Efficiency statistics of a quantum Otto cycle

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The stochastic efficiency [G. Verley *et al.*, Nat. Commun. **5**, 4721 (2014)] was introduced to evaluate the performance of energy-conversion machines in microscale. However, such an efficiency generally diverges when no heat is absorbed while work is produced in a thermodynamic cycle. As a result, any statistical moments of the efficiency do not exist. In this study, we come up with a different version of the definition for the stochastic efficiency (called the scaled fluctuating efficiency) which is always finite. Its mean value is equal to the conventional efficiency and higher moments characterize the fluctuations of the cycle. In addition, the fluctuation theorems are reexpressed via the efficiency. For working substance satisfying the equipartition theorem, we clarify that the thermodynamic uncertainty relation for the scaled fluctuating efficiency is valid in an Otto engine. To demonstrate our general discussions, the efficiency statistics of a quantum harmonic-oscillator Otto engine is systematically investigated. The probability that the scaled fluctuating efficiency surpasses the Carnot efficiency is explicitly obtained. This work may shed new insight for optimizing micromachines with fluctuations.

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I. INTRODUCTION

For a heat engine operating between a hot and cold reservoir, the conventional efficiency is defined by the ratio of the output work and the heat absorbed from the hot reservoir, which characterizes the performance of the engine. As the size of the engine decreases, the thermal fluctuations [1,2] and quantum fluctuations [3,4] become more significant. From the point of view of stochastic thermodynamics, the work, heat, and entropy of microscopic systems are all stochastic quantities. Hence, at a microscopic level, it is natural to expect that the efficiency, introduced to evaluate the ability of energy conversion for various thermal machines, is also a stochastic quantity.

Recently, the stochastic efficiency, defined as the ratio of the stochastic output work and the stochastic heat absorbed from the hot reservoir in a cycle, has been widely studied for classical heat engines [5–9]. This stochastic efficiency is also applied to a quantum Otto cycle in Refs. [10,11]. However, such a definition of the stochastic efficiency seems weird for the following three reasons. (1) The mean value of the stochastic efficiency is not equal to the conventional efficiency in general. In contrary, the mean values of stochastic work, heat, and entropy are equal to their counterparts in the conventional thermodynamics. (2) The efficiency approaches infinity with a nonzero probability. Such a result is due to the possibility that no heat is absorbed from the hot reservoir while work is produced in one realization of the cycle [11]. (3) Due to the divergent efficiency distribution, any moments of the efficiency are ill defined [9]. Thus one fails to evaluate the performance of the heat engine by the moments of this version of stochastic efficiency.

To avoid such weirdness, meanwhile, to evaluate the fluctuations in a practical heat engine, we come up with a different version of the stochastic efficiency (called the scaled fluctuating efficiency) through a scale transformation of the stochastic work. Then, the fluctuation theorems [1–4,12,13] are reexpressed via the efficiency. Moreover, the thermodynamic uncertainty relation (TUR) [14,15] for the scaled fluctuating efficiency is investigated. For the working substance satisfying the equipartition theorem, we obtain the TUR for a quantum Otto cycle in the quasistatic limit (a general proof) and in a finite-time Otto cycle (numerical simulations). As a specific example, we apply the scaled fluctuating efficiency to study a quantum harmonic-oscillator Otto cycle. We find that both the probability that the efficiency surpasses the Carnot efficiency and the probability that the efficiency is negative increase as the temperatures of the reservoirs decrease.

This paper is arranged as follows. In Sec. II, we introduce the quantum Otto cycle and the joint distribution of input work and absorbed heat from the hot reservoir. In Sec. III, the scaled fluctuating efficiency is given. The fluctuation theorems and the TUR are also reexpressed via the scaled fluctuating efficiency. In Sec. IV, we demonstrate our general discussions in a quantum Otto cycle with the harmonic oscillator being the working substance. And we systematically investigate the statistics of the scaled fluctuating efficiency. Section V is the summary and discussion.

II. JOINT DISTRIBUTION OF WORK AND HEAT IN A QUANTUM OTTO CYCLE

As illustrated in Fig. 1, we consider a quantum Otto cycle which involves four strokes: two adiabatic processes and two

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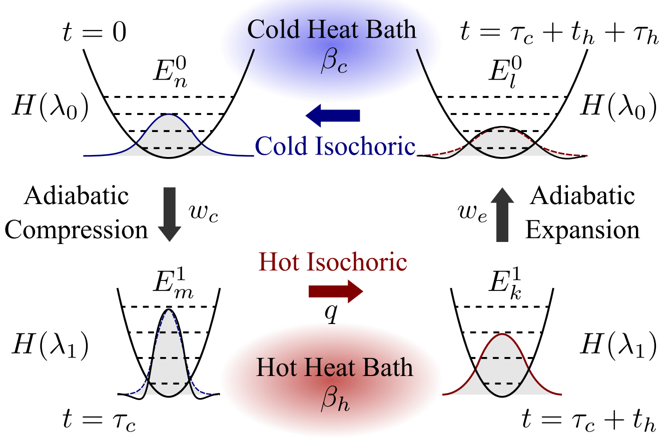

FIG. 1. Schematic of a finite-time quantum Otto cycle.

isochoric processes [16,17]. In the adiabatic compression (expansion) process, the Hamiltonian of the working substance is changed from $H(\lambda_0)$ to $H(\lambda_1)$ [from $H(\lambda_1)$ to $H(\lambda_0)$] during time τ_c (τ_h) through a time-dependent parameter λ. In the two isochoric processes, during time t_h (t_c), the working substance contacts a hot (cold) reservoir at the inverse temperature β_h (β_c) with fixed λ. For simplicity, we assume that a complete thermalization is achieved in the two isochoric processes. Namely, the working substance is thermal equilibrium with the corresponding reservoir at the end of each isochoric process.

The stochastic work and heat in a quantum Otto cycle are defined under the two-point measurement scheme [10]. At time $t=0$, τ_c, τ_c+t_h, $\tau_c+t_h+\tau_h$ ($t=0$ is the initial time of the adiabatic compression process), we apply the projective measurements of energy on the working substance according to the corresponding instantaneous Hamiltonian. Then, the stochastic work w_c (w_e) in the adiabatic compression (expansion) process and the stochastic absorbed heat q in the hot isochoric process are defined as

$$\begin{aligned} w_c &= E_m^1 - E_n^0, \\ q &= E_k^1 - E_m^1, \\ w_e &= E_l^0 - E_k^1, \end{aligned} \tag{1}$$

where $E_n^0, E_m^1, E_k^1, E_l^0$ are the measured energy of the four projective measurements corresponding to the times $t=0$, τ_c, τ_c+t_h, $\tau_c+t_h+\tau_h$, respectively (n, m, k, l denote the corresponding quantum numbers). Thus the joint probability distribution $P(w,q)$ of the total stochastic input work $w=w_c+w_e$, and the stochastic absorbed heat from the hot reservoir q is given by

$$P(w,q) = \sum_{n,m,k,l} \delta\big(w - E_m^1 + E_n^0 - E_l^0 + E_k^1\big)\delta\big(q - E_k^1 + E_m^1\big) \times |{}_1\langle m|U_c|n\rangle_0|^2 |{}_0\langle l|U_e|k\rangle_1|^2 \frac{e^{-\beta_c E_n^0 - \beta_h E_k^1}}{Z^0(\beta_c)Z^1(\beta_h)}, \tag{2}$$

where U_c, U_e are the unitary evolution operators corresponding to the compression and expansion processes, $|j\rangle_{0(1)}$ is the eigenstate of the Hamiltonian $H(\lambda_0)$ [$H(\lambda_1)$], and $Z^0(\beta_c) = \mathrm{Tr}[e^{-\beta_c H(\lambda_0)}]$, $Z^1(\beta_h) = \mathrm{Tr}[e^{-\beta_h H(\lambda_1)}]$ are the partition functions corresponding to the equilibrium states at $t=0$ and $t=\tau_c+t_h$, respectively.

III. SCALED FLUCTUATING EFFICIENCY

A. Definition

We define the scaled fluctuating efficiency of a (classical) quantum heat engine through a scale transformation of the stochastic work as

$$\eta = -\frac{w}{\langle q\rangle}, \tag{3}$$

where $\langle\cdot\rangle$ denotes the mean value over numerous measurements, i.e.,

$$\langle q\rangle = \int dw\, dq\, P(w,q)q. \tag{4}$$

For heat engines, the denominator is always nonzero ($\langle q\rangle > 0$), so the scaled fluctuating efficiency η in Eq. (3) is finite. Moreover, it follows from Eq. (4) that $\langle\eta\rangle = -\langle w\rangle/\langle q\rangle$, which is just the conventional efficiency.

From the joint distribution $P(w,q)$, the distribution of the scaled fluctuating efficiency $P(\eta)$ is obtained by

$$P(\eta) = \int dw\, dq\, P(w,q)\delta(\eta + w/\langle q\rangle). \tag{5}$$

The fluctuation of the scaled fluctuating efficiency is determined by the output work, which characterizes the reliability of the heat engine.

For practical calculation of the joint distribution of work and heat, we show the characteristic function of $P(w,q)$ [Eq. (2)] in the following:

$$\chi(u,v) \equiv \left\langle e^{iuw+ivq}\right\rangle = \chi_c(u,v)\chi_h(u,v), \tag{6}$$

where

$$\chi_c(u,v) = \frac{\mathrm{Tr}[U_c^\dagger e^{(iu-iv)H(\lambda_1)} U_c e^{(-iu-\beta_c)H(\lambda_0)}]}{Z^0(\beta_c)}, \tag{7}$$

$$\chi_h(u,v) = \frac{\mathrm{Tr}[U_e^\dagger e^{iuH(\lambda_0)} U_e e^{(-iu+iv-\beta_h)H(\lambda_1)}]}{Z^1(\beta_h)}. \tag{8}$$

Thus we find that the characteristic function of the joint distribution of work and heat is associated with the product of two transformed characteristic functions of work in the adiabatic process. The cumulant moments of work and heat are obtained from $\chi(u,v)$, such as the average input work

$$\langle w\rangle = -i\frac{\partial \ln\chi(u,v)}{\partial u}\bigg|_{u=v=0}, \tag{9}$$

the average heat absorbed form the hot reservoir

$$\langle q\rangle = -i\frac{\partial \ln\chi(u,v)}{\partial v}\bigg|_{u=v=0}, \tag{10}$$

and the variance of input work

$$\Delta w^2 = -\frac{\partial^2 \ln\chi(u,v)}{\partial u^2}\bigg|_{u=v=0}, \tag{11}$$

where $\Delta(\cdot) = \sqrt{\langle\cdot^2\rangle - \langle\cdot\rangle^2}$ denotes the standard deviation.

B. Fluctuation theorems

Fluctuation theorems indicate the equality relation in a general nonequilibrium process. Using the results of Refs. [18,19], we can reexpress the fluctuation theorems in terms of the scaled fluctuating efficiency for the Hamiltonian of the working substance involving time-reversal symmetry:

$$\left\langle e^{-\delta s(\eta,q)}\right\rangle = 1, \tag{12}$$

$$P(-\eta\langle q\rangle, q) = P_R(\eta\langle q\rangle, -q)e^{\delta s(\eta,q)}, \tag{13}$$

where $\delta s(\eta, q) = \beta_c(\eta_C q - \eta\langle q\rangle)$ is the total stochastic entropy production expressed in terms of η and q and $\eta_C = 1 - \beta_h/\beta_c$ is the Carnot efficiency. The subscript R denotes the reverse process of the cycle (the clockwise direction in Fig. 1). Then, using Jensen's inequality $e^{\langle x\rangle} \leqslant \langle e^x\rangle$, we have $\langle\eta\rangle \leqslant \eta_C$ for heat engines ($\langle q\rangle > 0$), which is the second law of thermodynamics. It is worth mentioning that this inequality is not sharp. In fact, for quantum systems without energy-level crossing when changing the parameter λ, we obtain a sharper inequality $\langle\eta\rangle \leqslant \eta_O$ as a result of the minimum work principle [20], where η_O is the Otto efficiency, i.e., the efficiency of an Otto cycle in the quasistatic limit (see Appendix A).

C. Thermodynamic uncertainty relation

Since the fluctuation theorems always imply the genralized thermodynamic uncertainty relation (TUR) [21], it follows from Eq. (13) that

$$\frac{\Delta\eta^2}{\langle\eta\rangle^2} \geqslant f(\langle\delta s\rangle), \tag{14}$$

where $f(x) = \text{csch}^2[g(x/2)]$ and $g(x)$ is the inverse function of $x\tanh x$. Equation (14) expresses a trade-off between the relative fluctuation of the efficiency and the dissipation quantified through the entropy production in a cycle. When $\langle\delta s\rangle \to 0$, $f(\langle\delta s\rangle) \approx 2/\langle\delta s\rangle$, which reproduces the TUR [14,15] for the efficiency. Since the scaled fluctuating efficiency is a scale transformation of the stochastic work, the TURs for the efficiency and work are equal.

For the spectra of the working substance with scale property, i.e., $E_n^1 = E_n^0/\epsilon$ [10] (ϵ is n independent), the general expression of the joint characteristic function [Eq. (6)] is obtained in the quasistatic limit (see Appendix B). From Eqs. (9) and (11), we obtain (the Boltzmann constant $k_B = 1$ here and after)

$$\begin{aligned}\langle w\rangle &= \eta_O T_c\left(\frac{\sigma_c}{1-\eta_O} - \frac{\sigma_h}{1-\eta_C}\right),\\ \Delta w^2 &= \eta_O^2 T_c^2\left[\frac{C_c}{(1-\eta_O)^2} + \frac{C_h}{(1-\eta_C)^2}\right],\end{aligned} \tag{15}$$

where T_c (T_h) is the temperature of the cold (hot) reservoir, $\eta_O = 1-\epsilon$, $\sigma_c \equiv E_c/T_c$ ($\sigma_h \equiv E_h/T_h$), E_c (E_h) is the internal energy of the working substance corresponding to the equilibrium state at $t = 0$ ($t = \tau_c + t_h$), and $C_c \equiv \partial E_c/\partial T_c$ ($C_h \equiv \partial E_h/\partial T_h$) is the heat capacity at constant volume. In addition, the average heat absorbed from the hot reservoir and the average entropy production of the cycle follow as $\langle q\rangle = -\langle w\rangle/\eta_O$ and $\langle\delta s\rangle = \beta_c\langle q\rangle(\eta_C - \eta_O)$, respectively. If the working substance satisfies the equipartition theorem $E_c \propto T_c$ and $E_h \propto T_h$ in the high-temperature limit, one has

$$\frac{\Delta\eta^2}{\langle\eta\rangle^2} = \frac{\Delta w^2}{\langle w\rangle^2} = \frac{1}{\langle\delta s\rangle}\left(\frac{1-\eta_O}{1-\eta_C} + \frac{1-\eta_C}{1-\eta_O}\right) \geqslant \frac{2}{\langle\delta s\rangle}, \tag{16}$$

with the equal sign saturated at $\eta_O = \eta_C$. The inequality (16) is consistent with the TUR in steady states [14,15] or in a specific Otto cycle [22].

Moreover, due to the third law of thermodynamics, $\langle\delta s\rangle \to 0$ in the low-temperature limit. Using the property of the function $f(x)$ in Eq. (14), the TUR for the efficiency is also reproduced in the low-temperature limit. Consequently, we expect that the TUR for efficiency is valid for an arbitrary temperature under these conditions. For a finite-time cycle, we numerically study the TUR in a specific model below.

IV. QUANTUM HARMONIC-OSCILLATOR HEAT ENGINE

In this section, we illustrate our general discussions above with a specific example: the working substance of the Otto cycle is modeled as a quantum harmonic oscillator. The frequency of such an oscillator is changed from ω_0 to ω_1 ($\omega_1 > \omega_0$) in the adiabatic compression process. Recently, the stochastic thermodynamics of this model has been widely studied. For example, the work statistics in the adiabatic process is studied in Refs. [23,24] and the TUR in the Otto cycle is studied in Ref. [22]. In the quasistatic limit, a delta distribution for the previous definition of the stochastic efficiency is reported in Ref. [10].

According to Refs. [23,24], $\chi_c(u, v)$ and $\chi_h(u, v)$ of the joint characteristic function in Eq. (6) are explicitly obtained as (see Appendix C for detailed derivation)

$$\chi_c(u, v) = 2\sinh\left(\frac{\beta_c\omega_0}{2}\right)\{2\cos[(u-v)\omega_1]\cos[(u-i\beta_c)\omega_0] + 2Q_c\sin[(u-v)\omega_1]\sin[(u-i\beta_c)\omega_0] - 2\}^{-\frac{1}{2}} \tag{17}$$

and

$$\chi_h(u, v) = 2\sinh\left(\frac{\beta_h\omega_1}{2}\right)\{2\cos(u\omega_0)\cos[(u-v-i\beta_h)\omega_1] + 2Q_h\sin(u\omega_0)\sin[(u-v-i\beta_h)\omega_1] - 2\}^{-\frac{1}{2}}, \tag{18}$$

where $Q_{c(h)} \geqslant 1$ is the corresponding nonadiabatic factor [25,26] and we have set $\hbar = 1$ for simplicity. The equal sign is hold when the quantum adiabatic condition is satisfied. In the following, we study the statistics of the scaled fluctuating efficiency of the Otto cycle in different circumstances.

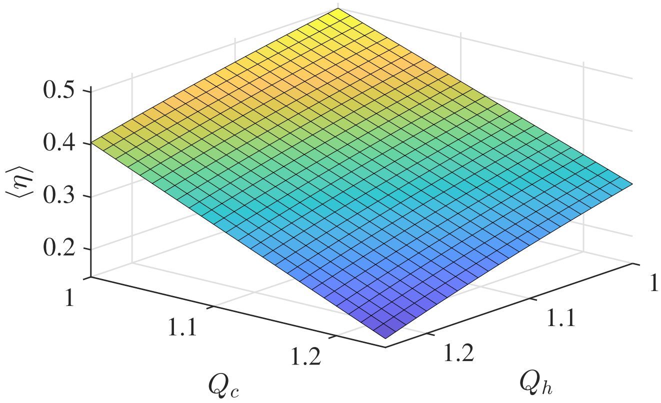

FIG. 2. Average efficiency as the function of Q_c and Q_h. In this figure, we use $\eta_O = 0.5$, $T_h = 1$, $\eta_C = 0.8$, and $\omega_1 = 1$.

A. Average efficiency and efficiency distribution of the heat engine

Combining Eqs. (9), (10), (17), and (18), the average efficiency is obtained as (see Appendix D)

$$\langle\eta\rangle = \frac{-\langle w\rangle}{\langle q\rangle} = \eta_O - \frac{\epsilon \sum_{\alpha=h,c} \vartheta_\alpha (Q_\alpha - 1)}{\vartheta_h - Q_c\vartheta_c}, \tag{19}$$

where $\eta_O = 1 - \epsilon$ ($\epsilon = \omega_0/\omega_1$) is the Otto efficiency achieved in the quantum adiabatic case with $Q_{c,h} = 1$, and $\vartheta_h \equiv \coth(\beta_h\omega_1/2)$, $\vartheta_c \equiv \coth(\beta_c\omega_0/2)$. Equation (19) indicates that the nonadiabatic effect decreases the average efficiency of the engine, which is demonstrated in Fig. 2.

On the other hand, the variance of the efficiency is obtained from Eq. (11) as

$$\Delta\eta^2 = \frac{\Delta w^2}{\langle q\rangle^2} = \frac{\eta_O^2\left(\vartheta_h^2 + \vartheta_c^2 - 2\right) + 2\left[\vartheta_h^2\left(Q_h^2 - 1\right)\epsilon^2 + \vartheta_c^2\left(Q_c^2 - 1\right) - \epsilon \sum_{\alpha=h,c}(Q_\alpha - 1)\left(\vartheta_\alpha^2 - 1\right)\right]}{(\vartheta_h - Q_c\vartheta_c)^2}. \tag{20}$$

In the quasistatic limit, the variance of efficiency $\Delta\eta^2_{\mathrm{adi}}$ accordingly becomes

$$\Delta\eta^2_{\mathrm{adi}} = \frac{\eta_O^2\left(\vartheta_h^2 + \vartheta_c^2 - 2\right)}{(\vartheta_h - \vartheta_c)^2}. \tag{21}$$

It is worth mentioning that the efficiency fluctuation does not vanish for a quantum harmonic-oscillator Otto cycle in the quasistatic limit, while the fluctuation of the previous version of the stochastic efficiency vanishes in this case [10].

It is shown in Fig. 3(a) that, in the quasistatic limit ($Q_{c,h} = 1$), the efficiency fluctuation decreases as the temperature increases, which is consistent with the TUR for efficiency [Eq. (16)] since $\langle\delta s\rangle$ increases with the temperature increase. In the nonadiabatic case, the efficiency fluctuation as the function of Q_c and Q_h is illustrated in Fig. 3(b), which reflects the enhancement of the fluctuation due to the nonadiabatic driving.

The efficiency distribution is obtained with $\chi(u, v)$ by the discrete Fourier transform. With different chosen parameters, we plot the efficiency distribution in Fig. 4 (adiabatic case) and Fig. 5 (nonadiabatic case) with the black dots. As comparisons, the Otto efficiency ($\eta_O = 0.5$) and Carnot efficiency ($\eta_C = 0.8$) are respectively represented with the blue dash-dotted line and the red dotted line. And we show the probability that the scaled fluctuating efficiency of an Otto cycle surpasses the Carnot efficiency in the figure. By comparing Fig. 4(a) ($T_h = 10$) with Fig. 4(b) ($T_h = 1$) or Fig. 5(a) ($T_h = 10$) with Fig. 5(b) ($T_h = 1$), one can infer that lower temperature leads to greater probability of the heat engine surpassing the Carnot efficiency. Meanwhile, the lower temperature increases the probability of the engine to be useless, namely, the engine outputs negative stochastic work. The greater fluctuation at lower temperature is illustrated via the TUR. Since $2/\langle\delta s\rangle \to \infty$ at low temperature due to the third law of thermodynamics and $\langle\eta\rangle$ is finite, $\Delta\eta^2$ should also go to infinity.

We would like to emphasize that, since the expressions of the characteristic functions [Eqs. (17) and (18)] are infinitely differentiable, the decay of the distribution in Fig. 5 should be faster than any power-law function. This property is different from the distribution of the previous stochastic efficiency with $\pm\infty$ values in Ref. [10] and ensures the existence of the variance of the efficiency, which indicates the fluctuation of the cycle.

B. Finite-time performance of the heat engine

To further explore the finite-time performance of the cycle, we first analyze the explicit time dependence of the nonadiabatic factors for a specific protocol. For an adiabatic process with frequency changed from ω_i to ω_f during time $t \in [0, \tau]$, the time dependence of the frequency of the harmonic oscillator is [22,27,28]

$$\omega(t) = \frac{\omega_i}{(\omega_i/\omega_f - 1)t/\tau + 1}. \tag{22}$$

Then, the nonadiabatic factor $Q(\tau)$ is obtained as (see Appendix E for detailed derivation)

$$Q(\tau) = 1 + \frac{1 - \cos\left[\sqrt{a^2\tau^2 - 1}\,\ln(\omega_f/\omega_i)\right]}{a^2\tau^2 - 1}, \tag{23}$$

where

$$a \equiv \frac{2\omega_f\omega_i}{\omega_f - \omega_i}. \tag{24}$$

As shown in Fig. 6, the nonadiabatic factor $Q(\tau)$ (blue solid line) oscillates with the driving time τ, reflecting the quantum coherence effect in the nonadiabatic transition. The orange dashed line represents $Q(\tau) = 1$, which is achieved for the quantum adiabatic driving or with some special values of τ [27,28].

In the following, we adopt the protocol of Eq. (22) for the finite-time adiabatic processes in the Otto cycle; then we use the explicit form of $Q(\tau)$ given in Eq. (23) to study the power at maximum efficiency (PME) and efficiency at maximum power (EMP) of the cycle. In this sense, the nonadiabatic factors become $Q_{c(h)} = Q(\tau_{c(h)})$, and then the average power $\langle P(\tau_c, \tau_h)\rangle \equiv -\langle w\rangle/(\tau_h + \tau_c)$ and the efficiency $\langle\eta(\tau_c, \tau_h)\rangle$ of

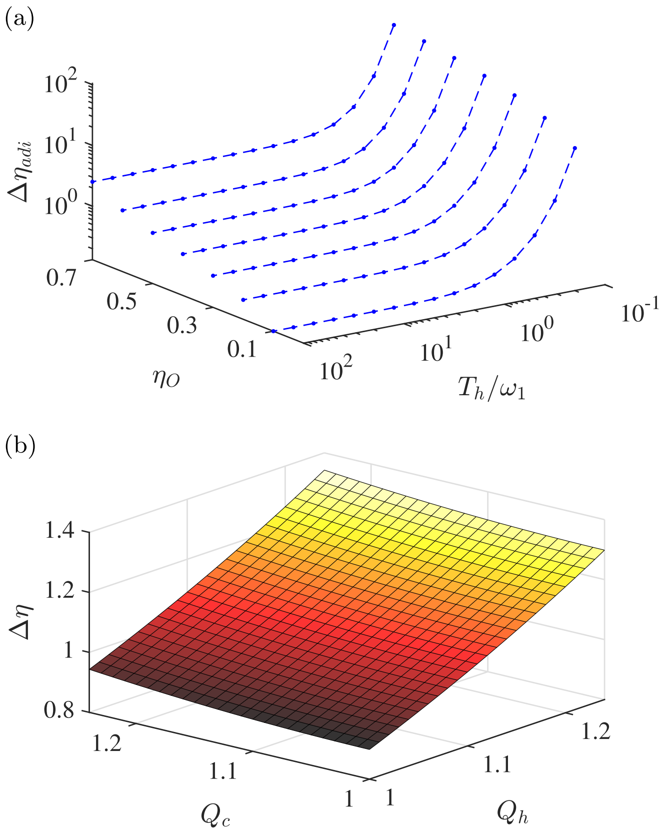

FIG. 3. Efficiency fluctuation. (a) Efficiency fluctuation as the function of temperature T_h with different η_O in the quasistatic limit. (b) Efficiency fluctuation as the function of Q_c and Q_h, where $\eta_O = 0.5$ and $T_h = 1$ are chosen. In this figure, $\eta_C = 0.8$ and $\omega_1 = 1$ are fixed.

the Otto engine are respectively

$$\langle P(\tau_c, \tau_h)\rangle = \frac{\omega_1}{2} \frac{\{[Q(\tau_c) - \epsilon]\vartheta_c - [1 - \epsilon Q(\tau_h)]\vartheta_h\}}{\tau_h + \tau_c} \quad (25)$$

and

$$\langle \eta(\tau_c, \tau_h)\rangle = \eta_O - \frac{\epsilon \sum_{\alpha=h,c} \vartheta_\alpha [Q(\tau_\alpha) - 1]}{\vartheta_h - Q(\tau_c)\vartheta_c}, \quad (26)$$

where the total duration of the two isochoric processes, i.e., $t_c + t_h$, is assumed to be much smaller than $\tau_c + \tau_h$ and is thus ignored.

Since $Q_{c(h)} = 1$ can be achieved within finite time (called the shortcut to adiabaticity [27,29]), the average efficiency of some cycles approaches the Otto efficiency η_O with nonvanishing power. These cycles happen to have the special operation time sets (τ_c^*, τ_h^*) corresponding to $Q_h(\tau_h^*) = Q_c(\tau_c^*) = 1$. With the help of Eq. (23), one finds the special operation time follows as

$$\tau_{h,c}^* = \frac{\eta_O}{\omega_0}\sqrt{\frac{1}{4} + \left(\frac{n_{h,c}\pi}{\ln \epsilon}\right)^2}, \quad (27)$$

with $n_{h,c} = 1, 2, 3, \ldots$. Therefore, the PME is

$$P(\tau_c^*, \tau_h^*) = \frac{\omega_0\omega_1(\vartheta_h - \vartheta_c)}{2\sum_{\alpha=h,c}\sqrt{1/4 + (n_\alpha\pi/\ln\epsilon)^2}}. \quad (28)$$

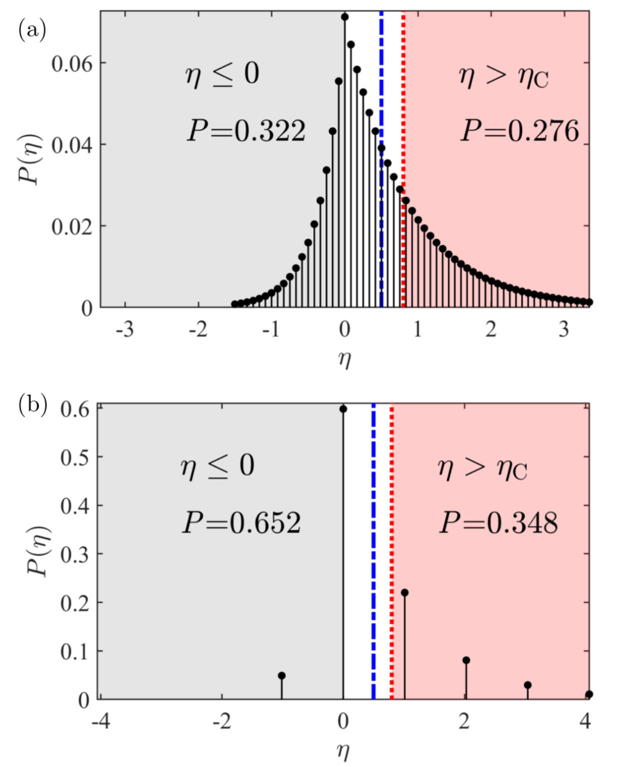

FIG. 4. Efficiency distribution in the adiabatic case with $Q_h = Q_c = 1$. The probability distribution of the scaled fluctuating efficiency is plotted with the black dots. The Otto efficiency and Carnot efficiency are respectively represented with the blue dash-dotted line and the red dotted line. The (gray) area in the left side denotes the negative work output regime; the (light red) area in the right side of the red dotted line represents the regime of $\eta > \eta_C$. The parameters are chosen as (a) $T_h = 10$, $T_c = 2$ and (b) $T_h = 1$, $T_c = 0.2$. In this figure, we choose $\omega_0 = 0.5$, $\omega_1 = 1$, and the Carnot efficiency is fixed at 0.8.

It should be noted that, in the usual finite-time thermodynamic cycles, the PME generally approaches zero [30–34]. Here, thanks to the special protocol we have chosen to realize the quantum adiabatic process in finite time, the current quantum Otto cycle outputs nonzero or even relatively large power (comparable to the maximum power) when the Otto efficiency is reached. Obviously, the maximum $P(\tau_c^*, \tau_h^*)$ is

$$P_{\max}(\tau_c^*, \tau_h^*) = \frac{\omega_0\omega_1(\vartheta_h - \vartheta_c)}{2\sqrt{1 + (2\pi/\ln\epsilon)^2}}, \quad (29)$$

which is achieved at $n_c = n_h = 1$. Besides, the second largest and third largest power are reached at $(n_c = 1, n_h = 2)$ and $(n_c = 2, n_h = 2)$, respectively. For $(n_\alpha\pi/\ln\epsilon)^2 \gg 1/4$, Eq. (28) can be approximated as

$$P(\tau_c^*, \tau_h^*) \approx \frac{\omega_0\omega_1 \ln\epsilon(\vartheta_h - \vartheta_c)}{2\pi(n_c + n_h)}, \quad (30)$$

which shows that $P(\tau_c^*, \tau_h^*)$ is a monotonically decreasing quasicontinuous function of n_c and n_h.

In addition, the EMP of this Otto engine as the function of η_O is illustrated in Fig. 7, where the ratio $\eta_O/\eta_C = 0.8$ is fixed. As shown in this figure, the EMP of our cycle (blue solid curve) is found to surpass the upper bound,

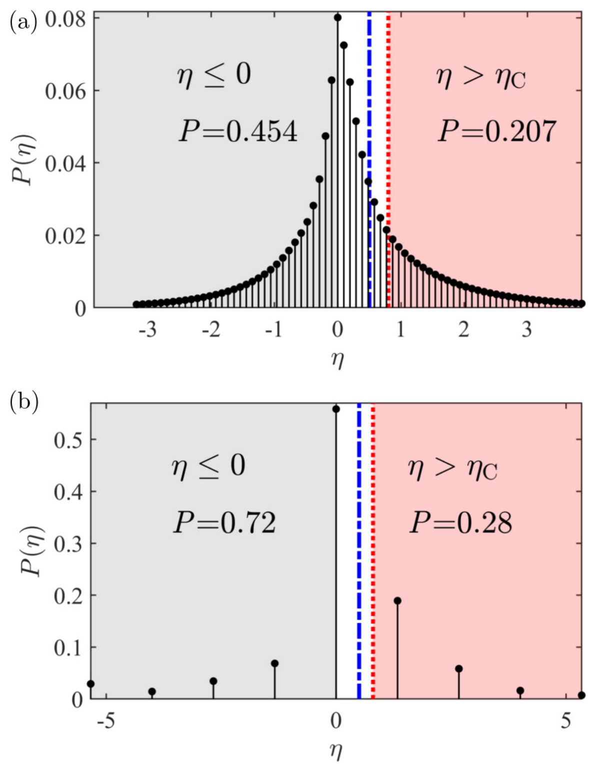

FIG. 5. Efficiency distribution in the nonadiabatic case with $Q_h = Q_c = 1.2$. The probability distribution of the scaled fluctuating efficiency is plotted with the black dots. The Otto efficiency and Carnot efficiency are respectively represented with the blue dash-dotted line and the red dotted line. The (gray) area in the left side denotes the negative work output regime; the (light red) area in the right side of the red dotted line represents the regime of $\eta > \eta_C$. The parameters are chosen as (a) $T_h = 10$, $T_c = 2$ and (b) $T_h = 1$, $T_c = 0.2$. In this figure, we choose $\omega_0 = 0.5$, $\omega_1 = 1$, and the Carnot efficiency is fixed at 0.8.

$\eta_+ = 2\eta_O/(3 - \eta_O)$ (black dashed curve), of the Otto cycle's EMP without considering the oscillation of the output work [34]. Meanwhile, the blue solid curve is also located above the Curzon-Ahlborn efficiency [35] $\eta_{\mathrm{CA}} = 1 - \sqrt{1 - \eta_{\mathrm{C}}}$ (red dashed curve), which is a typical EMP of the finite-time

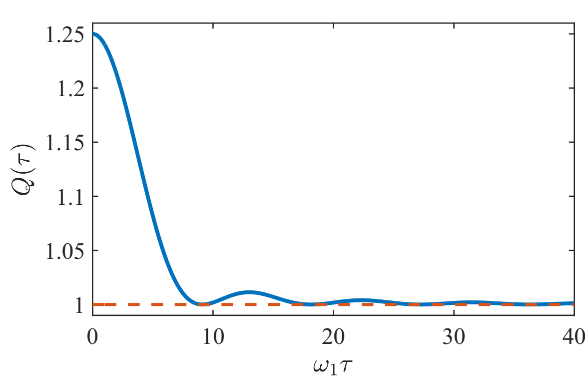

FIG. 6. Time dependence of the nonadiabatic factor. In this figure, the blue solid curve represents $Q(\tau)$ in Eq. (E7); the orange dashed line is $Q(\tau) = 1$. The initial and final frequencies of the harmonic oscillator in the adiabatic process are chosen as $\omega_0 = 0.5$ and $\omega_1 = 1$.

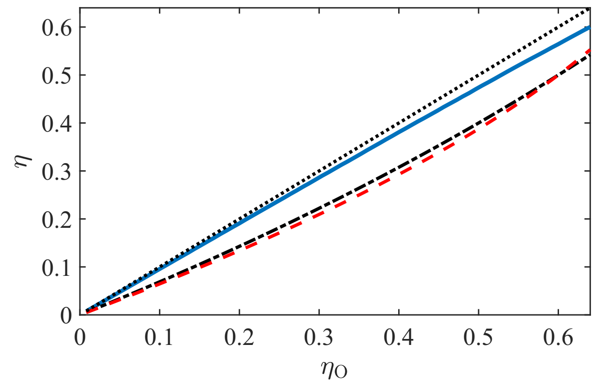

FIG. 7. Efficiency at maximum power of the Otto engine as the function of η_O. In this figure, $\eta_O/\eta_{\mathrm{C}} = 0.8$ is fixed. The blue solid curve represents the EMP of the Otto engine. The black dash-dotted curve is the upper bound for EMP, $\eta_+ = 2\eta_O/(3 - \eta_O)$, of the Otto cycle obtained in Ref. [34] without considering the oscillation of work. The red dashed curve represents the Curzon-Ahlborn efficiency $\eta_{\mathrm{CA}} = 1 - \sqrt{1 - \eta_{\mathrm{C}}}$ and the black dotted line represents the Otto efficiency.

Carnot cycle. This indicates that the quantum effect of the working substance in the Otto cycle is conducive to improving the EMP [34,36].

C. Thermodynamic uncertainty relation (TUR) for the scaled fluctuating efficiency

Because the spectra of a quantum harmonic oscillator have scale property and the system follows the equipartition theorem in the high-temperature limit, we conclude that, in the quasistatic limit, the TUR [Eq. (16)] is valid according to the discussions in Sec. III C.

For the nonadiabatic driving cycle, the results are shown in Fig. 8. The TUR [Eq. (16)] is still valid since $\frac{\Delta\eta^2}{\langle\eta\rangle^2}/(\frac{2}{\langle\delta s\rangle}) \equiv \psi$ increases monotonically with Q_h and Q_c. Here, without loss of generality, we take Q_h as the independent variable in the figure. On the contrary, the TUR may be violated due to the incomplete thermalization in the isochoric processes [22]. One can conclude from Fig. 8(a) that higher temperature makes ψ lower. Moreover, as shown in Fig. 8(b), when $Q_h \to 1$, ψ is closer to 1 in the case with $\eta_O = 0.7$. This is consistent with the discussions in Sec. III C that the condition for $\psi \to 1$ is $\eta_O \to \eta_C$.

V. SUMMARY AND DISCUSSION

In this paper, we come up with the scaled fluctuating efficiency for heat engine in microscale. The moments of the efficiency always exist, and its mean value is equal to the conventional efficiency. Moreover, the fluctuation theorems are reexpressed via the scaled fluctuating efficiency. For spectra of the working substance with scale property, the statistics of the efficiency is fully determined by the partition functions of the working substance in the quasistatic limit. Importantly, we reveal the connection between the TUR and the equipartition theorem.

For a quantum Otto cycle with a harmonic oscillator being the working substance, we obtain the exact expression of the joint characteristic function of work and heat. We find that

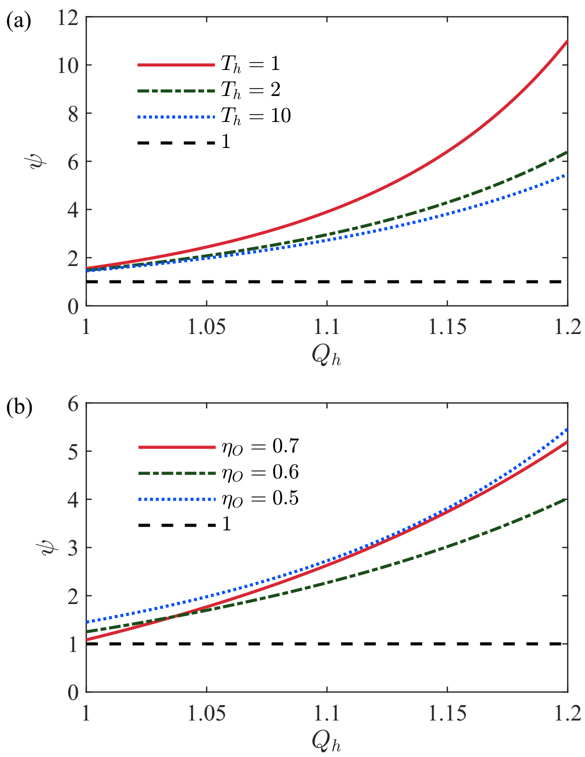

FIG. 8. Thermodynamic uncertainty relation for the scaled fluctuating efficiency with different nonadiabatic factors. (a) $T_h = 1, 2, 10$, $\eta_O = 0.5$; (b) $T_h = 10$, $\eta_O = 0.5, 0.6, 0.7$. In this figure, $Q_c = 1$, $\eta_C = 0.8$, and $\omega_1 = 1$ are fixed parameters.

the Otto efficiency can be reached with a finite output power (the power at maximum efficiency) with some special duration and the EMP surpasses the upper bound obtained in Ref. [34]. It is worth mentioning that the definition in Eq. (3) is not the unique definition of the stochastic efficiency which satisfies the requirements above. For other definitions, the fluctuation of the stochastic efficiency may not be only determined by the output work but by a mixture of the output work and the absorbed heat, e.g., to define the stochastic efficiency as $[-w + a(w - \langle w\rangle) + b(q - \langle q\rangle)]/\langle q\rangle$ with two optional dimensional parameters a, b. In the current work, we choose $a = b = 0$. Consequently, the stochastic efficiency fully reflects the influence of the fluctuation of output work, while cannot explicitly reflect the fluctuation of the absorbed heat of the engine. Especially for $a = 0$, $b = \eta_C$, the efficiency is a linear combination of work and heat, which is a linear function of the stochastic entropy production [Eq. (13)].

The theoretical predictions of the current study can be tested on some state-of-art experiments, such as the Brownian particle system [8] and trapped ion system [37]. As a direct extension, similar to extension efficiency, the coefficient of performance of a refrigerator can be defined as the ratio of the stochastic released heat to the average input work. Then, the statistics of a stochastic refrigerator can be further discussed [38]. Besides, it is expected that the many-body effect of the working substance [25,27,39–43] and the influences of the control protocols for the cycle [33,44–46] on the efficiency statistics and TUR will be taken into consideration in future investigations.

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APPENDIX A: PROOF OF $\langle\eta\rangle \leqslant \eta_O$

As a result of the minimum work principle [20], when the energy levels do not cross during the driving, the average work under finite-time driving $\langle w_c(e)\rangle$ is not less than it under quantum adiabatic driving $\langle w_c(e)\rangle_{\text{adi}}$. Namely, $\delta w_{c(e)} \equiv \langle w_{c(e)}\rangle - \langle w_{c(e)}\rangle_{\text{adi}} \geqslant 0$. Thus the average efficiency of a quantum Otto cycle with the complete thermalization satisfies

$$\begin{aligned}\langle\eta\rangle &= -\frac{\langle w\rangle}{\langle q\rangle} = -\frac{\langle w_c\rangle_{\text{adi}} + \langle w_e\rangle_{\text{adi}} + \delta w_c + \delta w_e}{\langle q\rangle_{\text{adi}} - \delta w_c}\\ &\leqslant -\frac{\langle w_c\rangle_{\text{adi}} + \langle w_e\rangle_{\text{adi}}}{\langle q\rangle_{\text{adi}}} = \eta_O,\end{aligned} \tag{A1}$$

for $\langle q\rangle_{\text{adi}} > -\langle w_c\rangle_{\text{adi}} - \langle w_e\rangle_{\text{adi}} > 0$ and $\delta w_c \geqslant 0, \delta w_e \geqslant 0$, where $\langle q\rangle_{\text{adi}}$ denotes the heat absorbed from the hot reservoir in the quasistatic limit.

APPENDIX B: JOINT CHARACTERISTIC FUNCTION FOR A QUANTUM OTTO CYCLE IN THE QUASISTATIC LIMIT

According to Eq. (6), the expression of the joint characteristic function $\chi(u, v)$ is obtained by a transformation of the characteristic function of work $\chi_c(u) \equiv \chi_c(u, 0)$ and $\chi_h(u) \equiv \chi_h(u, 0)$, i.e.,

$$\begin{aligned}\chi_c(u, v) &= \chi_c(u)|_{u\to u-v,\beta_c\to\beta_c+iv},\\ \chi_h(u, v) &= \chi_h(u)|_{\beta_h\to\beta_h-iv},\end{aligned} \tag{B1}$$

with scale property ($E_n^1 = E_n^0/\epsilon$); in the quasistatic limit, the expressions of the characteristic function $\chi_c(u)$ and $\chi_h(u)$ read

$$\begin{aligned}\chi_c(u) &= \sum_n \frac{e^{-\beta_c E_n^0}}{Z^0(\beta_c)} e^{iu\left(E_n^1 - E_n^0\right)}\\ &= \sum_n \frac{e^{-[\beta_c - iu(\epsilon^{-1}-1)]E_n^0}}{Z^0(\beta_c)}\\ &= \frac{Z^0[\beta_c - iu(\epsilon^{-1}-1)]}{Z^0(\beta_c)}\end{aligned} \tag{B2}$$

and

$$\begin{aligned}\chi_h(u) &= \sum_n \frac{e^{-\beta_h E_n^1}}{Z^1(\beta_h)} e^{iu\left(E_n^0 - E_n^1\right)}\\ &= \sum_n \frac{e^{-[\beta_c - iu(\epsilon-1)]E_n^1}}{Z^1(\beta_h)}\\ &= \frac{Z^1[\beta_h - iu(\epsilon-1)]}{Z^1(\beta_h)}.\end{aligned} \tag{B3}$$

Then, it follows from Eq. (B1) that the joint characteristic function $\chi(u, v)$ reads

$$\chi(u,v) = \frac{Z^0[\beta_c + iv - i(u-v)(\epsilon^{-1}-1)]Z^1[\beta_h - iv - iu(\epsilon-1)]}{Z^0(\beta_c)Z^1(\beta_h)}. \tag{B4}$$

APPENDIX C: JOINT CHARACTERISTIC FUNCTION FOR A QUANTUM HARMONIC-OSCILLATOR HEAT ENGINE

For a harmonic oscillator with time-dependent frequency in an adiabatic process during time $[0, \tau]$, the Hamiltonian is

$$H(t) = \frac{p^2}{2m} + \frac{1}{2}m\omega(t)^2x^2. \tag{C1}$$

Then, the characteristic functions of work $\chi_c(u)$, $\chi_h(u)$ (see Appendix B) read [23,24]

$$\chi_c(u) = 2\sinh\left(\frac{\beta\omega_0}{2}\right)\{2\cos(u\omega_1)\cos[(u-i\beta_c)\omega_0] + 2Q_c\sin(u\omega_1)\sin[(u-i\beta_c)\omega_0] - 2\}^{-\frac{1}{2}}, \tag{C2}$$

$$\chi_h(u) = 2\sinh\left(\frac{\beta\omega_1}{2}\right)\{2\cos(u\omega_0)\cos[(u-i\beta_h)\omega_1] + 2Q_h\sin(u\omega_0)\sin[(u-i\beta_h)\omega_1] - 2\}^{-\frac{1}{2}}, \tag{C3}$$

where

$$Q_c = \frac{\omega_1}{2\omega_0}\left[y_1(\tau_c)^2 + y_2(\tau_c)^2 + \frac{\dot{y}_1(\tau_c)^2 + \dot{y}_2(\tau_c)^2}{\omega_1^2}\right], \tag{C4}$$

the overhead dot denotes the time derivative, and y_1 and y_2 are the two general solutions of the classical harmonic oscillator, i.e.,

$$\ddot{y}(t) + \omega(t)^2y(t) = 0, \tag{C5}$$

with the initial value $\{y_1(0), y_2(0), \dot{y}_1(0), \dot{y}_2(0)\} = \{1, 0, 0, \omega_0\}$. Similarly, the expression of Q_h is given by the replacement $\omega_0 \leftrightarrow \omega_1, t \in [0, \tau_c] \rightarrow t \in [\tau_c + t_h, \tau_c + t_h + \tau_h]$.

Then, the expression of $\chi(u, v)$ is obtained by $\chi(u, v) = \chi_c(u, v)\chi_h(u, v)$, where [Eq. (B1)]

$$\begin{aligned}
\chi_c(u,v) &= \chi_c(u)|_{u\rightarrow u-v,\beta_c\rightarrow\beta_c+iv} \\
&= 2\sinh\left(\frac{\beta\omega_0}{2}\right)\{2\cos[(u-v)\omega_1]\cos[(u-i\beta_c)\omega_0] + 2Q_c\sin[(u-v)\omega_1]\sin[(u-i\beta_c)\omega_0] - 2\}^{-\frac{1}{2}}, \\
\chi_h(u,v) &= \chi_h(u)|_{\beta_h\rightarrow\beta_h-iv} \\
&= 2\sinh\left(\frac{\beta\omega_1}{2}\right)\{2\cos(u\omega_0)\cos[(u-v-i\beta_h)\omega_1] + 2Q_h\sin(u\omega_0)\sin[(u-v-i\beta_h)\omega_1] - 2\}^{-\frac{1}{2}}.
\end{aligned}$$

APPENDIX D: AVERAGE EFFICIENCY AND EFFICIENCY FLUCTUATION

The average output work and average absorbed heat per cycle can be obtained using Eqs. (9), (10), (17), and (18) as

$$-\langle w\rangle = -\frac{1}{2}[(\omega_1Q_c - \omega_0)\vartheta_c - (\omega_1 - \omega_0Q_h)\vartheta_h] \tag{D1}$$

and

$$\langle q\rangle = \frac{\omega_1}{2}(\vartheta_h - Q_c\vartheta_c), \tag{D2}$$

where

$$\vartheta_h \equiv \coth\frac{\beta_h\omega_1}{2}, \quad \vartheta_c \equiv \coth\frac{\beta_c\omega_0}{2}. \tag{D3}$$

Substituting Eqs. (D1) and (D2) into Eq. (3), the average efficiency is obtained as

$$\langle\eta\rangle = \frac{(\omega_1 - \omega_0Q_h)\vartheta_h - (\omega_1Q_c - \omega_0)\vartheta_c}{\omega_1(\vartheta_h - Q_c\vartheta_c)}. \tag{D4}$$

Then, Eq. (D4) is further expressed as Eq. (19) with η_O as

$$\langle\eta\rangle = \eta_O - \frac{\epsilon\sum_{\alpha=h,c}\vartheta_\alpha(Q_\alpha - 1)}{\vartheta_h - Q_c\vartheta_c}. \tag{D5}$$

In addition, the variance of work is obtained from Eq. (11) as

$$\Delta w^2 = \frac{\omega_1^2}{4}(1-\epsilon)^2(\vartheta_h^2 + \vartheta_c^2 - 2) + \frac{\omega_1^2}{2}\left[\vartheta_h^2(Q_h^2 - 1)\epsilon^2 + \vartheta_c^2(Q_c^2 - 1) - \epsilon\sum_{\alpha=h,c}(Q_\alpha - 1)(\vartheta_\alpha^2 - 1)\right]. \tag{D6}$$

Then, the variance of the efficiency, $\Delta\eta^2 = \Delta w^2/\langle q\rangle^2$, illustrated in Eq. (20) is obtained by using Eq. (D6) and Eq. (D2).

APPENDIX E: EXPLICIT TIME DEPENDENCE OF THE NONADIABATIC FACTOR

For the specific driving protocol in Eq. (22), the time dependence of the nonadiabatic factor in Eq. (23) can be directly calculated from its definition [Eq. (C4)] [27]. Here, we present another approach with respect to the internal energy of the working substance. It follows from Ref. [22] that the evolution of the harmonic oscillator in an adiabatic process during time $t \in [0, \tau]$ can be described by a linear differential equation as

$$\frac{d}{dt}\vec{\phi}(t) = \mathcal{M}(t)\vec{\phi}(t). \tag{E1}$$

Here,

$$\vec{\phi}(t) \equiv [\langle H(t)\rangle \quad \langle L(t)\rangle \quad \langle D(t)\rangle]^{\mathrm{T}}, \tag{E2}$$

where $\langle\cdot\rangle$ denotes the ensemble average with respect to the density matrix of the oscillator and T denotes the matrix transpose. $H(t) = p^2/(2m) + m\omega^2(t)x^2/2$, $L(t) = p^2/(2m) - m\omega^2(t)x^2/2$, and $D(t) = \omega(t)(xp + px)/2$ are respectively the Hamiltonian, the Lagrangian, and the generator of the scale transformation. The time-dependent matrix reads

$$\mathcal{M}(t) = \begin{pmatrix} \dot{\omega}/\omega & -\dot{\omega}/\omega & 0 \\ -\dot{\omega}/\omega & \dot{\omega}/\omega & -2\omega \\ 0 & 2\omega & \dot{\omega}/\omega \end{pmatrix}. \tag{E3}$$

The general solution of Eq. (E1) follows as

$$\vec{\phi}(\tau) = \mathscr{T}_{\leftarrow} \exp\left[\int_0^\tau \mathcal{M}(t)dt\right]\vec{\phi}(0), \tag{E4}$$

where $\mathscr{T}_{\leftarrow}$ denotes the time-ordered operation. For the specific protocol in Eq. (22), the matrix $\mathcal{M}(t)$ is independent of t. For the thermal equilibrium initial state, $\vec{\phi}(0) = [\langle H(0)\rangle \quad 0 \quad 0\ ]^{\mathrm{T}}$, we find

$$\vec{\phi}(\tau) = \begin{pmatrix} \dfrac{-\frac{\omega_f}{\omega_i}\left[\left(\frac{\omega_f}{\omega_i}\right)^{-\sqrt{1-a^2\tau^2}} + \left(\frac{\omega_f}{\omega_i}\right)^{\sqrt{1-a^2\tau^2}} - 2a^2\tau^2\right]}{2(a^2\tau^2-1)} \\ \dfrac{\left(\frac{\omega_f}{\omega_i}\right)^{1-\sqrt{1-a^2\tau^2}}\left[1-\left(\frac{\omega_f}{\omega_i}\right)^{2\sqrt{1-a^2\tau^2}}\right]}{2\sqrt{1-a^2\tau^2}} \\ \dfrac{a\tau\left(\frac{\omega_f}{\omega_i}\right)^{1-\sqrt{1-a^2\tau^2}}\left[1-\left(\frac{\omega_f}{\omega_i}\right)^{\sqrt{1-a^2\tau^2}}\right]^2}{2(a^2\tau^2-1)} \end{pmatrix}\langle H(0)\rangle, \tag{E5}$$

where $a = 2\omega_i\omega_f/(\omega_f - \omega_i)$. Thus the internal energy of the system at $t = \tau$ is

$$\langle H(\tau)\rangle = \frac{a^2\tau^2 - \cos[\sqrt{a^2\tau^2 - 1}\ln(\omega_f/\omega_i)]}{a^2\tau^2 - 1}\frac{\omega_f}{\omega_i}\langle H(0)\rangle. \tag{E6}$$

Consequently, the nonadiabatic factor is obtained by [25]

$$Q(\tau) = \frac{\langle H(\tau)\rangle}{\langle H\rangle_{\mathrm{adi}}} = 1 + \frac{\{1 - \cos[\sqrt{a^2\tau^2 - 1}\ln(\omega_f/\omega_i)]\}}{a^2\tau^2 - 1}, \tag{E7}$$

where $\langle H\rangle_{\mathrm{adi}} = \langle H(a\tau \to \infty)\rangle = \langle H(0)\rangle\omega_f/\omega_i$ is the internal energy of the system at the end of the process under quantum adiabatic driving. In the short-time limit $a\tau \to 0$ and long-time limit $a\tau \to \infty$, it is easy to check that

$$\lim_{a\tau\to 0} Q(\tau) = 1 + \frac{[\ln(\omega_f/\omega_i)]^2}{2}, \qquad \lim_{a\tau\to\infty} Q(\tau) = 1. \tag{E8}$$

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