Alices and Bobs are independent observers, $P(\vec{x}_i, \vec{y}_j) = P(\vec{x}_i)P(\vec{y}_j)$ and $P(\vec{x}_i) = P(\vec{y}_j) = 1/n$ for unbiased inputs with n the number of measurement settings. Defining the correlation observable as

$$\hat{W}_{(\vec{x}_i,\vec{y}_j)} = \sum_{a_1,a_2,\vec{x}_{i'},b_1,b_2,\vec{y}_{j'}} a_i b_j P(\vec{x}_{i'}, \vec{y}_{j'})\hat{H}_{a_1,a_2|\vec{x}_1,\vec{x}_2} \otimes \hat{H}_{b_1,b_2|\vec{y}_1,\vec{y}_2}, \tag{7}$$

we can obtain the correlation

$$C_{(\vec{x}_i,\vec{y}_j)} = \mathrm{Tr}\big[\hat{W}_{(\vec{x}_i,\vec{y}_j)}\rho_{AB}\big]. \tag{8}$$

The definition of Eq. (4) can also be used for multiple observers with the generalized definition

$$\begin{aligned}&\hat{H}_{a_1,\ldots,a_N|\vec{x}_1,\ldots,\vec{x}_N} \\ &\quad = \hat{M}_{a_1|\vec{x}_1}^{\dagger}\cdots\hat{M}_{a_{N-1}|\vec{x}_{N-1}}^{\dagger}\hat{\Pi}_{a_N|\vec{x}_N}\hat{M}_{a_{N-1}|\vec{x}_{N-1}}\cdots\hat{M}_{a_1|\vec{x}_1}\end{aligned} \tag{9}$$

and $\hat{H}_{b_1,\ldots,b_N|\vec{y}_1,\ldots,\vec{y}_N}$ is defined in the same way as above.

The situation of EPR steering in the two-sided sequential case is more complicated compared to Bell nonlocality due to the asymmetry of EPR steering. As a demonstration, here the calculations are limited only to the case of two Alices and two Bobs, in which we ask whether or not Alice1-Bob1 and Alice2-Bob2 can demonstrate EPR steering simultaneously with Alice2 and Bob2 performing projective measurements. EPR steering quantity S and corresponding classical bound B for n measurement settings [4] can be defined

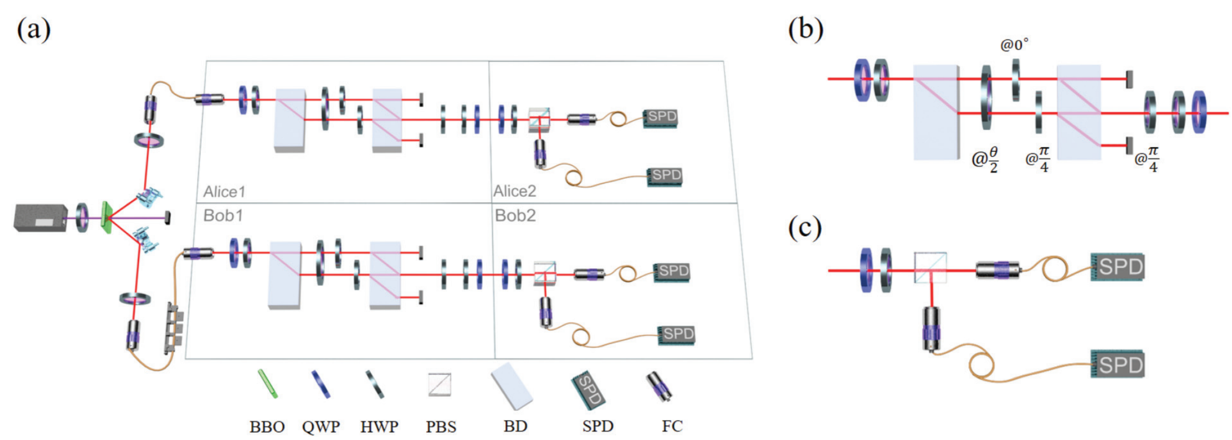

FIG. 2. Experimental setup. (a) Polarization-entangled photon pairs are generated via the type I phase-matching spontaneous parametric down-conversion process by pumping a joint β-barium-borate (BBO) crystal with a 404 nm semiconductor laser. Signal and idler photons are then distributed to Alices and Bobs with Alice1 and Bob1 performing optimal weak measurements and Alice2 and Bob2 performing projective measurements. (b) Setup for realizing optimal weak measurement. (c) Setup for realizing projective measurement. BBO: β-barium-borate; QWP: quarter-wave plate; HWP: half-wave plate; PBS: polarization beam splitter; BD: beam displacer; SPD: single-photon detector; FC: fiber coupler.

as [26]

$$S_n = \frac{1}{n}\left|\sum_{m=1}^{n} C_{(\vec{x}^m,\vec{y}^m)}\right|,$$

$$B_n = \max_{\{A_m\}}\left\{\lambda_{\max}\left(\frac{1}{n}\sum_{m=1}^{n} A_m\hat{\sigma}_m^B\right)\right\}, \tag{10}$$

whereas $S_n > B_n$ refutes any local hidden states theory. Here $A_m \in \{-1, 1\}$ represents Alice's declared result for the mth measurement setting of Bob's and $\lambda_{\max}(\hat{O})$ denotes the largest eigenvalue of $\hat{O}$. Detailed calculations give

$$S_{n_1}^{A_1-B_1} = G_{A_1}G_{B_1}. \tag{11}$$

After the measurement of Alice1 and Bob1, the state becomes

$$\rho_{\vec{k}}^{A1-B1} = \frac{1}{n_1}\sum_{\vec{k}}\sum_{i,j\in\{+,-\}}\left(\hat{M}_{\vec{k}}^i\otimes\hat{M}_{\vec{k}}^j\right)|\Psi\rangle_{AB}\langle\Psi|\left(\hat{M}_{\vec{k}}^i\otimes\hat{M}_{\vec{k}}^j\right)^\dagger, \tag{12}$$

where $\vec{k}$ denotes the measurement direction of Alice1 and Bob1. Then the detailed form of the steering quantity between Alice2 and Bob2 can be obtained:

$$S_{n_2}^{A_2-B_2} = 1 - 4\left(1-\frac{1}{2}F_{A_1}F_{B_1}\right)\frac{\sum_{k,l}|\langle k^+|l^+\rangle|^2|\langle k^-|l^+\rangle|^2}{n_1n_2} - 2F_{A_1}F_{B_1}\frac{\sum_{k,l}\mathrm{Re}[\langle l^-|k^+\rangle\langle l^+|k^-\rangle\langle k^-|l^-\rangle\langle k^+|l^+\rangle]}{n_1n_2}, \tag{13}$$

where $\sum_{k,l}$ denotes double summation $\sum_{k=1}^{n_1}\sum_{l=1}^{n_2}$ where n_1 and n_2 are numbers of measurement settings for Bob1 and Bob2, respectively, and $\{|k^\pm\rangle\}$ ($\{|l^\pm\rangle\}$) is the measurement basis of A1-B1 (A2-B2). Since the distributed state is the singlet state $|\psi^-\rangle_{AB}$, the measurement directions of Alice and Bob are chosen to be opposite to maximize S_n. When measurement settings are settled $S_n^{A_2-B_2}$ is only determined by $F_{A_1}F_{B_1}$. Consider the case of $n = 3$ in which measurement settings are chosen as $\{X, Y, Z\}$ and measurement strength θ is the same for Alice1 and Bob1; we can obtain that $S_3^{A_1-B_1} = G^2$, $S_3^{A_2-B_2} = 1 - 2G^2/3$ with $G = \cos 2\theta$. The corresponding classical bound is $B_3 = 1/\sqrt{3}$ and thus Alice1-Bob1 and Alice2-Bob2 can both demonstrate EPR steering when $G \in (0.7598, 0.7962)$. In practice, larger n is needed to obtain more violations. It should be emphasized here that in the case in which one-sided sequential multiple EPR steering refers to multiple Bobs aim at steering the state of one Alice, all Bobs have to choose the same measurement settings [12,20,23] to steer Alice simultaneously. In the two-sided sequential case, however, Bobs aim at steering the corresponding Alices respectively and their choice of measurement settings is thus independent of each other.

III. EXPERIMENTAL REALIZATION

We now describe the experimental setup to observe nonlocality sharing among four observers. As shown in Fig. 2(a), a 404 nm semiconductor laser with 100 mW power is used to pump a joint β-barium-borate (BBO) crystal to produce the polarization-entangled photon pairs via the type I phase-matching spontaneous parametric down-conversion process [27]. By adjusting wave plates placed before the BBO crystal, the singlet state $|\Psi^-\rangle = (|H\rangle|V\rangle - |V\rangle|H\rangle)/\sqrt{2}$ is generated where $|H\rangle$ and $|V\rangle$ refer to horizontal and vertical polarization states, respectively. The fidelity of the entangled pair state is measured to be $98.76 \pm 0.08\%$ [28]. Each half of the entangled pair is coupled into different optical fibers and then distributed to Alices and Bobs with Alice1 and Bob1 performing optimal weak measurements and Alice2 and Bob2 performing projective measurements. Coincidence events between four detectors are registered by avalanche photodiode single-photon detectors and a coincidence counter. The joint probability distributions for different measurement settings and outcomes are extracted from these coincidence counts within 10 s integral time.

As the core part of the experimental setup, Fig. 2(b) realizes optimal weak measurements described by Kraus operators $\hat{M}_{\pm 1|\vec{k}}$ in Eq. (1). The basic idea of the setup is to first transform the measurement basis $\{|k^+\rangle, |k^-\rangle\}$ into basis $\{|H\rangle, |V\rangle\}$ via the basis converter consisting of a quarter-wave plate and a half-wave plate (HWP). The interference between two beam displacers (BDs) then realizes optimal weak measurements $\hat{M}_{\pm 1|\vec{z}}$ with $\hat{\sigma}_{\vec{z}} \equiv |H\rangle\langle H| - |V\rangle\langle V|$ [9,25]. Lastly, another basis converter is used to transform $\{|H\rangle, |V\rangle\}$ back into the measurement basis. To be specific, an input state $|\phi_k\rangle = \alpha|k^+\rangle + \beta|k^-\rangle$ passes the basis converter placed before BD1 and becomes $|\phi_z\rangle = \hat{R}|\phi_k\rangle = \alpha|H\rangle + \beta|V\rangle$. Photons with polarization $|H\rangle$ are deflected down after passing BD, while nothing happens for $|V\rangle$. BD1 can be used to couple the path and polarization degrees of freedom of photons that $\hat{U}_{\mathrm{BD}}|\phi_z\rangle = \alpha|H\rangle|d\rangle + \beta|V\rangle|u\rangle$ with $|d\rangle, |u\rangle$ representing the down and the up path between two BDs, respectively. With operations of HWPs the state of photons before BD2 can be written as $\alpha(\cos\theta|V\rangle + \sin\theta|H\rangle)|d\rangle + \beta(\cos\theta|V\rangle + \sin\theta|H\rangle)|u\rangle$. Since only the middle path out of BD2 is retained, the components $|V\rangle|d\rangle$ and $|H\rangle|u\rangle$ in state $|\varphi\rangle$ are postselected and the path degree of freedom is eliminated. With a HWP fixed at $\pi/4$ placed after BD2 the state of photons becomes $\alpha\cos\theta|H\rangle + \beta\sin\theta|V\rangle = \hat{M}_{+1|\vec{z}}|\phi_z\rangle$. With another basis converter applied subsequently, the full setup completes the $\hat{M}_{+1|\vec{k}}$ operation corresponding to the $+1$ outcome of the measurement.. By adjusting the HWP after BD1 from $\theta/2$ to $\pi/4 - \theta/2$, operation $\hat{M}_{-1|\vec{k}}$ corresponding to the -1 outcome is realized [9].

We experimentally explore nonlocality sharing in the two-sided sequential case with two Alices and two Bobs and the results are given in Fig. 3. We first choose $n = 6$ measurement settings in the EPR steering scenario such that $B_{n=6} = 0.5393$ [26]. Then we consider the case that Alice1 and Bob1 measure with the $n = 6$ setting but Alice2 and Bob2 measure with the $n = 10$ setting. In the $(\theta_{A_1}, \theta_{B_1})$ parameter space we have chosen different points with equal strength that $\theta_{A_1} = \theta_{B_1} \in \{\pi/36, \pi/12, 0.34, 5\pi/36, 7\pi/36\}$. Specifically, the double EPR steering simultaneously can be clearly observed when $\theta_{A_1} = \theta_{B_1} = 0.34$. The measured nonlocality quantities support theoretical predictions with errors mainly coming from the Poisson distribution of photon counting and the imperfection of optical elements. It is clearly shown that while Alice1-Bob1 and Alice2-Bob2 cannot demonstrate Bell nonlocality simultaneously (see Appendix), they can both demonstrate EPR steering with the proper choice of measurement strength. It is interesting to point out that Alice2 and Bob2 can demonstrate one-way EPR steering if Bob1 performs no measurement and Alice1 preforms proper weak measurement [29].

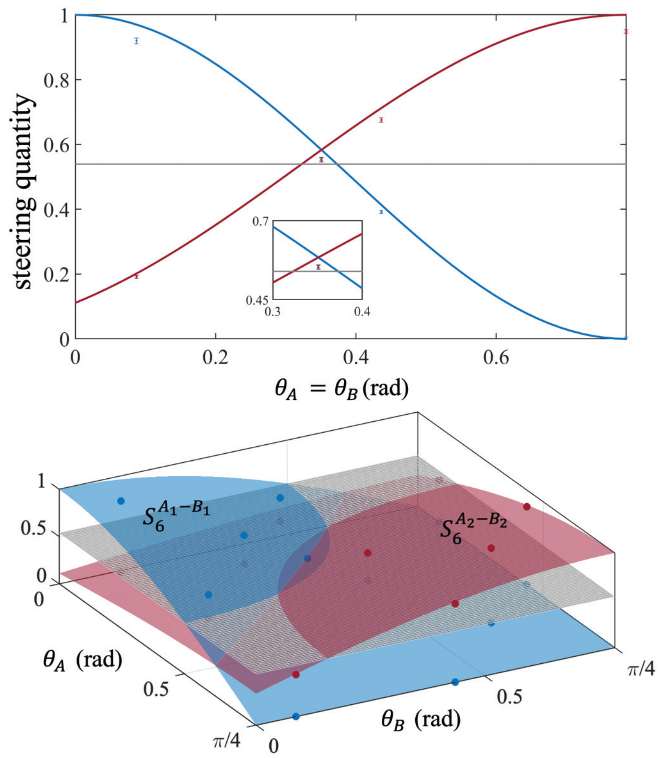

FIG. 3. Experimental results. The steering quantities $S_6^{A_1-B1}$, $S_6^{A_2-B_2}$, and $S_{10}^{A_2-B_2}$ are measured with different measurement strengths of Alice1 and Bob1 [$G_{A_1} = \cos(2\theta_{\mathrm{A}})$ and $G_{B_1} = \cos(2\theta_{\mathrm{B}})$]. In the upper panel, the results, that Alice1's and Bob1's measurement strengths are equal, are presented. The blue line and dots are the theoretical prediction and experimental results of $S_6^{A_1-B_1}$, respectively. The red line represents the theoretical predictions of $S_n^{A_2-B_2}$ with $n = \{6, 10\}$, and the corresponding experimental results are denoted by the red rhombus and green triangle that almost overlap. The two horizontal lines are the bounds of $B_6 = 0.5393$ and $B_{10} = 0.5236$. The maximal simultaneous violation is observed when $\theta_A = \theta_B = 0.34$ and the violation values are 0.0493 ± 0.0029 of $S_6^{A_1-B_1}$, 0.0485 ± 0.0020 of $S_6^{A_2-B_2}$, and 0.0636 ± 0.0023 of $S_{10}^{A_2-B_2}$. The error bars come from the Poissonian distribution of photon count that are too small to present in the figure. In the lower panel, the more experimental results with different G_{A_1} and G_{B_1} are presented. The blue and red surfaces denote the theoretical values of $S_6^{A_1-B_1}$ and $S_6^{A_2-B_2}$, and the blue and red dots are the corresponding experimental results. The gray plane denotes the bound $B_6 = 0.5393$.

IV. DISCUSSION AND CONCLUSION

In summary, we have explored theoretically and experimentally nonlocality sharing in the two-sided sequential case with one entangled pair distributed to multiple Alices and Bobs. We obtain the explicit formula that relates sequential optimal weak measurements and Bell quantity including the one-sided sequential case as a special situation. For the one-sided sequential case, it has been shown there exists measurement protocols to demonstrate arbitrarily many Bell-CHSH inequality violations with biased inputs [7] or unequal sharpness measurement to various Bobs [22]. It would be interesting to investigate whether or not such measurement protocols exist in the two-sided sequential case. Due to the asymmetry of Alice and Bob and the freedom of choosing measurement settings, it remains an open question that whether or not there exists an elegant analytical formula for EPR steering. Specifically, it would be interesting to investigate whether or not more than two pairs of Alice-Bob can demonstrate EPR steering simultaneously in the two-sided sequential case. Using an entangled photon pair, we experimentally verify the case of two Alices and two Bobs in

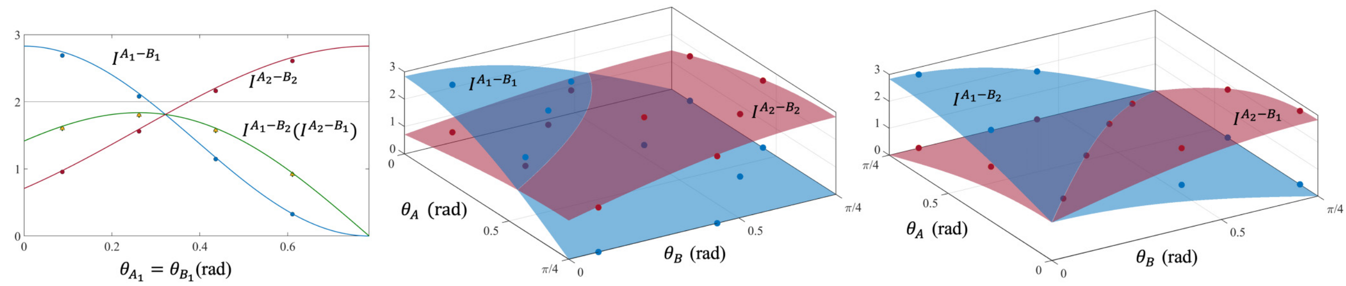

FIG. 4. Experimental results of the sequential Bell test. The results verified Eq. (A2) in which $G_{A_2}=G_{B_2}=1$ due to Alice2 and Bob2 performing projective measurements. Double Bell-CHSH inequality violations are observed only when Alice1 or Bob1 perform almost no measurement such that the situation is equivalent to the one-sided sequential case. When Alice1 and Bob1 adopt the *same measurement strength*, double Bell-CHSH inequality violations cannot be obtained.

which Alice1 and Bob1 perform optimal weak measurements and Alice2 and Bob2 perform projective measurements. For Alice1 and Bob1 adopting the same measurement strength, we observed double EPR steering simultaneously while it is shown that double Bell-CHSH inequality violations cannot be obtained. The results presented here not only shed new light on the understanding of the interplay between quantum measurement and nonlocality, but also may have important applications such as unbounded randomness certification [11,30–32], randomness access code [33,34] and one-sided device independent quantum key distribution [35–38].

Note added. Recently, a more general conclusion about Bell-CHSH inequality in a sequential measurement structure has been noted in [24].

ACKNOWLEDGMENTS

M.-J.H. acknowledges H. M. Wiseman, Eric Calvencanti, and Michael J. W. Hall for valuable discussions. Part of the theoretical work was done during his visit to Griffith University. J.Z. acknowledges X.-J. Ye, Y. Xiao, S. Cheng, and R. Wang for helpful discussions. This work is funded by the National Natural Science Foundation of China (Grant No. 92065113) and Anhui Initiative in Quantum Information Technologies.

APPENDIX A: BELL NONLOCALITY IN THE CASE OF TWO-SIDED SEQUENTIAL MEASUREMENTS

Bell quantity I in the Bell test scenario is defined as

$$I=|C_{(\vec{x}^0,\vec{y}^0)}+C_{(\vec{x}^0,\vec{y}^1)}+C_{(\vec{x}^1,\vec{y}^0)}-C_{(\vec{x}^1,\vec{y}^1)}|, \tag{A1}$$

whereas $I>2$ refutes any local hidden variables theory (Fig. 4). Measurement settings are usually chosen as $\vec{x}^0=Z$, $\vec{x}^1=X$, $\vec{y}^0=(X-Z)/\sqrt{2}$, $\vec{y}^1=-(X+Z)/\sqrt{2}$ to reach the Tsirelson's bound of $2\sqrt{2}$. In the case of two Alices and two Bobs where optimal weak measurements are performed by observers, calculations based on the method of Ref. [7] give

$$\begin{aligned} I^{A_1-B_1} &= 2\sqrt{2}G_{A_1}G_{B_1},\\ I^{A_2-B_2} &= \frac{\sqrt{2}}{2}(1+F_{A_1})G_{A_2}(1+F_{B_1})G_{B_2},\\ I^{A(B)_1-B(A)_2} &= \sqrt{2}G_{A(B)_1}(1+F_{B(A)_1})G_{B(A)_2}. \end{aligned} \tag{A2}$$

Due to symmetry configuration, the Bell quantities of Alice1-Bob2 and Bob1-Alice2 have the same form. The results are compatible with the one-sided sequential case obtained by Silva *et al.* if Alice1 performs no measurement with $F_{A_1}=1, G_{A_1}=0$ and Alice2 and Bob2 perform projective measurements with $G_{A_2}=G_{B_2}=1$. It can be shown from the above equations that double Bell-CHSH inequality violations happen only when Alice1 or Bob1 performs almost no measurement and the situation is very close to the one-sided case. Furthermore, Alice1-Bob1 and Alice2-Bob2 cannot demonstrate Bell nonlocality simultaneously. For the more general case with arbitrary N Alices and M Bobs, the explicit analytical form of Bell quantity for arbitrary Alice and Bob also can be derived. It is concluded that no more than double Bell-CHSH inequality violations can be obtained in this scenario under the unbiased input condition.

Similarly, in the case of the two-sided sequential case in which one entangled pair is distributed to arbitrarily many N Alices and M Bobs for sequential optimal weak measurements, the Bell quantity for arbitrary Alice and Bob, under the unbiased input condition, satisfies

$$\begin{aligned} I^{A_r-B_s} = &\frac{2\sqrt{2}}{2^{(r-1)}\times 2^{(s-1)}}(1+F_{A_1})\cdots(1+F_{A_{r-1}})G_{A_r}\\ &\times(1+F_{B_1})\cdots(1+F_{B_{s-1}})G_{B_s} \end{aligned} \tag{A3}$$

with $r\leqslant N$, $s\leqslant M$, and $G_{A_N}=G_{B_M}=1$ if the last Alice and Bob perform projective measurements. For $N=1$, it naturally reduces to the one-sided sequential case with one Alice and multiple Bobs.

[1] A. Einstein, B. Podolsky, and N. Rosen, Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?, Phys. Rev. **47**, 777 (1935).

[2] J. S. Bell, On the Einstein-Podolsky-Rosen paradox, Phys. Phys. Fiz. **1**, 195 (1964).

[3] N. Brunner, D. Cavalcanti, S. Pironio, V. Scarani and S. Wehner, "Bell nonlocality," Rev. Mod. Phys. **86**, 419 (2014).

[4] H. M. Wiseman, S. J. Jones, and A. C. Doherty, Steering, Entanglement, Nonlocality, and the Einstein-Podolsky-Rosen Paradox, Phys. Rev. Lett. **98**, 140402 (2007).

[5] R. Uola, A. C. S. Costa, H. C. Nguyen, and O. Gühne, Quantum steering, Rev. Mod. Phys. **92**, 015001 (2020).

[6] S. J. Jones, H. M. Wiseman, and A. C. Doherty, Entanglement, Einstein-Podolsky-Rosen correlations, Bell nonlocality, and steering, Phys. Rev. A **76**, 052116 (2007).

[7] R. Silva, N. Gisin, Y. Guryanova, and S. Popescu, Multiple Observers Can Share the Nonlocality of Half of an Entangled Pair by Using Optimal Weak Measurements, Phys. Rev. Lett. **114**, 250401 (2015).

[8] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, Proposed Experiment to Test Local Hidden-Variable Theories, Phys. Rev. Lett. **23**, 880 (1969).

[9] M.-J. Hu, Z.-Y. Zhou, X.-M. Hu, C.-F. Li, G.-C. Guo, and Y.-S. Zhang, Observation of non-locality sharing among three observers with one entangled pair via optimal weak measurement, npj Quantum Inf. **4**, 63 (2018).

[10] M. Schiavon, L. Calderaro, M. Pitaluga, G. Vallone, and P. Villoresi, Three-observer Bell inequality violation on a two-qubit entangled state, Quantum Sci. Technol. **2**, 015010 (2017).

[11] F. J. Curchod, M. Johansson, R. Augusiak, M. J. Hoban, P. Wittek, and A. Acín, Unbounded randomness certification using sequences of measurements, Phys. Rev. A **95**, 020102(R) (2017).

[12] S. Sasmal, D. Das, S. Mal, and A. S. Majumdar, Steering a single system sequentially by multiple observers, Phys. Rev. A **98**, 012305 (2018).

[13] S. Datta and A. S. Majumdar, Sharing of nonlocal advantage of quantum coherence by sequential observers, Phys. Rev. A **98**, 042311 (2018).

[14] K. Mohan, A. Tavakoli, and N. Brunner, Sequential random access codes and self-testing of quantum measurement instruments, New J. Phys. **21**, 083034 (2019).

[15] H. W. Li, Y. S. Zhang, X. B. An, Z. F. Han, and G. C. Guo, Three-observer classical dimension witness violation with weak measurement, Commun. Phys. **1**, 10 (2018).

[16] X. B. An, H. W. Li, Z. Q. Yin, M. J. Hu, W. Huang, B. J. Xu, S. Wang, W. Chen, G. C. Guo, and Z. F. Han, Experimental three-party quantum random number generator based on dimension witness violation and weak measurement, Opt. Lett. **43**, 3437 (2018).

[17] G. Foletto, L. Calderaro, A. Tavakoli, M. Schiavon, F. Picciariello, A. Cabello, P Villoresi, and G. Vallone, Experimental Certification of Sustained Entanglement and Nonlocality after Sequential Measurements, Phys. Rev. Appl. **13**, 044008 (2020).

[18] A. G. Maity, D. Das, A. Ghosal, A. Roy, and A. S. Majumdar, Detection of genuine tripartite entanglement by multiple sequential observers, Phys. Rev. A **101**, 042340 (2020).

[19] A. Kumari and A. K. Pan, Sharing nonlocality and nontrivial preparation contextuality using the same family of Bell expressions, Phys. Rev. A **100**, 062130 (2019).

[20] A. Shenoy H., S. Designolle, F. Hirsch, R. Silva, N. Gisin, and N. Brunner, Unbounded sequence of observers exhibiting Einstein-Podolsky-Rosen steering, Phys. Rev. A **99**, 022317 (2019).

[21] C. L. Ren, T. F. Feng, D. Yao, H. F. Shi, J. L. Chen, and X. Q. Zhou, Passive and active nonlocality sharing for a two-qubit system via weak measurements, Phys. Rev. A **100**, 052121 (2019).

[22] P. J. Brown, and R. Colbeck, Arbitrarily Many Independent Observers Can Share the Nonlocality of a Single Maximally Entangled Qubit Pair, Phys. Rev. Lett. **125**, 090401 (2020).

[23] Y.-H. Choi, S. Hong, T. Pramanik, H.-T Lim, Y.-S. Kim, H Jung, S.-W. Han, S. Moon, and Y.-M. Cho, Demonstration of simultaneous quantum steering by multiple observers via sequential weak measurements, Optica **7**, 675 (2020).

[24] S. Cheng, L. Liu, and M. J. W. Hall, Limitations on sharing Bell nonlocality between sequential pairs of observers, Phys. Rev. A **104**, L060201 (2021).

[25] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University, Cambridge, NY, 2000).

[26] D. J. Saunders, S. J. Jones, H. M. Wiseman, and G. J. Pryde, Experimental EPR-steering using Bell-local states, Nat. Phys. **6**, 845 (2010).

[27] P. G. Kwiat, E. Waks, A. G. White, I. Appelbaum, and P. H. Eberhard, Ultrabright source of polarization-entangled photons, Phys. Rev. A **60**, 773(R) (1999).

[28] D. F. V. James, P. G. Kwiat, W. J. Munro, and A. G. White, Measurement of qubits, Phys. Rev. A **64**, 052312 (2001).

[29] Y. Xiao, X.-J. Ye, K. Sun, J.-S. Xu, C.-F. Li, and G.-C. Guo, Demonstration of Multisetting One-Way Einstein-Podolsky-Rosen Steering in Two-Qubit Systems, Phys. Rev. Lett. **118**, 140404 (2017).

[30] B. Coyle, E. Kashefi, and M. J. Hoban, Certified randomness from steering using sequential measurements, Cryptography **3**, 27 (2019).

[31] J. Bowles, F. Baccari, and A. Salavrakos, Bounding sets of sequential quantum correlations and device-independent randomness certification, Quantum **4**, 344 (2020).

[32] G. Foletto, M. Padovan, M. Avesani, H. Tebyanian, P. Villoresi, and G. Vallone, Experimental test of sequential weak measurements for certified quantum randomness extraction, Phys. Rev. A **103**, 062206 (2021).

[33] D. Das, A. Ghosal, S. Kanjilal, A. G. Maity, and A. Roy, Unbounded pairs of observers can achieve quantum advantage in random access codes with a single pair of qubits, Phys. Rev. A **104**, L060602 (2021).

[34] G. Foletto, L. Calderaro, G. Vallone, and P. Villoresi, Experimental demonstration of sequential quantum random access codes, Phys. Rev. Res. **2**, 033205 (2020).

[35] C. Branciard, E. G. Cavalcanti, S. P. Walborn, V. Scarani, and H. M. Wiseman, One-sided device-independent quantum key distribution: Security, feasibility, and the connection with steering, Phys. Rev. A **85**, 010301(R) (2012).

[36] A. J. Bennet, D. A. Evans, D. J. Saunders, C. Branciard, E. G. Cavalcanti, H. M. Wiseman, and G. J. Pryde, Arbitrarily Loss-Tolerant Einstein-Podolsky-Rosen Steering Allowing a Demonstration over 1 km of Optical Fiber with No Detection Loophole, Phys. Rev. X **2**, 031003 (2012).

[37] N. Walk, S. Hosseini, J. Geng, O. Thearle, J. Yan Haw, S. Armstrong, S. M. Assad, J. Janousek, T. C. Ralph, T. Symul, H. M. Wiseman, and P. K. Lam, Experimental demonstration of Gaussian protocols for one-sided device-independent quantum key distribution, Optica **3**, 634 (2016).

[38] T. Gehring *et al.*, Implementation of continuous-variable quantum key distribution with composable and one-sided-device-independent security against coherent attacks, Nat. Commun. **6**, 8795 (2015).