# Nonclassical properties and Anderson localization of quantum states in coupled waveguides

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We consider the propagation of light beams through disordered lattices of coupled waveguides searching for Anderson localization and investigating the evolution of nonclassical properties of injected quantum states. We assume that the beam is initially in a variety of states, such as the complementary coherent state, the reciprocal binomial state, and the polynomial state. The statistical properties of the evolved states were analyzed numerically as functions of the localization and delocalization parameters averaged over many realizations of disorder. We also numerically reconstruct the Wigner function of the output state. Interestingly, we find that high values of disorder tend to preserve the quantum properties of some input states when we look at the input waveguide despite of the coupling between it and the neighboring waveguides.

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### I. INTRODUCTION

Nonclassical properties of quantized light field and its generation, i.e., "quantum states engineering," are essential ingredients of quantum optics and quantum information. Indeed, nonclassical states perform a crucial role in many potential applications such as quantum teleportation [1-3], quantum cryptography [4], quantum computation and quantum communication [5], and quantum internet [6]. In this direction, periodic photonic systems have emerged as a platform to engineer new light field structures, presenting numerous significant technological advances [7-10]. On the other hand, the study of nonperiodic photonic structures, by using small defects in periodic lattices [11-13] or disordered and quasiperiodic structures [14–18], has demonstrated a diversity of optical effects in the presence of Anderson localization, such as engineering of random lasers [19] and structurally colored materials with precisely controllable wavelength and angular dependence of scattering [20]. Recently, it was demonstrated how to tune and freeze disorder in photonic crystals by using percolation lithography [21].

Anderson localization—i.e., the suppression of transport due to destructive interference of the many paths associated with coherent multiple scattering from the modulation of a disordered potential [22]—has been experimentally observed in microwaves [23], light diffusive media [14,24], photonic crystals [15,16], Bose-Einstein condensates [25,26], sound waves [27], optical fiber arrays [28], etc. Inspired by these experimental investigations, many theoretical studies have been proposed by considering the system in the presence of disordered potentials (see, for example, Refs. [29–39]).

Anderson localization of nonclassical light was investigated for propagation in an array of waveguides in which neighboring waveguides are evanescently coupled and disorder can be added in a controlled manner [40]. Specifically, that work investigated the consequences of using both sub-Poissonian and super-Poissonian input light on the characteristics of Anderson localization, verifying the enhancement in fluctuations of localized light and superbunching due to the medium's disorder. Also, an important consequence of sub-Poissonian statistics of the incoming light is to quench the total fluctuations at the output [40]. The system employed in Ref. [40] is similar to that used in Ref. [16] to experimentally investigate the evolution of linear and nonlinear waves in the presence of Anderson localization. Moreover, the disordered one-dimensional waveguide lattice was also used to experimentally investigate an extensive list of phenomena: the signature of a localization phase transition for light by directly measuring wave transport inside the lattice [41], quantum correlations between noninteracting particles evolving simultaneously in a disordered medium [42], Hanbury-Brown-Twiss correlations of Anderson-localized waves [43], the control of the polarization state of coherent light propagating through an optically thick multiple-scattering medium by controlling only the spatial phase of the incoming field with a spatial light modulator [44], the coherent manipulation of two-photon path-entangled states by multimode interference in multimode waveguides [45], the observation of topological phase transitions in photonic quasicrystals [46], the observation of ensemble-averaged quantum correlations between pathentangled photons undergoing Anderson localization [47], the two-photon Anderson localization in a quadratic waveguide array with the emergence of off-diagonal disorder [48], etc.

Here, inspired by the results obtained in Ref. [40], we numerically investigate the propagation of light beams, previously prepared in nonclassical states of the electromagnetic field, propagating in disordered lattices of waveguides and undergoing Anderson localization. Our goal is to verify the influence of Anderson localization on the statistical properties of previously prepared input light field (nonclassical states). To this end, we assume the beam in a variety of states, namely, the complementary coherent state [49], reciprocal binomial state [50], polynomial state [51], thermal state, coherent state, and squeezed state [52].

The rest of the paper is organized as follows: The theoretical model is considered in next section, where we present the dynamical model of the system in Sec. II A, the characterization of the quantum states under consideration in Sec. II B, and the numerical methods in Sec. II C. In Sec. III we present the numerical results and our analyses. We conclude the paper in Sec. IV.

### **II. LINEAR ARRAY OF WAVEGUIDES**

The system we consider is a one-dimensional finite array of monomodal waveguides where prescribed input states of the electromagnetic field can propagate. Classically, fields propagating in the array of waveguides are coupled through evanescent waves passing over their boundary barriers; on the quantum level, one says that photons can coherently tunnel between neighboring waveguides so that the quantum state represents the overall-overlapping superposition of the modes of the waveguides [16]. The field can be injected into one or a few waveguides and disorder can be implemented in the array either by randomly adjusting the spacing among the parallel waveguides along the x direction, the propagation being in the z direction, or by randomly fixing the thicknesses of the waveguides. This kind of system has been constructed on an AlGaAs substrate [53,54] and direct identification and measurements of Anderson localization of states have been performed.

#### A. Theoretical model

The electromagnetic energy density in the array of waveguides, assuming that all media are linear and nonmagnetic, and that the relevant evanescent overlap occurs only between neighboring waveguides, is given by

$$\mathcal{H} = \sum_{j} \left( \frac{1}{2} \epsilon_0 n^2 \mathbf{E}_j^2 + \frac{1}{2\mu_0} \mathbf{B}_j^2 \right) + \sum_{(j,j')} \left( \frac{1}{2} \epsilon_0 n^2 \mathbf{E}_j \cdot \mathbf{E}_{j'} + \frac{1}{2\mu_0} \mathbf{B}_j \cdot \mathbf{B}_{j'} \right), \quad (1)$$

where *n* is the refraction index, which is a function of the transversal coordinates (x, y), and (j, j') denote next-neighbor pairs. We consider monochromatic fields propagating along the *z* direction.  $\mathbf{E}_j$  and  $\mathbf{B}_j$  are the electric and magnetic fields of the single mode supported by the *j*th waveguide.

At the quantum level, the field mode in the *j*th waveguide is written in terms of photon-annihilation and photon-creation operators,  $a_j$  and  $a_j^{\dagger}$ , respectively, and the evolution of the system is dictated by a Hamiltonian in the form [42]

$$H = \sum_{j} \beta_{j} a_{j}^{\dagger} a_{j} + \sum_{j} [C_{j+1,j} a_{j+1}^{\dagger} a_{j} + C_{j,j+1} a_{j}^{\dagger} a_{j+1}], \quad (2)$$

where  $\beta_j$  is the propagation constant associated with the *j*th waveguide and  $C_{j+1,j}$  and  $C_{j,j+1}$  are coupling coefficients between nearest-neighbor waveguides. These parameters depend on the refraction index profile as well as on the geometry and spacing of each waveguide; they could be obtained by

integrating the classical equation (1) using the proper form for the field modes [55]. Notice that we are considering fields with sufficiently low intensities to make nonlinear effects negligible; also, in this paper unless stated to the contrary, we use  $\hbar = 1$  and c = 1. The creation and annihilation operators satisfy the commutation relations

$$[a_j, a_l] = 0, \quad [a_j^{\dagger}, a_l^{\dagger}] = 0, \quad [a_j, a_l^{\dagger}] = \delta_{jl},$$
 (3)

and we assume the existence of eigenstates  $|\psi\rangle$  such that  $n_j|\psi\rangle = a_j^{\dagger}a_j|\psi\rangle = n_j|\psi\rangle$ , where  $n_j$  is the photon number of the *j*th waveguide.

We will analyze arrays where we can fix a constant coupling (tunneling rate) between neighbor waveguides,  $C_{j+1,j} = C_{j,j+1} = C$ , and introduce disorder by taking the coefficients  $\beta_j$  as random variables with zero-mean Gaussian distributions; the model then becomes isomorphic to the one-dimensional quantum tight-binding model used by Anderson [22], with  $\beta_j$  being the on-site energy, and the system should then present localization of states. This assertion is experimentally feasible since these coefficients are related with the guide geometry, which can be appropriately adjusted [41,43] to spatially modulate the index of refraction n(x). In this case, the Heisenberg equations can be written as [16,41,43,47,56]

$$i\frac{\partial a_j}{\partial z} = [a_j, H] = \beta_j a_j + C(a_{j+1} + a_{j-1}), \qquad (4)$$

where z = ct/n, i.e., measurements of intensity distribution at position *z* give the time evolution along the array.

Now we search for a solution of the Heisenberg equations (4), depending on the initial input state. Since the Heisenberg equations are linear in the annihilation (or for the creation) operators, it can be solved by finding the Green's function in such way that

$$a_j(z) = \sum_l G_{jl}(z) a_l(0),$$
 (5)

where  $a_l(0)$  correspond to the input state (at z = 0) into the *l*th waveguide. The Green's function correlates fields in the *j*th and the *l*th waveguides at all positions *z*. By inserting Eq. (5) into Eq. (4), one gets the following set of first-order differential equations for the Green's functions,

$$i\frac{\partial G_{jl}}{\partial z} = \beta_j G_{jl} + C(G_{j+1,l} + G_{j-1,l}).$$
(6)

From our assumptions,  $G_{jl}(z)$  depend on the parameters  $\beta_j$ , which vary randomly, and *C* that remains fixed; this is usually referred to as diagonal disorder. These equations can be solved numerically with great precision; taking specific distributions of  $\{\beta_j\}$  and a given value of *C*, solutions are obtained just depending on the initial input state.

We can work with a great simplification if we consider that the input field  $|\Psi_{in}\rangle$  is injected into only one waveguide [40], which we label by  $j_0$ , that is,  $a_j(0)|\Psi_{in}\rangle = 0$  for all  $j \neq j_0$ . Thus, considering the light injection only into the  $j_0$ th waveguide, the mean-field intensity output by waveguide j as a function of the lattice length z is given by

$$\langle I_j(z)\rangle = \langle a_j^{\dagger}(z)a_j(z)\rangle = \langle |G_{j,j_0}(z)|^2 \rangle \langle a_{j_0}^{\dagger}a_{j_0} \rangle, \qquad (7)$$

where the mean value of the Green's functions is a standard statistical mean over several realizations for different values of  $\{\beta_j\}$ , while the mean  $\langle a_{j_0}^{\dagger}a_{j_0}\rangle$  represents the quantum expectation value of the number operator  $a_{j_0}^{\dagger}(0)a_{j_0}(0)$ , which depends only on the input state. Also, we can calculate the correlation of the intensities between the output of two waveguides *j* and *l* to be given by

$$\langle I_j(z)I_l(z)\rangle = \langle |G_{j,j_0}(z)|^2 |G_{l,j_0}(z)|^2 \rangle \langle a_{j_0}^{\dagger 2} a_{j_0}^2 \rangle.$$
(8)

Another important quantity to qualify the statistics of photons propagating in the array is the second-order correlation function defined, for the *j*th waveguide, by

$$g_{j}^{(2)}(z) = \frac{\left\langle a_{j}^{\dagger 2}(z)a_{j}^{2}(z)\right\rangle}{\langle a_{j}^{\dagger}(z)a_{j}(z)\rangle^{2}} = \frac{\left\langle |G_{j,j_{0}}(z)|^{4}\right\rangle}{\langle |G_{j,j_{0}}(z)|^{2}\rangle^{2}} \frac{\left\langle a_{j_{0}}^{\dagger 2}a_{j_{0}}^{2}\right\rangle}{\langle a_{j_{0}}^{\dagger}a_{j_{0}}\rangle^{2}}.$$
 (9)

The correlation function  $g^{(2)}(z)$  indicates whether states evolve in a Poissonian way, or if they present either bunching or antibunching  $[g^{(2)}(z) > 1 \text{ and } g^{(2)}(z) < 1$ , respectively] in their photon distribution. In any case, all these quantities depend on the initial state in the input of the array of waveguides. The expressions for other quantities used to characterize the output field are presented in the results section.

#### B. Characterization of the input quantum states

In Ref. [40], a theoretical study of Anderson localization of light in an array of waveguides was presented, using coherent, thermal, and squeezed states at the input, to investigate the effects of nonclassicality. Here, we extend the work of Ref. [40] by considering different input states that show more general quantum statistics, and by presenting the evolution in the Wigner representation. We consider specifically the complementary coherent state (*CCS*) [49], the reciprocal binomial state (*RBS*) [50], and the polynomial state (*PS*) [51] as input states. These exotic states were proposed in the context of the projection synthesis method and allow for the measurement of the Husimi Q function, the phase distribution, and the quadrature variances of field states [57]. They present peculiar statistical properties and it is interesting to investigate how they evolve along the array.

The complementary coherent state [49] is written in the number basis as

$$|\Psi_{CCS}(\alpha, N)\rangle = \aleph_{CCS} e^{-|\alpha|^2/2} \frac{2^{-N}}{\sqrt{N!}} \sum_{k=0}^{N} \sqrt{k!} \alpha^{*N-k} e^{ik\pi/2} |k\rangle,$$
(10)

where the normalization constant  $\aleph_{CCS}$  is given by

$$\aleph_{CCS}^{2}(\alpha, N) = \frac{N! e^{|\alpha|^{2}} 2^{2N}}{\sum_{k=0}^{N} k! |\alpha|^{2(N-k)}}.$$
(11)

For these states we find the mean number of photons,

$$\langle a_{j_0}^{\dagger} a_{j_0} \rangle_{CCS} = \frac{\sum_{k=0}^{N} k! k |\alpha|^{2(N-k)}}{\sum_{k=0}^{N} k! |\alpha|^{2(N-k)}},$$
(12)

and the mean  $\langle a_{j_0}^{\dagger 2} a_{j_0}^2 \rangle$  is given by

$$\left[a_{j_0}^{\dagger 2}a_{j_0}^2\right]_{CCS} = \frac{\sum_{k=0}^N k! k(k-1) |\alpha|^{2(N-k)}}{\sum_{k=0}^N k! |\alpha|^{2(N-k)}}.$$
 (13)

The reciprocal binomial state [50], written as

$$|\Psi_{RBS}(\phi, N)\rangle = \aleph_{RBS} \sum_{k=0}^{N} {\binom{N}{k}}^{\frac{1}{2}} \exp\left[ik\left(\phi - \frac{\pi}{2}\right)\right]|k\rangle,$$
(14)

has normalization constant  $\aleph_{RBS}$  given by

$$\aleph_{RBS}^2 = \left[\sum_{k=0}^N \binom{N}{k}\right]^{-1}.$$
 (15)

For the *RBS*, we find

$$\langle a_{j_0}^{\dagger} a_{j_0} \rangle_{RBS} = \aleph_{RBS}^2 \sum_{k=0}^N \binom{N}{k} k$$
(16)

and

$$\left\{a_{j_0}^{\dagger 2}a_{j_0}^2\right\}_{RBS} = \aleph_{RBS}^2 \sum_{k=0}^N \binom{N}{k} k(k-1).$$
 (17)

The polynomial state [51] is defined by

$$|\Psi_{PS}(x,N)\rangle = \aleph_{PS} \sum_{k=0}^{N} {\binom{N}{k}}^{-\frac{1}{2}} \frac{H_{N-k}(x/\sqrt{2})e^{\frac{ik\pi}{2}}}{\sqrt{(2N-2k-1)!!}} |k\rangle, \quad (18)$$

where  $H_{N-k}(y)$  is a Hermite polynomial and  $k!! = k(k - 2)(k - 4)\cdots$  is the double factorial symbol, with the normalization constant  $\aleph_{PS}$  written as

$$\aleph_{PS}^{2} = \left[\sum_{k=0}^{N} {\binom{N}{k}}^{-1} \frac{H_{N-k}^{2}(x/\sqrt{2})}{(2N-2k-1)!!}\right]^{-1}.$$
 (19)

For the polynomial state we find

$$\langle a_{j_0}^{\dagger} a_{j_0} \rangle_{PS} = \Re_{PS}^2 \sum_{k=0}^N \binom{N}{k}^{-1} \frac{H_{N-k}^2(x/\sqrt{2})}{(2N-2k-1)!!} k, \qquad (20)$$

and

$$\langle a_{j_0}^{\dagger 2} a_{j_0}^2 \rangle_{PS} = \aleph_{PS}^2 \sum_{k=0}^N \binom{N}{k}^{-1} \frac{H_{N-k}^2(x/\sqrt{2})}{(2N-2k-1)!!} k(k-1).$$
(21)

For sake of comparison, we also investigate the input states used in Ref. [40]: thermal states (TS), mixed states with density matrix

$$\rho_{TS} = \frac{1}{1+\bar{n}} \sum_{n=0}^{\infty} \left(\frac{\bar{n}}{1+\bar{n}}\right)^n |n\rangle \langle n|, \qquad (22)$$

for which  $\langle a_{j_0}^{\dagger}a_{j_0}\rangle_{TS} = \bar{n}$  and  $\langle a_{j_0}^{\dagger 2}a_{j_0}^2\rangle_{TS} = 2\bar{n}^2$ ; coherent states (CS),

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle, \qquad (23)$$

for which  $\langle a_{j_0}^{\dagger}a_{j_0}\rangle_{CS} = |\alpha|^2$  and  $\langle a_{j_0}^{\dagger 2}a_{j_0}^2\rangle_{CS} = |\alpha|^4$ ; and squeezed-vacuum states,

$$|\psi_{SS}(|\zeta|e^{\phi})\rangle = \frac{1}{\cosh|\zeta|} \sum_{n=0}^{\infty} (-e^{i\phi} \tanh|\zeta|)^n \frac{\sqrt{(2n)!}}{2^n n!} |2n\rangle,$$
(24)

with squeezing parameter  $\zeta = |\zeta| e^{i\phi}$ , for which  $\langle a_{j_0}^{\dagger} a_{j_0} \rangle_{SS} = \sinh^2 |\zeta|$  and  $\langle a_{j_0}^{\dagger 2} a_{j_0}^2 \rangle_{CS} = \sinh^2 |\zeta| (1 + 3 \sinh^2 |\zeta|)$ .

## C. Computational method

Since we are assuming that the input light is injected only in the  $j_0$ th waveguide, all relevant quantities depend only on the Green's functions  $G_{j,j_0}$ , which satisfy

$$i\frac{\partial G_{j,j_0}}{\partial z} = \beta_j G_{j,j_0} + C(G_{j+1,j_0} + G_{j-1,j_0}).$$
(25)

To get solutions numerically, these equations were discretized and solved by using the Crank-Nicolson method [58] with step size  $\Delta z = 0.001$ . In the numerical analysis, we consider an array with 101 waveguides, which we label from 1 to 101 with light injected only in the waveguide  $j_0 = 51$ . Thus the initial condition is that  $G_{51,51}(0) = 1$  while  $G_{j,51}(0) = 0$  for all  $j \neq$ 51; additionally, since the waveguide array is finite containing M waveguides (here 101), we add the boundary conditions  $G_{0,l}(z) = G_{M+1,l}(z) = 0$  for all l = 1, 2, ..., M.

Following Ref. [40], we assume that the random coefficients  $\beta_j$  are independent of each other and follow a zero-mean Gaussian probability distribution of the form

$$P(\beta) = \frac{1}{\sqrt{2\pi\,\Delta^2}} \exp\left(\frac{-\beta^2}{2\,\Delta^2}\right),\tag{26}$$

with the variance  $\Delta^2$  measuring the disorder in the waveguide array. Although the Gaussian distribution is more natural, numerical tests show that the results do not depend on the specific distribution of the disordered parameter, but only on its variance. For sake of simplicity but without loss of generality, we fix the interaction parameter C = 1 and take  $\Delta/C$ quantifying the arrangement disorder.

To get the  $\beta_j$  coefficients, we used the Box-Müller method [59] that requires the generation of two random numbers with uniform distribution, which was done using a congruence method. For each set { $\beta_j$ } of disorder parameters, Eqs. (25) are solved numerically (fixing *C*) to find the relevant Green's functions. The functions  $G_{j,j_0}(z)$  depend on the disorder parameters  $\beta_j$  and on the coupling parameter *C*, and completely describe the dynamical evolution of any state injected into the waveguide array, only through the 51st waveguide. Naturally, to average over the sets of the random coefficients  $\beta_j$  one has to consider a reasonable number of realizations of disorder, different sets { $\beta_j$ }. We observed that the values of the output quantities do not change significantly for around 500 realizations of disorder. Here we take 1000 realizations in each simulation.

In our study, we take all the input states with the same energy, that is the same mean number of photons, specifically  $n_{in}(0) = \langle a_{51}^{\dagger}(0)a_{51}(0) \rangle = 10$ . To present the results obtained by our simulations, we choose five states from the families we have shown in Sec. II B, namely: two *CCS* states, named *CCS*<sub>1</sub> [=| $\Psi_{CCS}(0.1414, 10)$ ] and *CCS*<sub>2</sub> [=| $\Psi_{CCS}(1.916, 11)$ ]; two *PS* states, referred to as *PS*<sub>1</sub> [=| $\Psi_{PS}(0.374, 12)$ ] and *PS*<sub>2</sub> [=| $\Psi_{RBS}(\phi, N)$ , we set N = 20 and  $\phi = 0$ . For completeness and comparison, we also consider as input states a thermal





FIG. 1. Mean photon number distribution at the output of the array of waveguides (numbered as n = 1, 2, 3, ..., 101) for two values of propagation distance z: solid (blue) curves are for outputs at z = 5, while dashed (black) curves correspond to outputs at z = 20. The disorder parameters were fixed as (a)  $\Delta/C = 0$ , (b)  $\Delta/C = 0.1$ , (c)  $\Delta/C = 0.5$ , and (d)  $\Delta/C = 1$ . These plots were obtained by averaging over 1000 realizations, fixing the number of photons in the injected (input) state (z = 0, at the 51st waveguide)  $n_{in} = \langle a_{51}^{\dagger}(0)a_{51}(0) \rangle = 10$ .

state (*TS*) with  $\bar{n} = 10$ , a coherent state (*CS*) with  $|\alpha|^2 = 10$ , and a squeezed-vacuum state (*SS*) with  $\sinh^2 |\zeta| = 10$ .

#### **III. NUMERICAL RESULTS**

We start with two classical collective quantities of the propagated field: the light intensity and the participation number. Neither depend on the quantum characteristics of the input state and show the output behavior of all the waveguides composing the array. While the output intensity can show if localization has happened for a fixed propagation distance, the participation number shows how stable the localization or the dispersion are along the propagation. In the sequence, we focus on the quantum characteristics of the electromagnetic field output by one single waveguide ( $j_0$ ), for which we analyze quantum fluctuation quantities as well as the state of the propagated field through its Wigner function and photon number distribution.

As explicitly shown in Eq. (7), which is valid whenever light is injected into the  $j_0$ th waveguide, the average output intensity of the waveguide array depends on the average number of photons of the input state (here fixed as  $n_{in} = 10$ ) and on the degree of disorder of the system, carried by the random  $\beta$  coefficients, which is manifested by the Green's functions; the output intensity, that is the mean number of photons at the end of the array, does not depend on any other characteristics of the input state.

The average output intensity, as distributed among the 101 waveguides of the array, is shown in Fig. 1 for some values of  $\Delta/C$  and two values of propagation distance z. As mentioned before,  $\Delta/C$  measures the arrangement disorder, i.e., as the value of  $\Delta$  increases the coefficients  $\beta_j$  become more distinct from each other. In the absence of disorder [Fig. 1(a)], the



FIG. 2. The same as in Fig. 1, but now using (a)  $\Delta/C = 1.5$  and (b)  $\Delta/C = 5$ .

light suffers only the standard dispersion. However, as the disorder is increased [Figs. 1(b)-1(d)], a narrow peak of intensity around the input ( $j_0$ th, number 51) waveguide emerges, which characterizes the localization of the solution. For ratios  $\Delta/C$  greater then 1.5, the output-intensity profile does not change anymore with the increasing of the propagation distance, which is shown by the exact overlapping of the solid line (z = 5) and dashed line (z = 20) in Fig. 2.

In Fig. 3 we present, using a logarithmic scale, the mean number of photons for some values of disorder  $(\Delta/C)$  and considering two values of propagation distance, z = 5 and z = 20, respectively. The choice of using a logarithmic scale is justified by the fact that an exponential decay of the light intensity, in the waveguides different from the  $j_0$ th one, would signal the appearance of Anderson localization. We find that, in all cases, there exist two regions of decreasing exponentials and, except in the case of  $\Delta/C = 1.0$ , the pattern holds for both propagation distances. It should be mentioned, however, that plots of the output intensities (mean number of photons) may not be sufficient to state whether the light beam presents Anderson localization, since the system might have yet a small diffusion, which would become more evident by increasing the propagation distance.

To further investigate the occurrence of Anderson location, we can also calculate the participation number defined by, and



FIG. 3. The same as in Fig. 1, but now on a logarithm scale and using the values (a)  $\Delta/C = 1$ , (b)  $\Delta/C = 3$ , (c)  $\Delta/C = 5$ , and (d)  $\Delta/C = 10$ .



FIG. 4. Participation number as a function of propagation distance for various values of the disorder parameter: (a)  $\Delta/C = 0$ , 0.1 and 0.5; (b)  $\Delta/C = 1$ , 1.5, and 2; (c)  $\Delta/C = 2.5$ , 2.8, and 3; (d)  $\Delta/C = 3.5$ , 4, and 5; corresponding to solid-line (blue), dashedline (black) and dotted-line (red), respectively, in all cases. Results were obtained after averaging over 1000 realizations.

given in our case by,

$$\mathcal{P}(z) = \frac{\left(\sum_{j} \langle a_{j}^{\dagger}(z)a_{j}(z) \rangle\right)^{2}}{\sum_{j} \langle a_{j}^{\dagger}(z)a_{j}(z) \rangle^{2}}$$
$$= 1 + \frac{\sum_{j \neq k} \langle |G_{j,51}(z)|^{2} \rangle \langle |G_{k,51}(z)|^{2} \rangle}{\sum_{j} \langle |G_{j,51}(z)|^{2} \rangle^{2}}.$$
 (27)

We see that the participation number is not only independent of the mean number of photons of the input state but is actually completely independent of the electromagnetic field mode injected in the  $j_0$ th waveguide of the array, the number 51; it depends only on the waveguide array itself and its disorder encoded in the Green's functions.

The participation number indicates in how many waveguides there are photons as a function of propagation distance; thus, if  $\mathcal{P}(z)$  increases, it means that the beam remains scattering among the waveguides along propagation. It should be emphasized that, in the case of no disorder, the dispersion of  $\mathcal{P}(z)$  is linear, as indicated by Fig. 4(a). Also, for a very small amount of disorder,  $\Delta/C = 0.1$ , we find that the diffusion occurs faster than in the absence of disorder, while for  $\Delta/C = 0.5$  dispersion still occurs, but with a growth rate smaller than that in the case of no disorder. In all other cases reported in Figs. 4(b)-4(d), for larger values of  $\Delta/C$ , the plots of  $\mathcal{P}(z)$  stabilize as z increases, characterizing localization of the light in the array. Notice that some fluctuations are still observed, but on a small scale when compared with 1, which is the smallest value that can characterize a single waveguide.

Figures 5 and 6 show the second-order correlation function  $g^{(2)}$  of the field output by waveguide number 51, as a function of the disorder degree for different input states, calculated with Eq. (9), taking propagation distances z = 5 and z = 20, respectively. We first notice that, for all input states analyzed, taking measurements for a short propagation length (z = 5,



FIG. 5. Second-order correlation function of the output state in the 51st waveguide, at propagation distance z = 5, as a function of the disorder parameter for different input states. Panel (a) displays the curves corresponding to the states  $CCS_1$  in circles (blue),  $CCS_2$  in boxes (black), *RBS* in pentagon (cyan), *CS* in triangles (red), *PS*<sub>1</sub> in down triangles (gray), and *PS*<sub>2</sub> in diamond (green). For comparison, panel (b) also presents the states *SS* in circles (magenta) and *TS* in boxes (orange). All results were obtained with an average over 1000 realizations.

Fig. 5), the second-order correlation function increases as the disorder degree  $(\Delta/C)$  raises and reaches a maximum for  $\Delta/C = 1$  and then decreases to values greater than the value of  $g^{(2)}$  without disorder. Similarly, for longer arrays (z = 20, Fig. 6), one finds peaks of  $g^{(2)}$  somewhat higher than those of Fig. 5 but occurring at a much lower disorder degree and rapidly decaying for large disorder degree. Yet interestingly, for the propagation distance z = 20, we find that only the  $CCS_1$  state returns to the  $g^{(2)} < 1$  regime for large values of disorder. Note also that the curves for the  $CCS_2$  and the *RBS* states are nearly coincident due to the fact that, for these states,  $\langle a_{51}^{\dagger}(0)a_{51}^2(0) \rangle$  are very close, 94.78 and 95, respectively.

The variance of the output field intensity,  $(\Delta I)^2 = \langle I^2 \rangle - \langle I \rangle^2$ , of the 51st waveguide, when the input field is injected only in it, is obtained directly from Eqs. (7) and (8) as

$$\begin{split} [\Delta I_{51}(z)]^2 &= \langle |G_{51,51}(z)|^4 \rangle \langle a_{51}^{\dagger 2}(0) a_{51}^2(0) \rangle \\ &+ \langle |G_{51,51}(z)|^2 \rangle \langle a_{51}^{\dagger}(0) a_{51}(0) \rangle \\ &- \langle |G_{51,51}(z)|^2 \rangle^2 \langle a_{51}^{\dagger}(0) a_{51}(0) \rangle^2. \end{split}$$
(28)

To investigate the influence of the disorder on the intensity variance of the output states, in Figs. 7(a) and 7(b) we show these variances as functions of the disorder degree  $\Delta/C$ , at z = 5 and z = 20, respectively, for different input states injected into the 51st waveguide. We see that, for all states discussed, when the degree of disorder increases, the variances increase, some of them tending to stabilize, as for states  $PS_1$ ,  $PS_2$ , and CS, while others, like the cases of the states



FIG. 6. The same as in Fig. 5, but now with propagation distance z = 20.



FIG. 7. Variance of the output intensity of the 51st waveguide as a function of the disorder parameter for the propagation distances (a) z = 5 and (b) z = 20. We display the states  $CCS_1$  in circles (blue),  $CCS_2$  in boxes (black), *RBS* in pentagon (cyan), *CS* in triangles (red), *PS*<sub>1</sub> in down triangles (gray), and *PS*<sub>2</sub> in diamond (green), respectively.

 $CCS_1$ ,  $CCS_2$ , and *RBS*, reach a maximum and then decrease for large values of the disorder parameter, the decreasing rate being bigger for the state  $CCS_1$ . It is interesting to notice that the patterns of  $(\Delta I)^2$  practically do not change when one compares the results for the propagation lengths z = 5 and z = 20, Figs. 7(a) and 7(b), respectively. Here, likewise for Figs. 5 and 6, the curves for the  $CCS_2$  and the *RBS* states are almost coincident.

For the sake of comparison, in Table I we present the values of  $g^{(2)}$  and  $\Delta I^2$  for the injected states and their corresponding output values in waveguide 51 after a propagation distance z = 20 in an array with  $\Delta/C = 10$ . It is clear that CCS<sub>1</sub> is the only state to output  $g^{(2)} < 1$ , while PS<sub>2</sub> is the only state whose intensity variance decreases with propagation. From the input intensity variance, we see that CCS<sub>1</sub> is the closest to a number state and the farther from a classical state in terms of  $g^{(2)}$ , it helps understanding why it is the only one preserving nonclassicality after propagation.

#### Wigner representation

We can also study the influence of disorder in the propagation of light in the waveguide array using the Wigner representation of the quantum states. The Wigner function (WF) of the output state in the 51st waveguide is given by

$$W_{51}(\alpha, z) = \frac{1}{\pi} \int d^2 \xi \exp(\alpha \xi^* - \alpha^* \xi) \, \chi_{51}(\xi, z), \qquad (29)$$

TABLE I. Second-order correlation function and variance of the field intensity of the 51st waveguide for the initial states and propagated states with  $\Delta/C = 10$ .

	$g^{(2)}$		$\Delta I^2$	
State	z = 0	z = 20	z = 0	z = 20
$\overline{CCS_1}$	0.90	0.98	$2 \times 10^{-3}$	6.9
$CCS_2$	0.95	1.05	4.8	10.4
RBS	0.95	1.03	5.0	10.5
$PS_1$	1.08	1.18	18.2	20.0
PS <sub>2</sub>	1.17	1.28	27.1	26.4

with the symmetrically ordered characteristic function given by

$$\chi_{51}(\xi, z) = \operatorname{Tr}[\rho_{51}(0)e^{\xi a_{51}^{\top}(z) - \xi^* a_{51}(z)}],$$
(30)

where  $a_{51}(z) = G_{51,51}(z)a_{51}(0)$  and the input state is given by  $\rho_{51}(0) = |\psi_{51}\rangle\langle\psi_{51}|$ .

We take  $|\psi_{51}\rangle = \sum_{n=0}^{N} c_n |n\rangle$ , which is the form of the states described in Sec. II B. Then, the integral defining  $W_{51}(\alpha, z)$  can be performed analytically, and writing  $\alpha = x + iy$ , we obtain

$$W_{51}^{(N)}(\alpha, z) = 2e^{-2(x^2+y^2)} \sum_{t=0}^{N} \sum_{j=0}^{t} \sum_{n=0}^{N} \sum_{k=0}^{n} c_n^* c_t [G_{51,51}(z)]^j \times [G_{51,51}^*(z)]^k \frac{\sqrt{t!n!}}{(n-k)!} \delta_{(n-k),(t-j)} I_{jk}(x, y), \quad (31)$$

where

$$I_{jk}(x,y) = \sum_{l=0}^{k} \sum_{s=0}^{j} \frac{(-1)^{2j+k} i^{-(l+s)}}{(k-l)! l! (j-s)! s!} I_p(y) I_q(-x), \quad (32)$$

with p = l + s and q = k - l + j - s and

$$I_r(u) = (2u)^r + \sum_{m=2}^{e(r)} \frac{(-1)^{\frac{m}{2}} 2^{-\frac{m}{2}} r!}{(\frac{m}{2})! (r-m)!} (2u)^{r-m}, \qquad (33)$$

where the summation is over even integers and e(r) is the largest even integer not greater than r, i.e., e(r) = r if r is even and e(r) = r - 1 for r odd.

We can use the WF, given by Eq. (31), to investigate the propagation of states through the array, looking at the output state in the 51st waveguide, when the input is a truncated state in the number basis,  $\sum_{n=0}^{N} c_n |n\rangle$ ; Figs. 8 and 9 show examples of this case. In Fig. 8, we present the WF when the input state is the  $CCS_1$ , in Fig. 8(a) for the input state (z = 0), and in Figs. 8(b)-8(d) for the output states in the 51st waveguide, for the length z = 20, considering the disorder parameters  $\Delta/C = 0, 1, \text{ and } 7$ , respectively. We find that, in absence of disorder  $[\Delta/C = 0;$  Fig. 8(b)], the WF is similar to the one of the vacuum state; this is also confirmed by the photon number distribution, as shown below. But, for small values of the disorder parameter, Figs. 8(b) and 8(c), the WF profiles present very important differences when compared with that for the input state, showing that dispersion prevails. However, by increasing the disorder parameter, the WF of the output state becomes similar to that one for the input state; clearly this behavior is directly linked to a strong localization, but not necessarily of Anderson type.

In Fig. 9(a), we present the WF for the input state  $PS_1$  and, in Figs. 9(b)–9(d), for the output state of the 51st waveguide at z = 20 with the same values of the disorder parameter as in Fig. 8,  $\Delta/C = 0$ , 1, and 7, respectively. Here, we observe that, similarly to the results obtained for the  $CCS_1$  (Fig. 8), when considering a zero disorder [Fig. 9(b)] or a small disorder [here characterized by the value of the disorder parameter  $\Delta/C = 1$ , Fig. 9(c)], the output state of the 51st waveguide at z = 20 presents a configuration for the WF similar to that of the vacuum state. Clearly, the weak disorder is not enough to





FIG. 8. Panel (a) shows the Wigner function of the  $CCS_1$  input state, while panels (b)–(d) show the WF of the output state in the 51st waveguide, at the propagation distance z = 20, considering the disorder parameters  $\Delta/C = 0$ , 1, and 7, respectively. The plots consist on an average over 100 realizations considering the step size of dx = dy = 0.1.

ensure that the WF remains in the same shape, due to the interaction of the main guide (51st) with the neighboring guides. On the other hand, as the disorder parameter increases [for example, with  $\Delta/C = 7$  displayed in Fig. 9(d)] the WF is now preserved, which can be observed when comparing the panels (a) and (d) of Figs. 8 and 9. In other words, by increasing the disorder parameter, the WFs become more robust against



FIG. 9. The same as in Fig. 8, but now considering the  $PS_1$  as the input state.



FIG. 10. PND of the output state of the 51st waveguide shows (a) the input state  $CCS_1$  and output state (at z = 20) for  $\Delta/C = 0$  and (b) the output states for  $\Delta/C = 1$ , 3, and 7. These results correspond to an average over 100 realizations.

changes due to variations of the coupling between neighbor waveguides. Also, it should be pointed out that, in all cases, due to the computational cost the number of realizations to produce the output state was reduced to 100.

Finally, we can also use the WF [Eq. (31)] to obtain the photon number distribution (PND) of the output state of the 51st waveguide, given by

$$P_{51}^{(N)}(n,z) = \frac{1}{\pi} \int d^2 \alpha \, W_{51}^{(N)}(\alpha,z) \, W_n(\alpha), \qquad (34)$$

where  $W_n(\alpha)$  is the WF for the number state  $|n\rangle$ . The integral in Eq. (34) is calculated numerically and, as before, the number of realizations to get the proper averages was reduced to 100.

In Fig. 10 we show the PND in the 51st waveguide when the input state is the  $CCS_1$  and the length of the array is z =20. In Fig. 10(a), we plot together the PND of the input  $CCS_1$ state and the output PND  $[P_{51}^{(N)}(n, 20)]$  for null disorder, for comparison. Interestingly, the PND of the state  $CCS_1$ , which has  $n_{CCS_1} = 10$ , is very close to that of the number state  $|10\rangle$ ,  $P_{|10\rangle}(n) = \delta_{n,10}$ . On the other hand, the output state of the 51st waveguide (at z = 20), in the absence of disorder, is close to that for the vacuum state; this fact reinforces the results for the Wigner functions presented in Figs. 8(b) and 9(b).

In Fig. 10(b) we show the PND for the output state in the 51st waveguide considering three different values of the disorder parameter,  $\Delta/C = 0$ , 1, and 7. We clearly see that the increasing of disorder tends to favor the output PND to become closer to that of the input state; that is, disorder tends to preserve the PND.

In Fig. 11 we present the results of the PND, for the same parameters as in Fig. 10, but considering the input state given by the  $PS_1$ . Now we see, from Fig. 11(a), that the PND of the input  $PS_1$  state differs significantly from the PND for the



realizations to a 101 waveguides focusing in the injection and detection of waves in the middle one the 51st waveguide. The on site

waves in the middle one, the 51st waveguide. The on-site energy parameters  $\beta_j$  were assumed to be independent of each other and randomly taken following a zero-mean Gaussian distribution with variance  $\Delta^2$ , while the coupling between neighbor waveguides were fixed as constant *C*; the disorder parameter was defined as  $\Delta/C$ .

We investigated some quantities that qualify the nature of the statistics of the state. First, we verified that increasing the degree of impurity of the lattice, the mean photon-number distribution tends to concentrate around the waveguide where the input state is injected, with an exponential decay of the light intensity in the others, for any injected state with a given mean number of photons, here  $n_{in} = 10$ ; although this is not a definitive indication of localization, it does signalize it.

We also analyzed the participation number, indicating in how many waveguides there were photons, which shows the effects of disorder in the array and is totaly independent of the input state. We find, as the disorder parameter is increased, from no-disorder to a high disorder regime, the participation number changes from a linear increase with the propagation distance, characteristic of dispersion, to a flat behavior with very small fluctuations which represents a localization pattern.

We investigated second-order quantities as the  $g^{(2)}$  function and the output intensity variance at the central waveguide. We observe for all input states that, although localized, the average output state presents a classical behavior relatively to the bunching feature. However, as the disorder increases, the  $g^{(2)}$  function decreases and even presents antibunching again for the complementary coherent state and high disorder. The variance in the output intensity at the central waveguide increases with disorder and stabilizes to a final value for most of the input states investigated.

To investigate how the propagation through the lattice changes the input state, it is not enough to analyze quantities quantum averaged over the input state. It can be noticed, for example, when we look at the results for  $CCS_2$  and RBS, two different states with different features, although presenting nearly the same values of  $g^{(2)}$  and intensity variance at the output. To tackle this question we reconstructed the output state by means of its Wigner function, which could also be used to obtain the output probability distribution of number of photons. We observed a preservation of the characteristics of the Wigner function for high disorder parameter values, although

FIG. 11. The same as in Fig. 10 but now considering the input state being the  $PS_1$ .

number state, distinctly to the case of the  $CCS_1$ , although we still have  $n_{PS_1} = 10$ . However, also in agreement with the case of the  $CCS_1$ , we see in Fig. 11(b) that the PND of the output state is clearly closer to that of the input state as greater is the value of  $\Delta/C$ .

### **IV. CONCLUSION**

We have discussed the evolution of quantum states of the electromagnetic field propagating through a disordered plane waveguide array. Specifically, we have analyzed the propagation of three truncated states (in the number basis), namely, the complementary coherent state, the reciprocal binomial state, and the polynomial state; we also considered a thermal state, a coherent state, and a squeezed-vacuum state, for comparison.

In our numerical calculations, we considered arrays with

we also notice a suppression of negative values in average. Moreover, as the output probability distribution of number of photons changes with the features of the array as well as with the input state, a well designed array of coupled waveguides could be used to produce new states of the electromagnetic field. Classically, it is well known that conveniently designing the detuning and coupling coefficients of a waveguide array can shape the output intensity of light [60]. What we observed is that it can also be used to shape the quantum state output by a single waveguide.

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