


Generalized phase-space description of nonlinear Hamiltonian systems and Harper-like dynamics

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Phase-space features of the Wigner flow for generic one-dimensional systems with a Hamiltonian $H^W(q, p)$ constrained by the $\partial^2 H^W / \partial q \partial p = 0$ condition are analytically obtained in terms of Wigner functions and Wigner currents. Liouvillian and stationary profiles are identified for thermodynamic (TD) and Gaussian quantum ensembles to account for exact corrections due to quantum modifications over a classical phase-space pattern. General results are then specialized to the Harper Hamiltonian system, which, besides working as a feasible test platform for the framework here introduced, admits a statistical description in terms of TD and Gaussian ensembles, where the Wigner flow properties are all obtained through analytical tools. Quantum fluctuations over the classical regime are therefore quantified through probability and information fluxes whenever the classical Hamiltonian background is provided. Besides allowing for a broad range of theoretical applications, our results suggest that such a generalized Wigner approach works as a probe for quantumness and classicality of Harper-like systems in a framework which can be extended to any quantum system described by Hamiltonians in the form of $H^W(q, p) = K(p) + V(q)$.

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I. INTRODUCTION

The Weyl-Wigner (WW) [1–4] phase-space formalism of quantum mechanics (QM) encompasses the dynamics of quantum systems and offers not only an enlarged but also an equivalent description of QM in terms of quasiprobability distribution functions of position and momentum coordinates. It provides subtle insights into the boundaries between quantum and classical physics as well as a straightforward access to quantum information issues [5–8]. Without affecting the predictive power of QM, the WW formalism can also be thought of as the bridge between operator methods and path-integral techniques [9–11] encoded by a Weyl transform operation over a quantum operator \hat{O} , which is defined by

$$\begin{aligned} O^W(q, p) &= 2 \int_{-\infty}^{+\infty} ds \exp(2i p s / \hbar) \langle q - s | \hat{O} | q + s \rangle \\ &= 2 \int_{-\infty}^{+\infty} dr \exp(-2i q r / \hbar) \langle p - r | \hat{O} | p + r \rangle. \end{aligned} \quad (1)$$

For \hat{O} identified as a density matrix operator, $\hat{\rho} = |\psi\rangle\langle\psi|$, the Weyl transform operator $O^W(q, p)$ results in the so-called Wigner quasiprobability distribution

function,

$$\begin{aligned} h^{-1} \hat{\rho} &\rightarrow W(q, p) \\ &= (\pi \hbar)^{-1} \int_{-\infty}^{+\infty} ds \exp(2i p s / \hbar) \psi(q - s) \psi^*(q + s), \end{aligned} \quad (2)$$

which can also be identified as the Fourier transform of the off-diagonal elements of the associated density matrix $\hat{\rho}$, where $h = 2\pi \hbar$ is the Planck constant. As expected, it presumes a consistent probability distribution interpretation constrained by the normalization condition over $\hat{\rho}$, that is, $\text{Tr}_{(q,p)}[\hat{\rho}] = 1$.

Notwithstanding its major role in quantum optics, which can be extended to the context of plasma and nuclear physics [12,13], the WW phase-space framework is conveniently considered in the analysis of scattering and decoherence effects in solid-state physics, for instance, in the context of semiconductor transport-process phenomenology [14,15]. In addition, the formalism also provides interesting perspectives for the interpretation of the wave-function collapse [7,16,17]; for the paradigmatic understanding of standard QM [18–20], which includes a generalized correspondence between uncertainty relations and quantum observables [21–24]; and even for the investigation of more general scenarios, from quantum chaos [25] to quantum cosmology [26].

From a strictly theoretical perspective [8,27,28], the WW formalism admits a fluid equivalence of the phase-phase information flow, which encodes all the information provided by a quantum density-matrix operator. Starting with the density-matrix operator $W(q, p)$, one can map the quantum phase-space ensemble dynamics and its associated Wigner currents to describe how quantum fluctuations quantitatively

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affect probability distributions and information flows [8,27]. By construction, it results in a continuity equation which is reduced to the Liouville equation in the classical domain. Quantum fluctuations coupled to nonlinear effects driven by higher-order derivatives of the quantum-mechanical potential introduce distortions (expressed in powers of \hbar) over the Liouvillian flow, which set the boundaries between classical and quantum regimes [26–29].¹ Another relevant aspect involving the classical-to-quantum correspondence is related to the phase-space probability and information-flow interpretations devised by Wigner [1], which establish the correspondence between classical mechanics and classical thermodynamics (TD) on the same level as that between QM and TD. That is, the Maxwell-Boltzmann distribution of classical statistical mechanics $\exp[-H^W(q, p)/k_B\mathcal{T}]$, where H is the Hamiltonian, k_B is the Boltzmann constant, and \mathcal{T} is the temperature, can be statistically treated on equal footing to the quantum version described by $\sum_n \exp(-E_n/k_B\mathcal{T})W_n(q, p)$, where n is a quantum number index for eigenenergies E_n and Wigner-related eigenfunctions $W_n(q, p)$.

Given the universal features of the WW formalism described above, our proposal in this work is to extend the formulation of the QM in phase space to nonlinear Hamiltonian systems generically described by a Hamiltonian of the form

$$H^W(q, p) = K(p) + V(q), \quad (3)$$

which is evidently constrained by the $\partial^2 H^W / \partial q \partial p = 0$ condition, where $K(p)$ and $V(q)$ are arbitrary functions of p and q , respectively. The point here is that QM states are based on Hilbert spaces and operators implemented through the Schrödinger equation, whereas classical mechanics is geometrically defined on symplectic manifolds, whose dynamic trajectories are described by Hamilton's equations. For more general Hamiltonians, like $H^W(q, p) = K(p) + V(q)$, implementing the *Hamiltonian function* through an eigensystem $H^W \psi_n = E_n \psi_n$ is sometimes unfeasible. That is often the case for $K(p)$ and $V(q)$ simultaneously described by non-polynomial functions of p and q , respectively, for which the corresponding Schrödinger-like eigensystem cannot be solved, even numerically.

Alternatively, when the dynamics is described by a *Hamiltonian constraint*, as opposed to a Hamiltonian function, the WW formalism can be implemented through a probability flux-continuity equation, through which the fluid analogy and the analytical computability of the WW formalism provide solutions for quantum ensembles, as opposed to quantum states. Therefore, at least for (TD and Gaussian) quantum ensembles, the classical-quantum limits of exact quantum solutions can be tested by means of (quantum) probability distributions and information quantifiers, all obtained from the WW formalism.

¹Of course, it is well known that the Wigner function also exhibits nonclassical patterns which return negative values to the corresponding distribution function. Parallel frameworks which include Husimi Q [3,30] and Glauber-Sudarshan representations [31–34] or even the more specific optical tomographic probability representation [35–37] of QM are used so that we have an adequate interpretation of the negative quasiprobability distributions.

Hence, our starting point is to consider the Wigner flow framework for Hamiltonian systems of the form $H^W(q, p) = K(p) + V(q)$, for which the description of probability currents and the construction of the Hamiltonian-related (quantum) information quantifiers are expected. Once developed, the framework will be applied to Harper-type [38,39] systems, which naturally encompass nonlinearity properties arising from the momentum coordinate p . In this case, for $K(p)$ identified by a sinusoidal form $\cos(p/p_0)$, which is therefore distinct from the standard Schrödinger (quadratic momentum) formulation, it will be possible to confront classical and quantum scenarios in the mentioned Hamiltonian context.

The so-called Harper Hamiltonian originally described the effect of a uniform magnetic field on a conduction-band metal [38], where a tight-binding approximation for symmetric cubic crystals was assumed. The Harper-Hofstadter extension of the formalism [38,40,41] was generalized to lattice systems in the presence of a gauge field, where ground-level chiral many-body states emerge from the system populated with interacting particles [40]. In the past few years, the Harper Hamiltonian has been experimentally implemented for neutral particles in optical lattices through a laser-assisted tunneling mechanism, which effectively probes the expansion of the atoms in the lattice, where a potential-energy gradient is provided either by gravity or by magnetic fields [42]. The laser-assisted tunneling processes introduce single-site and single-particle-resolved controls which engender an image platform that enables one to build chiral systems and then to tune the particle number within a chiral state atom by atom. This allows for controlling the lattice size [42,43] exhibiting the band structure displayed by the so-called Hofstadter's butterfly [38,39]. The semiclassical interpretation of the original tight-binding approach for the Harper model results in a classical effective one-dimensional Hamiltonian that reads

$$H_H^W(q, p) = \pm[2\pi\alpha \cos(p/p_0) + 2\pi\alpha \cos(q/q_0)] \quad (4)$$

for an isotropic q - p phase space where the associated operators \hat{q} and \hat{p} are driven by QM-like (non)commuting relations, $[\hat{q}, \hat{p}] = i2\pi\alpha p_0 q_0$, with α identified as the Peierls phase [38,39,43], which works as an effective modulation of the reduced Planck constant \hbar in a periodic closed phase-space trajectory. Recently, the elementary properties exhibited by the Harper-Hamiltonian, Eq. (4), have also motivated research on the transition from order to chaos, through a kind of *kicked* time-dependent Harper map [44–47]. Concerns regarding the quantum topological issues of Harper-type models, Floquet systems exhibiting the Hall effect due to electromagnetic fields [48], and the extended Aubry-André-Harper formulation for classifying topological states [49] have also been investigated. Besides simulation and experimental feasibilities, Harper-like systems driven by the classical correspondence with $H_H(q, p)$ from (4) fill the conditions for $H^W(q, p) = K(p) + V(q)$ introduced above. Therefore, once the prescribed phase-space description of nonlinear (q and p) Hamiltonian systems is established, the Harper platform might be relevant, as a test, for describing the interplay between microscopic-quantum and macroscopic-classical realities.

Given the above assumptions, the outline of this paper is as follows. Section II is concerned with the foundations of

the WW framework in order to derive stationarity and quantifying Liouvillian properties related to the extended Wigner framework for $H^W(q, p) = K(p) + V(q)$ Hamiltonians. In Sec. III, the results are specialized to TD ensembles, which admit a precise interpretation of stationarity conditions in the context of the classical-to-quantum correspondence and are physically appealing in relation to statistical mechanics. The same analysis is extended to Gaussian ensembles in Sec. IV in a Wigner flow framework in which the overall quantum fluctuations over a classical phase-space trajectory can be computed in terms of a convergent infinite series expansion over \hbar^{2n} . Through exact analytical results, the Gaussian framework accounts for complete quantum corrections of the Hamiltonian classical profile, providing a natural approach to establish accurate correspondence between classical and quantum scenarios through the information quantum correlation quantifiers obtained from Wigner currents. In Sec. V, the Harper Hamiltonian is considered as a test platform for the derived extended phase-space framework. Stationarity and Liouvillianity conditions as well as parameters for classical-to-quantum correspondence are then specialized to TD and Gaussian ensembles in which the Wigner flow is driven by the Harper Hamiltonian. These are all explicitly obtained in terms of analytical expressions for the Wigner currents for both TD and Gaussian ensembles. Our conclusions and the outlook for further research are presented in Sec. VI.

II. STATIONARITY AND LIOUVILLIANITY IN THE EXTENDED WIGNER FRAMEWORK

Based on the symmetries of the Heisenberg-Weyl group of translations, the Wigner phase-space quasidistribution function associated with a density operator $\hat{\rho}$ in the form of an overlap integral, Eq. (2), was proposed by Wigner's seminal work [1] when accounting for quantum corrections to TD equilibrium states. The Wigner function's most elementary property concerns its marginal distributions, which return position and momentum distributions upon integration over the momentum and position coordinates, respectively,

$$|\psi(q)|^2 = \int_{-\infty}^{+\infty} dp W(q, p) \leftrightarrow |\varphi(p)|^2 = \int_{-\infty}^{+\infty} dq W(q, p), \quad (5)$$

such that the Fourier transform of the respective wave functions,

$$\varphi(p) = (2\pi\hbar)^{-1/2} \int_{-\infty}^{+\infty} dq \exp(ipq/\hbar) \psi(q), \quad (6)$$

is the property intrinsically based on the Hilbert space features of the Schrödinger QM that suppresses the coexistence of positive-definite position and/or momentum probability distributions.

The connection of the Wigner function to the matrix operator QM [see Eqs. (1) and (2)] allows for computing the averaged values of quantum observables described by generic operators \hat{O} evaluated through an overlap integral over the infinite volume described by the phase-space coordinates q and p as [1,4]

$$\langle O \rangle = \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dq W(q, p) O^W(q, p), \quad (7)$$

which corresponds to the trace of the product between $\hat{\rho}$ and \hat{O} , $\text{Tr}_{\{q,p\}}[\hat{\rho}\hat{O}]$. Moreover, the statistical aspects inherent in the $W(q, p)$ definition also admit extensions from pure states to statistical mixtures, through which, for instance, the replacement of $O^W(q, p)$ by $W(q, p)$ in Eq. (7) leads to the quantum purity computed through an analogous trace operation, $\text{Tr}_{\{q,p\}}[\hat{\rho}^2]$, which reads

$$\text{Tr}_{\{q,p\}}[\hat{\rho}^2] = 2\pi\hbar \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dq W(q, p)^2, \quad (8)$$

satisfying the pure-state constraint, $\text{Tr}_{\{q,p\}}[\hat{\rho}^2] = \text{Tr}_{\{q,p\}}[\hat{\rho}] = 1$.

In addition, flow properties of the Wigner function $W(q, p) \rightarrow W(q, p; t)$ are also connected to the Hamiltonian dynamics. These properties are described by a vector flux [8,27,28] $\mathbf{J}(q, p; t)$ decomposed into the phase-space coordinate directions \hat{q} and \hat{p} as $\mathbf{J} = J_q \hat{q} + J_p \hat{p}$ in order to reproduce a flow field connected to the Wigner-function dynamics through the continuity equation [3,4,8,27,28]

$$\partial_t W + \partial_q J_q + \partial_p J_p = 0, \quad (9)$$

where the shortened notation for partial derivatives is set as $\partial_a \equiv \partial/\partial a$. In this case, for a nonrelativistic Hamiltonian operator $H(\hat{Q}, \hat{P})$, from which the Weyl transforms yield

$$H(\hat{Q}, \hat{P}) = \frac{\hat{p}^2}{2m} + V(\hat{Q}) \rightarrow H^W(q, p) = \frac{p^2}{2m} + V(q), \quad (10)$$

one has [3,4,8,27]

$$J_q(q, p; t) = \frac{p}{m} W(q, p; t) \quad (11)$$

and

$$J_p(q, p; t) = - \sum_{\eta=0}^{\infty} \left(\frac{i\hbar}{2} \right)^{2\eta} \frac{1}{(2\eta+1)!} [\partial_q^{2\eta+1} V(q)] \partial_p^{2\eta} W(q, p; t), \quad (12)$$

with $\partial_a^s \equiv (\partial/\partial a)^s$, from which one notices that the above-identified series expansion contributions from $\eta \geq 1$ introduce the quantum corrections which distort classical trajectories. In fact, for Eq. (12), the suppression of the $\eta \geq 1$ contributions results in a classical Hamiltonian description of the phase-space probability distribution dynamics in terms of classical Wigner currents,

$$J_q^C(q, p; t) = +(\partial_p H^W) W(q, p; t) \quad (13)$$

and

$$J_p^C(q, p; t) = -(\partial_q H^W) W(q, p; t), \quad (14)$$

which, once substituted into Eq. (9), return the (classical) Liouville equation, with the classical phase-space velocity identified by $\mathbf{v}_{\xi(C)} = \dot{\xi} = (\dot{q}, \dot{p}) \equiv (\partial_p H^W, -\partial_q H^W)$, with $\nabla_{\xi} \cdot \mathbf{v}_{\xi(C)} = \partial_q \dot{q} + \partial_p \dot{p} = 0$, where dots denote the time derivative d/dt . Likewise, for a quantum current parameterized by $\mathbf{J} = \mathbf{w} W$, where the Wigner phase-space velocity \mathbf{w} is the quantum analog of $\mathbf{v}_{\xi(C)}$, a suitable divergent behavior

is identified by

$$\nabla_{\xi} \cdot \mathbf{w} = \frac{W \nabla_{\xi} \cdot \mathbf{J} - \mathbf{J} \cdot \nabla_{\xi} W}{W^2} \quad (15)$$

since $\nabla_{\xi} \cdot \mathbf{J} = W \nabla_{\xi} \cdot \mathbf{w} + \mathbf{w} \cdot \nabla_{\xi} W$ [8]. Therefore, Wigner-function stationarity and Liouvillian behaviors are straightforwardly identified and quantified in terms of Eqs. (9) and (15) by $\nabla_{\xi} \cdot \mathbf{J} = 0$ and $\nabla_{\xi} \cdot \mathbf{w} = 0$, respectively.

This Wigner current constraint, $\mathbf{J} = \mathbf{w} W$, indeed brings some relevant aspects of the topological properties of the Wigner flow. Considering the integral version of the above-mentioned stationary and Liouvillian conditions, these properties can be captured by the introduction of a *substantial derivative* [27,50,51], written as

$$\frac{D}{Dt} \int_V dV f(W) \equiv \int_V dV \left(\frac{D}{Dt} + \nabla_{\xi} \cdot \mathbf{v}_{\xi} \right) f(W), \quad (16)$$

which evinces the conservative features associated with the elementary vector unity of the phase space ξ surrounded by an infinitesimal volume $dV \equiv dp dq$. For the case of a volume V enclosed by the two-dimensional comoving closed surface \mathcal{C} , it gives rise to the usual tool for quantifying the flux of information through \mathcal{C} [27], which is depicted by the trajectory obtained from the classical velocity $\mathbf{v}_{\xi(\mathcal{C})}$, which can be seen from Eq. (16), when $f(W)$ is identified by a power of W or W^{β} ,

$$\begin{aligned} & \frac{D}{Dt} \left(\int_{V_{\mathcal{C}}} dV W^{\beta} \right) \\ &= \int_{V_{\mathcal{C}}} dV \left[\frac{D}{Dt} (W^{\beta}) + W^{\beta} \nabla_{\xi} \cdot \mathbf{v}_{\xi(\mathcal{C})} \right] \\ &= \int_{V_{\mathcal{C}}} dV \left[\partial_t (W^{\beta}) + \nabla_{\xi} \cdot (\mathbf{v}_{\xi(\mathcal{C})} W^{\beta}) \right] \\ &= \int_{V_{\mathcal{C}}} dV \left[-\beta W^{\beta-1} \nabla_{\xi} \cdot \mathbf{J} + \nabla_{\xi} \cdot (\mathbf{v}_{\xi(\mathcal{C})} W^{\beta}) \right] \\ &= - \int_{V_{\mathcal{C}}} dV \left[(\beta - 1) W^{\beta} \nabla_{\xi} \cdot \mathbf{w} \right. \\ & \quad \left. + \nabla_{\xi} \cdot (\mathbf{J} W^{\beta-1} - \mathbf{v}_{\xi(\mathcal{C})} W^{\beta}) \right] \\ &= - \int_{V_{\mathcal{C}}} dV (\beta - 1) W^{\beta} \nabla_{\xi} \cdot \mathbf{w} - \oint_{\mathcal{C}} d\ell W^{\beta-1} (\mathbf{J} \cdot \mathbf{n}), \quad (17) \end{aligned}$$

with the unitary vector \mathbf{n} satisfying $\mathbf{n} \cdot \mathbf{v}_{\xi(\mathcal{C})} = 0$.

For $\beta = 0$, Eq. (17) results in

$$\frac{D}{Dt} V = \int_{V_{\mathcal{C}}} dV \nabla_{\xi} \cdot \mathbf{w} - \oint_{\mathcal{C}} d\ell \mathbf{w} \cdot \mathbf{n} = 0, \quad (18)$$

which reflects the phase-space volume conservation, given that Green's theorem states that

$$\int_{V_{\mathcal{C}}} dV \nabla_{\xi} \cdot \mathbf{w} = \oint_{\mathcal{C}} d\ell \mathbf{w} \cdot \mathbf{n}, \quad (19)$$

which, of course, vanishes for $\nabla_{\xi} \cdot \mathbf{w} = 0$ in a kind of integral version of the Liouville condition. Analogously, by identifying the phase-space volume integrated probability as

$$\sigma_{(\mathcal{C})} = \int_{V_{\mathcal{C}}} dV W \quad (20)$$

for $\beta = 1$, Eq. (17) results in

$$\frac{D}{Dt} \sigma_{(\mathcal{C})} = - \oint_{\mathcal{C}} d\ell \mathbf{J} \cdot \mathbf{n}, \quad (21)$$

which reflects the probability conservation for stationary states, given that Green's theorem now states that

$$\int_{V_{\mathcal{C}}} dV \nabla_{\xi} \cdot \mathbf{J} = \oint_{\mathcal{C}} d\ell \mathbf{J} \cdot \mathbf{n}. \quad (22)$$

As proposed, the above set of stationarity and Liouvillianity quantifying tools will be evaluated in the extended Wigner framework. In this case, explicit expressions for the Wigner current components $\mathbf{J} = J_q \hat{q} + J_p \hat{p}$ must be deduced for QM Hamiltonians generically identified by the Weyl transform (3).

In what follows such an extended form of the Wigner continuity equation is obtained by deducing the time dependence of W . As a guide, our departing point is the von Neumann equation for the state-density operator $\hat{\rho} = |\psi\rangle\langle\psi|$, obtained from [3]

$$\partial_t \hat{\rho} = i\hbar^{-1} [\hat{\rho}, H] \equiv \partial_t^{(K)} \hat{\rho} + \partial_t^{(V)} \hat{\rho}, \quad \partial_t^{(A)} \hat{\rho} = i\hbar^{-1} [\hat{\rho}, \mathcal{A}], \quad (23)$$

which can then be separately evaluated in momentum and position representations for $\mathcal{A} \equiv K(\hat{P})$, $V(\hat{Q})$. Hence, we use the Wigner-function properties from Eq. (2) to transform each contribution into its respective Wigner representation (see Ref. [3] for nonrelativistic QM):

$$\begin{aligned} \partial_t^{(K)} \langle p|\rho|p' \rangle &= i\hbar^{-1} \langle p|\rho|p' \rangle [K(p') - K(p)] \Rightarrow \partial_t^{(K)} W(q, p; t) \\ &= i\hbar^{-1} (\pi\hbar)^{-1} \int_{-\infty}^{+\infty} dr \rho_{(p-r; p+r)}^{W, \varphi} \exp(-2iqr/\hbar) [K(p+r) - K(p-r)], \quad (24) \end{aligned}$$

where $\rho_{(p-r; p+r)}^{W, \varphi} \equiv \langle p-r|\rho|p+r \rangle$ has been identified by $\varphi(p-r) \varphi^*(p+r)$, and

$$\begin{aligned} \partial_t^{(V)} \langle q|\rho|q' \rangle &= i\hbar^{-1} \langle q|\rho|q' \rangle [V(q') - V(q)] \Rightarrow \partial_t^{(V)} W(q, p; t) \\ &= i\hbar^{-1} (\pi\hbar)^{-1} \int_{-\infty}^{+\infty} ds \rho_{(q-s; q+s)}^{W, \psi} \exp(2ips/\hbar) [V(q+s) - V(q-s)], \quad (25) \end{aligned}$$

where $\rho_{(q-s; q+s)}^{W, \psi} \equiv \langle q-s|\rho|q+s \rangle$ has been identified by $\psi(q-s) \psi^*(q+s)$. Now, by noticing that

$$K(p+r) - K(p-r) = 2 \sum_{\eta=0}^{\infty} \frac{r^{2\eta+1}}{(2\eta+1)!} \partial_p^{2\eta+1} K(p) \quad (26)$$

and

$$V(q+s) - V(q-s) = 2 \sum_{\eta=0}^{\infty} \frac{s^{2\eta+1}}{(2\eta+1)!} \partial_q^{2\eta+1} V(q) \quad (27)$$

and identifying the auxiliary variables r and s by $+i(\hbar/2)\partial_q$ [see Eq (24)] and $-i(\hbar/2)\partial_p$ [see Eq (25)], respectively, one recovers an equivalent Wigner continuity equation cast in the form of Eq. (9), i.e., with the stationarity quantifier explicitly given by

$$\partial_t W = \sum_{\eta=0}^{\infty} \frac{(-1)^\eta \hbar^{2\eta}}{2^{2\eta} (2\eta+1)!} \{ [\partial_q^{2\eta+1} V(q)] \partial_p^{2\eta+1} W - [\partial_p^{2\eta+1} K(p)] \partial_q^{2\eta+1} W \}, \quad (28)$$

from which one has

$$J_q(q, p; t) = + \sum_{\eta=0}^{\infty} \left(\frac{i\hbar}{2} \right)^{2\eta} \frac{1}{(2\eta+1)!} [\partial_p^{2\eta+1} K(p)] \partial_q^{2\eta} W(q, p; t) \quad (29)$$

and

$$J_p(q, p; t) = - \sum_{\eta=0}^{\infty} \left(\frac{i\hbar}{2} \right)^{2\eta} \frac{1}{(2\eta+1)!} [\partial_q^{2\eta+1} V(q)] \partial_p^{2\eta} W(q, p; t), \quad (30)$$

which encompass all the contributions for quantum corrections. Analogously, to fully capture the quantum effects over the classical Hamiltonian regime described by currents in the form of (13) and (14), the Liouvillianity quantifier [such as that from Eq. (15)] is expressed by

$$\nabla_\xi \cdot \mathbf{w} = \sum_{\eta=1}^{\infty} \frac{(-1)^\eta \hbar^{2\eta}}{2^{2\eta} (2\eta+1)!} \left\{ [\partial_p^{2\eta+1} K(p)] \partial_q \left[\frac{1}{W} \partial_q^{2\eta} W \right] - [\partial_q^{2\eta+1} V(q)] \partial_p \left[\frac{1}{W} \partial_p^{2\eta} W \right] \right\}. \quad (31)$$

Finally, as will be pointed out in the following sections, the above-presented forms for quantifiers of stationarity and Liouvillian regimes provide fundamental information about quantum ensembles, which is essential for discriminating classical from quantum behaviors.

III. THERMODYNAMIC ENSEMBLES

The Wigner function is also an essential tool for defining quantum projectors for TD ensembles. Once it is assumed that the spectral decomposition of the quantum system driven by the Hamiltonian (3) cannot be obtained, the stationary solution of the Wigner continuity equation [see Eq. (9)] for the corresponding quantum TD ensemble can be computed through a perturbative expansion over the classical Maxwell-Boltzmann distribution [1,52].

To find quantum corrections to the classical TD regime, one solves a closed equilibrium equation which indeed exhibits the response of the quantum system to the thermal equilibrium for relatively high temperatures. It yields the corresponding Wigner stationary distribution $W_{St}(q, p; \beta)$, where $\beta = 1/k_B T$. In fact, one departs from the classical distribution described by

$$W_0(q, p; \beta) = [\hbar \mathcal{Z}_0(\beta)]^{-1} \exp[-\beta H^W(q, p)], \quad (32)$$

with the partition function identified by

$$\mathcal{Z}_0(\beta) = \hbar^{-1} \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dp \exp[-\beta H^W(q, p)], \quad (33)$$

and notices that the stationary solution of Eq. (9) is iteratively obtained through a series expansion [1,52],

$$W_{St}^{(2N)}(q, p; \beta) = \sum_{\eta=0}^N \hbar^{2\eta} W_{2\eta}(q, p; \beta), \quad (34)$$

truncated at order $O(\hbar^{2N})$, from which

$$\lim_{N \rightarrow \infty} W_{St}^{(2N)}(q, p; \beta) = W_{St}(q, p; \beta).$$

The assumption that the TD equilibrium prevails everywhere in the phase space ensures that a closed-equation resolution procedure for determining, order by order, the contributions from $W_{2\eta}$ applies [1,52]. Given the arbitrariness of $K(p)$ and $V(q)$, in this first analysis, the quantum corrections will be constrained to contributions up to order $O(\hbar^2)$. In this case, one identifies $W_2(q, p; \beta)$ with $\chi_{(q, p; \beta)} W_0(q, p; \beta)$, where $\chi_{(q, p; \beta)}$ is a multiplicative parameter that factorizes the corrections over $W_0(q, p; \beta)$ and computes the time derivative of the stationary solution for $N=1$, $W_{St}^{(2)}(q, p; \beta) = W_0(q, p; \beta) + \hbar^2 W_2(q, p; \beta)$, in order to obtain

$$\begin{aligned} \partial_t W_{St}^{(2)} &= \partial_t W_0 + \hbar^2 \partial_t W_2 \\ &= (1 + \hbar^2 \chi_{(q, p; \beta)}) \partial_t W_0 + \hbar^2 W_0 \partial_t \chi_{(q, p; \beta)}. \end{aligned} \quad (35)$$

Up to order $O(\hbar^2)$, the contribution from $\partial_t \chi_{(q, p; \beta)}$ to the above expression can be expressed by

$$\begin{aligned} \partial_t \chi_{(q, p; \beta)} &= -\{\chi, H\}_{PB} \\ &= \partial_q V(q) \partial_p \chi_{(q, p; \beta)} - \partial_p K(p) \partial_q \chi_{(q, p; \beta)}. \end{aligned} \quad (36)$$

where PB stands for the *Poisson brackets*. Likewise, now from Eq. (28), one writes

$$\begin{aligned} \partial_t W_0 &= -\nabla_\xi \cdot \mathbf{J} = [\partial_q V(q) \partial_p W_0 - \partial_p K(p) \partial_q W_0] \\ &\quad - \frac{\hbar^2}{24} \{ [\partial_q^3 V(q)] \partial_p^3 W_0 - [\partial_p^3 K(p)] \partial_q^3 W_0 \} \\ &\quad + O(\hbar^4), \end{aligned} \quad (37)$$

where, due to the classical nature of W_0 , the contribution from classical currents vanishes, i.e.,

$$\partial_q V(q) \partial_p W_0 - \partial_p K(p) \partial_q W_0 = -\{W_0, H\}_{PB} = 0.$$

In the next step, after identifying the auxiliary derivatives given by $\partial_q W_0 = -\beta W_0 \partial_q V(q)$ and $\partial_p W_0 = -\beta W_0 \partial_p K(p)$, one notices that

$$\partial_q^2 W_0 = -\beta W_0 \partial_q^2 V(q) + \beta^2 W_0 [\partial_q V(q)]^2, \quad (38)$$

$$\partial_p^2 W_0 = -\beta W_0 \partial_p^2 K(p) + \beta^2 W_0 [\partial_p K(p)]^2 \quad (39)$$

and that

$$\partial_q^3 W_0 = -\beta W_0 \partial_q^3 V(q) + 3\beta^2 W_0 [\partial_q^2 V(q)] \partial_q V(q) - \beta^3 W_0 [\partial_q V(q)]^3, \quad (40)$$

$$\partial_p^3 W_0 = -\beta W_0 \partial_p^3 K(p) + 3\beta^2 W_0 [\partial_p^2 K(p)] \partial_p K(p) - \beta^3 W_0 [\partial_p K(p)]^3, \quad (41)$$

which can all be inserted into Eq. (37). Then, through the insertion of Eqs. (36) and (37) into Eq. (35), one obtains

$$\begin{aligned} \partial_t W_{St}^{(2)} &= \hbar^2 W_0 \left(\partial_q V(q) \partial_p \chi_{(q,p;\beta)} - \partial_p K(p) \partial_q \chi_{(q,p;\beta)} - \frac{\beta^2}{8} \{ [\partial_q^3 V(q)] [\partial_p^2 K(p)] \partial_p K(p) - [\partial_p^3 K(p)] [\partial_q^2 V(q)] \partial_q V(q) \} \right. \\ &\quad \left. + \frac{\beta^3}{24} \{ [\partial_q^3 V(q)] [\partial_p K(p)]^3 - [\partial_p^3 K(p)] [\partial_q V(q)]^3 \} \right) + O(\hbar^4). \end{aligned} \quad (42)$$

Finally, from the above result, the stationarity condition over $W_{St}^{(2)}$, $\partial_t W_{St}^{(2)} = 0$, results in

$$\chi_{(q,p;\beta)} = -\frac{\beta^2}{8} \partial_q^2 V(q) \partial_p^2 K(p) + \frac{\beta^3}{24} \{ \partial_q^2 V(q) [\partial_p K(p)]^2 + \partial_p^2 K(p) [\partial_q V(q)]^2 \}, \quad (43)$$

which gives

$$W_{St}^{(2)}(q, p; \beta) = \frac{\mathcal{Z}_0(\beta)}{\mathcal{Z}_{St}(\beta)} W_0(q, p; \beta) \left[1 + \hbar^2 \left(\frac{\beta^3}{24} \{ \partial_q^2 V(q) [\partial_p K(p)]^2 + \partial_p^2 K(p) [\partial_q V(q)]^2 \} - \frac{\beta^2}{8} \partial_q^2 V(q) \partial_p^2 K(p) \right) \right], \quad (44)$$

where the multiplying factor $\mathcal{Z}_0(\beta)/\mathcal{Z}_{St}(\beta)$ has been introduced in order to reflect the modifications of the associated partition functions as well as to guarantee the QM unitarity properties. Of course, for $K(p) \sim p^2$, one recovers the original Wigner results [1,52].

To summarize, the same systematic corrections can be used to compute the Wigner currents which, up to order $\mathcal{O}(\hbar^2)$, are written as

$$J_q^{(2)}(q, p; t) = + \left(\partial_p K(p) (1 + \hbar^2 \chi_{(q,p;\beta)}) - \frac{\hbar^2}{24} \partial_p^3 K(p) \{ \beta^2 [\partial_q V(q)]^2 - \beta \partial_q^2 V(q) \} \right) W_0, \quad (45)$$

and

$$J_p(q, p; t) = - \left(\partial_q V(q) (1 + \hbar^2 \chi_{(q,p;\beta)}) - \frac{\hbar^2}{24} \partial_q^3 V(q) \{ \beta^2 [\partial_p K(p)]^2 - \beta \partial_p^2 K(p) \} \right) W_0, \quad (46)$$

which can also be recast in the forms

$$J_q(q, p; t) \approx + \left(\partial_p K(p) - \frac{\hbar^2}{24} \partial_p^3 K(p) \{ \beta^2 [\partial_q V(q)]^2 - \beta \partial_q^2 V(q) \} \right) W_{St} \quad (47)$$

and

$$J_p(q, p; t) \approx - \left(\partial_q V(q) - \frac{\hbar^2}{24} \partial_q^3 V(q) \{ \beta^2 [\partial_p K(p)]^2 - \beta \partial_p^2 K(p) \} \right) W_{St}. \quad (48)$$

Naturally, according to the truncation criterion, the Wigner perturbed function contributions due to $W_{2\eta}(q, p; \beta)$ for η higher than 1 could also be evaluated in terms of the derivatives of $K(p)$ and $V(q)$. Different from Hamiltonian systems that, on the face of the quadratic momentum contributions, are quantum mechanically driven by the nonrelativistic

Schrödinger equation, in the scope of quantum and classical TD, the nonlinear effects from $K(p)$ give rise to a non-Gaussian behavior of the momentum distribution, even if the equilibrium phase-space (Maxwell-Boltzmann) distribution function is factorable in the position and momentum variables. It is independent of higher-order quantum correction terms

of W_{St} , which can be exhaustively computed to any order in powers of \hbar .

From now on, aiming to understand the decoupling of quantum corrections from nonlinear effects, a more comprehensive view of the phase-space dynamics can be obtained if it is cast in terms of a dimensionless version of $H^W(q, p)$ from Eq. (3),

$$\mathcal{H}(x, k) = \mathcal{K}(k) + \mathcal{V}(x). \quad (49)$$

In this case, $\mathcal{H}(x, k)$ is written in terms of dimensionless variables $x = (m\omega\hbar^{-1})^{1/2}q$ and $k = (m\omega\hbar)^{-1/2}p$, such that $\mathcal{H} = (\hbar\omega)^{-1}H$, $\mathcal{V}(x) = (\hbar\omega)^{-1}V((m\omega\hbar^{-1})^{-1/2}x)$, and $\mathcal{K}(k) = (\hbar\omega)^{-1}K((m\omega\hbar)^{1/2}k)$, where m is a mass-scale parameter and ω is an arbitrary angular frequency. Therefore, the Wigner function can be cast into the dimensionless form $\mathcal{W}(x, k; \omega t) \equiv \hbar W(q, p; t)$, with \hbar absorbed by $dpdq \rightarrow \hbar dx dk$ integrations, i.e.,

$$\begin{aligned} \mathcal{W}(x, k; \tau) &= \pi^{-1} \int_{-\infty}^{+\infty} dy \exp(2iky) \phi(x-y; \tau) \phi^* \\ &\quad \times (x+y; \tau), \end{aligned} \quad (50)$$

with $y = (m\omega\hbar^{-1})^{1/2}s$ and $\tau = \omega t$. In this case, the position and momentum wave functions, $\phi(x, \tau)$ and $\psi(q; t)$, are con-

sistently normalized by

$$\int_{-\infty}^{+\infty} dx |\phi(x; \tau)|^2 = \int_{-\infty}^{+\infty} dq |\psi(q; t)|^2 = 1. \quad (51)$$

The Wigner currents are now given by

$$\begin{aligned} \mathcal{J}_x(x, k; \tau) &= + \sum_{\eta=0}^{\infty} \left(\frac{i}{2}\right)^{2\eta} \frac{1}{(2\eta+1)!} [\partial_k^{2\eta+1} \mathcal{K}(k)] \\ &\quad \times \partial_x^{2\eta} \mathcal{W}(x, k; \tau), \end{aligned} \quad (52)$$

$$\begin{aligned} \mathcal{J}_k(x, k; \tau) &= - \sum_{\eta=0}^{\infty} \left(\frac{i}{2}\right)^{2\eta} \frac{1}{(2\eta+1)!} [\partial_x^{2\eta+1} \mathcal{V}(x)] \\ &\quad \times \partial_k^{2\eta} \mathcal{W}(x, k; \tau), \end{aligned} \quad (53)$$

so that $\omega \partial_x \mathcal{J}_x \equiv \hbar \partial_q J_p(q, p; t)$ and $\omega \partial_k \mathcal{J}_k \equiv \hbar \partial_p J_p(q, p; t)$, through which the dimensionless continuity equation is written in terms of the redefined phase-space coordinates $\xi = (x, k)$ as

$$\partial_\tau \mathcal{W} + \partial_x \mathcal{J}_x + \partial_k \mathcal{J}_k = \partial_\tau \mathcal{W} + \nabla_\xi \cdot \mathcal{J} = 0. \quad (54)$$

Turning back to the TD ensembles, the stationary Wigner function from Eq. (44) can be written in a dimensionless form,

$$\mathcal{W}_{St}^{(2)}(x, k; \beta) = \frac{\mathcal{Z}_0(\beta)}{\mathcal{Z}_{St}(\beta)} \mathcal{W}_0(x, k; \beta) \left[1 + \left(\frac{\beta\hbar\omega}{24} \{ \partial_x^2 \mathcal{V}(x) [\partial_k \mathcal{K}(k)]^2 + \partial_k^2 \mathcal{K}(k) [\partial_x \mathcal{V}(x)]^2 \} - \frac{(\beta\hbar\omega)^2}{8} \partial_x^2 \mathcal{V}(x) \partial_k^2 \mathcal{K}(k) \} \right) \right], \quad (55)$$

in this case up to order $O((\beta\hbar\omega)^3)$. From Eq. (55), it is interesting to note that, for quantum approached TD ensembles, despite the asymptotic stationarity, the non-Liouvillian feature is shown by the nonvanishing value of $\nabla_\xi \cdot \mathbf{w}$, which is written in the dimensionless form

$$\begin{aligned} \omega^{-1} \nabla_\xi \cdot \mathbf{w} &= \frac{(\beta\hbar\omega)^2}{12} [\partial_k^3 \mathcal{K}(k) \partial_x^2 \mathcal{V}(x) \partial_x \mathcal{V}(x) \\ &\quad - \partial_x^3 \mathcal{V}(x) \partial_k^2 \mathcal{K}(k) \partial_k \mathcal{K}(k)]. \end{aligned} \quad (56)$$

Of course, for the exact quantum description, only evinced in the case when the system spectral decomposition is known, the quantum propagator (Green's function) for a time-independent dimensionless Hamiltonian \mathcal{H} can be expressed by

$$\Delta(x, t; x', 0) = \langle x | \exp(-i\tau \hat{\mathcal{H}}) | x' \rangle, \quad (57)$$

which, for a canonical ensemble in equilibrium with a heat reservoir at temperature \mathcal{T} , corresponds to the inception of an associated thermal density matrix $\rho(x, x'; \beta\hbar\omega)$, obtained [2,3] by replacing the above related time dependence τ by $-i\beta\hbar\omega$. One thus would have

$$\begin{aligned} \rho(x, x'; \beta\hbar\omega) &= \Delta(x, -i\beta; x', 0) \\ &= \sum_n \exp(-\beta\hbar\omega \varepsilon_n) \psi_n^*(x) \psi_n(x'), \end{aligned} \quad (58)$$

where the Hamiltonian eigenstates $\psi_n(x)$ and eigenvalues ε_n are implicitly given by $\mathcal{H} \psi_n(x) = \varepsilon_n \psi_n(x)$.

In the coordinate representation, the functional $\rho(x, x'; \beta\hbar\omega)$ can have its delocalization aspects parameterized by displacement relations, i.e., $x \rightarrow x-y$ and $x' \rightarrow x+y$, so that the y-Fourier transform of $\rho(x-y, x+y; \beta\hbar\omega)$ is identified with a thermalized phase-space probability distribution,

$$\begin{aligned} \Omega(x, k; \beta) &= \pi^{-1} \int_{-\infty}^{+\infty} dy \exp(2iky) \rho(x-y, x+y; \beta\hbar\omega), \end{aligned} \quad (59)$$

which satisfies the Bloch equation

$$\frac{\partial \Omega}{\partial \beta} = -\hat{\mathcal{H}} \Omega = -\Omega \hat{\mathcal{H}}, \quad \Omega(\beta=0) \equiv \mathbb{I}. \quad (60)$$

One thus identifies a correspondence between $\rho(x-y, x+y; \beta\hbar\omega)$ and $\mathcal{Z}^{-1} \exp(-\beta\hbar\omega \hat{\mathcal{H}}) \equiv \mathcal{Z}^{-1} \Omega$, from which a large set of systematic TD and statistical results follows from the definition of the partition function as a trace-related functional given by $\mathcal{Z} \equiv \mathcal{Z}(\beta) = \text{Tr}[\exp(-\beta\hbar\omega \hat{\mathcal{H}})]$. Therefore, from Eq. (59), the thermalized (QM) Wigner function can be written as

$$\begin{aligned} \mathcal{W}_\Omega(x, k; \beta) &= (\mathcal{Z} \pi)^{-1} \int_{-\infty}^{+\infty} dy \\ &\quad \times \exp(2iky) \rho(x-y, x+y; \beta), \end{aligned} \quad (61)$$

normalized by

$$\begin{aligned} \mathcal{Z}(\beta) &= \text{Tr}[\exp(-\beta\hbar\omega\hat{\mathcal{H}})] = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dk \Omega(x, k; \beta) \\ &= \sum_{n=0}^{\infty} \exp(-\beta\hbar\omega \varepsilon_n). \end{aligned} \quad (62)$$

Finally, as matter of completeness, it is worth mentioning that results from Eqs. (44)–(56) can also be interpreted as an extension [from k^2 to generic $\mathcal{K}(k)$ Hamiltonian kinetic contributions] of the semiclassical truncated Wigner approximation (TWA). The TWA is shown by noticing that classical trajectories are identified by trajectories along which the Wigner function is conserved [2,53,54], which is indeed also supported by Wigner's seminal proposal [1]. In particular, the TWA is also constrained by the non-positive-definite property of the Wigner function such that the corresponding outcome function cannot be interpreted as a probability of a particular realization of the boundary conditions [53]. As pointed out in the broader context of phase-space quantum dynamics [53], several slightly different frameworks have been historically considered to circumvent such an issue, even in the context of k^2 -dependent Hamiltonians.

IV. GAUSSIAN ENSEMBLES

Gaussian ensembles are at the core of information issues in continuous-variable QM due to their manipulability properties for building vacuum states, thermal states, and coherent quantum states [55], as well as describing atomic ensembles [56]. Besides its theoretical appeal and its experimental relevance in quantum optics and low-dimensional physics, the phase-space representation of Gaussian ensembles can also be helpful in establishing the bridge between classical and quantum dynamics. In fact, Gaussian Wigner functions have been parametrically worked out in order to describe sets of measured data [57,58] correlated to the issues of classical-quantum correspondence [59], to probe some hypotheses of quantum chaos [60], and to understand the quantum and classical correlations emerging from noncommutative QM [17,19,20,23,24].

Given that the WW phase-space QM is closely connected to the information content of a quantum system [55,61–65], the evaluation of decoherence, stationarity, and nonclassicality aspects arising from the dynamical behavior of Gaussian ensembles might be relevant in describing more accurately the boundaries between classical and quantum regimes. Considering that the phase-space framework presented in Sec. II can be applied to a relevant set of nonlinear Hamiltonian systems, which includes the Harper Hamiltonian [38], in the following, stationarity and Liouvillianity quantifiers for Gaussian ensembles driven by $H^W(q, p)$, Eq. (3), will be analytically computed.

One thus considers a Gaussian Wigner function written as

$$G_\gamma(q, p) = \frac{\gamma^2}{\pi\hbar} \exp\left[-\frac{\gamma^2}{\hbar}\left(\frac{q^2}{a^2} + a^2 p^2\right)\right], \quad (63)$$

where, in the case of harmonic states, the parameter a is identified with $a = (m\omega)^{-1}$ for the mass scale m and the arbitrary angular frequency ω previously introduced. Again,

the problem can be recast into a dimensionless configuration with the Gaussian distribution written as

$$\mathcal{G}_\gamma(x, k) = \hbar G_\gamma(q, p) = \frac{\gamma^2}{\pi} \exp[-\gamma^2(x^2 + k^2)], \quad (64)$$

which leads to the following associated Wigner flow contributions:

$$\begin{aligned} \partial_x \mathcal{J}_x(x, k; \tau) &= + \sum_{\eta=0}^{\infty} \left(\frac{i}{2}\right)^{2\eta} \frac{1}{(2\eta+1)!} [\partial_k^{2\eta+1} \mathcal{K}(k)] \partial_x^{2\eta+1} \mathcal{G}_\gamma(x, k), \end{aligned} \quad (65)$$

$$\begin{aligned} \partial_k \mathcal{J}_k(x, k; \tau) &= - \sum_{\eta=0}^{\infty} \left(\frac{i}{2}\right)^{2\eta} \frac{1}{(2\eta+1)!} [\partial_x^{2\eta+1} \mathcal{V}(x)] \partial_k^{2\eta+1} \mathcal{G}_\gamma(x, k) \end{aligned} \quad (66)$$

once the Hamiltonian form from Eq. (49) has been assumed. From Gaussian relations with Hermite polynomials of order n , H_n , one has

$$\partial_\zeta^{2\eta+1} \mathcal{G}_\gamma(x, k) = (-1)^{2\eta+1} \gamma^{2\eta+1} H_{2\eta+1}(\gamma\zeta) \mathcal{G}_\gamma(x, k) \quad (67)$$

for $\zeta = x, k$, which can be reintroduced in Eqs. (65) and (66) to lead to potentially convergent series expansions. These allow for recasting the Wigner flow expressions into an analytical form, which accounts for the overall quantum modifications, i.e., for η from 1 to ∞ into Eqs. (65) and (66). In particular, for the quantum systems where \mathcal{V} and \mathcal{K} derivatives can be recast in the form

$$\partial_x^{2\eta+1} \mathcal{V}(x) = \lambda_{(x)}^{2\eta+1} \nu(x), \quad (68)$$

$$\partial_k^{2\eta+1} \mathcal{K}(k) = \mu_{(k)}^{2\eta+1} \kappa(k), \quad (69)$$

with λ, ν, μ , and κ identified as arbitrary auxiliary functions, it can be straightforwardly verified that, once they are substituted into Eqs. (65) and (66), the above expressions lead to

$$\begin{aligned} \partial_x \mathcal{J}_x(x, k; \tau) &= (+2i)\kappa(k) \mathcal{G}_\gamma(x, k) \sum_{\eta=0}^{\infty} \left(\frac{i\gamma\mu_{(k)}}{2}\right)^{2\eta+1} \\ &\quad \times \frac{1}{(2\eta+1)!} H_{2\eta+1}(\gamma x), \end{aligned} \quad (70)$$

$$\begin{aligned} \partial_k \mathcal{J}_k(x, k; \tau) &= (-2i)\nu(x) \mathcal{G}_\gamma(x, k) \\ &\quad \times \sum_{\eta=0}^{\infty} \left(\frac{i\gamma\lambda_{(x)}}{2}\right)^{2\eta+1} \frac{1}{(2\eta+1)!} H_{2\eta+1}(\gamma k). \end{aligned} \quad (71)$$

Finally, by noticing that

$$\sum_{\eta=0}^{\infty} H_{2\eta+1}(\gamma\zeta) \frac{s^{2\eta+1}}{(2\eta+1)!} = \sinh(2s\gamma\zeta) \exp[-s^2], \quad (72)$$

one gets

$$\begin{aligned} \partial_x \mathcal{J}_x(x, k; \tau) &= -2\kappa(k) \sin(\gamma^2 \mu_{(k)} x) \\ &\quad \times \exp(+\gamma^2 \mu_{(k)}^2 / 4) \mathcal{G}_\gamma(x, k), \end{aligned} \quad (73)$$

$$\partial_k \mathcal{J}_k(x, k; \tau) = +2v(x) \sin(\gamma^2 \lambda_{(x)} k) \times \exp(+\gamma^2 \lambda_{(k)}^2 / 4) \mathcal{G}_\gamma(x, k), \quad (74)$$

which, as prescribed, points to a convergent series for the stationarity quantifier $\nabla_\xi \cdot \mathcal{J}$. In addition, depending on the explicit form of the Hamiltonian, it can be manipulated to give the Liouvillian quantifier $\omega^{-1} \nabla_\xi \cdot \mathbf{w}$ and the complete pattern of the associated Wigner flow.

V. THE HARPER-LIKE DYNAMICS

First introduced by Harper [38] for parametrizing the behavior of electrons coupled to magnetic fields in a two-dimensional lattice, the so-called Harper Hamiltonian framework turned into an outstanding tool for several applications [66,67]. These include the description of fractal structures related to the Hofstadter spectral decomposition connected to the phenomenology of the quantum Hall effect [68] and of the quantum-mechanical topological phases which emerge in the context of designing ultracold-atom platforms for producing synthetic gauge fields and topological structures for neutral atoms [69,70].

From an effective perspective, Harper-like models can be reduced to a one-dimensional Hamiltonian description of nearest-neighbor couplings with a sinusoidal modulation of the on-site energies expressed by a Hamiltonian constraint given by

$$H \psi_n = -A_k (e^{+i\vartheta} \psi_{n+1} + e^{-i\vartheta} \psi_{n-1}) - A_x \cos(2\pi\alpha n + \theta) \psi_n, \quad (75)$$

where A_x and A_k correspond to the coupling magnitude and the modulation [42], respectively, and the phases θ and ϑ are related to the wave number in two dimensions.

By observing that the displacement of quantum numbers exhibited by $\psi_{n\pm 1}$ in Eq. (75) is associated with localized states in adjacent sites, quantum states described by $\psi_{n\pm 1} \sim \psi(x \pm a) \equiv \exp(\pm i k a) \psi(q)$ can be parameterized by the action of the translation operators $\exp(\pm i k a)$. According to the application of the dimensionless momentum operator k , for a coordinate correspondence given by $(x, a) \rightarrow (2\pi\alpha n, 2\pi\alpha)$, which is also dimensionless, the Hamiltonian (75) admits a semiclassical representation reduced to the form of Eq. (4). One can, indeed, demonstrate that Eq. (4) is derived from Eq. (75) [39] for corresponding coordinate operators \hat{x} and \hat{k} ,

satisfying $[\hat{x}, \hat{k}] = i 2\pi\alpha$. The so-called Peierls phase parameter $2\pi\alpha$ plays the role of an effective Planck constant through the condition $2\pi\alpha \equiv 1$ in the dimensionless variables arising from the canonical commutation relation $[\hat{x}, \hat{k}] = i$. It is also relevant to notice that questions related to quantum chaos are more suitably addressed in the case of $A_k = A_x = 2\pi\alpha \equiv 1$, through which the transition from integrable to chaotic Hamiltonian regimes is more clearly evinced.

Of course, the phase-space representation of the electron dynamics in a two-dimensional crystal is also covered by Eq. (75), which, in order to fit the nonlinear properties described by $\mathcal{H}(x, k) = \mathcal{K}(k) + \mathcal{V}(x)$, ignoring the global relative sign and admitting a phenomenological variation driven by an arbitrary parameter v^2 , can now be cast into the dimensionless form

$$\mathcal{H}_H(x, k) = \cos(k) + v^2 \cos(x), \quad (76)$$

which exhibits (time-dependent) cyclic analytical solutions. This Hamiltonian works as a feasible platform for identifying classicality and quantumness through the Wigner flow deviations from stationarity and Liouvillian regimes, according to the formalism introduced in the previous sections. The classical properties of the Harper-like Hamiltonian (76) are depicted in Fig. 1, where the phase-space trajectories associated with the corresponding lattice designs are identified for several values of the classical energy $\mathcal{H}_H \rightarrow \epsilon$ and for the phenomenological parameter v .

A. Results for TD ensembles

By identifying $\mathcal{K}(k)$ and $\mathcal{V}(x)$ with $\cos(k)$ and $v^2 \cos(x)$, respectively, as in Eq. (76), one straightforwardly writes the Harper-like associated classical distribution as

$$\mathcal{W}_0(x, k; \beta_{\hbar\omega}) = [\mathcal{Z}_0(\beta_{\hbar\omega})]^{-1} \exp\{-\beta_{\hbar\omega}[\cos(k) + v^2 \cos(x)]\}, \quad (77)$$

with the phase-space domain reduced to the intervals $x \in [-\pi, \pi]$ and $k \in [-\pi, \pi]$, where $\beta_{\hbar\omega} = \beta \hbar\omega$, from which the partition function can be obtained as

$$\begin{aligned} \mathcal{Z}_0(\beta_{\hbar\omega}) &= \int_{-\pi}^{+\pi} dx \int_{-\pi}^{+\pi} dk \exp\{-\beta_{\hbar\omega}[\cos(k) + v^2 \cos(x)]\} \\ &= 4\pi^2 I_0(\beta_{\hbar\omega}) I_0(v^2 \beta_{\hbar\omega}), \end{aligned} \quad (78)$$

where I_n is the modified Bessel function of order n . After the mathematical manipulations which result in Eq. (55), one obtains

$$\mathcal{W}_{St}^{(2)}(x, k; \beta_{\hbar\omega}) = \frac{\mathcal{Z}_0(\beta_{\hbar\omega})}{\mathcal{Z}_{St}(\beta_{\hbar\omega})} \mathcal{W}_0(x, k; \beta) \left(1 - \left\{ \frac{(\beta_{\hbar\omega})^3}{24} [v^2 \cos(k) \sin^2(x) + \cos(x) \sin^2(k)] + \frac{(\beta_{\hbar\omega})^2}{8} \cos(k) \cos(x) \right\} \right), \quad (79)$$

with

$$\mathcal{Z}_{St}(\beta_{\hbar\omega}) = 4\pi^2 I_0(\beta_{\hbar\omega}) I_0(v^2 \beta_{\hbar\omega}) - \frac{1}{24} (\beta_{\hbar\omega})^2 I_1(\beta_{\hbar\omega}) I_1(v^2 \beta_{\hbar\omega}) \quad (80)$$

and with the associated Wigner currents given by

$$\begin{aligned} \mathcal{J}_x^{(2)}(x, k; \tau) &= -\sin(k) \left(1 - \frac{\beta_{\hbar\omega}^3 v^2}{24} [v^2 \cos^2(k) \sin(x) + \sin(k) \cos^2(x)] \right. \\ &\quad \left. + \frac{1}{24} \{ \beta_{\hbar\omega} v^2 \cos(x) + \beta_{\hbar\omega}^2 [m^4 \sin^2(x) - 3v^2 \cos(k) \cos(x)] \} \right) \mathcal{W}_0, \end{aligned} \quad (81)$$

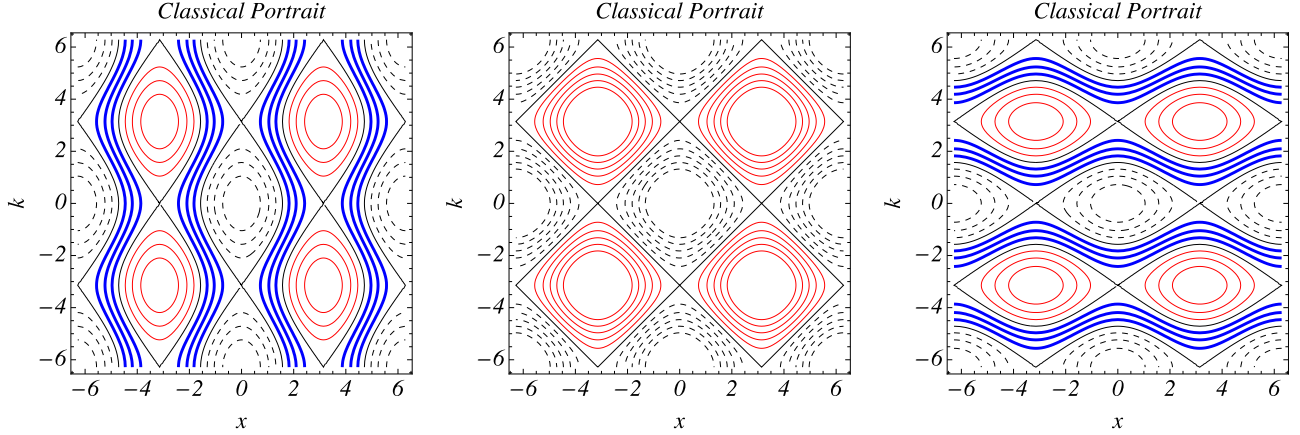


FIG. 1. Classical portrait of Harper Hamiltonians. Phase-space trajectories and corresponding lattice designs are for $\text{Max}\{v^2 - 1, 0\} < |\epsilon| < v^2 + 1$ associated with closed trajectories for $\epsilon > 0$ (black dashed lines), for $\epsilon < 0$ (red thin lines), and for $0 < |\epsilon| < v^2 - 1$, corresponding to open trajectories (blue thick lines), when they exist. The threshold (open-closed) value is given by $|\epsilon| = v^2 - 1$. The plots are for $v^2 = 2$ (left), with $|\epsilon| = 5/2, 2, 3/2, \dots, 0$; $v^2 = 1$ (middle), with $|\epsilon| = 5/2, 2, 3/2, \dots, 0$; and $v^2 = 1/2$ (right), with $|\epsilon| = 5/2, 2, 3/2, \dots, 0$.

$$\mathcal{J}_k^{(2)}(x, k; \tau) = v^2 \sin(x) \left(1 - \frac{\beta_{\hbar\omega}^3}{24} v^2 [v^2 \cos^2(k) \sin(x) + \sin(k) \cos^2(x)] \right. \\ \left. + \frac{1}{24} \{ \beta_{\hbar\omega} \cos(k) + \beta_{\hbar\omega}^2 [\sin^2(k) - 3v^2 \cos(k) \cos(x)] \} \right) \mathcal{W}_0, \quad (82)$$

which leads to the Liouillianity quantifier given by

$$\omega^{-1} \nabla_{\xi} \cdot \mathbf{w} = \frac{(\beta_{\hbar\omega})^2}{12} \sin(x) \sin(k) [v^4 \cos(x) - v^2 \cos(k)], \quad (83)$$

The partition function of classical and quantum [$O(\hbar^2)$] stationary ensembles can be considered when obtaining the TD variables, which include the corresponding quantum purity $\mathcal{P}_{(\beta_{\hbar\omega})}$ [71], the (dimensionless) internal energy $\mathcal{E}_{(\beta_{\hbar\omega})}$ [$\equiv E/(\hbar\omega)$], and the (dimensionless) heat capacity $\mathcal{C}_{(\beta_{\hbar\omega})}$ ($\equiv C/k_B$), given, respectively, by

$$\mathcal{P}_{(\beta_{\hbar\omega})} = \frac{\mathcal{Z}(2\beta_{\hbar\omega})}{\mathcal{Z}^2(\beta_{\hbar\omega})}, \quad \mathcal{E}_{(\beta_{\hbar\omega})} = -\frac{\partial}{\partial \beta_{\hbar\omega}} \ln [\mathcal{Z}(\beta_{\hbar\omega})], \\ \mathcal{C}_{(\beta_{\hbar\omega})} = \beta_{\hbar\omega}^2 \left(\frac{\partial}{\partial \beta_{\hbar\omega}} \right)^2 \ln [\mathcal{Z}(\beta_{\hbar\omega})].$$

From results depicted in Fig. 2, one notices that quantum contributions recover the meaning of quantum purity and work to constrain its maximum values to be lower than unity (pure state). The results also exhibit corrections to \mathcal{E} and \mathcal{C} which work fine only for $0 < \beta_{\hbar\omega} \lesssim 1$ in an interval that is unaffected by the choice of the Peierls phase parameter $2\pi\alpha$, set equal to unity.

Besides the natural interpretation of the quantum contributions which lead to an overall suppression of classical values, it is evident that negative values of \mathcal{E} have no classical analogs in the Schrödinger theory for Hamiltonians with quadratic kinetic terms. The internal energy of a real neutral gas depends on how the temperature \mathcal{T} appears in the equation of state. In general, negative values of \mathcal{E} mean that interaction contributions dominate over thermal kinetic ones, which decrease as β increases. In the same sense, despite evident modifications

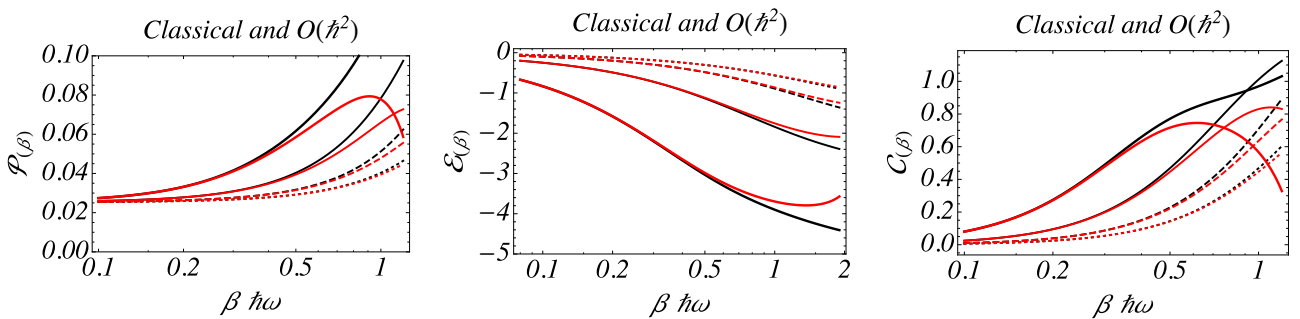


FIG. 2. (a) Purity $\mathcal{P}_{(\beta_{\hbar\omega})}$, (b) internal energy $\mathcal{E}_{(\beta_{\hbar\omega})}$, and (c) heat capacity $\mathcal{C}_{(\beta_{\hbar\omega})}$ as a function of $\beta_{\hbar\omega}$ for classical (black lines) and quantum [$O(\hbar^2)$, red lines] stationary regimes. The results are for $v^2 = 1/\sqrt{2}$ (dotted lines), 1 (dashed lines), $\sqrt{2}$ (thin solid lines), and 2 (thick solid lines).

for \mathcal{C} , a more accurate quantum behavior could be described only by accounting for higher-order terms in the \hbar^2 expansion. One notices that the results are not qualitatively modified by the parameter ν , which is set equal to 1 for the evaluation of the Wigner current maps (see Fig. 3).

To complete the TD ensemble analysis, the non-Liouvillian behavior is depicted in Fig. 3. The results are reduced to the central site, but they can be periodically reproduced from $x, k \in (-\pi, \pi)$ to $x, k \in (-\pi \pm 2n\pi, \pi \pm 2n\pi)$ (with $n \in \mathbb{Z}$) due to the lattice symmetry properties. For $\beta_{\hbar\omega} \ll 1$ (first row), quantum fluctuations are highly suppressed by the thermal fluctuations and cannot be identified through the patterns of the Wigner function (left column) and of the corresponding Wigner flow (middle column), which is depicted by the light-dark background color in Fig. 3. The Wigner flow stagnation points are identified by orange and blue crossing lines, where $\mathcal{J}_x^{(2)} = \mathcal{J}_k^{(2)} = 0$, and can be interpreted as a consequence of the $O(\hbar^2)$ quantum contributions which are evinced for $\beta_{\hbar\omega} \gtrsim 1$. Blue (orange) contour lines denote the threshold for the reversal of the Wigner flow in the x (k) direction. The quantum fluctuations are identified by clockwise and counterclockwise vortices (winding numbers equal to $+1$ and -1), separatrix intersections, and saddle points (winding number equals to zero). The contra-flux fringes (bounded by blue and orange lines) emerge to compensate the retarded evolution of the quantum flux. The classical profile does not exhibit such an overall local compensation phenomenon. For comparison reasons, the classical trajectory is shown as a collection of black thin lines, which do not exhibit any local features like the above-mentioned ones. The non-Liouvillian quantifier is depicted in the right column in Fig. 3, corresponding to the quantum TD ensemble Wigner function for $\beta_{\hbar\omega} = 0.1$ (first row), 1 (second row), 3 (third row), and 5 (fourth row), from which one notices that the quantum effects are much more evident for lower temperatures (increasing $\beta_{\hbar\omega}$ values).

B. Results for Gaussian ensembles

For Gaussian ensembles driven by $\mathcal{K}(k) = \cos(k)$ and $\mathcal{V}(x) = \nu^2 \cos(x)$, exact analytical results can be obtained. First, one identifies $\lambda_{(x)}^{2\eta+1} = \mu_{(k)}^{2\eta+1} = (-1)^{\eta+1}$ from Eqs. (68) and (69), which leads to $\mu = \lambda = i$, $\nu(x) = i\nu^2 \sin(x)$, and $\kappa(k) = i \sin(k)$. Once they are placed into Eqs. (73) and (74), one can write

$$\partial_x \mathcal{J}_x(x, k; \tau) = +2 \sin(k) \sinh(\gamma^2 x) \exp(-\gamma^2/4) \mathcal{G}_\gamma(x, k), \tag{84}$$

$$\begin{aligned} \partial_k \mathcal{J}_k(x, k; \tau) &= -2\nu^2 \sin(x) \sinh(\gamma^2 k) \\ &\times \exp(-\gamma^2/4) \mathcal{G}_\gamma(x, k) \end{aligned} \tag{85}$$

as a result of the convergent series expansion equations (70) and (71). This means that a Gaussian ensemble driven by the Harper-like Hamiltonian can be considered in order to compare classical and quantum regimes, providing the exact analytic expression for the quantum scenario in the phase space. The integrated Wigner currents obtained from Eqs. (73)

and (74) thus read

$$\begin{aligned} \mathcal{J}_x(x, k; \tau) &= +\frac{\gamma}{2\sqrt{\pi}} \sin(k) \exp(-\gamma^2 k^2) \\ &\times \{\text{erf}[\gamma(x - 1/2)] - \text{erf}[\gamma(x + 1/2)]\}, \end{aligned} \tag{86}$$

$$\begin{aligned} \mathcal{J}_k(x, k; \tau) &= -\frac{\gamma}{2\sqrt{\pi}} \nu^2 \sin(x) \exp(-\gamma^2 x^2) \\ &\times \{\text{erf}[\gamma(k - 1/2)] - \text{erf}[\gamma(k + 1/2)]\}, \end{aligned} \tag{87}$$

written in terms of error functions $\text{erf}(\dots)$, and of course, the components of the quantum-like velocity, \mathbf{w} , w_x , and w_k , are obtained by replacing $-k^2 \leftrightarrow +x^2$ in the exponential function of the respective equations, (86) and (87) (multiplied by $\pi\gamma^{-2}$).

The departure configuration ($\tau = 0$) of the Gaussian Wigner flow pattern described above is depicted in Fig. 4, where the density plots for stationarity and Liouvillianity quantifiers, $\nabla_\xi \cdot \mathcal{J}$ and $\omega^{-1} \nabla_\xi \cdot \mathbf{w}$, are shown.

The stationarity quantifier $\nabla_\xi \cdot \mathcal{J}$ is depicted according to the background color scheme, in which lighter regions correspond to nonvanishing local contributions to $\partial_t \mathcal{G}_\gamma(x, k)$. In spite of the x - k symmetry provided by the Gaussian distribution, one should notice that stationarity and Liouvillianity are decoupled from each other as their associated quantifiers do not exhibit the same local pattern.

Finally, once the above quantifiers, $\nabla_\xi \cdot \mathcal{J}$ and $\omega^{-1} \nabla_\xi \cdot \mathbf{w}$, are integrated over the volume enclosed by the classical path \mathcal{C} , the quantum effects are shown to average out. This shows that, despite providing the phase-space local identification of quantum effects, Gaussian ensembles, in the case of a Harper-Hamiltonian system, suppress the local vortices from the quantum Wigner pattern and keep the flow of probability and information properties quantitatively equivalent to those of the classical pattern.

VI. CONCLUSIONS

The phase-space framework for describing the classical-quantum behavior of systems driven by one-dimensional nonlinear Hamiltonians described by $H^W(q, p) = K(p) + V(q)$ (constrained to $\partial^2 H^W / \partial q \partial p = 0$) was introduced and analytically scrutinized through their Wigner flow properties. Generalized Liouvillian and stationary properties were established for TD and Gaussian quantum ensembles to quantify the quantum modifications over the classical phase-space Hamiltonian solutions. For TD ensembles, a natural extension of the Wigner approach for perturbatively constructing stationary Wigner functions was provided to include the Wigner procedure as a particular solution related to the Hamiltonian kinetic contribution identified by $K(p) = p^2/2m$. Following a parallel procedure, for Gaussian ensembles, the overall quantum distortion over a classical phase-space trajectory was obtained in terms of convergent infinite series expansions over \hbar^2 .

As noticed from previous works, the WW framework, now in its generalized version, can work as a probe for quantumness and classicality for an enlarged class of Hamiltonian systems which includes out-of-the-ordinary contributions for the kinetic term $K(p) \neq p^2/2m$. As an implementation example, our results were specialized to a Harper-like Hamiltonian

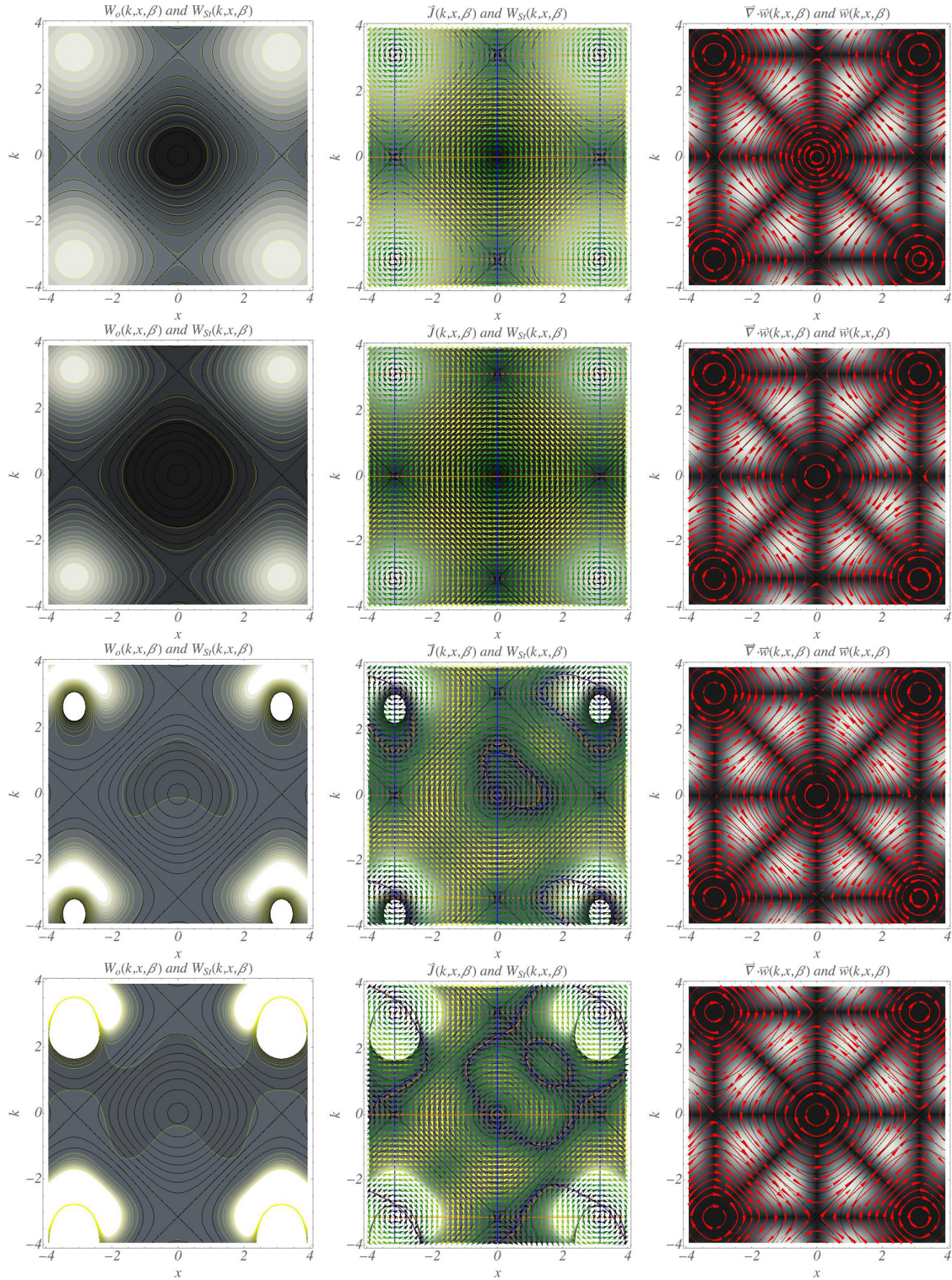


FIG. 3. Left: Classical (black contours) and QM-corrected (yellow contours) Wigner-function profiles, $\mathcal{W}_o(x, k; \beta_{ho})$ and $\mathcal{W}_{Si}^{(2)}(x, k; \beta_{ho})$. Middle: Features of the Wigner flow for the TD ensemble in the x - k plane. Blue contour lines are for $J_x^{(2)}(x, k; \beta_{ho}) = 0$, and orange contour lines are for $J_k^{(2)}(x, k; \beta_{ho}) = 0$. The green-yellow vector arrow color scheme shows the [quantum $O(\hbar^2)$] Wigner current profiles, with the domains of quantum fluctuations bounded by blue and orange lines. Darker green arrows correspond to intenser fluxes. Right: Liouvillian quantifier $\nabla_{\xi} \cdot \mathbf{w}$ superposed by the normalized quantum velocity field representation (red arrows) $\mathbf{w}/|\mathbf{w}|$. The background color scheme (from darker regions, $\nabla_{\xi} \cdot \mathbf{w} \sim 0$, to lighter regions, $\nabla_{\xi} \cdot \mathbf{w} > 0$) reinforces the approximated Liouvillian behavior for the central region. The results are for $v^2 = 1$ and $\beta_{ho} = 0.1$ (first row), 1 (second row), 3 (third row), and 5 (fourth row).

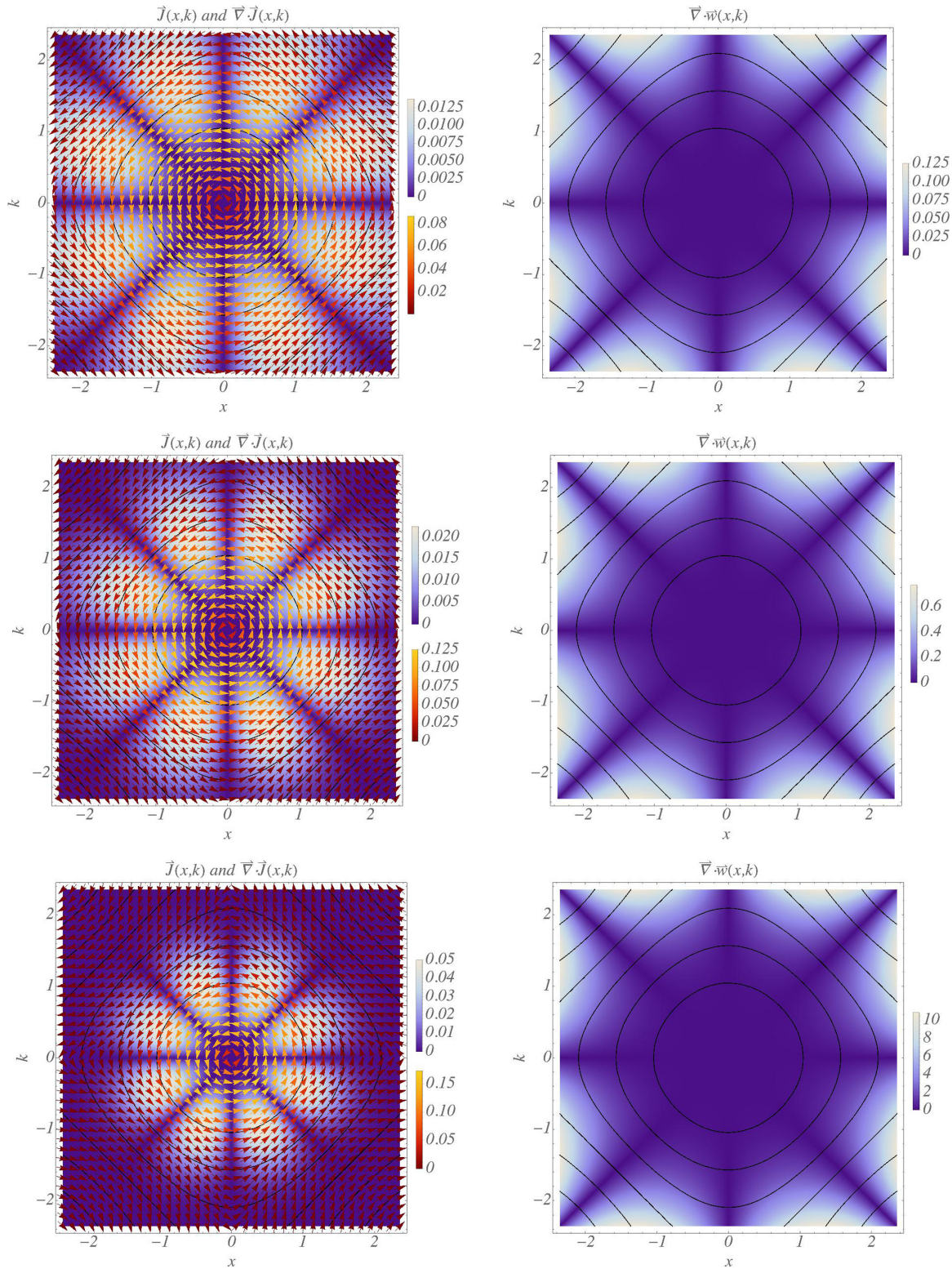


FIG. 4. Left: Features of the Wigner flow for the Gaussian ensemble in the x - k plane. At $\tau = 0$, Gaussian ensembles do not exhibit either vortices or stagnation points, in a kind of camouflage of the quantum corrections. The stationarity quantifier $\nabla_{\xi} \cdot \vec{J}$ and the modulus of \vec{J} are described, respectively, according to the light- and dark-blue background and yellow-red background color schemes. The results are for the increasing spreading characteristic of the Gaussian function for $\gamma = 1/\sqrt{2}$ (first row), 1 (second row), and $\sqrt{2}$ (third row). Peaked Gaussian distributions ($\gamma = \sqrt{2}$) localize the quantum distortions which result in nonstationarity. Right: Liouvillian quantifier $\omega^{-1} \nabla_{\xi} \cdot \vec{w}$, depicted through the light- and dark-blue background color scheme, from darker regions, $\nabla_{\xi} \cdot \vec{w} \sim 0$, to lighter regions, $\nabla_{\xi} \cdot \vec{w} > 0$. In both columns, classical patterns are shown as a collection of black lines.

system which, besides working as an experimental platform first identified in Harper's approach for crystal electrons in the presence of magnetic fields [38], admits a statistical description in terms of TD and Gaussian ensembles, like what was here considered. However, our extended Wigner phase-space approach is not restricted to Harper-like Hamiltonians. In fact, the Hamiltonian from Eq. (4) can be interpreted as a restrictive choice of the phenomenological parameters of the more general Aubry-André model [39,43], which includes additional linear contributions to a reparameterized form of $H^W(q, p)$, with $K(p) \rightarrow K(p) + \omega_p p$ and $V(q) \rightarrow V(q) + \omega_q q$, where momentum and position contribution coefficients $\omega_p(p)$ and $\omega_q(q)$ [see Eq. (4)] are the drivers of either bound-state or continuous spectrum regimes.

Beyond the condensed-matter physics examples, deformed Hamiltonians with $K(p) = \cosh(p/p_0)$, which are sometimes related to the Toda lattice theory [72], have been conjectured in the investigation of the Seiberg-Witten curve of an $N = 2$ Yang-Mills theory, where a quantization hypothesis follows from the correspondence between spectral theory and strings [73]. Following a similar prescription in which the hyperbolic behavior is dominant in the description of the competitive ecological equilibrium of populations, Lotka-Volterra-like systems [74], besides being extensively applied stochastic systems [75,76], can be treated via the dynamical equations arising from the Hamiltonian formulation described

(in dimensionless notation) by

$$\mathcal{H}(x, k) = x + k + e^{-x} + e^{-k}, \quad (88)$$

which yields the following classical equations of motion:

$$dx/d\tau = \{x, \mathcal{H}\}_{PB} = 1 - e^{-k}, \quad (89)$$

$$dp/d\tau = \{k, \mathcal{H}\}_{PB} = e^{-x} - 1, \quad (90)$$

whose phase-space trajectories can be extended to the non-commutative context, $[x, k] \neq 0$, through the WW formalism. This allows for quantifying (cf. Refs. [27,28]) related phase-space correlations and information flow aspects as well as quantumlike deviations from the well-known prey-predator classical system.

Finally, it is worth mentioning that the procedure discussed here can be equally specialized to the above-mentioned scenarios and could encompass more complex forms of non-Linear Hamiltonians. Clearly, an extended framework of phase-space QM that deserves further research.

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