Optimal quantum annealing: A variational shortcut-to-adiabaticity approach

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Suppressing unwanted transitions out of the instantaneous ground state is a major challenge in unitary adiabatic quantum computation. A recent approach consists in building counterdiabatic potentials approximated using variational strategies. In this contribution, we extend this variational approach to Lindbladian dynamics, having as a goal the suppression of diabatic transitions between pairs of Jordan blocks in quantum annealing. We show that, surprisingly, unitary counterdiabatic *Ansätze* are successful for dissipative dynamics as well, allowing for easier experimental implementations compared to Lindbladian *Ansätze* involving dissipation. Our approach not only guarantees improvements of open-system adiabaticity but also enhances the success probability of quantum annealing.

I. INTRODUCTION

Quantum control of noisy-intermediate scale quantum (NISQ) processors is an invaluable tool for the design of the ideal time dependence required to realize accurate quantum state transformations [1–7]. In quantum many-body systems, new methods of control have been developed [8–10], with applications ranging from state preparation [11] to optimization of hybrid quantum-classical algorithms [12–17].

Among the different approaches to quantum control, shortcuts to adiabaticity (STA) have gained increasing attention [18–24]. Given a time-dependent Hamiltonian H_0, the goal of STA is to design the evolution of the state in order to keep the system in one of the instantaneous eigenstates of H_0. A quantum system prepared in an eigenstate of its Hamiltonian at a time $t = 0$ will remain in the corresponding time-evolved eigenstate for all times even if the conditions for adiabaticity are violated [21,25]. The idea underlying STA is to compensate diabatic transitions between instantaneous energy eigenstates by suppressing nondiagonal terms of the Hamiltonian in the energy eigenbasis using a counterdiabatic potential.

While in few-body systems the success of STA has been unquestionable, its application to many-body dynamics has been initially hindered by the fact that exact counterdiabatic operators [25,26] are exceedingly difficult to realize experimentally, as they often involve nonlocal infinite-range interactions. In addition, they can only be computed if the full Hamiltonian spectrum is known, a requirement clearly impossible to satisfy in the many-body case. This issue has been solved in a breakthrough contribution [27] by using a variational approach to build approximate counterdiabatic operators. In [27–30], this approximate method has been applied successfully to many-body systems.

Applications to real-life problems require generalizing and implementing the variational approach to open quantum systems. This would be of great importance in the optimization of NISQ protocols in many-body quantum systems. A key example of this sort is quantum annealing (QA) [31–41], where the aim is to keep the system close to its ground state for the entire dynamics up to the annealing time τ. This procedure will eventually lead to the solution of an NP-hard problem [42,43]. When dissipation is present, the adiabatic timescale $\hbar/\Delta$, where Δ is the minimum spectral gap [44], has to be compared with the typical relaxation and decoherence timescales [45]. As a result, the adiabatic theorem is not sufficient to predict the success probability of QA. The challenge here is to find a strategy to minimize the effect of environment *and*, at the same time, avoid transitions to excited states.

In many relevant situations the dynamical evolution is governed by a Lindblad master equation. Quantum annealing is then realized by interpolating a starting and a target Lindbladian, whose zero-temperature instantaneous steady state (ISS) encodes the solution to the problem at hand. Diabatic and thermal transitions outside of the ISS manifold have to be minimized so as to realize high-fidelity quantum computation. Many attempts have been proposed in recent years [46–51] for related questions in unitary evolution [36,52–54].

An important leap forward in the field entails exploiting STA to advantageously design nonadiabatic protocols in the presence of dissipation. STA in open quantum systems have been studied in Refs. [55,56]. In this work, we focus on the quantum many-body case generalizing the variational approach of Ref. [27] to open systems. In particular, we derive a variational approach for Lindbladian dynamics and apply it to quantum annealing. This allows for the optimization of quantum driving in real-life scenarios by building approximate Lindbladian counterdiabatic (CD) operators which

satisfy locality constraints, in order to match with the actual experimental capabilities. Remarkably, in many relevant cases, it is sufficient to control the unitary part of the counterdiabatic terms. These are much more easily engineered experimentally and already realize good approximations of the exact counterdiabatic superoperator. We show that our control on the Hamiltonian considerably increases the ground state fidelity, therefore boosting the performance of quantum annealing.

This paper is organized as follows. In Sec. II, we discuss exact CD driving in Lindbladian dynamics adopting a superoperator representation [57]. In Sec. III, we present our variational approach of the search for the open-system CD superoperator. Our approach resembles the unitary case of Ref. [27]; however, there are important differences due to the fact that the generators of the dynamics are not Hermitian. In Sec. IV, firstly we validate our analysis by applying our formalism to a single qubit in interaction with an Ohmic environment, and, secondly, we study the ferromagnetic p-spin model with $p = 3$ as a paradigmatic example of a quantum annealing protocol of a many-body system. We draw our conclusions in Sec. V.

II. TRANSITIONLESS LINDBLADIAN DYNAMICS

Let us consider a quantum state $|\psi(t)\rangle$ evolving according to the Schrödinger equation with a time-dependent Hamiltonian $H_0(t)$. At any time, the state $|\psi(t)\rangle$ can be decomposed as $|\psi(t)\rangle = \sum_\alpha c_\alpha(t)|d_\alpha(t)\rangle$ where $|d_\alpha(t)\rangle$ are instantaneous eigenvectors of $H_0(t)$. The adiabatic regime is realized for long timescales $\tau \gg \max\{\hbar/[\epsilon_\alpha(t) - \epsilon_\beta(t)]\}$, where $\epsilon_\alpha(t)$ are the instantaneous eigenvalues of $H_0(t)$. In this limit, each coefficient $c_\alpha(t)$ evolves independently of the others. As shown in Refs. [25,26] the same results can be achieved for any finite time τ, provided the generator of the dynamics is $H(t) = H_0(t) + H_{\rm cd}(t)$, where

$$H_{\rm cd}(t) = i\hbar \sum_{\alpha \neq \beta} \frac{\langle d_\beta(t)|\dot{H}_0(t)|d_\alpha(t)\rangle}{\epsilon_\alpha(t) - \epsilon_\beta(t)} |d_\beta(t)\rangle\langle d_\alpha(t)| \quad (1)$$

is the so-called CD potential.

In all practical scenarios where dissipation is present, the evolution of a quantum system is not unitary but can instead be modeled by a Lindblad equation for the reduced density operator $\rho(t)$,

$$\dot{\rho}(t) = \mathcal{L}(t)[\rho(t)], \quad (2)$$

where $\mathcal{L}(t)[\bullet]$ is a (time-dependent) completely positive tracepreserving (CPTP) map expressed in the Lindblad form

$$\mathcal{L}(t)[\rho] = -i[H(t), \rho] + \sum_k \gamma_k(t)\left(L_k \rho L_k^\dagger - \frac{1}{2}\{L_k^\dagger L_k, \rho\}\right), \quad (3)$$

where $H(t)$ is the Hermitian generator of the unitary part of the evolution and L_k are Lindblad operators with rates $\gamma_k(t) \geqslant 0$ for all t.

In the superoperator representation [58], first a basis $\{\sigma_i\}_{i=0}^{D^2-1}$ of Hermitian and traceless operators (including the identity $i = 0$) is chosen, where D is the Hilbert space dimension. This defines the Hilbert-Schmidt space, whose vectors

$\sigma_i \leftrightarrow |\sigma_i\rangle\rangle$ are orthonormal with respect to the scalar product $(\sigma_i, \sigma_j) \equiv \langle\langle\sigma_i|\sigma_j\rangle\rangle = \mathrm{Tr}(\sigma_i\sigma_j)/D$. In this basis, density matrices are represented as D^2-dimensional coherent vectors $|\rho\rangle\rangle = \sum_i r_i|\sigma_i\rangle\rangle$ with coefficients $r_i = \langle\langle\sigma_i|\rho\rangle\rangle$ and the Lindblad equation takes the form

$$|\dot{\rho}\rangle\rangle = \mathbb{L}_0(t)|\rho\rangle\rangle, \quad (4)$$

where $\mathbb{L}_0(t)$ is the Lindbladian supermatrix having components $[\mathbb{L}_0]_{jk} = \langle\langle\sigma_j|\mathcal{L}(t)[\sigma_k]\rangle\rangle$. In this paper, bold symbols are used to indicate supermatrices and double kets (or bras) are reserved for coherence vectors, whereas calligraphic letters indicate the action of superoperators in the original Hilbert space.

While formally identical to the Liouville equation, Eq. (4) generates a nonunitary evolution since, in general, $\mathbb{L}_0(t)$ is not anti-Hermitian and nondiagonalizable. However, it can always be brought to the Jordan canonical form (JF), which is unique up to permutations [59]. In this form, $\mathbb{L}_0(t)$ assumes a block-diagonal structure $\mathbb{L}_J(t) = \mathbb{V}^{-1}(t)\mathbb{L}_0(t)\mathbb{V}(t) = \mathrm{diag}[J_0(t), \ldots, J_{N-1}(t)]$, where $\mathbb{V}(t)$ is a similarity matrix and each Jordan block (JB) $J_\alpha(t)$ is given by $[J_\alpha(t)]_{ij} = \lambda_\alpha(t)\delta_{ij} + \delta_{i,j+1}$. Each JB is associated with different noncrossing time-dependent (complex) eigenvalues of $\mathbb{L}_0(t)$, denoted $\lambda_\alpha(t)$. Each eigenvalue has algebraic multiplicity N_α and geometric multiplicity equal to 1. If the Lindbladian has exactly D^2 eigenvectors, each JB is a one-dimensional matrix and $\mathbb{L}_J(t)$ is diagonal. This is the one-dimensional Jordan form (1DJF). In this case, the Lindblad supermatrix is diagonalizable with complex eigenvalues.

Given the above, $\mathbb{L}_0(t)$ does not generally yield a basis of eigenvectors. However, we can define a basis of right and left quasieigenvectors that solve the following problems:

$$\mathbb{L}_0(t)|\mathcal{D}_\alpha^{n_\alpha}\rangle\rangle = |\mathcal{D}_\alpha^{n_\alpha-1}\rangle\rangle + \lambda_\alpha(t)|\mathcal{D}_\alpha^{n_\alpha}\rangle\rangle, \quad (5)$$

$$\langle\langle\mathcal{E}_\alpha^{n_\alpha}|\mathbb{L}_0(t) = \langle\langle\mathcal{E}_\alpha^{n_\alpha+1}| + \lambda_\alpha(t)\langle\langle\mathcal{E}_\alpha^{n_\alpha}| \quad (6)$$

with $n_\alpha \in \{1, \ldots, N_\alpha\}$ ($|\mathcal{D}_\alpha^0\rangle\rangle$ and $\langle\langle\mathcal{E}_\alpha^{N_\alpha+1}|$ are null vectors). N_α is the algebraic multiplicity of the eigenvalue λ_α. These states are doubly orthogonal ($\langle\langle\mathcal{E}_\beta^n|\mathcal{D}_\alpha^m\rangle\rangle = \delta_{mn}\delta_{\alpha\beta}$) and decompose the identity as $\mathbb{I} = \sum_{\alpha=0}^{N-1}\sum_{n_\alpha=1}^{N_\alpha}|\mathcal{D}_\alpha^{n_\alpha}\rangle\rangle\langle\langle\mathcal{E}_\alpha^{n_\alpha}|$. In the 1DJF, these quasieigenstates become exact eigenstates ($N_\alpha = 1$).

The dynamics of an open quantum system is said to be adiabatic when the evolution of the density operator in its Hilbert-Schmidt space can be decomposed into decoupled Jordan subspaces associated with distinct, time-dependent, noncrossing eigenvalues of the Lindbladian supermatrix $\mathbb{L}_0(t)$ [58,60]. For any finite evolution time τ, it is possible to define transitionless dynamics generated by $\mathbb{L}(t) = \mathbb{L}_0(t) + \mathbb{L}_{\rm cd}(t)$ introducing a counterdiabatic superoperator $\mathbb{L}_{\rm cd}(t)$ that ensures Jordan blocks are not mixed. The detailed derivation of $\mathbb{L}_{\rm cd}(t)$ can be found in Refs. [55,58]. For 1DJF, it reads

$$\mathbb{L}_{\rm cd}(t) = \sum_{\alpha \neq \beta} \frac{\langle\langle\mathcal{E}_\beta(t)|\dot{\mathbb{L}}_0(t)|\mathcal{D}_\alpha(t)\rangle\rangle}{\lambda_\alpha(t) - \lambda_\beta(t)} |\mathcal{D}_\beta(t)\rangle\rangle\langle\langle\mathcal{E}_\alpha(t)|, \quad (7)$$

which generalizes the unitary CD operator [25,26] to the 1DJF open case with $N_\alpha = 1$, where $\langle\langle\mathcal{E}_\alpha(t)|$ and $|\mathcal{D}_\alpha(t)\rangle\rangle$ are the left and right eigenvectors of $\mathbb{L}_0(t)$ with eigenvalue $\lambda_\alpha(t)$,

respectively [55]. This formula is derived under the assumption that the Lindbladian spectrum is nondegenerate so that the denominators are nonzero. The explicit form of the CD Lindbladian in the case in which the JF is not 1D can be found in Ref. [58].

Evaluating the open-system CD superoperator would require the knowledge of the whole Lindbladian spectrum, a requirement that it is impossible to fulfill in many-body systems.

III. VARIATIONAL FORMULATION

First of all, we quickly review the variational formulation of the search for approximate CD operators in closed quantum systems. If the system Hamiltonian depends on time via a parameter $s = s(t)$, then the CD potential of Eq. (1) can be equivalently [27] written as $H_{cd}(t) = \dot{s}(t)A_s$, and the gauge potential A_s satisfies

$$[i\hbar H_0'(s) - [A_s, H_0(s)], H_0(s)] = 0, \qquad (8)$$

where the prime denotes derivative with respect to s. Solving this equation is equivalent to minimizing the functional $S(A_s^*) = \mathrm{Tr}[G_s^2(A_s^*)]$ with respect to A_s^*, where $G_s(A_s^*) = H_0'(s) + i[A_s^*, H_0(s)]/\hbar$. At this point one could express A_s^* as a combination of local operators and minimize the functional $S(A_s^*)$ to achieve an approximation of the CD potential. This variational approach to the CD driving, in the unitary case, has been successfully applied, achieving high-fidelity transitionless quantum driving, in Refs. [27–30].

We here present a variational approach to approximate $\mathbb{L}_{cd}$ without knowing the Lindbladian spectrum, generalizing Ref. [27] to open quantum systems. Assuming the Lindbladian depends on time via the control field $s = s(t)$, we can rewrite Eq. (7) as $\mathbb{L}_{cd}(t) = \dot{s}\mathbb{A}_s$, where $\mathbb{A}_s$ is the counterdiabatic gauge supermatrix. To formulate the search for $\mathbb{A}_s$ on a variational basis, we start from the Lindblad equation and rotate to the Jordan representation using the similarity superoperator $\mathbb{V}(t)$. The system density matrix in this basis reads $|\tilde{\rho}(t)\rangle\rangle = \mathbb{V}^{-1}(t)|\rho(t)\rangle\rangle$ and satisfies the adiabatic-frame Lindblad equation

$$\partial_t |\tilde{\rho}(t)\rangle\rangle = (\mathbb{L}_J - \tilde{\mathbb{L}}_{cd} - \tilde{\mathbb{L}}_d)|\tilde{\rho}(t)\rangle\rangle, \qquad (9)$$

where $\mathbb{L}_J(t) = \mathbb{V}^{-1}\mathbb{L}_0(t)\mathbb{V} = \mathrm{diag}[J_0, (t), \dots, J_{N-1}(t)]$ and J_α is the Jordan block associated with the eigenvalue λ_α [61], and $\tilde{\mathbb{L}}_{cd}(t) + \tilde{\mathbb{L}}_d(t) = -\partial_t(\mathbb{V}^{-1})\mathbb{V} = \mathbb{V}^{-1}\partial_t\mathbb{V}$. We have defined

$$\tilde{\mathbb{L}}_{cd}(t) = \sum_{\alpha \neq \beta} \sum_{n_\alpha n_\beta} |\mathcal{D}_\alpha^{n_\alpha}\rangle\rangle\langle\langle\mathcal{E}_\alpha^{n_\alpha}|\partial_t \mathcal{D}_\beta^{n_\beta}\rangle\rangle\langle\langle\mathcal{E}_\beta^{n_\beta}|, \qquad (10)$$

$$\tilde{\mathbb{L}}_d(t) = \sum_{\alpha} \sum_{n_\alpha n_\beta} |\mathcal{D}_\alpha^{n_\alpha}\rangle\rangle\langle\langle\mathcal{E}_\alpha^{n_\alpha}|\partial_t \mathcal{D}_\alpha^{n_\beta}\rangle\rangle\langle\langle\mathcal{E}_\alpha^{n_\beta}|. \qquad (11)$$

Note that the superoperator $\tilde{\mathbb{L}}_d(t)$ is only diagonal in the Jordan indices but not necessarily within each Jordan block.

It is easy to prove that the CD superoperator satisfies

$$\partial_t \mathbb{L}_0 + \mathbb{F}_{ad}(t) = [\mathbb{L}_{cd}(t) + \mathbb{L}_d(t), \mathbb{L}_0(t)], \qquad (12)$$

where quantities without the tilde are in the time-dependent basis and

$$\mathbb{F}_{ad}(t) = -\mathbb{V}\partial_t\mathbb{L}_J(t)\mathbb{V}^{-1} = -\sum_{\alpha n_\alpha} \partial_t \lambda_\alpha |\mathcal{D}_\alpha^{n_\alpha}\rangle\rangle\langle\langle\mathcal{E}_\alpha^{n_\alpha}|. \qquad (13)$$

This equation is equivalent to

$$[\mathbb{L}_0'(s) - [\mathbb{A}_s + \mathbb{A}_d, \mathbb{L}_0(s)], \mathbb{L}_0(s)] = 0, \qquad (14)$$

where $\mathbb{A}_d$ is defined via the equation $\mathbb{L}_d = \dot{s}\mathbb{A}_d$. Equation (14) is formally equivalent to the unitary case of Ref. [27]. However, the fact that the Jordan basis is not unitarily equivalent to the time-independent basis $\{\sigma_i\}$ poses a problem. Inspired by the unitary case, we can define a superoperator $\mathbb{G}_s(\mathbb{A}_{tot}^*) = \mathbb{L}_0'(s) - [\mathbb{A}_{tot}^*, \mathbb{L}_0(s)]$ with $\mathbb{A}_{tot}^* = \mathbb{A}_s^* + \mathbb{A}_d^*$ and notice that its diagonal elements in the Jordan basis do not depend on $\mathbb{A}_{tot}^*$ while its off-diagonal elements are zero when $\mathbb{A}_{tot}^* = \mathbb{A}_{tot} = \mathbb{A}_s + \mathbb{A}_d$. Thus, the Hilbert-Schmidt norm of $\mathbb{G}_s$ in the Lindbladian basis is minimized when $\mathbb{A}_{tot}^* = \mathbb{A}_{tot}$. However, this approach does not provide any insight as the trace in $\mathbb{S} = \mathrm{Tr}(\mathbb{G}_s^\dagger \mathbb{G}_s)$ has to be evaluated in the Jordan eigenbasis, requiring the full diagonalization of the Lindbladian.

Hence, we work directly with Eq. (14). An approximate solution to Eq. (14) might lead to a total generator of the system dynamics $\mathbb{L}(s) = \mathbb{L}_0(s) + \dot{s}\mathbb{A}_s$ that does not yield CPTP dynamics. This problem can be circumvented by simply restricting the variational minimization to a subspace of the supermatrix space that contains only physically valid superoperators. In particular, the Kraus representation theorem [62] and the works by Gorini *et al.* [63] and Lindblad [64] ensure that for finite-size or separable Hilbert spaces the most general form of a CPTP map is given by

$$\mathcal{A}_{tot}^*[\bullet] = -\frac{i}{\hbar}[A_s^*, \bullet]$$
$$+ \sum_i \gamma_i^2 \left(\Gamma_i \bullet \Gamma_i^\dagger - \frac{1}{2}\{\Gamma_i^\dagger \Gamma_i, \bullet\} \right), \qquad (15)$$

where $(A^*)^\dagger = A_s$ and Γ_i are Lindblad operators. By postulating this form, we can expand the superoperator $\mathbb{A}_{tot}^*$ as

$$\mathbb{A}_{tot}^* = \sum_i \epsilon_i \mathbb{A}_i^{uni} + \sum_i \gamma_i^2 \mathbb{A}_i^{diss} \equiv \sum_j \alpha_j \mathbb{A}_j, \qquad (16)$$

where we separated the unitary and dissipative contributions to $\mathbb{A}_{tot}^*$ so as to impose the positivity of the rates in the Lindblad form of Eq. (15). The map of Eq. (15) would require bath engineering, which might be daunting in practice. However, as proven in Ref. [55], there exist cases (e.g., when the system purity is unaffected by the dynamics) where the unitary part of this map is sufficient to express the exact CD superoperator of the Lindbladian dynamics, providing an important simplification. Furthermore, in Appendix B, we show that the closed-system variational formulation can be obtained as a limiting case of our more general approach when all maps are unitary.

Equation (14) can then be recast into $\vec{\mathbb{B}} \cdot \vec{X} = \mathbb{Y}$, where $X_i = \alpha_i$, $\mathbb{B}_i = [[\mathbb{A}_i, \mathbb{L}_0], \mathbb{L}_0]$, and $\mathbb{Y} = [\mathbb{L}_0', \mathbb{L}_0]$. Therefore, the search for α_i can be reformulated as a minimization problem of the form

$$\vec{X} = \arg\min_{\vec{X}^*} \| \vec{\mathbb{B}} \cdot \vec{X}^* - \mathbb{Y} \|^2, \qquad (17)$$

where the norm is defined as $\|\mathbb{Q}\| \equiv \mathrm{Tr}(\mathbb{Q}^\dagger \mathbb{Q})$. The outcome of this minimization does not depend on the choice of the starting state.

Lindbladians having a 1DJF are more likely to appear when modeling dissipative systems using weak-coupling master equations, therefore in the following we will focus on this case, where the superoperator $\mathbb{A}_d$ commutes with $\mathbb{L}_0(s)$ and Eq. (14) simplifies to

$$[\mathbb{L}'_0(s) - [\mathbb{A}_s, \mathbb{L}_0(s)], \mathbb{L}_0(s)] = 0. \tag{18}$$

First, we validate our method by applying it to a single qubit in interaction with an Ohmic environment. Then, we will focus on the ferromagnetic p-spin model, a prototypical many-body system that will showcase the power of our method to improve quantum annealing of complex systems.

IV. RESULTS

A. Single qubit in an Ohmic environment

We consider a single qubit in interaction with a thermal Ohmic bath. The qubit Hamiltonian reads

$$H_0(s) = -[1 - q(s)](\omega_x/2)\sigma_x - q(s)(\omega_z/2)\sigma_z, \tag{19}$$

where $\omega_x = \omega_z = 1$ GHz is the energy scale (in units in which $\hbar = 1$) and $q(s) = 6\,s^5 - 15\,s^4 + 10\,s^3$ with $s = t/\tau$. The minimum gap is $\Delta = 1/\sqrt{2}$ at $s = q = 1/2$.

The gauge potential in the absence of system-bath coupling can be found analytically using Eq. (18) and reads

$$A_s = -q'(s)/[1 - 2q(s) + 2q^2(s)]\,\sigma_y. \tag{20}$$

Let us now assume that the system is weakly coupled to a dissipative bath and can be described by a Lindblad master equation [45,65]. The system is coupled to the bath via $U = \sigma_z$, which induces dephasing, where the details of the system-bath coupling and of the Lindblad equation are reported in Appendix C. We consider a temperature of $T = 17$ mK $= 2.23$ GHz and a dimensionless qubit-bath coupling strength of $\eta g^2 = 1 \times 10^{-4}$. The weak-coupling Lindbladian has a zero eigenvalue ($\lambda_0 = 0$) for any given t, whose right eigenvector $|\mathcal{D}_0(t)\rangle\rangle$ is the ISS and corresponds to the thermal state $\exp[-\beta H(t)]/\mathrm{Tr}\{\exp[-\beta H(t)]\}$. The Lindbladian has a 1DJF for every choice of parameters, thus there are $D^2 = 4$ Jordan blocks whose eigenvalues are known explicitly [47].

We initialize the qubit in the ground state of the Hamiltonian at $s = 0$, i. e., the eigenstate of σ_x with eigenvalue $+1$. Then, we fix an annealing time τ and evolve the system according to the Lindblad equation. We calculate the ground state probability and also the projections of the time-evolved state onto the four Jordan blocks. When the unitary evolution is adiabatic, the populations of the energy eigenstates stay constant in time. This is a consequence of the fact that each eigenstate will only acquire a phase factor. As opposed to the unitary case, however, open-system adiabaticity does not imply constant projections onto the Jordan blocks, due to the fact that the eigenvalues of the Lindbladian are complex in general. In particular, only the projection onto the ISS will remain constant in time if the adiabatic condition is met, whereas the numerical values of the other blocks might

change. However, if some of the populations are zero at the beginning of the dynamics, they are bound to remain zero for the entire adiabatic dynamics and this is a measure of adiabaticity.

We consider three different values of τ, corresponding to three different regimes: $\tau = 1$, 10, and 100 ns. The value of $\tau = 1$ ns is the quench limit: in the absence of CD terms unitary and dissipative dynamics overlap, as the time is too short with respect to the typical timescales of the bath. For $\tau = 10$ ns, the fidelity in the unitary limit (calculated as the ground state probability at time τ) is close to 91%, and is similar in the presence of the environmental bath (90%) since the evolution time is again shorter than the relaxation timescale. In addition, the dissipative adiabatic criterion is violated, and thus we cannot follow each Jordan block adiabatically in the absence of a CD driving term. Indeed, it has been proven in Ref. [47] that, in order to follow the ISS with a maximum error in the norm of the evolved state of ϵ when $U = \sigma_z$, the annealing time must be chosen as $\tau \gtrsim C/\epsilon^3$ where C is a positive constant depending on the specific norm used. For $\tau = 100$ ns, the time evolution is almost adiabatic as shown in the following, and the CD plays a marginal role. Here, the annealing time is larger than the relaxation timescale and the ground state probability at the end of a dissipative evolution drops to 95%, as opposed to the unitary limit in which it is close to 100% up to numerical errors.

As a variational *Ansatz*, we take inspiration from the analytic unitary result of Eq. (20) and consider $\mathbb{A}_s \to \mathcal{A}_s^{\mathrm{test}}[\bullet] = -i[y\sigma_y, \bullet]$, where y is the variational parameter to be optimized minimizing Eq. (17).

In Fig. 1, we plot the populations of each JB $|\langle\langle \mathcal{E}_\alpha(s)|\rho(s)\rangle\rangle|$ as a function of the dimensionless time $s = t/\tau$. In the left-hand column, the qubit has been evolved by using $\mathbb{L}_0$ alone without including CD corrections. The center column shows the same quantities when the qubit has been evolved using $\mathbb{L}_0 + \dot{s}\mathbb{A}_s$ with the exact CD superoperator computed using Eq. (7). The right-hand column shows the populations when the qubit has been evolved using $\mathbb{L}_0 + \dot{s}\mathbb{A}_s^{\mathrm{test}}$. When there is no CD term, the populations of the last two Jordan blocks quickly grow as the two subspaces mix with the second block due to nonadiabatic transitions when τ violates the dissipative adiabatic condition. By contrast, these populations remain small when $\tau = 100$ ns. The exact CD potential decouples the dynamics of the last two JBs and the populations of these levels remain zero for all choices of τ. The population of the block with $\alpha = 1$ slightly decreases as a consequence of the fact that the (real) eigenvalue $\lambda_1(t)$ is small and negative. The variational CD superoperator successfully decouples the dynamics of the different blocks as well.

Adding the dissipative CD superoperator to the Lindbladian $\mathbb{L}_0$ increases the ground state probability as well. This is due to the fact that the target state is decomposed onto the first two Jordan blocks at $s = 1$ as well, thus the suppression of nonadiabatic transitions outside of these Jordan blocks allows one to reach the target state with more accuracy. This is shown in Fig. 2, where we plot the ground state probability $P_-(s) = \mathrm{Tr}[|d_0(s)\rangle \langle d_0(s)| \rho(s)]$ as a function of s, where $|d_0(s)\rangle$ is the instantaneous ground state of $H_0(s)$.
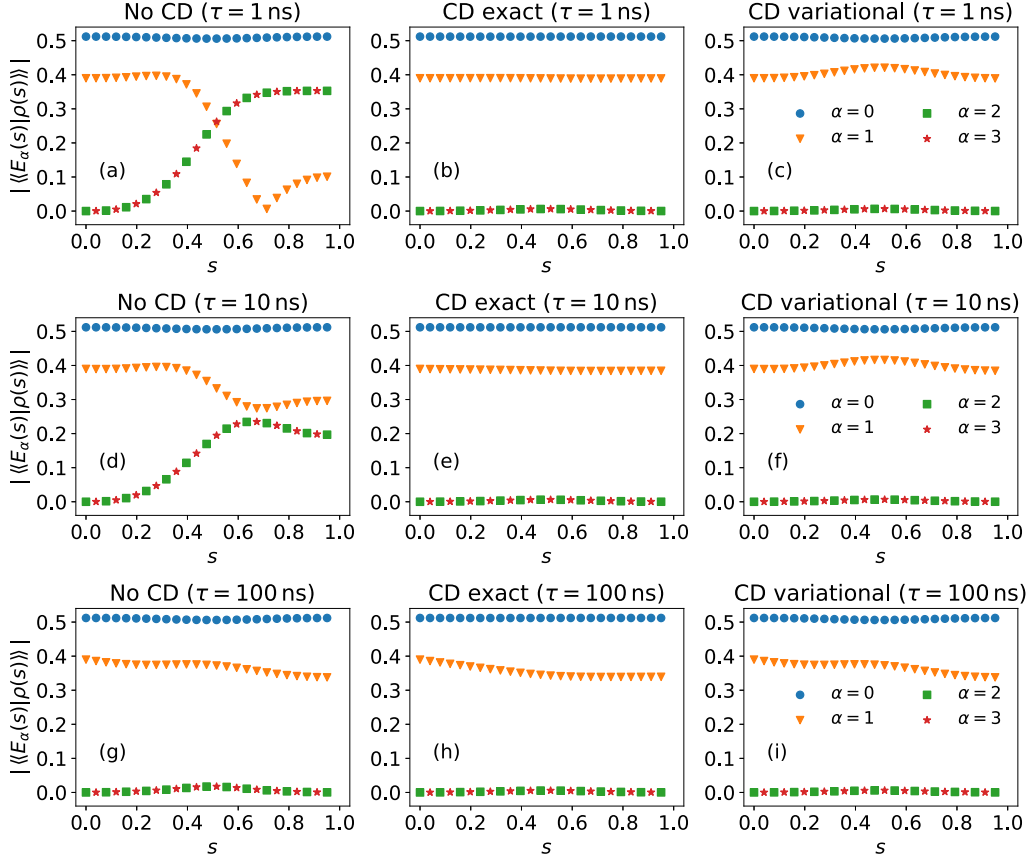
FIG. 1. Jordan blocks overlaps $|\langle\langle \mathcal{E}_\alpha(s)|\rho(s)\rangle\rangle|$ as a function of $s = t/\tau$, for a single qubit in interaction with an Ohmic environment. In panels (a), (d), and (g), we show results obtained without the CD superoperator. In panels (b), (e), and (h), we show results obtained with the exact CD superoperator of Eq. (7). In panels (c), (f), and (i), we show results obtained with the variational CD superoperator $\mathcal{A}_s[\bullet] = -i[y\sigma_y, \bullet]$, where y is the variational parameter found by minimizing Eq. (17). Panels (a)–(c): $\tau = 1$ ns; panels (d)–(f): $\tau = 10$ ns; panels (g)–(i): $\tau = 100$ ns.

B. Ferromagnetic *p*-spin model

Next, we consider the ferromagnetic *p*-spin model, whose Hamiltonian reads

$$H_p(s) = -\Gamma[1 - q(s)]\sum_{i=1}^{n}\sigma_i^x - \frac{J}{n^{p-1}}q(s)\left(\sum_{i=1}^{n}\sigma_i^z\right)^p \quad (21)$$

with $\Gamma = J = 1$ GHz and $q(s) = 6s^5 - 15s^4 + 10s^3$ with $s = t/\tau$. We consider $n = 3$ qubits with $p = 3$. The unitary dynamics of this system is easy to simulate due to the fact that the *p*-spin Hamiltonian commutes with the total

angular momentum $S^2 = S_x^2 + S_y^2 + S_z^2$ with $2S_\alpha = \sum_i \sigma_i^\alpha$ ($\alpha \in \{x, y, z\}$) at all times. In addition, the interesting states for QA, i. e., the paramagnetic ground state of S_x and the ferromagnetic ground state of S_z^p, both belong to the symmetry subspace corresponding to $S = n/2$ and $D = 2S + 1 = n + 1$, therefore numerical simulations of any unitary dynamics can be restricted to this D-dimensional space.

To preserve this symmetry, we consider a collective dephasing model where the whole system is collectively coupled to a single dephasing Ohmic bath via the total magnetization
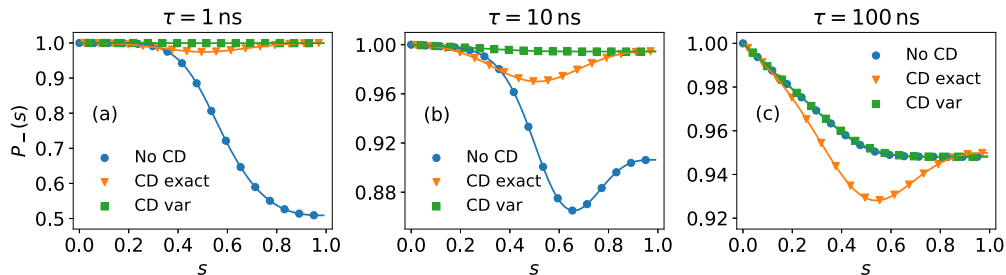

FIG. 2. Ground state probability as a function of $s = t/\tau$, for a single qubit in interaction with an Ohmic environment. Panel (a): $\tau = 1$ ns; panel (b): $\tau = 10$ ns; panel (c): $\tau = 100$ ns. Note the different scale on the vertical axis.

S_z. This form can appear experimentally when a qubit system is coupled to a long-wavelength mode of the bath, so that the qubit system is insensitive to spatial variations of the bath modes [37,38]. The coupling to the environment is modeled via the adiabatic master equation [65]. We consider a temperature of $T = 1/\beta = 17\,\text{mK} = 2.23\,\text{GHz}$ and dimensionless qubit-bath coupling strengths of $\eta g^2 = 1 \times 10^{-4}$ and 1×10^{-2}.

For $n = 3$, the Hilbert space dimension is $D = 4$. The 4×4 operator space is spanned by the basis of operators $\{\Sigma_i\}_{i=0}^{D^2-1}$, where Σ_0 is the identity and the remaining $D^2 - 1$ operators are Hermitian, traceless, and orthonormal. In particular, the basis operators are $(m > l)$ $(\Sigma_{2i-1})_{lm} = \sqrt{2}\delta_{l,m+1} + \sqrt{2}\delta_{l+1,m}$ and $(\Sigma_{2i})_{lm} = -i\sqrt{2}\delta_{l,m+1} + i\sqrt{2}\delta_{l+1,m}$ with $i = (m - l) + (l - 1)(2D - l)/2 \in \{1, 2, \ldots, 6\}$. The remaining three operators are $\Sigma_{13} = \text{diag}(1, 1, -1, -1)$, $\Sigma_{14} = \text{diag}(1, -1, 1, -1)$, and $\Sigma_{15} = \text{diag}(1, -1, -1, 1)$. The Lindbladian is diagonalizable and there are $D^2 = 16$ 1D Jordan blocks.

We prepare the p-spin system into the thermal state $|\mathcal{D}_0(0)\rangle\rangle$, i.e., the starting density matrix is $\rho(0) = \exp(2\beta\,\Gamma S_x)/\text{Tr}[\exp(2\beta\,\Gamma S_x)]$. For $\beta \to \infty$, this state corresponds to the ground state of the Hamiltonian in Eq. (21). At the beginning of the evolution, only the JB corresponding to the ISS steady state is populated. An adiabatic evolution will hence leave the system in the ISS of the Lindbladian at all times: $|\rho_{\text{ad}}(t)\rangle\rangle = |\mathcal{D}_0(t)\rangle\rangle$ for all t. In this setting, no transitions towards other Jordan blocks are allowed, since $\langle\langle\mathcal{E}_\alpha(t)|\rho_{\text{ad}}(t)\rangle\rangle \equiv \langle\langle\mathcal{E}_\alpha(t)|\mathcal{D}_0(t)\rangle\rangle = \delta_{\alpha,0}$. The density-matrix fidelity between the time-evolved state $|\rho(t)\rangle\rangle$ and the ISS will therefore provide a measure of adiabaticity. It reads

$$F(\rho_{\text{ad}}(s), \rho(s)) = \left(\text{Tr}\sqrt{\sqrt{\rho_{\text{ad}}(s)}\rho(s)\sqrt{\rho_{\text{ad}}(s)}}\right)^2. \quad (22)$$

We restrict ourselves to final times $\tau = 1$ and 10 ns [61].

In the unitary case the S_y operator breaks time-reversal invariance and is the zeroth-order term of a number of expansions such as the local ansatz [27], the nested commutators ansatz [28], or the cyclic ansatz [29]. When τ is very short, the environment does not have enough time to act and the dynamics are almost unitary, thus in this regime we expect S_y to be the most relevant part of the *Ansatz*. For longer evolutions, the environment kicks in and the dissipative part of the *Ansatz* might play a more important role in the suppression of diabatic transitions between pairs of Jordan blocks. In the following, we will show that this is indeed not the case and a unitary *Ansatz* for the CD superoperator is enough to decouple the system's Jordan blocks.

In order to highlight the different contributions to the variational CD operator, we here considered the following test Lindbladian:

$$\mathbb{A}_s^{\text{test}} \to \mathcal{A}_s^{\text{test}}[\bullet] = -i[b_1 S_y + b_2 S_y^3 + b_3(S_x S_y S_z + \text{H. c.}), \bullet]$$

$$+ \sum_{i=1}^{15} a_i^2 \left(\Sigma_i \bullet \Sigma_i - \frac{1}{2}\{\Sigma_i \Sigma_i, \bullet\}\right). \quad (23)$$

The first line of Eq. (23) describes the unitary part and is reminiscent of the cyclic *Ansatz* of Ref. [29], which is particularly successful in the unitary case of the p-spin model with $p = 3$.
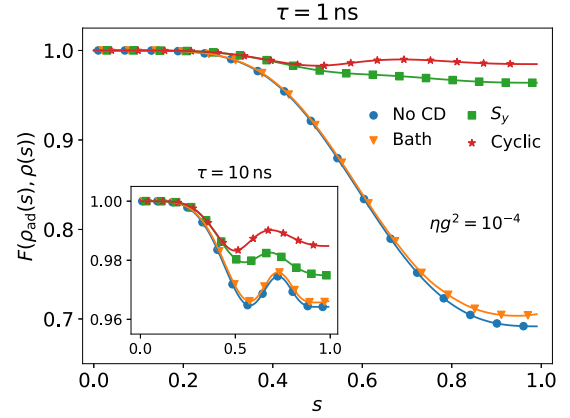
FIG. 3. Fidelity between the time-evolved density matrix $\rho(s)$ and the thermal state $\rho_{\text{ad}}(s)$ as a function of $s = t/\tau$, for $\eta g^2 = 1 \times 10^{-4}$. The different curves refer to the three *Ansätze* described in the main text.

On the one hand, the possible experimental implementation of this 3-local term is a nontrivial task. On the other hand, terms like this are likely to appear, for instance, when using the nested commutators *Ansatz* [28]. In addition, we only employ the cyclic *Ansatz* as a proof of principle: as we will show later on, our results remain valid even if we consider the simpler S_y. The second line describes the (diagonal) dissipative part, including all possible basis operators for this system in the (4×4)-dimensional operator space barring the identity Σ_0, thus we have a maximum of 18 variational parameters to optimize.

In the following calculations, we consider three special cases:

(1) Case Bath: $b_1 = b_2 = b_3 = 0$ so as to consider a purely dissipative ansatz.

(2) Case S_y: $a_i = 0$ $\forall i$ and $b_2 = b_3 = 0$, i. e., the *Ansatz* is unitary and only includes S_y.

(3) Case Cyclic: $a_i = 0$ $\forall i$ so as to consider the unitary cyclic *Ansatz*.

In Fig. 3, we plot the fidelity between the density matrix at the time s and the thermal density matrix at the same time, in these three cases, compared to the case with no CD superoperator, for $\tau = 1\,\text{ns}$ (main panel) and $\tau = 10\,\text{ns}$ (inset), and a coupling strength of $\eta g^2 = 1 \times 10^{-4}$. The efficiency of the *Ansatz* of Eq. (23) is mostly due to its unitary part, while the dissipative part plays a negligible role for both annealing times. As expected, for the shortest annealing time, the S_y term is responsible for the largest improvement in the fidelity. For $\tau = 10\,\text{ns}$, the fidelity is above 0.96 even in the absence of CD corrections: the instantaneous state is close to the ISS. It is remarkable that, even in this case, which should be governed by thermal processes, unitary CD superoperators allow improving the fidelity with the thermal density matrix as opposed to a purely dissipative *Ansatz*.

On the other hand, if we increase the system-environment coupling strength to $\eta g^2 = 1 \times 10^{-2}$, the dissipative part of the *Ansatz* starts to play a role as shown in Fig. 4. For $\tau = 1\,\text{ns}$ (main panel), the scenario is similar to that reported in Fig. 3 (main panel): for short annealing times, it is reasonable to expect that unitary CD superoperators would be
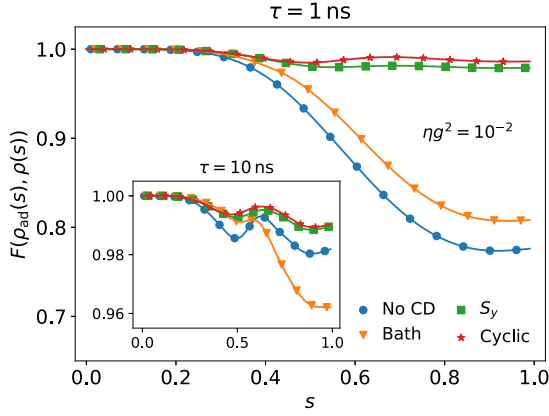
FIG. 4. Fidelity between the time-evolved density matrix $\rho(s)$ and the thermal state $\rho_{ad}(s)$ as a function of $s = t/\tau$, for $\eta g^2 = 1 \times 10^{-2}$. The different curves refer to the three *Ansätze* described in the main text.

the most effective ones since the environment will act on longer timescales. By contrast, the dissipative dynamics for $\tau = 10$ ns are close to being adiabatic, and the environment affects the dynamics significantly. This is evident from the inset of Fig. 4. Here we see that the unitary part of the *Ansatz* alone performs similarly to the case of $\eta g^2 = 1 \times 10^{-4}$. In addition, the cyclic and S_y *Ansätze* have similar performances, as opposed to $\eta g^2 = 1 \times 10^{-4}$. However, the striking difference is that the Bath *Ansatz* is detrimental instead in this case: a naive minimization of the *Ansatz* of Eq. (23) does not yield coefficients that satisfy the detailed balance Kubo-Martin-Schwinger condition [65]. Therefore, the time-evolved state departs from the ISS around the middle of the dynamics.

Having shown that our method is able to improve open-system adiabaticity, we now prepare the *p*-spin system in its ground state at $t = 0$ and evolve it using the approximate CD superoperator found previously. It can be easily done exploiting the fact that the variational minimization does not depend on the starting state. We show that we can additionally improve the success probability of QA, i. e., the ground state probability at $t = \tau$. Our results are summarized in Fig. 5, where we report the ground state probability at $t = \tau = 10$ ns for the three *Ansätze*, for $\eta g^2 = 1 \times 10^{-4}$ and 1×10^{-2}. Clearly, a unitary *Ansatz* for the gauge potential is more geared towards the optimization of QA. The inclusion
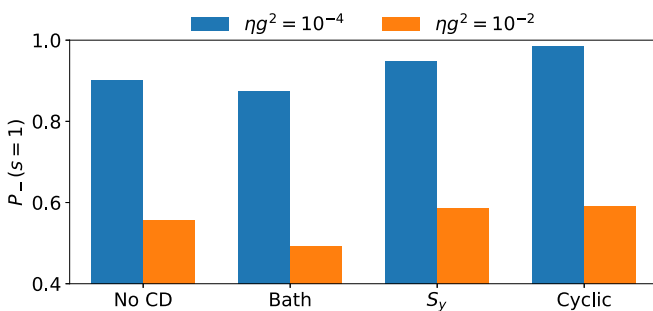

FIG. 5. Ground state probability at $t = \tau = 10$ ns for the various *Ansätze* considered in the main text, compared with the case without the CD superoperator.

of dissipation in the *Ansatz* negatively affects the ground state probability: a purely dissipative *Ansatz* (bath) decreases the success probability of QA, whereas we observe an enhancement in the ground state probability for the S_y and cyclic *Ansätze*, consistently with known results for unitary dynamics [29]. Thus, we stress that no quantum channel engineering is required in this scheme: controlling the unitary evolution of a dissipative quantum system alone can improve open- and closed-system adiabaticity.

V. CONCLUSIONS

In conclusion, we have formulated the search for dissipative counterdiabatic superoperators on variational grounds. We have applied our method to a relevant system for adiabatic quantum computation and we have shown that known unitary *Ansätze* for the counterdiabatic gauge potential offer an excellent compromise between open- and closed-system adiabaticity in that they are able to reduce the coupling between the Jordan blocks in which Lindbladians are decomposed and, at the same time, enhance the ground state probability at the end of the dynamics.

Our approach can be applied if the system dynamics can be expressed in the Lindblad form, which is always the case if the Markovian approximation holds. In general, the non-Markovian limit does not admit a generic form of the dynamical equation, hence methods have to be developed case by case. This is still an open point that we leave to future analysis.

APPENDIX A: UNITARY ADIABATIC THEOREM IN THE SUPEROPERATOR FORMALISM

We start from the Liouville equation for the density operator $\rho(t) = |\psi(t)\rangle \langle \psi(t)|$:

$$\dot{\rho}(t) = -\frac{i}{\hbar}[H_0(t), \rho(t)]. \tag{A1}$$

The adiabatic basis, in which $H_0(t)$ is diagonal, allows us to expand the ket state as $|\psi(t)\rangle = \sum_n c_n(t)|d_n(t)\rangle$. Inserting this decomposition into Eq. (A1) and remembering that $\dot{\rho} = |\dot{\psi}\rangle \langle \psi| + |\psi\rangle \langle \dot{\psi}|$, we can write

$$\sum_{nm} [\dot{c}_n(t)c_m^*(t) + c_n(t)\dot{c}_m^*] |d_n(t)\rangle \langle d_m(t)|$$

$$+ \sum_{nm} c_n(t)c_m^*(t)\big(|\dot{d}_n(t)\rangle \langle d_m(t)| + \text{H. c.}\big)$$

$$= -\frac{i}{\hbar} \sum_{nm} [\epsilon_n(t) - \epsilon_m(t)]c_n(t)c_m^*(t) |d_n(t)\rangle \langle d_m(t)| . \tag{A2}$$

If we now define the multi-index $\alpha = Dn + m \in \{0, 1, \ldots, D^2 - 1\}$, introduce the (coherence) operator basis $|d_n(t)\rangle \langle d_m(t)| \to |\mathcal{D}_\alpha\rangle\rangle$, and define the coefficient of

the coherence vector as $c_n(t)c_m^*(t) \to r_\alpha(t)$, we see that the unitary Lindbladian has a 1D Jordan representation with purely imaginary eigenvalues $-i[\epsilon_n(t) - \epsilon_m(t)]/\hbar \to \lambda_\alpha(t)$. The Lindbladian spectrum is always degenerate here since, when $n = m$, the corresponding λ_α is zero.

Suppressing the second term on the left-hand side of Eq. (A2) amounts to suppressing diabatic transitions between energy eigenstates, which in turn corresponds to suppressing transitions between 1D Jordan blocks.

APPENDIX B: VARIATIONAL UNITARY CD DRIVING AS A LIMITING CASE OF THE 1DJF

Here we show that the unitary variational formulation of CD driving [27] is contained into the 1DJF open system formulation when both the Lindbladian $\mathbb{L}_0(s)$ and the CD gauge potential $\mathbb{A}_s^*$ are unitary maps, i. e.,

$$\mathbb{L}_0(s) \to \mathcal{L}_s[\bullet] = -\frac{i}{\hbar}[H_0(s), \bullet], \tag{B1}$$

$$\mathbb{L}_0'(s) \to \mathcal{L}_s'[\bullet] = -\frac{i}{\hbar}[H_0'(s), \bullet], \tag{B2}$$

$$\mathbb{A}_s^* \to \mathcal{A}_s^*[\bullet] = -\frac{i}{\hbar}[A_s^*, \bullet]. \tag{B3}$$

In fact, in this case we have

$$\mathbb{G}_s \to \mathcal{G}_s[\bullet] = \mathcal{L}_s'[\bullet] - \mathcal{A}_s^*[\mathcal{L}_s[\bullet]] + \mathcal{L}_s[\mathcal{A}_s^*[\bullet]]$$

$$= -\frac{i}{\hbar}[H_0'(s), \bullet] + \frac{1}{\hbar^2}[A_s^*, [H_0(s), \bullet]]$$

$$- \frac{1}{\hbar^2}[H_0(s), [A_s^*, \bullet]]. \tag{B4}$$

Using the Jacobi identity $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$, the last two terms yield $[[A_s^*, H_0(s)], \bullet]/\hbar^2$, therefore Eq. (B4) is equivalent to

$$\mathbb{G}_s \to \mathcal{G}_s[\bullet] = -\frac{i}{\hbar}\Big[H_0'(s) + \frac{i}{\hbar}[A_s^*, H_0(s)], \bullet\Big]$$

$$\equiv -\frac{i}{\hbar}[G_s, \bullet], \tag{B5}$$

where G_s is defined in the main text. We now show that $\mathbb{S}$ is minimized if and only if S is minimized, i. e., $\mathbb{A}_s^* = \mathbb{A}_s \iff A_s^* = A_s$ [66]. To see this, it is sufficient to evaluate the matrix elements of $\mathbb{G}_s$ in the eigenbasis of the Lindbladian, which, as shown in Appendix A, in the case of unitary Lindbladian dynamics corresponds to the operator bases $\langle\langle\mathcal{E}_\alpha| \to |d_n(t)\rangle \langle d_m(t)|$ with $\alpha = Dn + m$ and $|\mathcal{D}_\beta\rangle\rangle \to |d_j(t)\rangle \langle d_k(t)|$ with $\beta = Dj + k$. We consider the case where there are no degeneracies in the Hamiltonian spectrum so that all Bohr frequencies with different indices are nonzero. We have

$$(\mathbb{G}_s)_{\alpha\beta} = \langle\langle\mathcal{E}_\alpha|\mathbb{G}_s|\mathcal{D}_\beta\rangle\rangle = \frac{1}{D}\mathrm{Tr}(\mathcal{E}_\alpha^\dagger \mathcal{G}_s[\mathcal{D}_\beta])$$

$$= -\frac{i}{D\hbar}\mathrm{Tr}(\mathcal{E}_\alpha^\dagger[G_s, \mathcal{D}_\beta])$$

$$= -\frac{i}{D\hbar}\mathrm{Tr}(G_s[\mathcal{D}_\beta, \mathcal{E}_\alpha^\dagger])$$

$$= -\frac{i}{D\hbar}\sum_{ab}(G_s)_{ab}[\mathcal{D}_\beta, \mathcal{E}_\alpha^\dagger]_{ba}$$

$$= -\frac{i}{D\hbar}\sum_{ab}(G_s)_{ab}(\delta_{bj}\delta_{km}\delta_{na} - \delta_{bm}\delta_{nj}\delta_{ka})$$

$$= -\frac{i}{D\hbar}[(G_s)_{nj}\delta_{km} - (G_s)_{km}\delta_{nj}]. \tag{B6}$$

Four cases must be distinguished:

(1) $n \neq j, k \neq m$. In this case, $(\mathbb{G}_s)_{\alpha\beta} = 0$.

(2) $n = j, k \neq m$. In this case, $(\mathbb{G}_s)_{\alpha\beta} = \frac{i}{D\hbar}(G_s)_{km}$.

(3) $n \neq j, k = m$. In this case, $(\mathbb{G}_s)_{\alpha\beta} = -\frac{i}{D\hbar}(G_s)_{nj}$.

(4) $n = j, k = m$ (so $\alpha = \beta$). In this case, $(\mathbb{G}_s)_{\alpha\alpha} = -\frac{i}{D\hbar}[(G_s)_{nn} - (G_s)_{mm}]$.

Therefore, we have

$$\mathbb{S} = \sum_{\alpha\beta}|\mathbb{G}_s|_{\alpha\beta}^2 = \sum_\alpha|\mathbb{G}_s|_{\alpha\alpha}^2 + \sum_{\alpha\neq\beta}|\mathbb{G}_s|_{\alpha\beta}^2$$

$$= \sum_{nm}\frac{1}{D^2}(\omega_{nm}')^2 + \frac{2}{D\hbar^2}\sum_{m\neq k}|G_s|_{km}^2. \tag{B7}$$

Therefore, $\mathbb{S}$ is minimized when $(G_s)_{km} = 0$ for $k \neq m$, that is, when $A_s^* = A_s$. Conversely, if $(\mathbb{G}_s)_{\alpha\beta} = 0$ with $\alpha \neq \beta$, then according to points (2) and (3) we must have $(G_s)_{km} = 0$ for $k \neq m$, thus the unitary variational approach to CD driving of Ref. [27] is a particular case of the more general Lindbladian formulation.

APPENDIX C: WEAK-COUPLING LIMIT LINDBLAD EQUATION

Consider a system-bath Hamiltonian of the form $H_{SB}(t) = H_0(t) + H_B + gU \otimes B$, where U is a system operator, $H_B = \sum_k \omega_k b_k^\dagger b_k$ is the Hamiltonian of the bath modeled as noninteracting bosons, and $B = \sum_k(b_k + b_k^\dagger)$. The system-bath coupling strength is g. The weak-coupling-limit adiabatic Lindbladian of Refs. [45,65] reads

$$\mathcal{L}_t[\bullet] = -i[H_0(t) + H_{\mathrm{LS}}(t), \bullet]$$

$$+ \sum_\omega \gamma[\omega(t)]\Big(\Gamma_\omega(t) \bullet \Gamma_\omega^\dagger(t) - \frac{1}{2}\{\Gamma_\omega^\dagger(t)\Gamma_\omega(t), \bullet\}\Big), \tag{C1}$$

where $\Gamma_\omega = \sum_{a,b:\epsilon_a-\epsilon_b=\omega}|\epsilon_a\rangle \langle\epsilon_a| U |\epsilon_b\rangle \langle\epsilon_b|$ are Lindblad operators, the Lamb shift is $H_{\mathrm{LS}}(t) = \sum_\omega \zeta[\omega(t)]\Gamma_\omega^\dagger(t)\Gamma_\omega(t)$, and the (Ohmic) spectral functions read

$$\gamma(\omega) = \frac{2\pi\eta g^2 \omega e^{-|\omega|/\omega_c}}{1 - e^{-\beta\omega}}, \tag{C2}$$

$$\zeta(\omega) = \mathrm{P.P.}\int_{-\infty}^\infty \frac{\gamma(\omega')}{\omega - \omega'}\frac{d\omega'}{2\pi}, \tag{C3}$$

with ηg^2 being a dimensionless parameter related to the system-bath coupling strength ($\eta g^2 \ll 1$), $\beta = 1/T$ is the inverse temperature ($k_B = 1$), and ω_c is a high-frequency cutoff that we fix to $\omega_c = 8\pi$ GHz. Equation (C1) assumes that the Born, Markov, and rotating wave approximations are valid, as a consequence of the separation between system and bath timescales [65].

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