Angular momentum transferred by the field of a moving point charge

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The flux of angular momentum of the electromagnetic field of an arbitrarily moving point charge is investigated. General equations for the transfer of angular momentum at arbitrary distance from the charge are obtained, and corresponding equations in the far-field approximation are derived. An explicit expression is obtained for the flux of angular momentum in the wave zone in terms of coordinates, velocity, and acceleration of the charge. The torque that would act on an object if it absorbs the incident radiation is calculated. It is shown that this torque is proportional to the curl of the stress tensor of the electromagnetic field; in the far-field approximation the torque is proportional to the curl of the Poynting vector. Application of the obtained formulas is illustrated by the example of a rotating dipole.

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I. INTRODUCTION

It is well known that the electromagnetic field of a point charge carries angular momentum [1]. The angular momentum is usually divided into a spin part associated with polarization and an orbital part associated with the helical wave front. A light beam with a helical wave front is usually referred to as a twisted light. The first theoretical and experimental research on twisted light or vortex radiation was devoted to the laser radiation modified by an astigmatic optical system, numerically computed holograms [2,3], or microscopic spiral phase plates. These publications have stimulated extensive studies of the vortex optical beam. A more extensive bibliography of the history in this field can be found in a recent review [4]. Interest in vortex radiation quickly spread into different areas of physics: transfer of information, interaction with atoms, and high-energy particle collision and radiation processes. The vortex light beams have opened a wide range of applications, such as spatial optical trapping of atoms or microscopic objects, phase-contrast microscopy, and nano- or microscale physics [4,5].

The first experiments on capture of the particles in traps and their rotation were carried out at the end of the last century [6,7] and are still being carried out with metal particles [8–10], birefringent particles [11], and dielectric spherical or spheroidal particles [12–14]. Most of these papers also contain a theoretical description of the interaction of a twisted laser beam with small particles. A theoretical description of twisted electrons interacting with electric and magnetic fields is presented in Ref. [15]. A detailed description of these and similar works is given in review papers [16–18]. Various methods are being developed to register the orbital angular momentum of radiation [19,20].

High-energy photons carrying angular momentum can be emitted by the vortex beams of charged particles. In recent years, attention to the x-ray vortices produced by high-energy particles has been boosted by the interest in microscopy and spectroscopy at the atomic and nanometric scale. Radiation in the x-ray range carrying the angular momentum was obtained by converting an x-ray beam [21] and by use of a helical undulator [22]. Various schemes of twisted photon beam production in undulators [23–25] and free-electron lasers [26] have been proposed. Cherenkov radiation and transition radiation emitted by vortex electrons were studied theoretically [27–29]. Radiation of high-energy charged particles channeled in solid and liquid crystals has been studied theoretically in recent papers [30–33]. X-ray vortex radiation has found numerous applications in both classical and quantum optics condensed matter, high-energy physics, optics, etc. (see the review in Ref. [34] and references therein).

Despite active experimental research, there is a lack of theoretical studies on transfer of angular momentum by the field of an arbitrarily moving charge. Most of the work related to the angular momentum of the electromagnetic field relates to the laser radiation. The available theoretical papers in the area of particle radiation cover the angular momentum of radiation only in some special cases, such as synchrotron radiation, radiation in crystals, or polarization radiation [25,27–33].

This paper aims to fill this gap. In Sec. II we obtain a general expression for the flux of angular momentum through a unit area in terms of the stress tensor of the field of a point charge. In Sec. III, we study the flux of angular momentum in the far-field approximation. Since the angular momentum essentially depends on the specific coordinate system, we consider two cases: when the coordinate origin is relatively far from the charge, and when the distance between the charge

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and the coordinate origin is much less than the distance to the observation point. Approximate expressions for the flux of angular momentum of a nonrelativistic charge are obtained in Sec. IV. Section V is devoted to study of the torque exerted on an area element due to the electromagnetic field. The obtained expressions were applied to calculate the angular momentum flux and the torque in the near field and far field of a rotating dipole in Sec. VI. Finally, Sec. VII is devoted to discussion of the results obtained.

II. THE FLUX OF ANGULAR MOMENTUM IN THE FIELD OF A POINT CHARGE

The flux of angular momentum $\boldsymbol{L}$ of an electromagnetic field across an infinitely large spherical surface is defined by the Poynting vector $\boldsymbol{P}$ as [35]

$$\frac{d\boldsymbol{L}}{dt} = \frac{1}{c}\oint(\boldsymbol{r}\times\boldsymbol{P})ds, \quad \boldsymbol{P} = \frac{c}{4\pi}(\boldsymbol{E}\times\boldsymbol{H}), \tag{1}$$

where $\boldsymbol{E}$ and $\boldsymbol{H}$ are the electric and magnetic fields, respectively, $\boldsymbol{r}$ is the radius vector, and c is the speed of light. This expression can be interpreted so that the angular momentum density $(\boldsymbol{r}\times\boldsymbol{P})/c^2$ is transported at the speed of light in the radial direction from the source charge. Equation

$$\frac{d\boldsymbol{L}}{ds\,dt} = \frac{1}{c}(\boldsymbol{r}\times\boldsymbol{P}) \tag{2}$$

is conventionally used for calculation of the angular momentum flux in the wave beam [36]. However, it is not applicable to the flux of angular momentum in the near field of a pointlike charge. In general, the angular momentum flux of an arbitrary electromagnetic field is defined by the energy-momentum tensor.

An explicit expression for the angular momentum flux in terms of electric and magnetic fields is obtained in Appendix A. It follows from Eq. (A4) that the flux of angular momentum through an area ds orthogonal to the radius vector $\boldsymbol{r}$ of arbitrary coordinate system is equal to

$$\frac{d\boldsymbol{L}}{dt\,ds} = \frac{1}{4\pi r}[(\boldsymbol{E}\times\boldsymbol{r})(\boldsymbol{E}\,\boldsymbol{r}) + (\boldsymbol{H}\times\boldsymbol{r})(\boldsymbol{H}\,\boldsymbol{r})]. \tag{3}$$

This expression takes a simpler form written in terms of the Maxwell stress tensor σ_{ij} (A2) in the spherical coordinate system $r,\ \theta,\ \varphi$ with unit vectors $\boldsymbol{e}_r, \boldsymbol{e}_\theta, \boldsymbol{e}_\varphi$

$$\frac{d\boldsymbol{L}}{dt\,ds} = r(\sigma_{12}\boldsymbol{e}_\varphi - \sigma_{13}\boldsymbol{e}_\theta). \tag{4}$$

The electric and magnetic fields of an arbitrary moving charge at a point $\boldsymbol{r}$ and at a time moment t are [37]

$$\boldsymbol{E} = \boldsymbol{E}_1 + \boldsymbol{E}_2, \quad \boldsymbol{H} = (\boldsymbol{R}\times\boldsymbol{E})/R, \tag{5}$$

$$\boldsymbol{E}_1 = \frac{eR^2\boldsymbol{\kappa}}{c(R-\boldsymbol{\beta}\boldsymbol{R})^3}, \quad \boldsymbol{E}_2 = \frac{e(1-\beta^2)(\boldsymbol{R}-R\boldsymbol{\beta})}{(R-\boldsymbol{\beta}\boldsymbol{R})^3},$$
$$\boldsymbol{\kappa} = [\boldsymbol{R}\times[(\boldsymbol{R}-R\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]]/R^2, \tag{6}$$

where $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ is the vector from the charge position to the point $\boldsymbol{r}$ (see Fig. 1), $\boldsymbol{r}' = \boldsymbol{r}'(t')$ is the charge position at the retarded time moment $t' = t - R/c$, e is the charge, $\boldsymbol{\beta} = \boldsymbol{v}/c,\ \boldsymbol{v} = \boldsymbol{v}(t')$ is the particle velocity, and the dot denotes the time derivative.

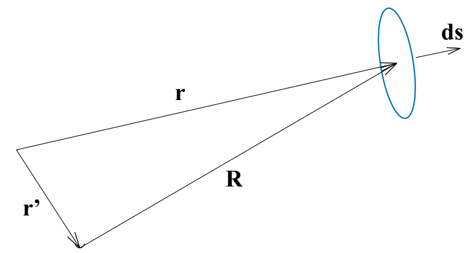

FIG. 1. Notations used.

Equations (3)–(6) determine the flux of angular momentum of the electromagnetic field at any distance from the charge. It is useful to find approximate equations for the angular momentum flux at large distance. The field $\boldsymbol{E}_1$ decreases with distance as $1/R$, and the field $\boldsymbol{E}_2$ decreases with distance as $1/R^2$. At large distances from the charge, where $R \gg c\dot{\beta}$, the electric field $\boldsymbol{E}_1$ and the corresponding magnetic field prevail. Therefore, when calculating the intensity of radiation, we can neglect the field $\boldsymbol{E}_2$ and the corresponding magnetic field.

In calculating the angular momentum at large distances, we must not, however, neglect the terms of order $1/R^2$, because in approximation $1/R$ the longitudinal components $(\boldsymbol{E}\,\boldsymbol{r})$ and $(\boldsymbol{H}\,\boldsymbol{r})$ vanish. The next section is devoted to calculation of the angular momentum flux in the far-field approximation.

III. FLUX OF ANGULAR MOMENTUM IN THE FAR-FIELD APPROXIMATION

Calculation of the angular momentum of radiation has one outstanding distinction compared with calculation of the intensity of radiation: The angular momentum depends significantly on the choice of the coordinate system. When calculating the intensity of radiation, two conditions are usually assumed to be satisfied: (i) $R \gg c\dot{\beta}$ (then the field $\boldsymbol{E}_2$ can be neglected) and (ii) $R, r \gg r'$ (see Fig. 1; then you can put $\boldsymbol{r} \approx \boldsymbol{R}$). However, in practice, there are cases when the second condition is not met. For example, when calculating the intensity of synchrotron radiation, it is convenient to choose the origin at the center of a circular orbit, but measurements are carried out at a distance comparable to or even less than the orbital radius, as, for example, in Fig. 2. Nevertheless, approximation (i), $R \gg c\dot{\beta}$ (the far-field or wave zone approximation), is sufficient to calculate the radiation intensity.

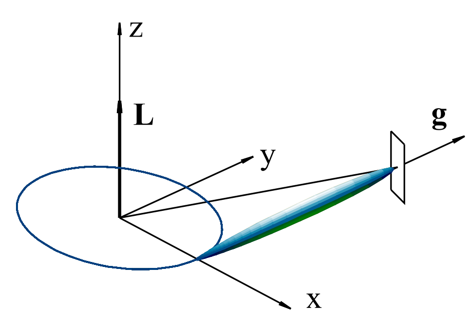

FIG. 2. Synchrotron radiation. The main part of the angular momentum transferred to the target is due to the radiation pressure $\boldsymbol{g}$ on the target, which causes the torque relative to the z axis.

A completely different situation arises when calculating the angular momentum of radiation. Two substantially different situations should be distinguished here: (i) The target is in the wave zone ($R \gg c\dot{\beta}$), but the distances R and r' are comparable; and (ii) the target is in the wave zone and $R, r \gg r'$. Let us consider these cases separately.

A. Angular momentum in the wave zone when $r' \sim r, R$

In this case, the longitudinal components $(\boldsymbol{E}\,\boldsymbol{r})$ and $(\boldsymbol{H}\,\boldsymbol{r})$ in Eq. (3) do not vanish in the approximation of order $1/R$. Hence we can neglect the field $\boldsymbol{E}_2$ and put $\boldsymbol{E} \approx \boldsymbol{E}_1$. Then the vectors of the electric and magnetic fields are mutually orthogonal and orthogonal to the direction of radiation $\boldsymbol{N} = \boldsymbol{R}/R$. If we put in Eq. (3) $\boldsymbol{H} = \boldsymbol{N} \times \boldsymbol{E}$, $(\boldsymbol{E}\boldsymbol{N}) = 0$, then it can be transformed to the form

$$\frac{d\boldsymbol{L}}{dt\,ds} = \frac{r}{4\pi}E^2(\boldsymbol{n}\times\boldsymbol{N})(\boldsymbol{n}\boldsymbol{N})ds = \frac{1}{c}(\boldsymbol{r}\times\boldsymbol{P})(\boldsymbol{N}\boldsymbol{n}). \quad (7)$$

Thus, if the distances r', r, and R are comparable in magnitude, then the flux of angular momentum in the radiation field can be considered as a mechanical torque due to the radiation pressure. The multiplier $(\boldsymbol{n}\boldsymbol{N})$ in Eq. (7) takes into account that the area $\boldsymbol{ds} = \boldsymbol{n}ds$ is oriented at an angle to the direction of radiation $\boldsymbol{N}$. In this case, the angular momentum of radiation can be definitely considered as the orbital angular momentum.

B. Angular momentum in the wave zone when $R, r \gg r'$

Let us consider the case when the charge moves in a region whose dimensions are much less than the distance R and we are interested in the angular momentum of the field relative to some point lying in the region of motion of the charge ($r' \ll R$). Then the scalar products $(\boldsymbol{E}\,\boldsymbol{r})$ and $(\boldsymbol{H}\,\boldsymbol{r})$ in Eq. (3) become small and decrease with distance as $1/R$. In this case, the contribution of the $\boldsymbol{E}_2$ field should also be taken into account. In the limit $r \to \infty$, Eq. (3) takes the form

$$\frac{d\boldsymbol{L}}{d\Omega dt} = \frac{r^3}{4\pi}[(\boldsymbol{E}_1\times\boldsymbol{n})(\boldsymbol{E}_2\,\boldsymbol{n}) + (\boldsymbol{H}_1\times\boldsymbol{n})(\boldsymbol{H}_2\boldsymbol{n})], \quad (8)$$

where $\boldsymbol{H}_i = (\boldsymbol{R}\times\boldsymbol{E}_i)/R$. Each term of the last equation decreases as $1/r^3$ as $r \to \infty$.

This expression can also be written in terms of the Poynting vector. If we make the substitution $\boldsymbol{H}_1 = \boldsymbol{n}\times\boldsymbol{E}_1$ in Eq. (8), then we get

$$\frac{d\boldsymbol{L}}{ds\,dt} = \frac{r}{4\pi}[\boldsymbol{E}_1(\boldsymbol{H}_2\boldsymbol{n}) - \boldsymbol{H}_1(\boldsymbol{E}_2\boldsymbol{n}) - \boldsymbol{n}(\boldsymbol{H}_2\boldsymbol{n})(\boldsymbol{E}_1\boldsymbol{n})]. \quad (9)$$

The first two terms decrease with distance as $1/r^3$, and the last term decreases with distance as $1/r^4$. Therefore it can be neglected. On the other hand, in the considered approximation,

$$\boldsymbol{n}\times\boldsymbol{P} = \frac{c}{4\pi}[\boldsymbol{E}_1(\boldsymbol{n}\boldsymbol{H}_2) - \boldsymbol{H}_1(\boldsymbol{n}\boldsymbol{E}_2)]. \quad (10)$$

Hence the flux of angular momentum in the wave zone in approximation $R, r \gg r'$ can be calculated through the Poynting vector

$$\frac{d\boldsymbol{L}}{ds\,dt} = \frac{1}{c}(\boldsymbol{r}\times\boldsymbol{P}) \quad (11)$$

in agreement with Eq. (2). This equation, in essence, coincides with Eq. (7), since in the expansion of the scalar product $(\boldsymbol{N}\boldsymbol{ds})$ in powers of r'/r the first term is equal to 1.

The flux of the modulus of the angular momentum of radiation can be related to the radiation intensity. Let us denote by α the angle between the vectors $\boldsymbol{n}$ and $\boldsymbol{P}$, and by I the radiation intensity (energy per unit time). Then

$$\frac{d|\boldsymbol{L}|}{ds\,dt} = \frac{1}{c}r\sin\alpha\frac{dI}{ds}. \quad (12)$$

Let us find an explicit expression for the angular momentum flux in terms of the charge position and its velocity and acceleration. Substituting $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{r}'$ into Eqs. (5), (6), and (8), and keeping the largest term in the expansion in powers of $1/r$, we obtain

$$\frac{d\boldsymbol{L}}{d\Omega dt} = \frac{e^2}{4\pi c(1-\boldsymbol{\beta}\boldsymbol{n})^5}\left[(1-\beta^2)(\boldsymbol{\kappa}\times\boldsymbol{n}) + \frac{\kappa^2(\boldsymbol{r}'\times\boldsymbol{n})}{c(1-\boldsymbol{\beta}\boldsymbol{n})}\right]. \quad (13)$$

Now, the vector $\boldsymbol{\kappa}$ contains only the main term of the expansion in powers of $1/r$:

$$\boldsymbol{\kappa} = [\boldsymbol{n}\times[(\boldsymbol{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}]]. \quad (14)$$

This equation determines the angular distribution of the angular momentum carried by radiation of an arbitrarily moving charge. The first term in Eq. (13) does not depend on the choice of the origin of the coordinate system, while the second one linearly depends on the position vector of the charge. In this sense, the first term can be associated with the spin of the radiation, and the second one can be associated with the orbital angular momentum. The classical theory, in principle, does not distinguish the spin and orbital parts of the total angular momentum. Nevertheless, there is active discussion on the issue of dividing the angular momentum of electromagnetic field into orbital and spin parts. See, for example, Refs. [23,38,39] and references therein. We will not plunge into this discussion here. Therefore we denote the studied total angular momentum just as the angular momentum of radiation.

IV. NONRELATIVISTIC APPROXIMATION

In the nonrelativistic approximation, $\beta \ll 1$. Assuming in Eq. (13) $\beta = 0$, we obtain

$$\frac{d\boldsymbol{L}}{d\Omega dt} = \frac{e^2}{4\pi c}(\boldsymbol{n}\times\dot{\boldsymbol{\beta}}). \quad (15)$$

Hence the angular momentum of the radiation is orthogonal to the vectors $\boldsymbol{n}$ and $\dot{\boldsymbol{\beta}}$. If the particle moves within a limited region of space, then the time-averaged flux of angular momentum transferred by radiation is zero. Actually, Eq. (15) describes the exchange of angular momentum between the near and far zones and therefore does not represent emission of angular momentum. Moreover, the integral of the angular momentum of the radiation over the solid angle is in this approximation zero.

In order to calculate the nonzero value of the angular momentum flux, it is necessary to take into account the next term in the expansion in β. As a result of expansion up to β^2, we obtain

$$\frac{d\boldsymbol{L}}{d\Omega dt} = \frac{e^2}{4\pi c}[(\boldsymbol{n}\times\dot{\boldsymbol{\beta}})(1+4\boldsymbol{\beta n}) + (\boldsymbol{n}\times\boldsymbol{\beta})(\boldsymbol{n}\dot{\boldsymbol{\beta}})]. \quad (16)$$

Let us find the time-average angular momentum flux. When averaging over time t, one should keep in mind that the right-hand side of the last expression depends on t' and that $dt = (1-\boldsymbol{\beta n})dt'$. Hence

$$\left\langle\frac{d\boldsymbol{L}}{d\Omega dt}\right\rangle = \frac{e^2}{4\pi c}\frac{1}{T}\int_0^T [(\boldsymbol{n}\times\dot{\boldsymbol{\beta}})(1+4\boldsymbol{\beta n}) + (\boldsymbol{n}\times\boldsymbol{\beta})(\boldsymbol{n}\dot{\boldsymbol{\beta}})](1-\boldsymbol{\beta n})dt'. \quad (17)$$

The average of the term linear in $\dot{\boldsymbol{\beta}}$ is zero, and averaging the remaining term gives

$$\left\langle\frac{d\boldsymbol{L}}{d\Omega dt}\right\rangle = \frac{e^2}{4\pi c}\langle 3(\boldsymbol{n}\times\dot{\boldsymbol{\beta}})(\boldsymbol{\beta n}) + (\boldsymbol{n}\times\boldsymbol{\beta})(\boldsymbol{n}\dot{\boldsymbol{\beta}})\rangle. \quad (18)$$

Integrating this expression over the solid angle, we obtain the total angular momentum emitted per unit time

$$\left\langle\frac{d\boldsymbol{L}}{dt}\right\rangle = \frac{2e^2}{3c}\langle\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}}\rangle. \quad (19)$$

The vector $(\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})$ indicates the direction of the instantaneous mechanical angular momentum of the particle. The last expression up to sign coincides with the well-known equation for the loss of angular momentum of a charged particle due to radiation friction [37].

V. TORQUE EXERTED BY THE FIELD

By definition, the angular momentum of the electromagnetic field depends on the choice of the coordinate system. Under certain conditions, as discussed in Sec. III A, radiation pressure can make a significant contribution to the angular momentum flux. For example, the orbital angular momentum of a photon of synchrotron radiation in an accelerator (Fig. 2) is equal to the product of the photon momentum and the orbital radius a:

$$L = \frac{\hbar\omega}{c}a \sim \gamma^3\hbar,$$

where $\omega \sim \gamma^3 c/a$ is the photon frequency and $\gamma = (1-\beta^2)^{-1/2}$ is the relativistic factor. In modern accelerators, the angular momentum flux can reach enormous values. In this case, as in many others, the angular momentum of the radiation relative to the geometric center of the trajectory is of interest. However, if we investigate the radiation of an unknown source, then the position of this geometric center is not determined. It can also be that the source of radiation is so far away that the entire area of radiation shrinks to a point, as is usually the case in astronomy. In these cases, Eq. (13) has little sense. Only the integral over the solid angle is important, because it determines the rate of loss of angular momentum by the charge due to radiation friction.

However, the "vortex radiation" has a property that can theoretically be measured without being bound to any particular coordinate system. To date, a large number of experiments cited above have been carried out in which a vortex laser beam drives the microparticles into rotation both on the laser beam axis and around the beam axis. Detailed discussion of the forces acting on small particles is presented in Refs. [40–42].

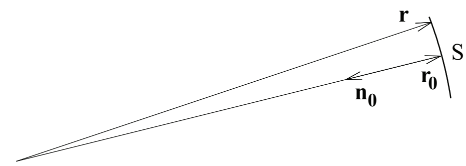

FIG. 3. The flux of the radial component of the angular momentum of the field through the area S.

The flux of the radial component of the angular momentum through an infinitely small area orthogonal to the radius vector is zero because $(\boldsymbol{Lr}) = 0$ as one can see in Eq. (3). However, we are going to show that if the field of the stress tensor is not uniform and the area is of finite dimension, this flux is not zero.

Consider a small area S orthogonal to the radius vector passing through the center of the area. The smallness of the area means that its dimensions are much less than r. Let us denote the coordinate of the center of the area by $\boldsymbol{r}_0$ as shown in Fig. 3. The unit normal to the area is $\boldsymbol{n}_0 = -\boldsymbol{r}_0/r_0$. Let us find the torque acting on the area due to the total flux of the angular momentum through the area. The physical meaning of the components of the stress tensor σ_{ij} defined by Eq. (A2) is that they represent forces acting per unit area. The diagonal elements represent pressure, and the off-diagonal element σ_{ij} is shear acting in the ith direction on a unit area with the jth normal.

Figure 4 schematically shows the lines of the shear force vector σ_{i1} on the area orthogonal to the radius vector $\boldsymbol{r}_0$. If the field of shears is not uniform, they produce a torque acting on the area. The flux of the radial component of the angular momentum through the area is given by integral

$$\frac{dL_n}{dt} = \int n_{0i}g_{i1}(\boldsymbol{r})ds = \int n_{0i}e_{ijm}x_j\sigma_{m1}(\boldsymbol{r})ds. \quad (20)$$

Here, $g_{ij}(\boldsymbol{r})$ is the tensor (A3), and index 1 corresponds to the radial coordinate. The flux of the radial component of the angular momentum through the center of the area is zero, since $n_{0i}e_{ijm}x_{0j} \sim \boldsymbol{n}_0\times\boldsymbol{r}_0 = 0$. However, as can be seen from Fig. 3, at a distance from the center, the cross product $\boldsymbol{n}_0\times\boldsymbol{r}$ is no longer zero.

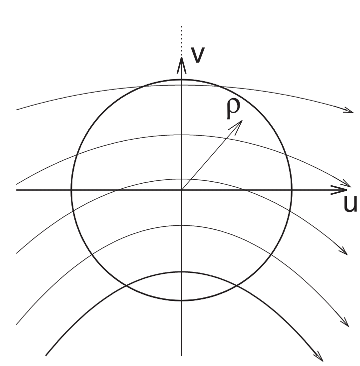

FIG. 4. Lines of force of the vector g_{i1} on the area S.

Let us expand the components of the tensor $\sigma_{m1}(\boldsymbol{r})$ in a Taylor series in the vicinity of the center. Further on, we use the spherical coordinate system $(x_1, x_2, x_3) = (r, \theta, \varphi)$. Let us introduce on the area mutually orthogonal coordinates u, v such that the coordinate lines of u and v coincide with the coordinate lines of θ and φ, respectively. Then

$$\sigma_{m1}(\boldsymbol{r}) = \sigma_{m1}(\boldsymbol{r}_0) + \left(u \frac{\partial \sigma_{m1}(\boldsymbol{r})}{\partial u}\bigg|_0 + v \frac{\partial \sigma_{m1}(\boldsymbol{r})}{\partial v}\bigg|_0 \right).$$

The vertical bar means that the value of derivatives is taken at the center. Next we substitute this in Eq. (20). The vector $\boldsymbol{n}_0$ has coordinates $n_{0i} = (n_{01}, -u/r, -v/r)$. For the sake of simplicity, assume that the area is a disk. Using polar coordinates on the surface of the disk

$$u = \rho \cos\psi, \quad v = \rho \sin\psi,$$

we transform the integral (20) to the form

$$\begin{aligned} \frac{dL_r}{dt} &= \pi \left(\frac{\partial \sigma_{31}(\boldsymbol{r})}{\partial u}\bigg|_0 - \frac{\partial \sigma_{21}(\boldsymbol{r})}{\partial v}\bigg|_0 \right) \int \rho^3 d\rho \\ &= \frac{1}{4\pi} S^2 \left(\frac{\partial \sigma_{31}(\boldsymbol{r})}{\partial x_2}\bigg|_0 - \frac{\partial \sigma_{21}(\boldsymbol{r})}{\partial x_3}\bigg|_0 \right), \end{aligned} \tag{21}$$

where S is the area of the disk. Thus the flux of the radial component of the angular momentum through the area is proportional to the square of its area and to the curl of σ_{ij}, taken with respect to the first index.

If the expression in parentheses is denoted by Ω_n,

$$\Omega_n = \left(\frac{\partial \sigma_{31}}{\partial x_2} - \frac{\partial \sigma_{21}}{\partial x_3} \right)\bigg|_0, \tag{22}$$

then the torque acting on the area which is orthogonal to the radius vector takes the form

$$\frac{dL_n}{dt} = \frac{1}{4\pi} S^2 \Omega_n. \tag{23}$$

The off-diagonal elements of the tensor σ_{ij} decrease with distance as $1/r^3$, and the curl of these components decreases as $1/r^4$; therefore the torque acting on an area of finite dimensions decreases with distance as $1/r^4$, as, for example, in the case of a rotating dipole [Eq. (31)]. This may cause the difficulties in measuring the torque of radiation at large distances.

Obviously, the calculations performed can be generalized to areas orthogonal to other coordinate lines. The curl of the stress tensor σ_{ij} is a tensor of the second rank

$$R_{ij} = e_{kli} \frac{\partial \sigma_{lj}}{\partial x_k}. \tag{24}$$

Using the diagonal elements of this tensor, we can compose a vector

$$\boldsymbol{\Omega} = \left(\frac{\partial \sigma_{31}}{\partial x_2} - \frac{\partial \sigma_{21}}{\partial x_3}, \frac{\partial \sigma_{12}}{\partial x_3} - \frac{\partial \sigma_{32}}{\partial x_1}, \frac{\partial \sigma_{23}}{\partial x_1} - \frac{\partial \sigma_{13}}{\partial x_2} \right), \tag{25}$$

which, by analogy with the hydrodynamics, is usually referred to as vorticity of the electromagnetic field [41,43,44]. Sometimes this term is used to refer to the curl of the Poynting vector of light [41,43]. The components of the vector $\boldsymbol{\Omega}$ designate the torque acting on an area of finite size, orthogonal to the corresponding axis. In an experiment with a twisted laser beam [11], the rotation of small particles away from the axis of the laser beam was observed. This is probably a manifestation of the nonzero vorticity of the laser beam in the off-axis direction.

Speaking of torque acting on an area, we mean the flux of the normal component of angular momentum through the area. Actually, the angular momentum absorbed by a real object depends significantly on the optical properties of the material and on diffraction at its edges.

Until now, we have not made any assumptions about the distance between the charge and the point of observation. Let us now find the asymptotic expressions for the torque of the field at distances $r, R \gg r'$. We introduce a vector $\boldsymbol{g}_\perp$ with components σ_{i1}. The vector $\boldsymbol{g}_\perp$ is orthogonal to the vector $\boldsymbol{n}$ and represents the shear force acting tangentially on the area orthogonal to the vector $\boldsymbol{n}$. Arguing as in the derivation of Eq. (8), we obtain a similar expression for the vector $\boldsymbol{g}_\perp$ in a spherical coordinate system

$$\boldsymbol{g}_\perp = -\frac{1}{4\pi}[\boldsymbol{E}_1(\boldsymbol{E}_2\boldsymbol{n}) + H_1(\boldsymbol{H}_2\boldsymbol{n})], \quad i = \theta, \varphi. \tag{26}$$

Bearing in mind that in this approximation $\boldsymbol{H}_1 = (\boldsymbol{n} \times \boldsymbol{E}_1)$ and taking into account the equality (10), we obtain

$$\boldsymbol{g}_\perp = \frac{1}{c}(\boldsymbol{n} \times (\boldsymbol{P} \times \boldsymbol{n})).$$

One can see from the last expression that the vector $\boldsymbol{g}_\perp$ is proportional to the transverse, with respect to $\boldsymbol{n}$, component of the Poynting vector. Therefore the lines shown in Fig. 4 can be interpreted as the field lines of the transverse component of the Poynting vector. Accordingly, the vorticity of radiation in the far-field approximation is

$$\Omega_n = \frac{1}{c}(\boldsymbol{n}\,\mathrm{rot}\boldsymbol{P}). \tag{27}$$

Equations (7), (11), and (27) show that in the far-field approximation, the angular momentum flux is determined by the transverse component of the Poynting vector and that the torque acting on an area of finite dimensions is proportional to the curl of the transverse component of the Poynting vector.

VI. ROTATING ELECTRIC DIPOLE

As a simple example, we consider the field produced by a rotating electric dipole. This is the simplest source of vortex radiation, and therefore it has attracted the attention of many authors. The rate of loss of angular momentum by an electric dipole is calculated in Sec. 72 of the textbook by Landau and Lifshitz [37]. Gough [45] found the intrinsic angular momentum of the radiation field of a rotating dipole and the flux of angular momentum in the far-field approximation. An experiment to measure the angular momentum flux from a rotating dipole was proposed by Vul'fson [46] and then implemented by Emile *et al.* [47,48]. The time-averaged flux of angular momentum through a radially oriented element of the spherical surface was calculated by Barnett [49].

A rotating dipole field is a good example to show the physical content of the angular momentum flux in the near and far field and the vorticity of radiation. A derivation of the main formulas is given in Appendix B. Here, we will consider only the field properties related to its angular momentum.

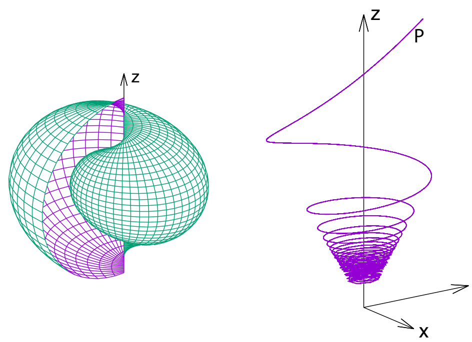

FIG. 5. The wave front of the field defined by Eqs. (B1) has a typical singularity on the z axis (left). One of the Poynting vector force lines (right).

Let the law of motion of the dipole vector $\boldsymbol{d}$ in the Cartesian coordinate system (x, y, z) be $\boldsymbol{d} = d(\cos\omega t, \sin\omega t, 0)$. The field of the rotating dipole (B1) obviously has the properties of a vortex field. The wave front given by the equation $\tau = \text{const}$ is shown in Fig. 5. Obviously, this surface is not orthogonal to the direction of propagation of radiation and has a singularity on the z axis.

Averaging the Poynting vector (B2) over time, we obtain

$$\langle P_r\rangle = \frac{d^2\omega^4}{8\pi c^3 r^2}(1+\cos^2\theta), \quad \langle P_\theta\rangle = 0,$$
$$\langle P_\varphi\rangle = \frac{d^2\omega}{4\pi r^5}(1+k^2r^2)\sin\theta. \tag{28}$$

The last equation shows that the Poynting vector is directed tangentially to the surface of a cone with apex at the origin and an angular opening θ. Figure 5 shows one of the lines of force of the Poynting vector. In the near zone, in the region of small ρ, the azimuthal component $\langle P_\varphi\rangle$ prevails. This component is responsible for the angular momentum of the field. It varies as $1/r^5$ at $r \to 0$ and falls off as $1/r^3$ in the far-field region. In the far zone prevails the radial component $\langle P_r\rangle$, which determines the radiation intensity and decreases as $1/r^2$.

Next we calculate the flux of angular momentum in the radial direction by use of Eq. (4). Averaging over time the stress tensor (B3), we obtain

$$\left\langle \frac{d\boldsymbol{L}}{dt\,ds}\right\rangle = \frac{d^2\sin\theta}{4\pi r^5}(\boldsymbol{e}_\varphi\cos\theta - \boldsymbol{e}_\theta\rho^3). \tag{29}$$

At small distances, the flux of the component L_φ prevails. At larger distances, only the θ component of the flux remains in the radiation [45]

$$\left\langle \frac{d\boldsymbol{L}}{dt\,d\Omega}\right\rangle = -\frac{d^2\omega^3}{4\pi c^3}\sin\theta\,\boldsymbol{e}_\theta. \tag{30}$$

Note that the angular momentum flux has its maximum in direction $\theta = \pi/2$ and is zero on the axis of rotation. Equation (30) can be obtained by use of Eq. (2) if the Poynting vector (28) is averaged over time and the term with the highest power of r is kept. However, at arbitrary distances, Eq. (2) is not correct.

Integration of Eq. (30) over the solid angle gives the well-known formula for the rate of loss of the angular momentum of the dipole due to radiation reaction

$$\left\langle \frac{d\boldsymbol{L}}{dt}\right\rangle = \frac{2d^2\omega^3}{3c^3}\hat{\boldsymbol{z}}.$$

In order to find the vorticity of the electromagnetic field, we calculate the radial component of the vector $\boldsymbol{\Omega}$. According to Eqs. (22) and (B3), we have

$$\Omega_n = \frac{1}{r\sin\theta}\left(\frac{\partial(\sigma_{31}\sin\theta)}{\partial\theta} - \frac{\partial\sigma_{21}}{\partial\varphi}\right) = \frac{d^2\omega^3}{2\pi c^3 r^4}\cos\theta. \tag{31}$$

This value, in contrast to the angular momentum flux, is maximum in the direction of the z axis, that is, in the direction of the vortex axis (see Fig. 5). In the equatorial plane xy, the vorticity is zero. This property resembles the polarization of radiation. It can be seen from Eqs. (B1) that in the direction of the z axis the field is circularly polarized, and in the equatorial plane it is linearly polarized, since

$$\frac{E_\varphi^2}{\cos^2\theta} + E_\theta^2 = \text{const}.$$

The sign of Ω_n changes when passing through the equatorial plane: The sign of the vorticity coincides with the sign of the projection of the angular velocity of rotation of the dipole onto the direction of radiation. We also note that Ω_n decreases with distance as $1/r^4$ and this dependence is the same at both large and small distances from the dipole.

Finally, we show that in the far-field approximation the vorticity of radiation can be calculated as the curl of the Poynting vector in accordance with the general formula (27). Indeed, taking the curl of $\boldsymbol{P}$ in Eqs. (B2) and averaging over time, we obtain

$$\langle(\boldsymbol{n}\,\mathrm{rot}\boldsymbol{P})\rangle = \frac{d^2\omega^3}{2\pi c^2 r^4}\cos\theta(1+\rho^{-2}), \tag{32}$$

which, generally speaking, is not the vorticity, but coincides with its exact value (31) in asymptotics $r \to \infty$, when the second term in (32) vanishes.

VII. DISCUSSION

We investigated the angular momentum of the electromagnetic field of an arbitrarily moving point charge. In the general case, the angular momentum flux is determined by the stress tensor of the electromagnetic field [Eqs. (3) and (4)]. From a practical point of view, the angular momentum flux at distances much greater than the characteristic wavelength of radiation is of interest. This area is usually referred to as a wave zone. We have shown that the flux of angular momentum in the wave zone is proportional to the vector product of the radius vector of the observation point by the Poynting vector [Eqs. (2) and (7)]. Since the angular momentum depends significantly on the choice of coordinate system, two limiting cases can be distinguished.

(i) The distance between the charge and the coordinate origin is comparable to the distance between the charge and the point of observation.

(ii) The distance between the charge and the origin is much less than the distance between the charge and the point of observation.

In the first case, the flux of the angular momentum of radiation can be interpreted as the pressure of radiation on a target that produces a torque around the origin of the coordinate system. In this case, one can take into account only the main part of electromagnetic field decreasing with distance as $1/r$ and assume that the vectors of the electric field, the magnetic field, and the Poynting vector are mutually orthogonal.

In the second case, it is necessary to take into account the components of the electromagnetic field that decrease with distance as $1/r^2$. Thus we take into account that the direction of the Poynting vector does not coincide with the direction of radiation, more precisely, with the direction of the radius vector of the observation point. In this case, the flux of the angular momentum of the radiation is determined by the component of the Poynting vector that is transverse to the radius vector. In the last case, we have obtained an explicit expression for the flux of angular momentum of radiation as a function of coordinates, velocity, and acceleration of the charge [Eq. (13)].

An important property of radiation is the torque acting on the object due to the electromagnetic field. This property allows us to use the radiation field as optical tweezers and optical traps [6,7,9,17]. From this point of view, it is of interest to find the flux of the radial component of the angular momentum through an area of finite dimensions. We have shown that this quantity, denoted as the vorticity of the radiation, is proportional to the curl of the stress tensor [Eq. (25)], and in the wave zone it is proportional to the curl of the Poynting vector [Eq. (27)].

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APPENDIX A: THE FLUX OF ANGULAR MOMENTUM

The angular momentum tensor $L^{\mu\nu}$ of the electromagnetic field is defined by the energy-momentum tensor $T^{\mu\sigma}$ as [37]

$$L^{\mu\nu} = \int (x^\mu T^{\nu\sigma} - x^\nu T^{\mu\sigma}) dS_\sigma, \tag{A1}$$

where dS_σ is the vector equal in magnitude to the area of a hypersurface element and normal to this element. The spatial components of the energy-momentum tensor form a three-dimensional stress tensor

$$\sigma_{ik} = \frac{1}{4\pi}\left[-E_i E_k - H_i H_k + \frac{1}{2}\delta_{ik}(E^2 + H^2)\right], \tag{A2}$$

with E_i and H_i being the components of electric and magnetic fields, respectively, and δ_{ik} being the Kronecker symbol. The time components determine the energy and momentum density of the electromagnetic field

$$T^{00} = \frac{E^2 + H^2}{8\pi}, \quad T^{0i} = \frac{1}{c}P_i, \quad \boldsymbol{P} = \frac{c}{4\pi}(\boldsymbol{E} \times \boldsymbol{H}),$$

where $\boldsymbol{P}$ is the Poynting vector.

Passing to three-dimensional notation, we introduce a three-dimensional angular momentum vector with components $L_i = \frac{1}{2}e_{ijk}L^{jk}$, where e_{ijk} is the unit antisymmetric symbol. The flux of the ith component of the vector $\boldsymbol{L}$ through the unit area orthogonal to the kth axis is determined by the three-dimensional tensor [37,40]

$$\begin{aligned} g_{ik} &= e_{ijm} x_j \sigma_{mk} \\ &= \frac{1}{4\pi}\left[-(\boldsymbol{r} \times \boldsymbol{E})_i E_k - (\boldsymbol{r} \times \boldsymbol{H})_i H_k + \frac{1}{2}(E^2 + H^2) e_{ijk} r_j\right]. \end{aligned} \tag{A3}$$

The flux of the angular momentum of the field through an arbitrarily oriented area $\boldsymbol{ds}$ is

$$\begin{aligned} \frac{d\boldsymbol{L}}{dt} = \frac{1}{4\pi}\Big[&-(\boldsymbol{r} \times \boldsymbol{E})(\boldsymbol{E}\, \boldsymbol{ds}) - (\boldsymbol{r} \times \boldsymbol{H})(\boldsymbol{H}\, \boldsymbol{ds}) \\ &+ \frac{1}{2}(E^2 + H^2)(\boldsymbol{r} \times \boldsymbol{ds})\Big]. \end{aligned} \tag{A4}$$

APPENDIX B: THE POYNTING VECTOR AND THE STRESS TENSOR FOR THE FIELD OF A ROTATING DIPOLE

The electromagnetic field of a rotating electric dipole in a spherical coordinate system (r, θ, φ) is as follows [37, §72]:

$$\begin{aligned} H_\theta &= \frac{d\omega}{r^2 c}(\cos\tau - \rho\sin\tau), \\ H_\varphi &= \frac{d\omega}{r^2 c}\cos\theta(\sin\tau + \rho\cos\tau), \\ E_r &= \frac{2d}{r^3}\sin\theta(\cos\tau - \rho\sin\tau), \\ E_\theta &= \frac{d}{r^3}\cos\theta(-\cos\tau + \rho\sin\tau + \rho^2\cos\tau), \\ E_\varphi &= \frac{d}{r^3}(-\sin\tau - \rho\cos\tau + \rho^2\sin\tau), \end{aligned} \tag{B1}$$

where

$$\rho = \frac{\omega r}{c}, \quad \tau = \omega t' - \varphi, \quad t' = t - \frac{r}{c}.$$

The Poynting vector

$$\boldsymbol{P} = \frac{c}{4\pi}(\boldsymbol{E} \times \boldsymbol{H})$$

has the components

$$\begin{aligned} P_r &= P_0[2\rho^3(\sin^2\tau + \cos^2\theta\cos^2\tau) - 2\rho^2\sin 2\tau\sin^2\theta + 2\rho\cos 2\tau\sin^2\theta + \sin 2\tau\sin^2\theta], \\ P_\theta &= 2P_0\sin\theta\cos\theta(\rho^2\sin 2\tau - 2\rho\cos 2\tau - \sin 2\tau), \\ P_\varphi &= 4P_0\sin\theta\left(\rho^2\sin^2\tau - \rho\sin 2\tau + \cos^2\tau\right), \end{aligned} \tag{B2}$$

where

$$P_0 = \frac{d^2\omega}{8\pi r^5}, \quad \rho = \frac{\omega r}{c}, \quad \tau = \omega t - kr - \phi.$$

The angular momentum flux is determined by the off-diagonal elements of the stress tensor (A2)

$$\begin{aligned}
\sigma_{12} =& \frac{d^2}{4\pi r^6} \sin 2\theta \left(\cos^2\tau - \rho \sin 2\tau - \rho^2 \cos 2\tau + \frac{1}{2}\rho^3 \sin 2\tau \right), \\
\sigma_{13} =& \frac{d^2}{4\pi r^6} \sin\theta (\sin 2\tau + 2\rho \cos 2\tau - 2\rho^2 \sin 2\tau + 2\rho^3 \sin^2\tau), \\
\sigma_{23} =& -\frac{d^2}{4\pi r^6} \cos\theta \left(\frac{1}{2} \sin 2\tau + \rho \cos 2\tau - \rho^2 \sin 2\tau \right).
\end{aligned} \tag{B3}$$

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