Internal transformations and internal symmetries in linear photonic systems

Cheng Guo ,[1,*] Zhexin Zhao,[2] and Shanhui Fan[2,†]
[1]*Department of Applied Physics, Stanford University, Stanford, California 94305, USA*
[2]*Ginzton Laboratory and Department of Electrical Engineering, Stanford University, Stanford, California 94305, USA*

(Received 8 October 2021; accepted 22 December 2021; published 14 February 2022)

Lorentz reciprocity, energy conservation, and time-reversal symmetry are three important global constraints of Maxwell's equations. Unlike time-reversal symmetry, Lorentz reciprocity and energy conservation usually are not considered as symmetries, i.e., they are not associated with operators. In this paper, we provide a unified treatment of these three global constraints from a perspective of internal symmetry. We define operators of transformations associated with each of these constraints, referred to as internal transformations. When Maxwell's equations are written as a linear system of equations, these internal transformations correspond to the operations of transpose, conjugate transpose, and conjugate of the system matrix, respectively. We show the three global constraints naturally follow from three fundamental identities of linear systems under the three matrix operations. We discuss the properties of electromagnetic fields and scattering matrices associated with these internal transformations. These internal transformations form the Klein four-group $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$, and the internal symmetry group of any photonic structure corresponds to one of the five subgroups of V_4. Our paper provides a theoretical foundation for further exploration of symmetries in photonic systems.

DOI: 10.1103/PhysRevA.105.023509

I. INTRODUCTION

In physics, the investigation of symmetries yields some of the most profound results. One prominent example is the standard model, a gauge quantum field theory containing the internal symmetries of $U(1) \times SU(2) \times SU(3)$ [1]. Photonic structures are physical systems designed to manipulate light, which have great importance in scientific and engineering applications. Photonic structures can possess many symmetries. A thorough investigation of these symmetries can lead to a deeper understanding of photonic systems. For example, rotational symmetry is crucial for understanding the modes in optical fibers [2]; mirror symmetry was exploited in designing photonic crystal add-drop filters [3]; parity-time symmetry [4], i.e., the invariance under the combined operation of parity and time reversal, has generated significant recent interest in photonics [5–8].

Photonic systems are also subject to a set of global constraints, including time-reversal symmetry, Lorentz reciprocity, and energy conservation. Understanding these global constraints has been very important for photonic design. Unlike time-reversal symmetry, however, Lorentz reciprocity and energy conservation usually are not considered as symmetries, i.e., one does not associate them with operators that transform Maxwell equations.

In this paper, we show that these three global constraints on photonic structures, including time-reversal symmetry, energy conservation, and Lorentz reciprocity, can be studied in a uniform way using the language of symmetry. We define operators of transformations associated with each of these constraints. To distinguish transformations and symmetries of different natures, we denote the transformations and symmetries associated with the global constraints as "internal" and the usual geometrical transformations and symmetries as "external". We discuss the properties of electromagnetic fields and scattering matrices associated with these internal transformations. We also show that these internal transformations form the Klein four-group $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$, and the internal symmetry group of any photonic structure corresponds to one of the five subgroups of V_4. Our paper provides a natural group theory framework to study the global constraints on photonic structures. It should be fruitful in future works to further elucidate the symmetry properties of photonic systems by incorporating both internal and external symmetries into a unified group theory.

We note that the three global constraints and their conditions on scattering matrices are known. One objective of our paper is to rederive these fundamental results in a unified way from the perspective of symmetry. Such a viewpoint elucidates the origins of these constraints from the fundamental identities of linear systems. It also leads to new results such as the Klein four-group classification.

The rest of this paper is organized as follows. In Sec. II, we provide the mathematical background. In Sec. III, we introduce operators of transformations associated with reciprocity, energy conservation, and time reversal, and discuss the implications of these internal transformations on the properties of electromagnetic fields. In Sec. IV, we discuss the implications of the internal transformations on scattering matrices. In Sec. V, we discuss the group theory structure associated with these internal transformations and symmetries. We conclude in Sec. VI.

*guocheng@stanford.edu
†shanhui@stanford.edu

II. MATHEMATICAL BACKGROUNDS

A. Fundamental identities of linear systems

We start by reviewing the relevant fundamental identities of linear systems. Additional details can be found in Ref. [9].

Consider an arbitrary linear system of equations, referred to as the "original" system:

$$Ax = b \tag{1}$$

where A is a complex matrix, and x and b are complex column vectors.

From the original system, one can define three related systems: (1) the transposed system

$$A^T x_1 = b_1, \tag{2}$$

(2) the c-transposed system

$$A^\dagger x_2 = b_2, \tag{3}$$

and (3) the conjugated system

$$A^* x_3 = b_3 \tag{4}$$

where A^T, $A^\dagger$, and A^* are the transpose, conjugate transpose, and conjugate of the matrix A. These operations are all involutory, i.e., $(A^T)^T = (A^\dagger)^\dagger = (A^*)^* = A$, thus each operation defines a mutual relation. A system of Eq. (1) is defined as self-transpose, self-c-transpose, or self-conjugate if and only if A is invariant under transpose, conjugate transpose, or conjugate, respectively.

We note the following fundamental relations between the original system and each of the three related systems.

1. Original and transposed system

From Eqs. (1) and (2),

$$x_1^T Ax = x_1^T b, \qquad x^T A^T x_1 = x^T b_1. \tag{5}$$

Since a complex scalar is invariant under transpose,

$$x_1^T Ax = \left(x_1^T Ax\right)^T = x^T A^T x_1. \tag{6}$$

Therefore,

$$x_1^T b - x^T b_1 = x_1^T Ax - x^T A^T x_1 = 0. \tag{7}$$

2. Original and c-transposed system

From Eqs. (1) and (3), similar to the procedure above, we have

$$x_2^\dagger b - (x^\dagger b_2)^* = x_2^\dagger Ax - (x^\dagger A^\dagger x_2)^* = 0. \tag{8}$$

3. Original and conjugated system

The complex conjugate of Eq. (1) yields

$$A^* x^* = (Ax)^* = b^*. \tag{9}$$

When a system is self-transpose, self-c-transpose, or self-conjugate, these identities Eqs. (7)–(9) reduce to the corresponding symmetry constraints of the system.

In the subsequent discussion, we will see that Eqs. (7), (8), and (9) have direct connections to Lorentz reciprocity, energy conservation, and time-reversal symmetry, respectively.

B. From linear systems to linear differential equations

Now we extend the above analysis of linear systems to linear differential equations.

Let V be an open subset of $\mathbb{R}^n$, let $\mathcal{F} = \{u : V \to \mathbb{C}^m\}$ be the space of complex vector functions on V, and let $\mathcal{A} : \mathcal{F} \to \mathcal{F}$ be a linear differential operator. Consider a system of complex linear partial differential equations (PDEs) [10]:

$$\mathcal{A}u = f, \tag{10}$$

where $f \in \mathcal{F}$ is given, and $u \in \mathcal{F}$ is unknown. To ensure a unique solution, one usually provides boundary conditions, denoted as $\mathcal{B}$, on some part Γ of ∂V. $(V, \Gamma, \mathcal{A}, f, \mathcal{B})$ is referred to as the original PDE problem.

A system of linear differential equations can be transformed into a system of linear algebraic equations through discretization such as by using finite-difference methods [11]. A detailed procedure can be found in Ref. [11]. Then V is discretized into D, and the PDE *together with* the boundary conditions are transformed into a system of linear equations:

$$Au = f. \tag{11}$$

(D, A, f) is referred to as the original algebraic problem.

Our analysis in Sec. II A applies to (D, A, f). We define the three related systems (D, A^T, f_1), $(D, A^\dagger, f_2)$, and (D, A^*, f_3). Then we take the continuum limit and transform them into three related PDE problems: the transposed PDE problem $(V, \Gamma_1, \mathcal{A}^T, f_1, \mathcal{B}_1)$, the c-transposed PDE problem $(V, \Gamma_2, \mathcal{A}^\dagger, f_2, \mathcal{B}_2)$, and the conjugated PDE problem $(V, \Gamma_3, \mathcal{A}^*, f_3, \mathcal{B}_3)$. The obtained linear differential operators $\mathcal{A}^T$, $\mathcal{A}^\dagger$, and $\mathcal{A}^*$ are defined as the transpose, the conjugate transpose, and the conjugate of $\mathcal{A}$. We establish the following relations between these linear differential operators by taking the continuum limit of Eqs. (7), (8), and (9):

$$\int_V [v^T \mathcal{A}u - u^T \mathcal{A}^T v]\, dV = 0, \tag{12}$$

$$\int_V [v^\dagger \mathcal{A}u - (u^\dagger \mathcal{A}^\dagger v)^*]\, dV = 0, \tag{13}$$

$$\mathcal{A}^* u^* = (\mathcal{A}u)^*, \tag{14}$$

which hold for all u that solves the original PDE problem and v that solves the related PDE problem. In particular, u and v must satisfy the respective partial differential equations *and the corresponding boundary conditions*; these boundary conditions may be different for the two problems.

While this procedure is cumbersome, it in principle yields $\mathcal{A}^T$, $\mathcal{A}^\dagger$, and $\mathcal{A}^*$ of any given $\mathcal{A}$ [9]. In practice, we have a much simpler method. Motivated by the discussions above, we use a set of relations related to Eqs. (12)–(14) as the alternative definitions of $\mathcal{A}^T$, $\mathcal{A}^\dagger$, and $\mathcal{A}^*$. Since the definition of a differential operator is independent of boundary conditions, it is useful to reformulate Eqs. (12)–(14) as

$$\int_V [v^T \mathcal{A}u - u^T \mathcal{A}^T v]\, dV = \text{boundary terms}, \tag{15}$$

$$\int_V [v^\dagger \mathcal{A}u - (u^\dagger \mathcal{A}^\dagger v)^*]\, dV = \text{boundary terms}, \tag{16}$$

$$\mathcal{A}^* u^* = (\mathcal{A}u)^*, \tag{17}$$

which holds for all u and v that satisfy the respective partial differential equations *without specific boundary conditions*. Here "boundary term" means the integration result depends solely on the values of u and v—and some of their derivatives—taken on the boundary ∂V. Equations (15) and (16) are called the "extended Green's identity" [9].

One can take Eqs. (15), (16), and (17) as alternative definitions of $\mathcal{A}^T$, $\mathcal{A}^\dagger$, and $\mathcal{A}^*$, respectively. It can be proved that such definitions are equivalent to the original definitions by discretization [9]. A linear differential operator $\mathcal{A}$ is defined to be self-transpose, self-c-transpose, and self-conjugate if $\mathcal{A} = \mathcal{A}^T$, $\mathcal{A} = \mathcal{A}^\dagger$, and $\mathcal{A} = \mathcal{A}^*$, respectively.

Here we briefly remark on the terminology. In the study of linear differential operators, one usually uses the term "adjoint" to refer to the transpose for a real operator and the c transpose for a complex operator. Our discussion involves both the transpose and c transpose of a complex operator. To avoid confusion, we adopt the more specific terms "transpose" and "c transpose" instead of "adjoint."

C. Symmetry properties of the curl operator

In this subsection, we focus on a special linear differential operator, the curl operator $\nabla\times$. The curl operator plays a central role in Maxwell's equations. It is a linear differential operator that acts on the space of three-dimensional complex vector fields over $\mathbb{R}^3$. Below, we show that it is self-transpose, self-c-transpose, and self-conjugate.

1. Self-transpose

From the basic vector formulas, for two complex vector fields $\boldsymbol{a}(\boldsymbol{r})$ and $\boldsymbol{b}(\boldsymbol{r})$,

$$\begin{aligned}\boldsymbol{a}^T\nabla\times\boldsymbol{b} - \boldsymbol{b}^T\nabla\times\boldsymbol{a} &\equiv \boldsymbol{a}\cdot(\nabla\times\boldsymbol{b}) - \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a})\\ &= \nabla\cdot(\boldsymbol{b}\times\boldsymbol{a}).\end{aligned} \tag{18}$$

Integrating Eq. (18) over an arbitrary volume $V \in \mathbb{R}^3$ enclosed by a surface S, we get

$$\int_V [\boldsymbol{a}^T\nabla\times\boldsymbol{b} - \boldsymbol{b}^T\nabla\times\boldsymbol{a}]\, dV = \oint_S (\boldsymbol{b}\times\boldsymbol{a})\cdot d\boldsymbol{S}. \tag{19}$$

Thus $\nabla\times$ is self-transpose by the definition in Eq. (15).

2. Self-c-transpose

From the basic vector formulas,

$$\begin{aligned}\boldsymbol{a}^\dagger\nabla\times\boldsymbol{b} - (\boldsymbol{b}^\dagger\nabla\times\boldsymbol{a})^* &\equiv \boldsymbol{a}^*\cdot(\nabla\times\boldsymbol{b}) - \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a}^*)\\ &= \nabla\cdot(\boldsymbol{b}\times\boldsymbol{a}^*).\end{aligned} \tag{20}$$

Integrating Eq. (20) over an arbitrary volume $V \in \mathbb{R}^3$ enclosed by a surface S, we get

$$\int_V [\boldsymbol{a}^\dagger\nabla\times\boldsymbol{b} - (\boldsymbol{b}^\dagger\nabla\times\boldsymbol{a})^*]\, dV = \oint_S (\boldsymbol{b}\times\boldsymbol{a}^*)\cdot d\boldsymbol{S}. \tag{21}$$

Thus $\nabla\times$ is self-c-transpose by the definition in Eq. (16).

3. Self-conjugate

For a complex vector $\boldsymbol{a}$,

$$(\nabla\times\boldsymbol{a})^* = \nabla\times(\boldsymbol{a}^*). \tag{22}$$

Thus $\nabla\times$ is self-conjugate by the definition in Eq. (17).

III. FUNDAMENTAL INTERNAL TRANSFORMATIONS AND SYMMETRIES

A. Maxwell's equations and three fundamental internal transformations

Now we apply the general mathematical theory in Sec. II to Maxwell's equations. This investigation naturally yields three fundamental internal transformations and symmetries.

For linear time-invariant systems, Maxwell's equations are

$$\begin{aligned}&\nabla\times\boldsymbol{E}(\omega,\boldsymbol{r}) = i\omega\boldsymbol{B}(\omega,\boldsymbol{r}),\\ &\nabla\times\boldsymbol{H}(\omega,\boldsymbol{r}) = -i\omega\boldsymbol{D}(\omega,\boldsymbol{r}) + \boldsymbol{J}(\omega,\boldsymbol{r}).\end{aligned} \tag{23}$$

Maxwell's equations must be complemented by constitutive relations, which defines a linear photonic system. In this paper, we consider a general linear photonic system made of any linear local inhomogeneous dispersive bianisotropic medium described by a 6×6 constitutive matrix $C(\omega,\boldsymbol{r})$:

$$\begin{aligned}\begin{pmatrix}\boldsymbol{D}(\omega,\boldsymbol{r})\\ \boldsymbol{B}(\omega,\boldsymbol{r})\end{pmatrix} &= C(\omega,\boldsymbol{r})\begin{pmatrix}\boldsymbol{E}(\omega,\boldsymbol{r})\\ \boldsymbol{H}(\omega,\boldsymbol{r})\end{pmatrix}\\ &= \begin{pmatrix}\varepsilon(\omega,\boldsymbol{r}) & \zeta(\omega,\boldsymbol{r})\\ \eta(\omega,\boldsymbol{r}) & \mu(\omega,\boldsymbol{r})\end{pmatrix}\begin{pmatrix}\boldsymbol{E}(\omega,\boldsymbol{r})\\ \boldsymbol{H}(\omega,\boldsymbol{r})\end{pmatrix},\end{aligned} \tag{24}$$

where ε, μ, ζ, and η are 3×3 matrices of electric permittivity, magnetic permeability, electric-magnetic coupling strength, and magnetoelectric coupling strength, respectively. We refer to $C(\omega,\boldsymbol{r})$ as the original physical system.

Substituting Eq. (24) in Eq. (23), we obtain

$$M\Phi = a \tag{25}$$

where

$$\begin{aligned}&M \equiv \begin{pmatrix}-\omega\varepsilon & i\omega\zeta + \nabla\times\\ -i\omega\eta + \nabla\times & -\omega\mu\end{pmatrix}, \quad \Phi \equiv \begin{pmatrix}\boldsymbol{E}\\ i\boldsymbol{H}\end{pmatrix},\\ &a \equiv \begin{pmatrix}i\boldsymbol{J}\\ 0\end{pmatrix}.\end{aligned} \tag{26}$$

We have omitted the arguments ω and $\boldsymbol{r}$ for brevity. We choose $(\boldsymbol{E}, i\boldsymbol{H})^T$ instead of $(\boldsymbol{E}, \boldsymbol{H})^T$ as the independent variables; such a choice is important to establish the connection of the three mathematical identities of Eqs. (7), (8), and (9), to the three global constraints for Maxwell's equations.

Now we can define the internal transformations and symmetries of linear photonic systems. To motivate our definition, we first consider a more familiar type of external symmetry: rotational symmetry. We start from an original system as described by $C(\omega,\boldsymbol{r})$ in Eq. (24). Suppose we apply a rotation to the system; the transformed system is then described by [12]

$$\tilde{C}(\omega,\boldsymbol{r}) = \begin{pmatrix}R\,\varepsilon(\omega,R^{-1}\boldsymbol{r})\,R^{-1} & R\,\zeta(\omega,R^{-1}\boldsymbol{r})\,R^{-1}\\ R\,\eta(\omega,R^{-1}\boldsymbol{r})\,R^{-1} & R\,\mu(\omega,R^{-1}\boldsymbol{r})\,R^{-1}\end{pmatrix}, \tag{27}$$

where R is a 3×3 orthogonal real matrix that describes the rotation in real space. We claim that the rotation R is a symmetry of the system if $\tilde{C}(\omega,\boldsymbol{r}) = C(\omega,\boldsymbol{r})$. The external symmetries, however, are not the only symmetries available for photonic structures. Below we discuss the transformations and symmetries associated with reciprocity, energy conservation, and time reversal. We show that the relevant transformations correspond to the transpose, c transpose, and conjugate operations, respectively.

1. Transposed system

Starting from the original system as described by Eq. (25), we consider its transposed system:

$$M^T\Phi_1 = a_1 \tag{28}$$

where

$$M^T \equiv \begin{pmatrix} -\omega\varepsilon^T & -i\omega\eta^T + \nabla\times \\ i\omega\zeta^T + \nabla\times & -\omega\mu^T \end{pmatrix}, \quad \Phi_1 \equiv \begin{pmatrix} \boldsymbol{E_1} \\ i\boldsymbol{H_1} \end{pmatrix},$$
$$a_1 \equiv \begin{pmatrix} i\boldsymbol{J_1} \\ \boldsymbol{0} \end{pmatrix}. \tag{29}$$

Hence the transposed system is described by a constitutive matrix:

$$C_1(\omega, \boldsymbol{r}) = \begin{pmatrix} \varepsilon^T & -\eta^T \\ -\zeta^T & \mu^T \end{pmatrix}. \tag{30}$$

We define the transformation $C(\omega, \boldsymbol{r}) \to C_1(\omega, \boldsymbol{r})$ as the transformation of reciprocity. A self-transpose system satisfying $C(\omega, \boldsymbol{r}) = C_1(\omega, \boldsymbol{r})$ is called *reciprocal*.

2. c-transposed system

Starting from the original system as described by Eq. (25), we consider its c-transposed system:

$$M^\dagger\Phi_2 = a_2 \tag{31}$$

where

$$M^\dagger \equiv \begin{pmatrix} -\omega\varepsilon^\dagger & i\omega\eta^\dagger + \nabla\times \\ -i\omega\zeta^\dagger + \nabla\times & -\omega\mu^\dagger \end{pmatrix}, \quad \Phi_2 \equiv \begin{pmatrix} \boldsymbol{E_2} \\ i\boldsymbol{H_2} \end{pmatrix},$$
$$a_2 \equiv \begin{pmatrix} i\boldsymbol{J_2} \\ \boldsymbol{0} \end{pmatrix}. \tag{32}$$

Hence the c-transposed system is described by a constitutive matrix

$$C_2(\omega, \boldsymbol{r}) = \begin{pmatrix} \varepsilon^\dagger & \eta^\dagger \\ \zeta^\dagger & \mu^\dagger \end{pmatrix}. \tag{33}$$

We define the transformation $C(\omega, \boldsymbol{r}) \to C_2(\omega, \boldsymbol{r})$ as the transformation of energy conservation. A self-c-transpose system satisfying $C(\omega, \boldsymbol{r}) = C_2(\omega, \boldsymbol{r})$ is called *energy conserving* or *lossless*.

3. Conjugated system

Starting from the original system as described by Eq. (25), we consider its conjugated system:

$$M^*\Phi_3 = a_3 \tag{34}$$

where

$$M^* \equiv \begin{pmatrix} -\omega\varepsilon^* & -i\omega\zeta^* + \nabla\times \\ i\omega\eta^* + \nabla\times & -\omega\mu^* \end{pmatrix}, \quad \Phi_3 \equiv \begin{pmatrix} \boldsymbol{E_3} \\ i\boldsymbol{H_3} \end{pmatrix},$$
$$a_3 = \begin{pmatrix} i\boldsymbol{J_3} \\ \boldsymbol{0} \end{pmatrix}. \tag{35}$$

Hence the conjugated system is described by a constitutive matrix:

$$C_3(\omega, \boldsymbol{r}) = \begin{pmatrix} \varepsilon^* & -\zeta^* \\ -\eta^* & \mu^* \end{pmatrix}. \tag{36}$$

We define the transformation $C(\omega, \boldsymbol{r}) \to C_3(\omega, \boldsymbol{r})$ as the transformation of time reversal. A self-conjugate system satisfying $C(\omega, \boldsymbol{r}) = C_3(\omega, \boldsymbol{r})$ is called *time-reversal symmetric*.

Our definitions of reciprocal, lossless, and time-reversal symmetric systems are identical to the standard definitions [13]. Our treatment, however, allows us to discuss these general constraints in terms of symmetry. A system is reciprocal, for example, if and only if the transformation of reciprocity is a symmetry of the system. We note that while reciprocity is well known, the discussion of reciprocity as a symmetry is not widely recognized.

B. Fundamental relations under fundamental internal transformations

Now we derive the three fundamental constraints on linear photonic systems as imposed by the three fundamental transformations discussed in the previous section. Our derivation directly uses the fundamental identities [Eqs. (15), (16), and (17)] as discussed in Sec. II A. The constraints that we derive are known in the literature. However, the connection of these constraints to the fundamental identities of linear systems has not been emphasized. Moreover, our derivation here provides a unified view of these constraints from the perspective of internal transformation and internal symmetry.

1. Original and transposed systems

$$\Phi_1^T M\Phi = \begin{pmatrix} \boldsymbol{E_1^T} & i\boldsymbol{H_1^T} \end{pmatrix} \begin{pmatrix} -\omega\varepsilon & i\omega\zeta + \nabla\times \\ -i\omega\eta + \nabla\times & -\omega\mu \end{pmatrix} \begin{pmatrix} \boldsymbol{E} \\ i\boldsymbol{H} \end{pmatrix}, \tag{37}$$

$$\Phi^T M^T \Phi_1 = (\boldsymbol{E}^T \quad i\boldsymbol{H}^T) \begin{pmatrix} -\omega\varepsilon^T & -i\omega\eta^T + \nabla\times \\ i\omega\zeta^T + \nabla\times & -\omega\mu^T \end{pmatrix} \begin{pmatrix} \boldsymbol{E_1} \\ i\boldsymbol{H_1} \end{pmatrix}. \tag{38}$$

Subtracting Eq. (38) from Eq. (37), we obtain

$$\begin{aligned} \Phi_1^T M\Phi - \Phi^T M^T\Phi_1 &= i\big(\boldsymbol{E_1^T}\nabla\times\boldsymbol{H} + \boldsymbol{H_1^T}\nabla\times\boldsymbol{E} - \boldsymbol{E}^T\nabla \\ &\quad \times \boldsymbol{H_1} - \boldsymbol{H}^T\nabla\times\boldsymbol{E_1}\big) \\ &= i\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H_1} - \boldsymbol{E_1}\times\boldsymbol{H}) \end{aligned} \tag{39}$$

where all the terms without $\nabla\times$ have been canceled.

On the other hand, from Eqs. (25) and (28),

$$\Phi_1^T M\Phi - \Phi^T M^T\Phi_1 = \Phi_1^T a - \Phi^T a_1 = i(\boldsymbol{E_1}\cdot\boldsymbol{J} - \boldsymbol{E}\cdot\boldsymbol{J_1}). \tag{40}$$

Combining Eqs. (39) and (40), we obtain

$$\nabla\cdot(\boldsymbol{E}\times\boldsymbol{H_1} - \boldsymbol{E_1}\times\boldsymbol{H}) = \boldsymbol{E_1}\cdot\boldsymbol{J} - \boldsymbol{E}\cdot\boldsymbol{J_1}. \tag{41}$$

We integrate Eq. (41) over an arbitrary volume V enclosed by a surface S to get

$$\oint_S (\boldsymbol{E} \times \boldsymbol{H_1} - \boldsymbol{E_1} \times \boldsymbol{H}) \cdot d\boldsymbol{S} = \int_V (\boldsymbol{E_1} \cdot \boldsymbol{J} - \boldsymbol{E} \cdot \boldsymbol{J_1})\, dV. \tag{42}$$

This is the well-known *generalized reciprocity theorem* [13,14]. It relates a pair of sources and fields in two mutually transposed systems. When a system is self-transpose (reciprocal), Eq. (42) reduces to the conventional *Lorentz reciprocity* theorem that relates a pair of sources and responses in that single system [15,16].

2. Original and c-transposed systems

$$\Phi_2^\dagger M \Phi = (\boldsymbol{E_2^\dagger} \quad -i\boldsymbol{H_2^\dagger})\begin{pmatrix} -\omega\varepsilon & i\omega\zeta + \nabla\times \\ -i\omega\eta + \nabla\times & -\omega\mu \end{pmatrix}\begin{pmatrix} \boldsymbol{E} \\ i\boldsymbol{H} \end{pmatrix}, \tag{43}$$

$$\Phi^\dagger M^\dagger \Phi_2 = (\boldsymbol{E^\dagger} \quad -i\boldsymbol{H^\dagger})\begin{pmatrix} -\omega\varepsilon^\dagger & i\omega\eta^\dagger + \nabla\times \\ -i\omega\zeta^\dagger + \nabla\times & -\omega\mu^\dagger \end{pmatrix}\begin{pmatrix} \boldsymbol{E_2} \\ i\boldsymbol{H_2} \end{pmatrix}. \tag{44}$$

Subtracting the conjugate of Eq. (44) from Eq. (43), we obtain

$$\begin{aligned} \Phi_2^\dagger M \Phi - (\Phi^\dagger M^\dagger \Phi_2)^* &= i(\boldsymbol{E_2^*} \cdot \nabla \times \boldsymbol{H} - \boldsymbol{H_2^*} \cdot \nabla \times \boldsymbol{E} + \boldsymbol{E} \cdot \nabla \times \boldsymbol{H_2^*} - \boldsymbol{H} \cdot \nabla \times \boldsymbol{E_2^*}) \\ &= -i\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H_2^*} + \boldsymbol{E_2^*} \times \boldsymbol{H}). \end{aligned} \tag{45}$$

On the other hand, from Eqs. (25) and (31),

$$\Phi_2^\dagger M \Phi - (\Phi^\dagger M^\dagger \Phi_2)^* = \Phi_2^\dagger a - (\Phi^\dagger a_2)^* = i(\boldsymbol{E_2^*} \cdot \boldsymbol{J} + \boldsymbol{E} \cdot \boldsymbol{J_2^*}). \tag{46}$$

Combining Eqs. (45) and (46), we obtain

$$-\nabla \cdot (\boldsymbol{E} \times \boldsymbol{H_2^*} + \boldsymbol{E_2^*} \times \boldsymbol{H}) = \boldsymbol{E_2^*} \cdot \boldsymbol{J} + \boldsymbol{E} \cdot \boldsymbol{J_2^*}. \tag{47}$$

We integrate Eq. (47) over an arbitrary volume V enclosed by a surface S to get

$$-\oint_S (\boldsymbol{E} \times \boldsymbol{H_2^*} + \boldsymbol{E_2^*} \times \boldsymbol{H}) \cdot d\boldsymbol{S} = \int_V (\boldsymbol{E_2^*} \cdot \boldsymbol{J} + \boldsymbol{E} \cdot \boldsymbol{J_2^*})\, dV. \tag{48}$$

This is the less-known *modified mutual energy theorem* [17]. It relates a pair of sources and fields in two mutually c-transposed systems. When a system is self-c-transpose (lossless), Eq. (48) reduces to the *mutual energy theorem* that relates a pair of sources and responses in that single system [18]. If one further chooses the pair of sources and responses to be identical ($\boldsymbol{J_2} = \boldsymbol{J}, \boldsymbol{E_2} = \boldsymbol{E}, \boldsymbol{H_2} = \boldsymbol{H}$), the mutual energy theorem reduces to the conventional *Poynting theorem* for lossless systems [19,20].

3. Original and conjugated systems

The complex conjugate of Eq. (25) yields

$$M^* \Phi^* = a^*. \tag{49}$$

The complex conjugate of Eq. (49) returns Eq. (25). Therefore,

$$M\Phi = a \iff M^*\Phi^* = a^*, \tag{50}$$

where

$$\Phi^* = \begin{pmatrix} \boldsymbol{E}^* \\ -i\boldsymbol{H}^* \end{pmatrix}, \qquad a^* = \begin{pmatrix} -i\boldsymbol{J}^* \\ \boldsymbol{0} \end{pmatrix}. \tag{51}$$

This is the *generalized time-reversal theorem*. It states that $(\boldsymbol{E}, \boldsymbol{H}, \boldsymbol{J})$ satisfy Maxwell's equations for an original system if and only if $(\boldsymbol{E}^*, -\boldsymbol{H}^*, -\boldsymbol{J}^*)$ satisfy Maxwell's equations for its conjugated system. When the system is self-conjugate (time-reversal symmetric), Eq. (50) reduces to the usual constraint from time-reversal symmetry [21,22].

In summary, the general relations between mutually transposed, c-transposed, and conjugated systems naturally yield the generalized theorems of reciprocity, energy conservation, and time-reversal symmetry, respectively.

IV. PHYSICAL IMPLICATIONS ON SCATTERING MATRICES

So far, we have introduced three fundamental internal transformations, defined three related systems, and derived three fundamental relations of the electromagnetic fields in the original and the related systems. These fundamental results have direct physical implications. In this section, we discuss their implications on scattering matrices.

A. Definition of scattering matrices

First, we introduce the definition of scattering matrices. Additional details can be found in Refs. [23,24]. As shown in Fig. 1, we consider a general linear time-invariant system characterized by $C(\omega, \boldsymbol{r})$ within a volume V enclosed by a surface ∂V. We assume there are no sources within V: $\boldsymbol{J} = \boldsymbol{0}$. The system is connected to its exterior by Q physical ports through ∂V. The physical ports are waveguides made of linear time-invariant media characterized by $C^{(m)}(\omega, \boldsymbol{r})$, $m = 1, \ldots, Q$, which are assumed to be reciprocal, lossless, and time-reversal symmetric, homogeneous along the propagation direction, and reflection symmetric under the mirror operation that reverses the propagation direction. Each physical port supports one or multiple guided modes, i.e., eigensolutions of Maxwell's equations. Counting all the Q ports, there are P incoming and P outgoing modes in total. We assume the field energy is exclusively carried by these guided modes outside ∂V. The surface ∂V is chosen such that all the physical ports are aligned normal to ∂V and different physical ports and their guided modes are assumed to be essentially nonoverlapping. For clarity, we use the indices m and n to enumerate the

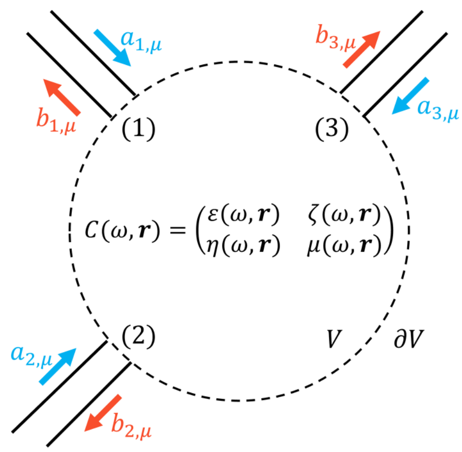

FIG. 1. A general optical circuit. It consists of a linear time-invariant system characterized by $C(\omega, \boldsymbol{r})$ within a volume V enclosed by a surface ∂V, which is connected to its exterior by Q physical ports ($Q = 3$ is shown). Each physical port is a waveguide consisting of a linear time-invariant medium that is homogeneous along the propagation direction and is reciprocal, lossless, time-reversal symmetric, and reflection symmetric under the mirror operation that reverses the propagation direction. Each port may support multiple modes, as labeled by μ. In total, there are P incoming and P outgoing modes in all the physical ports.

physical ports, μ and ν to enumerate the modes in a single physical port, and i and j to enumerate all the P modes. We use the superscript t to denote the transverse components of a vector field tangential to ∂V. In each physical port, we specify a Cartesian coordinate system such that $\boldsymbol{r}^t \equiv (x, y)$ are tangential to ∂V, and z is along the outgoing direction with $z = 0$ at ∂V.

Under these assumptions, one can construct a set of orthonormal bases $\{\boldsymbol{e}_i, \boldsymbol{h}_i\}$, normalized by unit energy flux. These basis modes can be chosen such that their transverse fields $\{\boldsymbol{e}_i^t(\boldsymbol{r}^t), \boldsymbol{h}_i^t(\boldsymbol{r}^t)\}$ are purely real. This choice has been used in Ref. [24]. A proof of this choice, for the waveguides satisfying the constraints as outlined above, can be found in Ref. [25]. These modes satisfy the orthonormal conditions:

$$\oint_{\partial V} d\boldsymbol{S} \cdot \boldsymbol{e}_i^t \times \boldsymbol{h}_j^t = \oint_{\partial V} d\boldsymbol{S} \cdot \boldsymbol{e}_j^t \times \boldsymbol{h}_i^t = -2\,\delta_{ij}. \tag{52}$$

Then the transverse fields of light in the physical ports outside ∂V can be expressed as

$$\begin{aligned}\boldsymbol{E}^t(\boldsymbol{r}^t, z) &= \sum_{i=1}^{P} (a_i e^{-i\beta_i z} + b_i e^{i\beta_i z}) \boldsymbol{e}_i^t(\boldsymbol{r}^t),\\ \boldsymbol{H}^t(\boldsymbol{r}^t, z) &= \sum_{i=1}^{P} (a_i e^{-i\beta_i z} - b_i e^{i\beta_i z}) \boldsymbol{h}_i^t(\boldsymbol{r}^t).\end{aligned} \tag{53}$$

By Eq. (53), the incoming and outgoing waves can be represented by complex vectors:

$$\boldsymbol{a} = [a_1, \ldots, a_P]^T, \qquad \boldsymbol{b} = [b_1, \ldots, b_P]^T, \tag{54}$$

where a_i and b_i are the complex coefficients of the ith incoming and outgoing modes, respectively, as determined by

$$a_i = \frac{1}{4} \oint_{\partial V} d\boldsymbol{S} \cdot \left[\boldsymbol{h}_i^t \times \boldsymbol{E}^t(z=0) - \boldsymbol{e}_i^t \times \boldsymbol{H}^t(z=0)\right], \tag{55}$$

$$b_i = \frac{1}{4} \oint_{\partial V} d\boldsymbol{S} \cdot \left[\boldsymbol{h}_i^t \times \boldsymbol{E}^t(z=0) + \boldsymbol{e}_i^t \times \boldsymbol{H}^t(z=0)\right]. \tag{56}$$

Since the system is linear time invariant and source free, the outgoing waves are completely determined by the incoming waves and by the system properties as characterized by $C(\omega, \boldsymbol{r})$. Hence, there is a linear relation between $\boldsymbol{a}$ and $\boldsymbol{b}$, which can be written in matrix form as

$$\boldsymbol{b} = S\boldsymbol{a}. \tag{57}$$

S is called a *scattering matrix*. S has a size of $P \times P$, and its element S_{ij} gives the transition amplitude for photons from jth basis mode to ith basis mode.

B. Constraints on scattering matrices under fundamental internal transformations

We have defined the scattering matrix S for an original system characterized by $C(\omega, \boldsymbol{r})$. From the original system, we can define the three related systems characterized by $C_k(\omega, \boldsymbol{r})$, $k = 1, 2, 3$, as defined by Eqs. (30), (33), and (36), respectively, within the same volume V enclosed by the same surface ∂V. The ports of these related systems are identical to those of the original system. Thus, we can choose the same orthonormal basis for light in the physical ports, and define the scattering matrices for the three related systems:

$$\boldsymbol{b}_k = S_k \boldsymbol{a}_k, \qquad k = 1, 2, 3 \tag{58}$$

where $\boldsymbol{a}_k, \boldsymbol{b}_k$, and S_k are the incoming amplitudes, outgoing amplitudes, and scattering matrices of the kth related systems, respectively.

A natural question is, what are the relations between S and S_k?

We answer this question using the fundamental relations in Sec. III B.

1. Original and transposed systems

Since there are no sources within V, Eq. (42) becomes

$$\oint_{\partial V} (\boldsymbol{E} \times \boldsymbol{H_1} - \boldsymbol{E_1} \times \boldsymbol{H}) \cdot d\boldsymbol{S} = 0. \tag{59}$$

We express the fields at the surface in terms of the incoming and outgoing amplitudes using the orthonormal basis of the physical ports, then perform the integration over the cross sections of each port. Using mode orthonormality [Eq. (52)], Eq. (59) becomes

$$\sum_{i=1}^{P} [(a_i + b_i)(a_{1,i} - b_{1,i}) - (a_{1,i} + b_{1,i})(a_i - b_i)] = 0, \tag{60}$$

which can be simplified as

$$\sum_{i=1}^{P} (b_i a_{1,i} - a_i b_{1,i}) = \boldsymbol{b}^T \boldsymbol{a}_1 - \boldsymbol{a}^T \boldsymbol{b}_1 = 0. \tag{61}$$

Substituting Eqs. (57) and (58) into Eq. (61), we obtain

$$\boldsymbol{a}^T(S^T - S_1)\boldsymbol{a}_1 = 0. \tag{62}$$

Since Eq. (62) holds for any $\boldsymbol{a}$ and $\boldsymbol{a}_1$, it requires

$$S_1 = S^T. \tag{63}$$

This is the relation between S_1 and S.

2. Original and c-transposed systems

Since there are no sources within V, Eq. (48) becomes

$$\oint_{\partial V}(\boldsymbol{E}\times\boldsymbol{H}_2^* + \boldsymbol{E}_2^*\times\boldsymbol{H})\cdot d\boldsymbol{S} = 0. \tag{64}$$

After the same expansion and integration procedure, Eq. (64) becomes

$$\sum_{i=1}^{P}[(a_i+b_i)(a_{2,i}-b_{2,i})^* + (a_{2,i}+b_{2,i})^*(a_i-b_i)] = 0, \tag{65}$$

which can be simplified as

$$\sum_{i=1}^{P}(a_{2,i}^*a_i - b_{2,i}^*b_i) = \boldsymbol{a}_2^\dagger\boldsymbol{a} - \boldsymbol{b}_2^\dagger\boldsymbol{b} = 0. \tag{66}$$

Substituting Eqs. (57) and (58) into Eq. (66), we obtain

$$\boldsymbol{a}_2^\dagger(I - S_2^\dagger S)\boldsymbol{a} = 0. \tag{67}$$

Since Eq. (67) holds for any $\boldsymbol{a}$ and $\boldsymbol{a}_2$, it requires

$$S_2 = (S^\dagger)^{-1}. \tag{68}$$

This is the relation between S_2 and S.

3. Original and conjugated systems

Since there are no sources within V, Eq. (50) becomes

$$M\Phi = 0 \iff M^*\Phi^* = 0, \tag{69}$$

which states that if $(\boldsymbol{E}, \boldsymbol{H})$ is a solution of an original system then $(\boldsymbol{E}_3, \boldsymbol{H}_3) = (\boldsymbol{E}^*, -\boldsymbol{H}^*)$ is a solution of the conjugated system. In particular, this holds for the fields at the surface ∂V. We express the fields at the surface in terms of the incoming and outgoing amplitudes using the orthonormal basis, and then the above condition becomes

$$a_{3,i} + b_{3,i} = (a_i + b_i)^*, \tag{70}$$

$$a_{3,i} - b_{3,i} = -(a_i - b_i)^*, \tag{71}$$

which can be simplified as

$$a_{3,i} = b_i^*, \qquad b_{3,i} = a_i^*. \tag{72}$$

In vector form,

$$\boldsymbol{a}_3 = \boldsymbol{b}^*, \qquad \boldsymbol{b}_3 = \boldsymbol{a}^*. \tag{73}$$

Substituting Eq. (73) into Eq. (58), we obtain

$$\boldsymbol{a}^* = S_3\boldsymbol{b}^*. \tag{74}$$

Substituting Eq. (57) into Eq. (74), we get

$$\boldsymbol{a}^* = S_3S^*\boldsymbol{a}^*. \tag{75}$$

Since Eq. (75) holds for any $\boldsymbol{a}$, it requires

$$S_3 = (S^*)^{-1}. \tag{76}$$

This is the relation between S_3 and S.

In summary, the transformations associated with reciprocity, energy conservation, and time reversal naturally yield the relations of the scattering matrices of mutually transposed, c-transposed, and conjugated systems, respectively. When the system is invariant under any of these internal transformations, these general relations become the corresponding symmetry constraints, as discussed in standard textbooks such as Ref. [23].

V. GROUP THEORY OF FUNDAMENTAL INTERNAL SYMMETRIES

So far, we discuss three fundamental internal transformations of reciprocity, energy conservation, and time reversal, which correspond to the transpose, c transpose, and conjugate of the matrix differential operator M, respectively. We define three fundamental internal symmetries of reciprocity, energy conservation, and time reversal if the system is invariant under the corresponding internal transformations.

We now show that the identity and the three internal transformations form the Klein four-group V_4. Consider the set of transformations $G = \{1, 1^T, 1^\dagger, 1^*\}$, which, respectively, denote the transformation of identity $C(\omega, \boldsymbol{r}) \to C(\omega, \boldsymbol{r})$, of reciprocity $C(\omega, \boldsymbol{r}) \to C_1(\omega, \boldsymbol{r})$, of energy conservation $C(\omega, \boldsymbol{r}) \to C_2(\omega, \boldsymbol{r})$, and of time reversal $C(\omega, \boldsymbol{r}) \to C_3(\omega, \boldsymbol{r})$. We emphasize that these operators act on the constitutive matrix field $C(\omega, \boldsymbol{r})$; they should be distinguished from operators in quantum mechanics, which act on Hilbert space of states. In our notation, 1 indicates that these purely internal transformations involve no external transformations; T, $\dagger$, and $*$ highlight the fact that the transformations of reciprocity, energy conservation, and time reversal correspond to the mathematical operations of transpose, c transpose, and conjugate, respectively.

One can easily check that the set G forms a group under the usual composition of transformations $\circ$. For example, $1^T \circ 1^\dagger = 1^*$. Moreover, the elements of 1^T, $1^\dagger$, and 1^* are of order 2, e.g., $(1^T)^2 = 1$, meaning that they are all involutory. Therefore, G is the Klein four-group, $G = V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$. This group is Abelian and can be generated by any of the two elements of order 2. The group V_4 provides a unified view of the relations among the three fundamental internal transformations and symmetries.

We visualize the V_4 group in two ways. Figure 2(a) shows its multiplication table. Figure 2(b) shows its Cayley diagram. Recall that a Cayley diagram is a directed graph [26]. Each node of the graph represents one element of the group, and each type of directed edge represents a generator. The direction of each edge is indicated by an arrowhead, which can be omitted if the edge is bidirected. In Fig. 2(b), there are

(a)

1	1^T	$1^\dagger$	1^*
1^T	1	1^*	$1^\dagger$
$1^\dagger$	1^*	1	1^T
1^*	$1^\dagger$	1^T	1

(b)

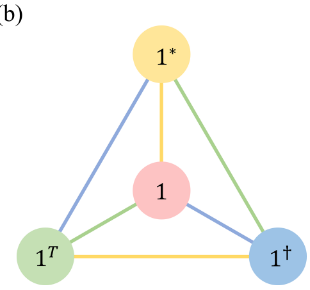

FIG. 2. (a) Multiplication table of $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$. (b) Cayley diagram of $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$. The green, blue, and yellow edges represent the action of 1^T, $1^\dagger$, and 1^*, respectively. (In grayscale, the green, blue, and yellow edges appear gray, dark gray, and light gray, respectively.) The edges are undirected since each action is involutory.

four nodes, each representing one of the four elements of the V_4 group; they are colored red for 1, green for 1^T, blue for $1^\dagger$, and yellow for 1^*. [These colors agree with those in Fig. 2(a).] The colored edges represent multiplying by the corresponding elements. For instance, the two green edges show that multiplying by 1^T maps 1 to 1^T and vice versa, and $1^\dagger$ to 1^* and vice versa.

The main results of our paper in Secs. II–IV can be succinctly summarized by annotating the Cayley diagram of V_4, as shown in Fig. 3. In this annotated Cayley diagram, the red, green, blue, and yellow nodes denote the original, transposed, *c*-transposed, and conjugated systems, respectively. In each node, we denote the matrix differential operator with M, constitutive matrix with C, and scattering matrix with S. The colored edges represent the transformations that connect two adjacent systems. The green, blue, and yellow edges denote 1^T, $1^\dagger$, and 1^*, respectively. This diagram provides a unified view of the three fundamental symmetries: Lorentz reciprocity, energy conservation, and time reversal.

Describing the internal symmetry in terms of V_4 allows us to classify any photonic system in terms of its internal

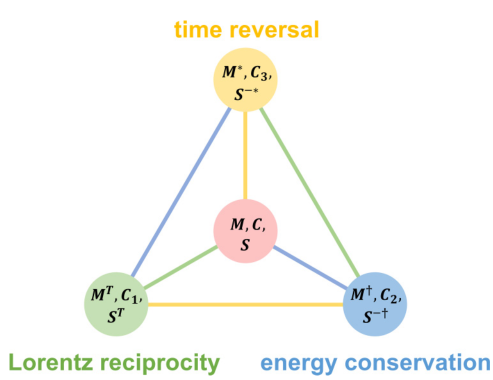

FIG. 3. Summary of the main results. The red, green, blue, and yellow nodes denote the original, transposed, *c*-transposed, and conjugated systems, respectively. The colored edges represent the transformations that connect two adjacent systems. The green, blue, and yellow edges denote 1^T, $1^\dagger$, and 1^*, respectively. (In grayscale, the green, blue, and yellow edges appear gray, dark gray, and light gray, respectively.)

symmetry. Any linear photonic system belongs to one and only one of the five subgroups of V_4.

(0) $H_0 = \{1\}$: no symmetry.
(1) $H_1 = \{1, 1^T\}$: reciprocal only.
(2) $H_2 = \{1, 1^\dagger\}$: energy conserving (lossless) only.
(3) $H_3 = \{1, 1^*\}$: time-reversal symmetric only.
(4) $H_4 = V_4$: reciprocal, energy-conserving, and time-reversal symmetric.

Here we provide examples of each class. H_0 includes lossy gyrotropic media such as yttrium iron garnets [27] or magnetic Weyl semimetals [28–30]. H_1 includes lossy nongyrotropic media. H_2 includes lossless gyrotropic media. A proposal of metamaterials belonging to H_3 can be found in Ref. [31]. H_4 includes vacuum.

VI. FINAL REMARKS AND CONCLUSION

In conclusion, we have provided a unified theory of the three global constraints—Lorentz reciprocity, energy conservation, and time-reversal symmetry—from the perspective of internal symmetry. We define the operators of transformations associated with each of these constraints, referred to as internal transformations. These internal transformations correspond to the mathematical operations of matrix transpose, conjugate transpose, and conjugate, respectively. We point out that the three global constraints naturally follow from three fundamental identities of linear systems under the three matrix operations. We discuss the properties of electromagnetic fields and scattering matrices associated with these internal transformations. We show that these internal transformations form the Klein four-group $V_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$, and the internal symmetry group of any photonic structure corresponds to one of the five subgroups of V_4.

Our paper provides a theoretical foundation for further exploration of symmetries in photonic systems. Our group theory can be readily extended to include other transformations and symmetries. In later works, we will use this theoretical framework to study additional internal transformations and symmetries. We will also study *external* and *compound* transformations and symmetries. We will summarize all these results into a larger group theory, where V_4 is only a subgroup. These results can be used to systematically study all symmetry properties of photonic structures.

Symmetries impose direct constraints on many physical properties of photonic systems. Classification of photonic systems by symmetries allows us to specify the fundamental constraints for each class systematically. Such an investigation has led to some fundamental results such as the adjoint Kirchhoff's law for all thermal emitters [32], and symmetry constraints on many-body radiative heat transfer [33]. More investigations are expected in the future.

ACKNOWLEDGMENT

This work is supported by the Simons Foundation.

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