Complementary relation between tripartite entanglement and the maximum steering inequality violation

Tian-Ze Dai, Yi Fan, and Liang Qiu*
School of Materials Science and Physics, China University of Mining and Technology, Xuzhou 221116, People's Republic of China

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We investigate the relation between tripartite entanglement of the three-qubit system and the maximum steering inequality violation of the reduced two-qubit states. Firstly, it is found that a single parameter family of entangled three-qubit pure states have the maximum steering inequality violation among all of the three-qubit pure states for a fixed amount of tripartite entanglement. The tripartite entanglement is quantified by the genuinely multipartite concurrence, generalized geometric measure, and tangle. Subsequently, the complementary relation between tripartite entanglement and the maximum steering inequality violation for an arbitrary three-qubit pure state can be established. Particularly, the result also holds for three-qubit mixed states if the entanglement measure is tangle. The complementary relations indicate that the maximum steering inequality violation of the reduced two-qubit system is at the expense of tripartite entanglement.

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I. INTRODUCTION

Quantum entanglement describes quantum correlations among distant parties that are completely forbidden in the classical regime [1]. Multipartite entangled states have vital applications in quantum information processing tasks [2–4]. A fundamental issue in quantum information science is to quantify the entanglement of the multipartite system. Various measures of multipartite entanglement have been proposed, such as the genuinely multipartite concurrence (GMC) [5,6], generalized geometric measure (GGM) [7–12], tangle [13], and concurrence fill [14].

Quantum steering lies between entanglement and Bell nonlocality [15]. The idea of Einstein-Podolsky-Rosen (EPR) steering was first introduced in the bipartite scenario by Schrödinger [16,17] in the context of the EPR argument [1]. Much later, a criterion for experimentally demonstrating the EPR argument using the Heisenberg uncertainty relation was proposed [18]. Steering describes a nontrivial trait of quantum physics that the local measurements on one side can "steer" the state on the other side. Quantum steering was rigorously and formally defined from the perspective of quantum information theory in 2007 [19]. Since then, quantum steering has attracted much attention in different fields [20]. The violations of steering criteria, which are obtained using correlations, state assemblages, and full information, can be used to detect quantum steering. For example, steering inequalities have been designed to observe steering [18,21–29]. Several experiments to demonstrate the effect of steering have been performed [29,30]. Steerable states are beneficial for randomness generation [31], subchannel discrimination [32], quantum information processing [33], and one-sided device-independent processing in quantum key distribution [34].

The correlation statistics of two-body subsystems are always used to infer the multipartite properties of a composite quantum system [35–41]. As quantum correlations and nonlocality both are essential as resources in information theory, it is interesting to establish the link between them. The complementarity between tripartite quantum correlations and the Bell-inequality violation in three-qubit states has been investigated in Ref. [42]. The relations between the reduced bipartite steering and bipartite as well as tripartite entanglement of the three-qubit states have been established [43]. Moreover, a complementarity relation is established between the capacity of multiport classical information transmission via quantum states and multiparty quantum correlation measures for three-qubit pure states; this is important because it establishes a connection between the multiparty entanglement content of multipartite quantum states and their ability to act as substrates in quantum information protocols [44]. Despite remarkable progress, how the correlation statistics of two-body subsystems depends on the multipartite properties of a composite system is still not clear. Particularly, comparatively little is known about the dependence of the reduced bipartite steering of a three-qubit state on tripartite entanglement of the three-qubit state. In this paper, we study the complementary relations between tripartite entanglement and the reduced bipartite steering in three-qubit states inspired by the results given in Refs. [42–44]. It is found that, among all the three-qubit pure states for a fixed amount of tripartite entanglement, the maximally steering inequality violating states give the maximum steering inequality violation. Therefore a complementary relation between tripartite entanglement and the maximum steering inequality violation can be obtained. The maximally steering inequality violating states lie at the boundary of the complementary relation. The measures of tripartite entanglement considered are the GMC, GGM, and tangle.

*cumtlqiu@163.com

This paper is organized as follows. In Sec. II, we briefly review the measures of tripartite entanglement and the three-setting linear steering inequality. Subsequently, the relations between tripartite entanglement and the steering inequality violation of the reduced bipartite states are investigated in Sec. III. The discussion and conclusion are given in Sec. IV.

II. MEASURES OF TRIPARTITE ENTANGLEMENT AND THE THREE-SETTING LINEAR STEERING INEQUALITY VIOLATION

The multipartite quantum state is genuinely entangled if it is not separable in any bipartite split. We present the measures of tripartite entanglement that will be required later in this paper. They are the GMC, GGM, and tangle. While the first two measures are based on the concept of distance to a relevant class of states, the third one is based on the concept of monogamy. The three-setting linear steering inequality used in this paper is the one given by Cavalcanti *et al.* [21], and we will also introduce it in this section.

A. GMC

In order to distinguish genuine multipartite entanglement from partial entanglement, Ma *et al.* defined a generalized concurrence, GMC, as an entanglement measure for a multipartite system [5]. The GMC of an n-partite pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_n$ with $\dim(\mathcal{H}_i) = d_i (i = 1, 2, \dots, n)$ is given as

$$\mathcal{C}(|\psi\rangle) = \min_{\mu_i} \sqrt{2\left[1 - \mathrm{Tr}\left(\rho^2_{A_{\mu_i}}\right)\right]}, \tag{1}$$

in which μ_i denotes the bipartition in the set of all possible bipartitions $\{A_i|B_i\}$. The GMC can be generalized to the case of mixed states via the convex roof construction

$$\mathcal{C}(\rho) = \inf_{\{p_i,|\psi_i\rangle\}} \sum_i p_i \mathcal{C}(|\psi_i\rangle), \tag{2}$$

where the infimum is over all possible decompositions $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$. The GMC is exactly the square root of the length of the shortest edge of the concurrence triangle for a three-qubit system [14].

B. GGM

The GGM, as a measure of genuine multipartite entanglement for pure states, is defined as the distance of the n-partite state $|\psi\rangle$ from the set of all multiparty states $|\varphi\rangle$ that are not genuinely entangled

$$\mathcal{G}(|\psi\rangle) = 1 - \max_{|\varphi\rangle} |\langle\varphi|\psi\rangle|^2. \tag{3}$$

Here, the maximization is done over all pure states that are not genuinely n-party entangled. In Refs. [7–9], it is shown that an equivalent mathematical expression of the GGM reads

$$\mathcal{G}(|\psi\rangle) = 1 - \max\left\{\lambda^2_{A:B} \middle| A \cup B = \{1, 2, \dots, n\},\ A \cap B = \emptyset\right\}, \tag{4}$$

where $\lambda_{A:B}$ is the maximal Schmidt coefficient in the $A:B$ split of the state $|\psi\rangle$.

C. Tangle

Monogamy of quantum correlations can be used to quantify the shareability of quantum correlations in multipartite systems [13]. Tangle, as an entanglement measure of a three-qubit system, is equal to the quantum monogamy score corresponding to the square of concurrence [42]

$$\tau(\rho_{ABC}) = C^2_{A:BC} - C^2_{AB} - C^2_{AC}, \tag{5}$$

where $C_{XY} (X, Y = A, B, C)$ is the concurrence of a two-qubit system and is defined as $C_{XY} = \max\{0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ with $\lambda_1, \dots, \lambda_4$ being the square roots of the eigenvalues of $\rho_{XY}[(\sigma_y \otimes \sigma_y)\rho^*_{XY}(\sigma_y \otimes \sigma_y)]$. ρ^*_{XY} is the complex conjugation of ρ_{XY}, and σ_y is the Pauli matrix. Tangle is always nonnegative due to the fact that the square of concurrence is monogamous.

For a three-qubit pure state given as $|\psi\rangle = \sum_{ijk} a_{ijk}|ijk\rangle$ in the standard basis, the tangle is [13]

$$\tau(|\psi\rangle) = 4|d_1 - 2d_2 + 4d_3|, \tag{6}$$

where

$$\begin{aligned} d_1 &= a^2_{000}a^2_{111} + a^2_{001}a^2_{110} + a^2_{010}a^2_{101} + a^2_{100}a^2_{011}, \\ d_2 &= a_{000}a_{111}a_{011}a_{100} + a_{000}a_{111}a_{101}a_{010} \\ &\quad + a_{000}a_{111}a_{110}a_{001} + a_{011}a_{100}a_{101}a_{010} \\ &\quad + a_{011}a_{100}a_{110}a_{001} + a_{101}a_{010}a_{110}a_{001}, \\ d_3 &= a_{000}a_{110}a_{101}a_{011} + a_{111}a_{001}a_{010}a_{100}. \end{aligned} \tag{7}$$

D. The three-setting linear steering inequality violation

Some steering inequalities, which are derived from the assumption of the local hidden states model, can indicate the occurrence of steering by the violation of them. As an example, Cavalcanti *et al.* proposed the linear steering inequality [21] to check whether a bipartite state is steerable from Alice to Bob

$$F_n(\rho_{AB}, \mu) = \frac{1}{\sqrt{n}}\left|\sum_{k=1}^{n} \langle A_k \otimes B_k\rangle\right| \leqslant 1, \tag{8}$$

where $\langle A_k \otimes B_k\rangle = \mathrm{Tr}(\rho_{AB}(A_k \otimes B_k))$, $A_k = \hat{a}_k \cdot \vec{\sigma}$, and $B_k = \hat{b}_k \cdot \vec{\sigma}$, with $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ being a vector composed of the Pauli matrices. $\hat{a}_k, \hat{b}_k \in \mathbb{R}^3$ are unit and orthonormal vectors. The set of measurements is given by $\{\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n, \hat{b}_1, \hat{b}_2, \dots, \hat{b}_n\}$. Obviously, Alice and Bob can perform n dichotomic measurements on their respective subsystems.

In the Hilbert-Schmidt representation, a two-qubit state can be given as

$$\rho_{AB} = \frac{1}{4}\left[I_2 \otimes I_2 + \vec{a} \cdot \vec{\sigma} \otimes I_2 + I_2 \otimes \vec{b} \cdot \vec{\sigma} + \sum_{i,j} t_{ij}\sigma_i \otimes \sigma_j\right]. \tag{9}$$

In the equation, $\vec{a}$ and $\vec{b}$ are the local Bloch vectors. $t_{ij} = \mathrm{Tr}(\rho_{AB}(\sigma_i \otimes \sigma_j))$, and $T_{AB} = [t_{ij}]$ is the correlation matrix.

For the case of the three measurement settings, i.e., $n = 3$, the three-setting linear steering inequality can be obtained from Eq. (8), and the state ρ_{AB} is F_3 steerable if and only if [43]

$$S_{AB} = \mathrm{Tr}\big(T_{AB}^{T}T_{AB}\big) > 1, \tag{10}$$

where the superscript T denotes the transpose of the correlation matrix T_{AB}. Actually, $S_{AB} = \sum_{i,j=1}^{3}\langle\sigma_i \otimes \sigma_j\rangle^2$ for two-qubit states [45]. Among the three reduced two-qubit states of a three-qubit state ρ_{ABC}, $S_{\max}(\rho_{ABC})$ is introduced to pick the one with the maximum steering inequality violation [43]

$$S_{\max}(\rho_{ABC}) = \max\{S_{AB}, S_{AC}, S_{BC}\}. \tag{11}$$

III. TRIPARTITE ENTANGLEMENT VERSUS THE MAXIMUM STEERING INEQUALITY VIOLATION

For bipartite pure states, the relation between S_{AB} and concurrence C_{AB} is $S_{AB} = 1 + 2C_{AB}^2$, which can be derived with methods similar to those used in Refs. [43,46]. The result indicates that the more entangled the state is, the more steerable the state is. However, we cannot infer anything about steering of the bipartite reduced states of a three-qubit pure state. In this section, the relations between tripartite entanglement of three-qubit pure states and the maximum steering inequality violation for two-qubit reduced states are established.

In order to obtain the results, we introduce the single parameter family of genuinely three-qubit entangled states, which we can call the maximally steering inequality violating states because they give the maximum steering inequality violation among all three-qubit pure states for a fixed amount of tripartite entanglement. The state is

$$|\psi\rangle_\alpha = \frac{1}{\sqrt{2+2\alpha^2}}[|000\rangle + \alpha(|010\rangle + |101\rangle) + |111\rangle], \tag{12}$$

where the state parameter $\alpha \in [0, 1]$. The state $|\psi\rangle_\alpha$ belongs to the Greenberger-Horne-Zeilinger (GHZ) class when $\alpha \in [0, 1)$. For $\alpha = 1$, it belongs to the W class having a zero tangle. States of this class are considered to be the maximally dense-coding-capable states [44] as well as the maximally Bell-inequality violating states [42] because they have the maximum multiport dense coding capacity and Bell-inequality violation for a fixed amount of tripartite quantum correlations.

The GHZ and W class states are two disjoint but complete subsets of genuinely three-qubit entangled pure states. Hence it is sufficient to establish the complementary relations for the GHZ and W class states [42].

The GHZ class states can be converted into the GHZ state using stochastic local quantum operations and classical communication (SLOCC) with nonzero probability and be characterized by parameters $\alpha_X (X = A, B, C)$, β, ϕ as

$$\begin{aligned}|\psi\rangle_{\mathrm{GHZ}} = \frac{1}{\sqrt{\kappa}}[&\cos\beta|000\rangle + e^{i\phi}\sin\beta(\cos\alpha_A|0\rangle + \sin\alpha_A|1\rangle)\\ &\times\otimes(\cos\alpha_B|0\rangle + \sin\alpha_B|1\rangle)\\ &\times\otimes(\cos\alpha_C|0\rangle + \sin\alpha_C|1\rangle)],\end{aligned} \tag{13}$$

where $\kappa = 1 + \cos\alpha_A\cos\alpha_B\cos\alpha_C\cos\phi\sin 2\beta$. $\alpha_X \in (0, \pi/2]$, $\beta \in (0, \pi/4]$, and $\phi \in [0, 2\pi)$.

Similarly, the W class states can be converted into the W state using SLOCC with nonzero probability and be given as

$$|\psi\rangle_{\mathrm{W}} = \sqrt{d}|000\rangle + \sqrt{a}|001\rangle + \sqrt{b}|010\rangle + \sqrt{c}|100\rangle, \tag{14}$$

where $a, b, c, d > 0$ and satisfy the normalizing condition $a + b + c + d = 1$.

A. GMC versus the maximum steering inequality violation

The relation between GMC and the maximum steering inequality violation for three-qubit pure states is derived in this section.

Lemma 1. For a three-qubit pure state $|\psi\rangle$, if the GMC obtains from, for example, an $A : BC$ split, then the maximum steering inequality violation $S_{\max}(|\psi\rangle) = S_{BC}(|\psi\rangle)$.

Proof. For the GHZ class states, the GMC is

$$\begin{aligned}\mathcal{C}(|\psi\rangle_{\mathrm{GHZ}}) = \min\Big\{&\sqrt{2\big[1 - \mathrm{Tr}\big(\rho_A^2\big)\big]}, \sqrt{2\big[1 - \mathrm{Tr}\big(\rho_B^2\big)\big]},\\ &\times\sqrt{2\big[1 - \mathrm{Tr}\big(\rho_C^2\big)\big]}\Big\},\end{aligned} \tag{15}$$

in which $\rho_X (X = A, B, C)$ are the reduced states of $|\psi\rangle_{\mathrm{GHZ}}$. The condition that the GMC is obtained from the bipartition $A : BC$ implies $\mathrm{Tr}(\rho_A^2) \geqslant \mathrm{Tr}(\rho_B^2)$, $\mathrm{Tr}(\rho_A^2) \geqslant \mathrm{Tr}(\rho_C^2)$. The results will deduce the conditions

$$\begin{aligned}&(\cos^2\alpha_A - \cos^2\alpha_B)\sin^2\alpha_C \geqslant 0,\\ &(\cos^2\alpha_A - \cos^2\alpha_C)\sin^2\alpha_B \geqslant 0.\end{aligned} \tag{16}$$

The steering inequality violations $S_{AB}(|\psi\rangle_{\mathrm{GHZ}})$, $S_{AC}(|\psi\rangle_{\mathrm{GHZ}})$, and $S_{BC}(|\psi\rangle_{\mathrm{GHZ}})$ for the reduced states ρ_{AB}, ρ_{AC}, and ρ_{BC} of the GHZ class states are

$$S_{AB}(|\psi\rangle_{\mathrm{GHZ}}) = \frac{\kappa^2 + (2\sin^2\alpha_A\sin^2\alpha_B - \sin^2\alpha_B\sin^2\alpha_C - \sin^2\alpha_A\sin^2\alpha_C)\sin^2 2\beta}{\kappa^2}, \tag{17}$$

$$S_{AC}(|\psi\rangle_{\mathrm{GHZ}}) = \frac{\kappa^2 + (2\sin^2\alpha_A\sin^2\alpha_C - \sin^2\alpha_A\sin^2\alpha_B - \sin^2\alpha_B\sin^2\alpha_C)\sin^2 2\beta}{\kappa^2}, \tag{18}$$

$$S_{BC}(|\psi\rangle_{\mathrm{GHZ}}) = \frac{\kappa^2 + (2\sin^2\alpha_B\sin^2\alpha_C - \sin^2\alpha_A\sin^2\alpha_B - \sin^2\alpha_A\sin^2\alpha_C)\sin^2 2\beta}{\kappa^2}. \tag{19}$$

Using the conditions given in Eq. (16), one can find

$$\begin{aligned}S_{BC}(|\psi\rangle_{\mathrm{GHZ}}) - S_{AB}(|\psi\rangle_{\mathrm{GHZ}}) &= \frac{3}{\kappa^2}(\sin^2\alpha_C - \sin^2\alpha_A)\sin^2\alpha_B\sin^2\beta \geqslant 0,\\ S_{BC}(|\psi\rangle_{\mathrm{GHZ}}) - S_{AC}(|\psi\rangle_{\mathrm{GHZ}}) &= \frac{3}{\kappa^2}(\sin^2\alpha_B - \sin^2\alpha_A)\sin^2\alpha_C\sin^2\beta \geqslant 0.\end{aligned} \tag{20}$$

Thus the maximum steering inequality violation $S_{\max}(|\psi\rangle_{\rm GHZ})$ is equal to $S_{BC}(|\psi\rangle_{\rm GHZ})$.

For the W class states, if the GMC is obtained from the bipartition $A:BC$, i.e., $\mathcal{C}(|\psi\rangle_{\rm W})=\sqrt{2[1-{\rm Tr}(\rho_A^2)]}$, one will get the following conditions:

$$(a+c)b-(a+b)c=a(b-c)\geqslant 0,$$
$$(b+c)a-(a+b)c=b(a-c)\geqslant 0. \tag{21}$$

The steering inequality violations $S_{AB}(|\psi\rangle_{\rm W})$, $S_{AC}(|\psi\rangle_{\rm W})$, and $S_{BC}(|\psi\rangle_{\rm W})$ for the reduced states ρ_{AB}, ρ_{AC}, and ρ_{BC} of the W class states are

$$S_{AB}(|\psi\rangle_{\rm W})=1+8bc-4a(b+c),$$
$$S_{AC}(|\psi\rangle_{\rm W})=1+8ac-4b(a+c),$$
$$S_{BC}(|\psi\rangle_{\rm W})=1+8ab-4c(a+b). \tag{22}$$

The conditions given in Eq. (21) will ensure that $S_{BC}(|\psi\rangle_{\rm W})\geqslant S_{AB}(|\psi\rangle_{\rm W})$ and $S_{BC}(|\psi\rangle_{\rm W})\geqslant S_{AC}(|\psi\rangle_{\rm W})$. Therefore the maximum steering inequality violation $S_{\max}(|\psi\rangle_{\rm W})=S_{BC}(|\psi\rangle_{\rm W})$.

One should note that a similar proof holds for the cases in which the GMC is obtained from the other two bipartitions no matter whether the three-qubit pure states considered are the GHZ or W class states. Hence the proof is completed. ■

With Lemma 1, one can prove the following theorem.

Theorem 1. If $|\psi\rangle$, which is a three-qubit pure state, has the same value of the GMC as that of the state $|\psi\rangle_\alpha$, the maximum steering inequality violations of the former and the latter satisfy the ordering $S_{\max}(|\psi\rangle_\alpha)\geqslant S_{\max}(|\psi\rangle)$.

Proof. If $\mathcal{C}(|\psi\rangle_{\rm GHZ})$ is obtained from the $A:BC$ split for the GHZ class states, the GMC of $|\psi\rangle_{\rm GHZ}$ is

$$\mathcal{C}(|\psi\rangle_{\rm GHZ})=\frac{1}{\kappa}\sqrt{(\cos^2\alpha_B\cos^2\alpha_C-1)\sin^2\alpha_A\sin^2 2\beta}. \tag{23}$$

From Lemma 1, the corresponding maximum steering inequality violation $S_{\max}(|\psi\rangle_{\rm GHZ})=S_{BC}(|\psi\rangle_{\rm GHZ})$, which is given in Eq. (19).

The GMC and the maximum steering inequality violation of the state $|\psi\rangle_\alpha$ are given as

$$\mathcal{C}(|\psi\rangle_\alpha)=\frac{1-\alpha^2}{1+\alpha^2}, \tag{24}$$

$$S_{\max}(|\psi\rangle_\alpha)=\frac{1+10\alpha^2+\alpha^4}{(1+\alpha^2)^2}, \tag{25}$$

respectively. The condition $\mathcal{C}(|\psi\rangle_{\rm GHZ})=\mathcal{C}(|\psi\rangle_\alpha)$ implies

$$\alpha^2=\frac{\kappa-\sqrt{(\cos^2\alpha_B\cos^2\alpha_C-1)\sin^2\alpha_A\sin^2 2\beta}}{\kappa+\sqrt{(\cos^2\alpha_B\cos^2\alpha_C-1)\sin^2\alpha_A\sin^2 2\beta}}. \tag{26}$$

Substituting α^2 into the expression of $S_{\max}(|\psi\rangle_\alpha)$, one can compare $S_{\max}(|\psi\rangle_\alpha)$ with $S_{\max}(|\psi\rangle_{\rm GHZ})$ and find $S_{\max}(|\psi\rangle_\alpha)\geqslant S_{\max}(|\psi\rangle_{\rm GHZ})$ through numerical calculation.

Under the assumption that the GMC is obtained from the bipartition $A:BC$, $\mathcal{C}(|\psi\rangle_{\rm W})$ is given as

$$\mathcal{C}(|\psi\rangle_{\rm W})=2\sqrt{(a+b)c}. \tag{27}$$

If $\mathcal{C}(|\psi\rangle_{\rm W})=\mathcal{C}(|\psi\rangle_\alpha)$, one will get

$$\alpha^2=\frac{1-\mathcal{C}(|\psi\rangle_{\rm W})}{1+\mathcal{C}(|\psi\rangle_{\rm W})}. \tag{28}$$

Substituting α^2 into the expression of $S_{\max}(|\psi\rangle_\alpha)$, one can compare $S_{\max}(|\psi\rangle_\alpha)$ with $S_{\max}(|\psi\rangle_{\rm W})$. Through numerical calculation, it is easily found that $S_{\max}(|\psi\rangle_\alpha)\geqslant S_{\max}(|\psi\rangle_{\rm W})$.

Similarly, the proof also holds for the cases when the GMC of GHZ or W class states is obtained from the other two bipartitions. Therefore the proof is completed. ■

From the expressions of the GMC and the maximum steering inequality violation of the state $|\psi\rangle_\alpha$, i.e., $\mathcal{C}(|\psi\rangle_\alpha)$ and $S_{\max}(|\psi\rangle_\alpha)$, one may note that $2\mathcal{C}^2(|\psi\rangle_\alpha)+S_{\max}(|\psi\rangle_\alpha)=3$. According to Theorem 1, this indicates that $S_{\max}(|\psi\rangle_\alpha)\geqslant S_{\max}(|\psi\rangle)$ when $\mathcal{C}(|\psi\rangle_\alpha)=\mathcal{C}(|\psi\rangle)$ if $|\psi\rangle$ is a three-qubit pure state. Thus the following complementary relation holds for the three-qubit pure states:

$$2\mathcal{C}^2(|\psi\rangle)+S_{\max}(|\psi\rangle)\leqslant 3. \tag{29}$$

The complementary relation suggests that the maximum steering inequality violation by the reduced bipartite states depends on the tripartite entanglement present in the tripartite system. From the relation, one may note that for all three-qubit pure states with a fixed amount of the maximum steering inequality violation S, the maximum value of the GMC of these states is $\sqrt{(3-S)/2}$.

B. GGM versus the maximum steering inequality violation

The complementary relation between GGM and the maximum steering inequality violation for three-qubit pure states is derived in this section.

Lemma 2. For a three-qubit pure state $|\psi\rangle$, if the GGM obtains from, for example, an $A:BC$ split, then the maximum steering inequality violation $S_{\max}(|\psi\rangle)=S_{BC}(|\psi\rangle)$.

Proof. For the GHZ class states, the GGM is $\mathcal{G}(|\psi\rangle_{\rm GHZ})=1-\max\{\lambda_A,\lambda_B,\lambda_C\}$ with $\lambda_X(X=A,B,C)$ being the maximum eigenvalues of the reduced states ρ_X of $|\psi\rangle_{\rm GHZ}$. Through straightforward calculation, λ_X is given as

$$\begin{aligned}\lambda_A(|\psi\rangle_{\rm GHZ})&=\frac{1}{2}\left(1+\sqrt{1+\frac{(\cos^2\alpha_B\cos^2\alpha_C-1)\sin^2\alpha_A\sin^2 2\beta}{\kappa^2}}\right),\\ \lambda_B(|\psi\rangle_{\rm GHZ})&=\frac{1}{2}\left(1+\sqrt{1+\frac{(\cos^2\alpha_A\cos^2\alpha_C-1)\sin^2\alpha_B\sin^2 2\beta}{\kappa^2}}\right),\\ \lambda_C(|\psi\rangle_{\rm GHZ})&=\frac{1}{2}\left(1+\sqrt{1+\frac{(\cos^2\alpha_A\cos^2\alpha_B-1)\sin^2\alpha_C\sin^2 2\beta}{\kappa^2}}\right).\end{aligned} \tag{30}$$

The fact that the GGM is obtained from the bipartition $A:BC$ implies $\lambda_A\geqslant\lambda_B$ and $\lambda_A\geqslant\lambda_C$, which give the same conditions as those expressed in Eq. (16). These conditions ensure that $S_{BC}(|\psi\rangle_{\rm GHZ})$ takes the biggest value among $S_{AB}(|\psi\rangle_{\rm GHZ})$, $S_{AC}(|\psi\rangle_{\rm GHZ})$, and $S_{BC}(|\psi\rangle_{\rm GHZ})$. Thus the maximum steering inequality violation $S_{\max}(|\psi\rangle_{\rm GHZ})=S_{BC}(|\psi\rangle_{\rm GHZ})$.

For the W class states, the corresponding maximum eigenvalues λ_X of the reduced states ρ_X of $|\psi\rangle_{\rm W}$ are

$$\lambda_A(|\psi\rangle_{\rm W}) = \tfrac{1}{2}[1+\sqrt{1-4(a+b)c}],$$
$$\lambda_B(|\psi\rangle_{\rm W}) = \tfrac{1}{2}[1+\sqrt{1-4(a+c)b}],$$
$$\lambda_C(|\psi\rangle_{\rm W}) = \tfrac{1}{2}[1+\sqrt{1-4(b+c)a}]. \quad (31)$$

When the GGM is obtained from the bipartition $A:BC$, one will get $\lambda_A(|\psi\rangle_{\rm W}) \geqslant \lambda_B(|\psi\rangle_{\rm W})$ and $\lambda_A(|\psi\rangle_{\rm W}) \geqslant \lambda_C(|\psi\rangle_{\rm W})$, which deduce the same conditions given in Eq. (21). These conditions will give the results $S_{BC}(|\psi\rangle_{\rm W}) \geqslant S_{AB}(|\psi\rangle_{\rm W})$, $S_{BC}(|\psi\rangle_{\rm W}) \geqslant S_{AC}(|\psi\rangle_{\rm W})$. Thus the maximum steering inequality violation $S_{\max}(|\psi\rangle_{\rm W}) = S_{BC}(|\psi\rangle_{\rm W})$.

Similar results will be given for the cases in which the GGM is obtained from the other two bipartitions. Therefore the proof is completed. ■

Based on the result given in Lemma 2, we can prove the following theorem.

Theorem 2. If $|\psi\rangle$, which is a three-qubit pure state, has the same value of the GGM as that of the state $|\psi\rangle_\alpha$, the maximum steering inequality violations of them satisfy the ordering $S_{\max}(|\psi\rangle_\alpha) \geqslant S_{\max}(|\psi\rangle)$.

Proof. For the GHZ class states, $\mathcal{G}(|\psi\rangle_{\rm GHZ}) = 1 - \lambda_A(|\psi\rangle_{\rm GHZ})$ if the GGM is obtained from the bipartition $A:BC$. With the result given in Lemma 2, the corresponding maximum steering inequality violation $S_{\max}(|\psi\rangle_{\rm GHZ})$ is equal to $S_{BC}(|\psi\rangle_{\rm GHZ})$ given in Eq. (19).

On the other hand, the GGM of the state $|\psi\rangle_\alpha$ is $\mathcal{G}(|\psi\rangle_\alpha) = \frac{1}{2} - \frac{\alpha}{1+\alpha^2}$, and the maximum steering inequality violation is given in Eq. (25). $\mathcal{G}(|\psi\rangle_{\rm GHZ}) = \mathcal{G}(|\psi\rangle_\alpha)$ implies

$$\alpha = \frac{1-\sqrt{1-(2\lambda_A(|\psi\rangle_{\rm GHZ})-1)^2}}{2\lambda_A(|\psi\rangle_{\rm GHZ})-1}. \quad (32)$$

Thus one can compare $S_{\max}(|\psi\rangle_\alpha)$ with $S_{\max}(|\psi\rangle_{\rm GHZ})$ by substituting α into $S_{\max}(|\psi\rangle_\alpha)$, and find $S_{\max}(|\psi\rangle_\alpha) \geqslant S_{\max}(|\psi\rangle_{\rm GHZ})$ through numerical calculation.

For the W class states, $\mathcal{G}(|\psi\rangle_{\rm W}) = 1 - \lambda_A(|\psi\rangle_{\rm W})$ when the GGM is obtained from the bipartition $A:BC$. The corresponding maximum steering inequality violation $S_{\max}(|\psi\rangle_{\rm W})$ is equal to $S_{BC}(|\psi\rangle_{\rm W})$ given in Eq. (22) based on the result of Lemma 2.

If the assumption $\mathcal{G}(|\psi\rangle_{\rm W}) = \mathcal{G}(|\psi\rangle_\alpha)$ is considered, the state parameter α of $|\psi\rangle_\alpha$ is equal to

$$\alpha = \frac{1-2\sqrt{(a+b)c}}{\sqrt{1-4(a+b)c}}. \quad (33)$$

Substituting α into $S_{\max}(|\psi\rangle_\alpha)$ and comparing it with $S_{\max}(|\psi\rangle_{\rm W})$, one can find $S_{\max}(|\psi\rangle_\alpha) \geqslant S_{\max}(|\psi\rangle_{\rm W})$.

Similarly, the proof also holds when one obtains the GGM of the GHZ or W class states from the other bipartitions. Thus the proof is completed. ■

One may also note that the GGM and the maximum steering inequality violation of the state $|\psi\rangle_\alpha$ saturate $8\mathcal{G}(|\psi\rangle_\alpha)[1-\mathcal{G}(|\psi\rangle_\alpha)] + S_{\max}(|\psi\rangle_\alpha) = 3$. The result of Theorem 2 implies $S_{\max}(|\psi\rangle) \leqslant S_{\max}(|\psi\rangle_\alpha)$ if $\mathcal{G}(|\psi\rangle) = \mathcal{G}(|\psi\rangle_\alpha)$ for the $|\psi\rangle$ being a three-qubit pure state. Thus the following complementary relation exists for the three-qubit pure states:

$$8\mathcal{G}(|\psi\rangle)[1-\mathcal{G}(|\psi\rangle)] + S_{\max}(|\psi\rangle) \leqslant 3. \quad (34)$$

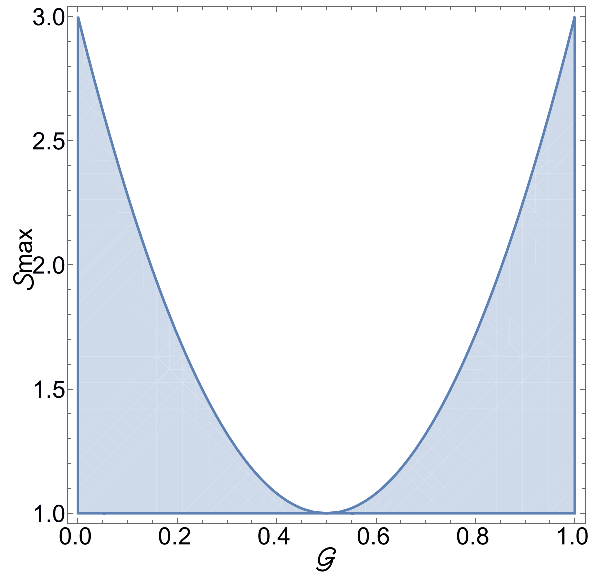

FIG. 1. For three-qubit pure states, the region in which the GGM and the maximum steering inequality violation satisfy the inequality given in Eq. (34).

Obviously, the complementary relation again suggests that the maximum steering inequality violation by the reduced bipartite states depends on the tripartite entanglement present in the tripartite system. If at least one of the reduced states of a three-qubit pure state is F_3 steerable, the GGM of the three-qubit pure state is larger than $\frac{1}{2}+\sqrt{\frac{S-1}{8}}$ or smaller than $\frac{1}{2}-\sqrt{\frac{S-1}{8}}$ for a fixed amount of the maximum steering inequality violation S. The complementary relation between the GGM and the maximum steering inequality violation given in Eq. (34) is plotted in Fig. 1. The maximally steering inequality violating states $|\psi\rangle_\alpha$ form the boundary of the region.

C. Tangle versus the maximum steering inequality violation

In this section, the complementarity between tangle and the maximum steering inequality violation for three-qubit states is derived.

Theorem 3. If $|\psi\rangle$, which is a three-qubit pure state, has the same value of tangle as that of the state $|\psi\rangle_\alpha$, i.e., $\tau(|\psi\rangle) = \tau(|\psi\rangle_\alpha)$, the maximum steering inequality violations of them satisfy the ordering $S_{\max}(|\psi\rangle_\alpha) \geqslant S_{\max}(|\psi\rangle)$.

Proof. The tangle of the GHZ class states is

$$\tau(|\psi\rangle_{\rm GHZ}) = \frac{\sin^2\alpha_A \sin^2\alpha_B \sin^2\alpha_C \sin^2 2\beta}{\kappa^2}. \quad (35)$$

The tangle of the state $|\psi\rangle_\alpha$ is

$$\tau(|\psi\rangle_\alpha) = 1 - \frac{4\alpha^2}{(1+\alpha^2)^2}. \quad (36)$$

The condition $\tau(|\psi\rangle_{\rm GHZ}) = \tau(|\psi\rangle_\alpha)$ implies

$$\alpha^2 = \frac{1+(\cos\alpha_A\cos\alpha_B\cos\alpha_C - \sin\alpha_A\sin\alpha_B\sin\alpha_C)\sin 2\beta}{1+(\cos\alpha_A\cos\alpha_B\cos\alpha_C + \sin\alpha_A\sin\alpha_B\sin\alpha_C)\sin 2\beta}. \quad (37)$$

Substituting α^2 into the expression of $S_{\max}(|\psi\rangle_\alpha)$ given in Eq. (25), one can compare $S_{\max}(|\psi\rangle_\alpha)$ with $S_{\max}(|\psi\rangle_{\mathrm{GHZ}})$. Through straightforward calculation, it is found that $S_{\max}(|\psi\rangle_\alpha) \geqslant S_{\max}(|\psi\rangle_{\mathrm{GHZ}})$.

For the W class states, $\tau(|\psi\rangle_{\mathrm{W}}) = 0$. For the state $|\psi\rangle_\alpha$, $\tau(|\psi\rangle_\alpha) = 0$ if and only if $\alpha = 1$. The corresponding maximum steering inequality violation $S_{\max}(|\psi\rangle_\alpha) = 3$ when $\alpha = 1$. One can straightforwardly compare $S_{\max}(|\psi\rangle_{\alpha=1})$ with $S_{AB}(|\psi\rangle_{\mathrm{W}})$, $S_{AC}(|\psi\rangle_{\mathrm{W}})$, and $S_{BC}(|\psi\rangle_{\mathrm{W}})$ and find $S_{\max}(|\psi\rangle_{\alpha=1}) \geqslant S_{\max}(|\psi\rangle_{\mathrm{W}})$. Thus the proof is completed. ■

Based on the expressions of the tangle $\tau(|\psi\rangle_\alpha)$ and the maximum steering inequality violation $S_{\max}(|\psi\rangle_\alpha)$, one has the complementary relation $2\tau(|\psi\rangle_\alpha) + S_{\max}(|\psi\rangle_\alpha) = 3$. Theorem 3 indicates that $S_{\max}(|\psi\rangle_\alpha) \geqslant S_{\max}(|\psi\rangle)$ when $\tau(|\psi\rangle_\alpha) = \tau(|\psi\rangle)$, and thus the following complementary relation holds for three-qubit pure states:

$$2\tau(|\psi\rangle) + S_{\max}(|\psi\rangle) \leqslant 3. \tag{38}$$

The complementary relation between the tangle and the maximum steering inequality violation can be extended to the case of three-qubit mixed states. In Ref. [47], the tangle of three-qubit mixed states has been defined by convex roof construction

$$\tau(\rho) = \min_{\{p_i, |\psi\rangle_i\}} \sum p_i \tau(|\psi\rangle_i). \tag{39}$$

The minimization is over all the pure state decompositions of ρ, i.e., $\rho = \sum_i p_i |\psi\rangle_i \langle\psi|$ with $p_i \geqslant 0$ and $\sum_i p_i = 1$. On the other hand, $S_{\max}(\rho)$ is convex under mixing [43]. Therefore the tangle $\tau(\rho)$ and the maximum steering inequality violation $S_{\max}(\rho)$ of an arbitrary three-qubit state ρ follow the following complementary relation:

$$2\tau(\rho) + S_{\max}(\rho) \leqslant 3. \tag{40}$$

The complementary relation indicates that the maximum steering inequality violation by the reduced bipartite states depends on the tripartite entanglement present in the tripartite system. From the relation, one could conclude that for all three-qubit pure states with a fixed amount of the maximum steering inequality violation S, the maximum value of the tangle of these states is $(3 - S)/2$.

IV. DISCUSSION AND CONCLUSIONS

In Ref. [43], the authors investigated relations between the maximum steering inequality violation in reduced two-qubit systems, different measures of bipartite entanglement of the reduced states, and tripartite entanglement of the three-qubit state. Particularly, they obtained a similar result to that given in Eq. (40) by straightforwardly calculating the reduced bipartite steering of a three-qubit state and the tangle of the three-qubit state. Here, we give the result with a different method. Firstly, we prove that the maximum steering inequality violation of the state $|\psi\rangle_\alpha$ is always greater than or equal to that of an arbitrary three-qubit pure state if the state $|\psi\rangle_\alpha$ and the three-qubit pure state have the same value of tangle. Then, we give the complementary relation between the maximum steering inequality violation and tangle for the state $|\psi\rangle_\alpha$. In the end, we obtain the complementary relation for the three-qubit pure state given in Eq. (38) and extend it to the case of the three-qubit mixed state. Thus the complementary relation between the maximum steering inequality violation and the tangle is the corollary of Theorem 3. Furthermore, different from tangle, which is based on the concept of monogamy, the other two measures of tripartite entanglement are based on the concept of distance to a relevant class of states, and the two measures are not considered in Ref. [43]. In addition, the tangle is not a good measure of genuine tripartite entanglement even for pure states because there exist a large number of pure states [for example, the W class states given in Eq. (14)] for which it becomes 0. Therefore the introduction of the other two measures of tripartite entanglement is necessary.

Steering lies between entanglement and Bell nonlocality, and the relations between steering and entanglement deserve investigation. In this paper, a single parameter family of entangled three-qubit pure states are introduced and are called the maximally steering inequality violating states due to the fact that the states have the maximum steering inequality violation among all of the three-qubit pure states for a fixed amount of tripartite entanglement. The GMC, GGM, and tangle are employed to quantify tripartite entanglement. Subsequently, the corresponding complementary relations between tripartite entanglement of three-qubit pure states and the maximum steering inequality violation in reduced two-qubit systems are established. In particular, the complementary relation can also hold for the three-qubit mixed states when the tangle is used to quantify tripartite entanglement. From the complementary relations, one could make the conclusion that the maximum steering inequality violation in two-qubit reduced systems is at the expense of tripartite entanglement of the three-qubit system, or vice versa. Our results can be used in a scenario where three parties share genuinely entangled systems to perform information-theoretic protocols among them and at the same time they might need steering between their subparts. In this regard, it is very useful to know which state is more entangled for a fixed amount of steering in the bipartite scenario.

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