Quantumness beyond entanglement: The case of symmetric states

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Nowadays, it is accepted that truly quantum correlations can exist even in the absence of entanglement. For the case of symmetric states, a physically trivial unitary transformation can alter a state from entangled to separable, and vice versa. We propose to certify the presence of quantumness via an average of a state's bipartite entanglement properties over all physically relevant modal decompositions. We investigate extremal states for such a measure: SU(2)-coherent states possess the least quantumness, whereas the opposite extreme is inhabited by states with maximally spread Majorana constellations.

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I. INTRODUCTION

Entanglement is commonly understood as the inability to describe the state of a compound system in terms of the states of its constituent parts [1]. As it stands, this concept may also be applied to different (classical) degrees of freedom of a physical system [2–4]. However, the possibility of performing separate measurements on two subsystems, which is a key aspect of entanglement, does not hold for these classical counterparts. Truly quantum entanglement exceeds our understanding of classical correlations [5–7] and can be ascribed to the intricacies of the measurement process in the quantum domain [8,9]. This is the main motivation that fueled the search for a complete characterization of correlations present in a state [10–20].

It is a repeated mantra that entanglement is a fragile, yet crucial resource for performing useful quantum tasks. However, a few *obiter dicta* are in order here. First, the presence of entanglement does not guarantee the quantumness of a state: in polarization optics, the canonical coherent states, agreed upon to be the most classical states, may be highly entangled from the naive viewpoint of entanglement between polarization modes [21,22]. Second, the presence of quantumness does not necessarily require entanglement: there exist separable states that nevertheless exhibit traits unparalleled in the classical world [23]. Third, significantly entangled states need not be fragile: for example, spin-squeezed states are highly entangled, yet particularly robust [24,25].

Photonic systems constitute a particularly versatile platform to implement quantum protocols. But, as optical fields can be decomposed in a variety of fundamental modes (i.e., as it is straightforward to change the partitioning of the Hilbert space into modes), the encoding of quantum information in photons is not unique. Actually, a mode transformation can alter a state from being entangled to being separable, and vice versa [26].[1] One might rightly argue that the physics of entanglement in this case should not change just by altering the basis [4], as changing the basis here is akin to producing entanglement by tilting one's head: a wave plate can enact this transformation. In other words, there is more to quantumness than entanglement: entanglement relies on a preferred decomposition of Hilbert space, whereas quantumness persists in all sensible decompositions. In consequence, a *bona fide* criterion of quantumness inspired by standard entanglement measures should assign no preference to any of these modal decompositions.

An alternative and popular quantification of quantumness is through the negativity of quasiprobability distributions [27–36]. This, however, is different from entanglement and is only applicable to physical systems with a particular set of dynamical variables and thus a particular mathematical structure. Our investigation is independent from these quasiprobability distributions and is therefore unconstrained by continuous variable systems.

[1]Mode transformations can, in principle, be done with arbitrary quantum states, but they are particularly simple to implement in photonic systems.

In this paper, we analyze the question of entanglement-inspired quantumness for the relevant case of pure two-mode symmetric states, which are permutationally invariant. In the standard Fock basis of the orthogonal modes, they can be written as a superposition,

$$|\psi\rangle = \sum_{n=0}^{2S} \psi_n \, |n\rangle_a \, |2S-n\rangle_b \,, \tag{1.1}$$

which shows that they contain exactly $2S$ excitations. In this decomposition over modes a and b, which may for example correspond to horizontally and vertically polarized states of light, the state is entangled if and only if more than a single coefficient ψ_n is nonzero. However, changing the modal decomposition through a physically trivial operation can change the state's entanglement properties, implying that entanglement alone is insufficient to fully characterize the quantumness of these states. In fact, these kinds of physically trivial operations cannot generate all types of entangled states, especially not the most useful entangled states, from separable ones [37]. We rectify this situation by taking advantage of the symmetric nature of these states to elucidate the entanglement properties that persist beyond a single modal decomposition.

Our treatment is equally applicable to the entanglement between two halves of a Bose-Einstein condensate (BEC) after it is split [38–41]. In general, the amount of entanglement present depends on the axis along which the BEC is split, so we provide a measure that does not prioritize any splitting axis.

The existence of a symmetry simplifies the mathematical description and makes the states experimentally interesting, largely because symmetrically manipulating the system generally requires fewer resources than addressing individual constituents. In particular, symmetric states are relevant to many experimental situations, such as spin squeezing [42]. These states are also numerically tractable, in that the size of their Hilbert spaces grows only linearly with S, as opposed to exponentially. For these reasons, there have been numerous attempts to characterize the entanglement properties of symmetric (i.e., bosonlike) states [43–52].

A considerable amount of work has since been done using a multipartite description of symmetric states. In this scenario, the Hilbert space of the systems is considered as a tensor product of $2S$ single-qubit Hilbert spaces. In such a partitioning, changing the modal decomposition as above amounts to a series of local operations and thus does not affect the overall entanglement properties [53–55]. This has consequently led to entanglement measures defined from the perspective of multipartite entanglement [56–58].

Still, it seems natural to address the entanglement properties of states such as Eq. (1.1) from a bipartite perspective to properly describe the quantumness found in, e.g., arbitrary spin-S systems. In addition, we should have a mode-independent quantification of the total quantumness present in such a system, as measured by a proper measure of entanglement. Here, we tackle this problem by averaging a bipartite entanglement measure over all modal decompositions to provide a covariant notion of quantumness. This rectifies the apparent equivalence between the bipartite entanglement properties of SU(2)-coherent states with $\psi_0 = 1$ and other spin projection eigenstates with $\psi_n = 1$ for some $n \neq 0, 2S$, giving a measure of quantumness that tracks the entanglement that persists through all physically equivalent modal decompositions.

Averaging entanglement over the unitary invariant measure on the space of pure states has been discussed before [59–61]. However, our measure, being SU(2) covariant, appears as a sum of multipole moments of a state, which allows us to connect quantumness to its geometrical properties. Exploiting the Majorana representation [62,63], the problem appears to be closely related to distributing points over the surface of the Bloch (or Poincaré) sphere. We recall that the question of distributing points uniformly over a sphere has not only inspired mathematical research [64,65], but it has been attracting the attention of physicists working in a variety of fields [66–78]. We find that the most quantum states have these points maximally spread, whereas the most classical states are the SU(2)-coherent states, which are represented by the most concentrated configuration: just a single point. This satisfies all of the *desiderata* for a bipartite entanglement measure that respects the SU(2) nature of symmetric states and should prove useful to the many applications in which the quantumness of spin-S states is tied to their advantages in quantum metrology and quantum information protocols.

II. SU(2)-COVARIANT MEASURE OF BIPARTITE ENTANGLEMENT

To assess the amount of entanglement present in a pure state (1.1), we shall use the linear entropy of the reduced density matrices ϱ_i ($i \in \{a, b\}$),

$$\mathcal{E}(|\psi\rangle) = 1 - \mathrm{Tr}\left(\varrho_i^2\right), \tag{2.1}$$

where $\varrho_a = \mathrm{Tr}_b(\varrho)$ (analogously for ϱ_b) and $\varrho = |\psi\rangle\langle\psi|$ is the density matrix of the total system. In terms of the Schmidt coefficients ψ_n, the linear entropy is given by [63]

$$\mathcal{E}(|\psi\rangle) = 1 - \sum_{n=0}^{2S} |\psi_n|^4, \tag{2.2}$$

where $\mathcal{E} = 0$ implies a separable state and $\mathcal{E} = \frac{2S}{2S+1}$ a fully entangled one, where the former has a single nonzero Schmidt coefficient and the latter, like Bell states, has $2S+1$ Schmidt coefficients with equal magnitude [79]. Since the linear entropy is an entanglement monotone for bipartite pure states, it fully characterizes the entanglement present in this case. Changing the mode decomposition changes the Schmidt coefficients of a state, thereby changing the linear entropy $\mathcal{E}$.

Transforming the modes is represented by a unitary transformation $R \in \mathrm{SU}(2)$. This can be written in the form

$$R(\theta, \phi) = \exp\left[\frac{\theta}{2}\left(e^{i\phi} a b^\dagger - e^{-i\phi} a^\dagger b\right)\right], \tag{2.3}$$

where a and b are the bosonic operators responsible for annihilating excitations in modes a and b, respectively. For example, the separable [SU(2)-coherent] state $|2S\rangle_a \, |0\rangle_b$ can be transformed by an SU(2) rotation into $|2S\rangle_{a+b} \, |0\rangle_{a-b} = R(\frac{\pi}{2}, 0) \, |2S\rangle_a \, |0\rangle_b$, where the new modes are annihilated by linear combinations of the original bosonic operators $a \pm b$. This rotated state can be expressed in the original mode decomposition as $|2S\rangle_{a+b}|0\rangle_{a-b} = 2^{-2S}\sum_{n=0}^{2S}\sqrt{\binom{2S}{n}}\,|n\rangle_a|2S-n\rangle_b$; the state is separable in one basis and highly entangled in the other.

To make an SU(2)-covariant measure that treats states such as $|2S\rangle_a|0\rangle_b$ and $|2S\rangle_{a+b}|0\rangle_{a-b}$ on the same footing, we average $\mathcal{E}$ over all of the relevant partitions of Hilbert space. Using the normalized Haar measure [80] dR for SU(2), our averaged entanglement measure reads

$$\bar{\mathcal{E}}(|\psi\rangle) = \int dR\,\mathcal{E}(R\,|\psi\rangle). \tag{2.4}$$

In the language of polarization, this is equivalent to averaging the entanglement found after passing through a random wave plate, thus giving no privilege to a particular basis, such as horizontal and vertical or diagonal and antidiagonal, for analyzing the entanglement.

The action of R on the coefficients ψ_n is not straightforward, so we instead evaluate this quantity by resorting to a parametrization of symmetric states that is better suited to describing SU(2) transformations. To this end, we start by expressing the density matrix ϱ as

$$\varrho = \sum_{K=0}^{2S}\sum_{q=-K}^{K}\varrho_{Kq}\,T_{Kq}, \tag{2.5}$$

where the irreducible tensors (also called polarization operators) associated with spin S are given by [81,82]

$$T_{Kq} = \sqrt{\frac{2K+1}{2S+1}}\sum_{m,m'=-S}^{S} C_{Sm,Kq}^{Sm'}\,|S,m'\rangle\langle S,m|, \tag{2.6}$$

with $C_{S_1m_1,S_2m_2}^{Sm}$ denoting the Clebsch-Gordan coefficients [83] that couple a spin S_1 and a spin S_2 to a total spin S and vanish unless the usual angular momentum coupling rules are satisfied: $0 \leqslant K \leqslant 2S$ and $-K \leqslant q \leqslant K$. These are $(2S+1)^2$ operators that constitute a basis of the space of linear operators acting on the Hilbert space and the correspondence between spin eigenstates and two-mode bosonic states is given by $|S,m\rangle = |S+m\rangle_a|S-m\rangle_b$. The expansion coefficients

$$\varrho_{Kq} = \mathrm{Tr}(\varrho\,T_{Kq}^{\dagger}) \tag{2.7}$$

are called the state multipoles and contain the complete information about the state sorted in the appropriate way: they are the Kth-order moments of the generators. Normalization dictates that $\varrho_{00} = 1/\sqrt{2S+1}$, and Hermiticity implies $\varrho_{Kq}^{*} = (-1)^q\varrho_{K-q}$.

Due to their very same definition, the multipoles inherit the proper transformation under SU(2); that is, if the state experiences the unitary transformation $\widetilde{\varrho} = R\,\varrho\,R^{\dagger}$, the multipoles transform as

$$\widetilde{\varrho}_{Kq} = \sum_{q'=-K}^{K} D_{q'q}^{K*}(R)\,\varrho_{Kq'}, \tag{2.8}$$

where $D_{q'q}^{K}(R)$ are the Wigner D-matrices [83].

The linear entropy of the transformed state can be computed via the reduced density matrix

$$\widetilde{\varrho}_a = \sum_{Kq}\sqrt{\frac{2K+1}{2S+1}}\,\widetilde{\varrho}_{Kq}\sum_{m=-S}^{S} C_{Sm,Kq}^{Sm}\,|S+m\rangle_{a\,a}\langle S+m|. \tag{2.9}$$

Then, using the orthogonality of the Clebsch-Gordan coefficients, the trace of the square of $\widetilde{\varrho}_a$ yields

$$\mathcal{E}(R\,|\psi\rangle) = 1 - \sum_{K,K'}\widetilde{\varrho}_{K0}\widetilde{\varrho}_{K'0}^{*}\delta_{KK'}. \tag{2.10}$$

We can then average over the rotations using properties of the D-matrices. To this end, we note that

$$\begin{aligned}\int dR\,\widetilde{\varrho}_{K0}\widetilde{\varrho}_{K'0}^{*} &= \sum_{q,q'}\varrho_{Kq}\varrho_{K'q'}^{*}\int dR\,D_{q0}^{K*}\,D_{q'0}^{K'}\\ &= \frac{\delta_{KK'}}{2K+1}\sum_{q=-K}^{K}|\varrho_{Kq}|^2.\end{aligned} \tag{2.11}$$

The averaged entanglement thus becomes

$$\bar{\mathcal{E}}(|\psi\rangle) = 1 - \sum_{K=0}^{2S}\frac{1}{2K+1}\sum_{q=-K}^{K}|\varrho_{Kq}|^2. \tag{2.12}$$

As the multipoles are directly accessible in the laboratory [84,85], $\bar{\mathcal{E}}$ allows for an experimental certification of quantumness [86]. Furthermore, $\bar{\mathcal{E}}$ involves all the moments, so it improves previous measures relying solely on the variances [87,88].

For the case of pure states that we are dealing with, we expand in the angular-momentum basis as $|\psi\rangle = \sum_m \psi_m\,|S,m\rangle$, so (2.12) takes the form

$$\bar{\mathcal{E}}(|\psi\rangle) = 1 - \frac{1}{2S+1}\sum_{K=0}^{2S}\sum_{q=-K}^{K}\left|\sum_{m,m'=-S}^{S} C_{Sm,Kq}^{Sm'}\psi_{m'}\psi_m^{*}\right|^2, \tag{2.13}$$

which is the quantumness measure that we advocate.

III. EXTREMAL STATES

The averaged linear entropy (2.13) can be regarded as a nonlinear functional of the density matrix. The higher the value of $\bar{\mathcal{E}}$, the greater the value of the average entanglement. Some pure states give the maximal value of $\mathcal{E}$ for a given partition, but no pure state achieves $\mathcal{E} = \frac{2S}{2S+1}$ for all partitions. Maximally mixed states, in contrast, give the maximum value of $\bar{\mathcal{E}}$, but linear entropy is only an entanglement measure for pure states. For this reason, we will restrict our investigation to pure states, using a geometrical picture that relates each state to a set of $2S+1$ points on the surface of a sphere.

We first try to ascertain states that minimize $\bar{\mathcal{E}}$. In Ref. [89], it was claimed that the cumulative multipolar distribution

$$\mathcal{A}_M \equiv \sum_{K=0}^{M}\sum_{q=-K}^{K}|\varrho_{Kq}|^2 \tag{3.1}$$

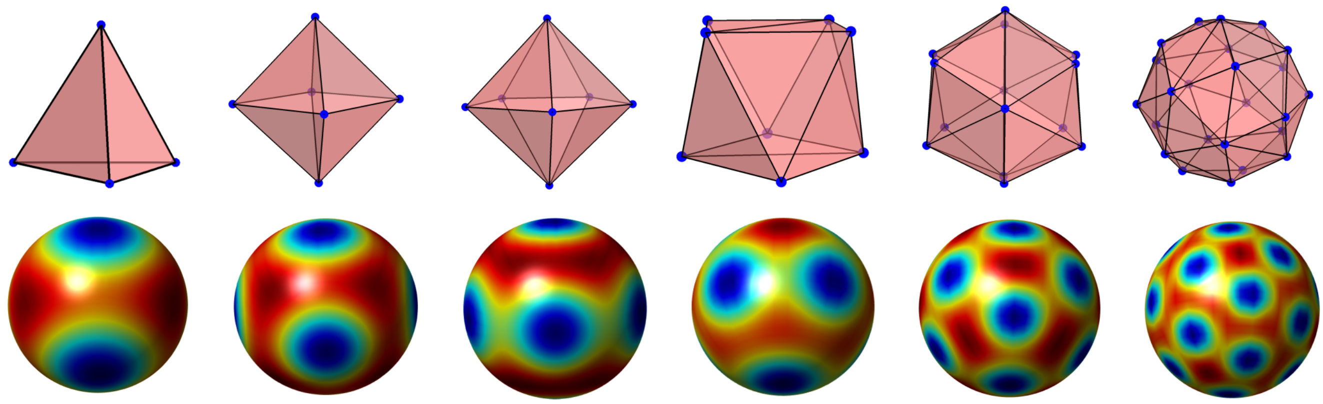

FIG. 1. Density plots of the SU(2) Q functions for the most quantum states, which extremize the bipartite entanglement averaged over all modal decompositions, for the cases $S = 2, 3, 7/2, 4, 6$, and 12 [from left to right; blue (dark disks) indicates the zero values and red (dark polygons) the maximal ones]. On top, we sketch the corresponding Majorana constellation for each state.

is maximal for SU(2)-coherent states for all $M \leqslant S$. These states are defined as [90]

$$|\theta, \phi\rangle = \frac{1}{(1+|\alpha|^2)^S} \exp(\alpha S_+)\,|S, -S\rangle\,, \tag{3.2}$$

where $S_\pm = S_x \pm iS_y$ are the ladder operators for SU(2) and the complex number α corresponds to the stereographic projection of the point (θ, ϕ) on the sphere; viz., $\alpha = \tan(\theta/2)e^{-i\phi}$. The monotonicity of the coefficients $1/(K+1)$ immediately implies that the SU(2)-coherent states minimize $\bar{\mathcal{E}}$, with a value of $\bar{\mathcal{E}}_{\text{coh}}$ determined by

$$1 - \bar{\mathcal{E}}_{\text{coh}} = \frac{1}{4S+1}\sum_{m=0}^{2S} \binom{2S}{m}^2 \binom{4S}{2m}^{-1} = \frac{\sqrt{\pi}\,\Gamma(2S+1)}{2\Gamma(2S+3/2)}. \tag{3.3}$$

This accords with many other quantumness indicators agreeing that SU(2)-coherent states, which correspond to a single point on the surface of the sphere, are the least quantum [21]. Other seemingly separable states with a single nonzero coefficient ψ_n have larger values of $\bar{\mathcal{E}}$, as can be computed explicitly from Eq. (2.13) to yield

$$1 - \bar{\mathcal{E}} = \frac{1}{2S+1}\sum_{K=0}^{2S} \left(C^{Sm}_{Sm,K0}\right)^2. \tag{3.4}$$

This sum grows as $|m|$ approaches S in the same way that the overlap between a vector of length S pointing at an angle $\varphi = \arcsin(m/S)$ from the horizontal added to a vector of length K pointing along the horizontal remains closest to the former vector when φ points toward the north or south pole. The average entanglement for a state $|S, m\rangle$ thus grows monotonically with $S - |m|$, demonstrating that this form of quantumness lifts the degeneracy between states $|S, m\rangle$ and SU(2)-coherent states $|S, S\rangle$ that is otherwise present when only their entanglement properties are evaluated.

Next, we concentrate on maximizing $\bar{\mathcal{E}}$. If we write the set of unknown normalized state amplitudes in Eq. (2.13) as $\psi_m = a_m + ib_m$ ($a_m, b_m \in \mathbb{R}$), we find that the maxima corresponds to a (quartic) polynomial program [91] that can be solved by standard methods. We provide a complete list of the numerical solutions for ψ_m found for different values of S up to 15 in Ref. [92].

Although the coefficients ψ_m completely characterize $|\psi\rangle$, they do not provide a lucid picture of the state. To this end, we will use the concept of Majorana representation [62,63], which maps every $(2S+1)$-dimensional pure state $|\psi\rangle$ into the polynomial

$$\psi(\theta, \phi) = \langle\theta, \phi|\psi\rangle \propto \sum_{m=-S}^{S} \sqrt{\frac{(2S)!}{(S-m)!(S+m)!}}\,\psi_m\,\alpha^{S+m}. \tag{3.5}$$

Up to a global unphysical factor, $|\psi\rangle$ is determined by the set $\{\alpha_i\}$ of the $2S$ complex zeros of $\psi(\theta, \phi)$, suitably completed by points at infinity if the degree of $\psi(\theta, \phi)$ is less than $2S$. A nice geometrical representation of $|\psi\rangle$ by $2S$ points on the unit sphere (often called the constellation) is obtained by an inverse stereographic map of $\{\alpha_i\}$. Two states with the same constellation are the same, up to a global phase. For example, the SU(2)-coherent states have all $2S$ of the "stars" in their constellation co-located at angular coordinates (θ, ϕ). Several decades after its conception, this stellar representation has recently attracted a great deal of attention in several fields [66–78].

Intimately related to the Majorana polynomial $\psi(\theta, \phi)$ is the SU(2) Q-function, defined as

$$Q(\theta, \phi) = |\psi(\theta, \phi)|^2. \tag{3.6}$$

Obviously, the stars $\{\alpha_i\}$ are also the zeros of $Q(\theta, \phi)$, so the Q-function is an attractive way to depict the state to help appreciate the symmetries of $|\psi\rangle$. It is not surprising that it has gained popularity in modern quantum information [21].

The Q-functions and the corresponding Majorana constellations for a few examples of extremal states are shown in Fig. 1, with many more given in Ref. [92]. The resulting constellations have the points symmetrically placed on the unit sphere, which agrees with other previous notions of quantumness, such as states of maximal Wehrl-Lieb entropy [93].

In special dimensions, the constellations show a remarkable additional degree of symmetry, some of which are summarized in Table I. In particular, we get constellations that coincide with the Platonic solids: they are optimal states for quantum communication [94] and for fundamental tests

TABLE I. Symmetries of the constellations associated to the maximal states for the values of S.

S	Group	Order	Constellation
1	C_2	2	
$\frac{3}{2}$	S_3	6	triangle
2	S_4	24	Platonic
$\frac{5}{2}$	D_{12}	12	triangle + poles
3	$C_2 \times S_4$	48	Platonic
$\frac{7}{2}$	D_{20}	20	pentagon + poles
4	D_{16}	16	twisted cube
5	D_{16}	16	twisted cube + poles
6	$C_2 \times A_5$	120	Platonic
7	D_{24}	24	twisted hexagon + poles
8	A_4	12	
12	S_4	24	

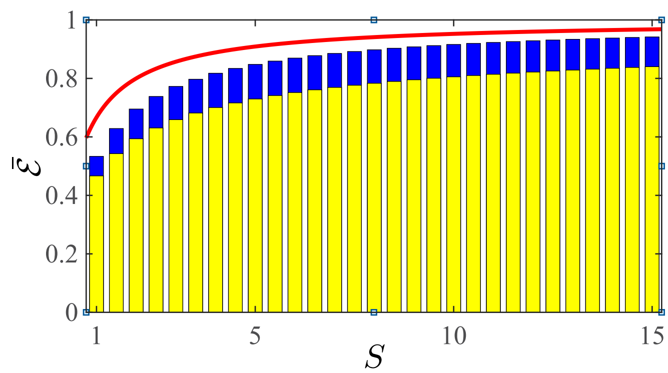

FIG. 2. Average entanglement $\bar{\mathcal{E}}$ for the states of maximal (upper, blue bars) and of minimal (lower, yellow bars) average entanglement as a function of S. The continuous red line on top of the bars represents the upper limit $\mathcal{E} = \frac{2S}{2S+1}$ attainable in (2.2).

of quantum mechanics [95]. Surprisingly, states whose constellations correspond to a twisted cube have higher average entanglement than those corresponding to a cube.

The optimal states have amazing features: for values of S such as 2, 3, 6, 8, and 12, they are maximally unpolarized [89] and they are optimal to estimate rotations about any axis [96], all because they have sufficiently isotropic angular momentum properties which are all optimized by symmetric states. For other values of S, they have highly spread constellations without having isotropic angular momentum properties; when $S = 11/2$ and $13/2$, for example, they are not even isotropic to first order. We again direct the interested reader to the full list given in Ref. [92].

Other criteria of quantumness have been considered in this context of symmetric states and maximally spread Majorana constellations. Among them, the Kings [89] and the Queens [97] of Quantumness seem to be closely related to our approach, where the former are states with maximally isotropic angular momentum properties and the latter are states that are maximally different from convex combinations of SU(2)-coherent states. For some dimensions, the optimal states turn out to be the same, but for others, they are different [21], highlighting the rich physics underlying symmetric states and sphere point picking.

In Fig. 2, we plot the value of the averaged entropy $\bar{\mathcal{E}}$ for the maximal states found numerically as a function of the dimension S. For comparison, we have also included the corresponding values for the minimal states, which correspond to coherent states. As we can appreciate, $\bar{\mathcal{E}}$ approaches the limit value of unity as S grows. One can easily guess that $\bar{\mathcal{E}} \sim 1 - 1/(2S)$, which shows that the higher the value of S, the more quantum the extremal state is.

IV. CONCLUDING REMARKS

In summary, we have comprehensively examined the notion of average entanglement for symmetric states, which is the physically relevant quantity for these states and is directly accessible for realistic experiments. We have proven that SU(2)-coherent states are minimal. Their opposite counterparts, maximizing the average entanglement, have interesting properties. Apart from their indisputable geometrical beauty, there surely is plenty of room for the application of these states.

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