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Annihilating and creating nonlocality without entanglement by postmeasurement information

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Nonlocality without entanglement (NLWE) is a nonlocal quantum phenomenon that arises in separable state discrimination. We show that the availability of the postmeasurement information about the prepared subensemble can affect the occurrence of NLWE in discriminating non-orthogonal nonentangled states. We provide a two-qubit state ensemble consisting of four nonorthogonal separable pure states and show that the postmeasurement information about the prepared subensemble can annihilate NLWE. We also provide another two-qubit state ensemble consisting of four nonorthogonal separable states and show that the postmeasurement information can create NLWE. Our result can provide a useful method to share or hide information using nonorthogonal separable states.

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I. INTRODUCTION

Whereas nonorthogonal quantum states cannot be perfectly discriminated, in general, we can always perfectly discriminate orthogonal quantum states by using appropriate measurements [1–4]. However, it is also known that there are some multiparty orthogonal nonentangled (separable) states that cannot be perfectly discriminated only by *local operations and classical communication* (LOCC) [5]. In other words, there exists some separable measurements that cannot be implemented by LOCC. In discriminating separable states of multiparty quantum systems, a phenomenon that can be achieved by global measurements but cannot be achieved only by LOCC is called *nonlocality without entanglement* (NLWE) [5–7].

In discriminating orthogonal separable states, NLWE occurs when the states cannot be perfectly discriminated by LOCC [5,8–12]. On the other hand, in the problem of discriminating nonorthogonal separable states, NLWE occurs when the globally optimal discrimination cannot be achieved by LOCC [6,7,13–15]. For example, the double trine ensemble in a two-qubit system that consists of three nonorthogonal separable states is known to show NLWE in their discrimination [6,7].

Recently, it was shown that there are some nonorthogonal states that can be perfectly discriminated when the *postmeasurement information* (PI) about the prepared subensemble is provided [16]. However, it is also known that there are some nonorthogonal states that are still impossible to be perfectly discriminated even if the PI about the prepared subensemble is available [17–19]. Therefore, in discriminating multiparty nonorthogonal separable states with the PI about the prepared subensemble, NLWE occurs when the globally optimal discrimination cannot be achieved by LOCC even with the help of PI. A natural question that can be raised here is whether the availability of the PI about the prepared subensemble can affect the occurrence of NLWE.

Here, we provide an answer to the question by showing that the PI about the prepared subensemble can annihilate or create NLWE in discriminating nonorthogonal separable states. We first consider an ensemble of two-qubit separable states having a NLWE phenomenon and show that the ensemble loses NLWN when the PI about the prepared subensemble is available, thus, *annihilating NLWE by PI*. We further consider a two-qubit ensemble of separable states without the NLWE phenomenon and show that PI can activate NLWE of the ensemble, therefore, *creating NLWE by PI*.

This paper is organized as follows. We first recall the definition and some properties about separable states and separable measurements in two-qubit systems. We further recall the definition of *minimum-error discrimination (ME)* [20–23], one representative state discrimination strategy, and provide some useful properties of ME depending on the availability of PI. As the main results of this paper, we provide a two-qubit state ensemble consisting of four nonorthogonal separable states and show that NLWE occurs in discriminating the states in the ensemble. With the same ensemble, we further show that the occurrence of NLWE in the state discrimination can be vanished when the PI about the prepared subensemble is available. Moreover, we provide another two-qubit state ensemble consisting of four nonorthogonal separable states and show that NLWE does not occur in discriminating the states of the ensemble. With the same ensemble, we further show the occurrence of NLWE in the state discrimination with the PI about the prepared subensemble.

II. QUANTUM STATE DISCRIMINATION IN TWO-QUBIT SYSTEMS

In two-qubit ($2\otimes 2$) systems, a state is described by a density operator ρ, that is, a positive-semidefinite operator $\rho\succeq 0$ having unit trace $\mathrm{Tr}\,\rho=1$, acting on a bipartite Hilbert

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space $\mathcal{H}=\mathbb{C}^2\otimes\mathbb{C}^2$. A measurement with m outcomes is expressed by a positive operator valued measure (POVM) $\{M_i\}_i$ that consists of m positive-semidefinite operators $M_i\succeq 0$ on $\mathcal{H}$ satisfying $\sum_i M_i=\mathbb{1}$, where $\mathbb{1}$ is the identity operator on $\mathcal{H}$. When $\{M_i\}_i$ is performed on a quantum system prepared with ρ, the probability that M_i is detected is $\mathrm{Tr}(\rho M_i)$ due to the Born rule.

A positive-semidefinite operator is called *separable* if it is a sum of positive-semidefinite product operators. Similarly, a measurement $\{M_i\}_i$ is called *separable* if M_i is separable for all i. In particular, a *LOCC measurement* is a separable measurement that can be implemented by LOCC [24].

An operator E on $\mathcal{H}$ is called *positive partial transpose* (PPT) [25,26] if

$$\mathrm{PT}(E)\succeq 0, \tag{1}$$

where $\mathrm{PT}(\cdot)$ is the partial transposition taken in the standard basis $\{|0\rangle,|1\rangle\}$ on the second subsystem (Although the PPT property does not depend on the choice of subsystem to be transposed, we take the second subsystem throughout this paper for simplicity). In two-qubit systems, PPT is a necessary and sufficient condition for a positive-semidefinite operator to be separable [26]. Thus, the set of all positive-semidefinite separable operators on $\mathcal{H}$ can be represented as

$$\mathrm{SEP}=\{E\,|\,E\succeq 0,\ \mathrm{PT}(E)\succeq 0\}. \tag{2}$$

The dual set to SEP is defined as

$$\mathrm{SEP}^*=\{A\,|\,\mathrm{Tr}(AB)\geqslant 0\ \forall B\in\mathrm{SEP}\}. \tag{3}$$

Since all elements of SEP are positive semidefinite, all positive semidefinite operators are in SEP*. We also note that all PPT operators are in SEP* because all elements of SEP are PPT and $\mathrm{Tr}(AB)=\mathrm{Tr}[\mathrm{PT}(A)\mathrm{PT}(B)]$ for any two operators A and B.

Throughout this paper, we only consider the situation of discriminating states from the state ensemble,

$$\mathcal{E}=\{\eta_i,\rho_i\}_{i\in\Lambda},\quad \Lambda=\{0,1,+,-\}, \tag{4}$$

where ρ_i is a $2\otimes 2$ separable state and η_i is the probability that state ρ_i is prepared.

The ensemble $\mathcal{E}$ can be seen as an ensemble consisting of two subensembles,

$$\mathcal{E}_0=\left\{\eta_i\Big/\sum_{j\in\mathsf{A}_0}\eta_j,\ \rho_i\right\}_{i\in\mathsf{A}_0},\quad \mathsf{A}_0=\{0,1\},$$
$$\mathcal{E}_1=\left\{\eta_i\Big/\sum_{j\in\mathsf{A}_1}\eta_j,\ \rho_i\right\}_{i\in\mathsf{A}_1},\quad \mathsf{A}_1=\{+,-\}, \tag{5}$$

where $\mathcal{E}_0$ and $\mathcal{E}_1$ are prepared with probabilities $\sum_{j\in\mathsf{A}_0}\eta_j$ and $\sum_{j\in\mathsf{A}_1}\eta_j$, respectively.

A. Minimum-error discrimination

Let us consider the state discrimination of $\mathcal{E}$ in Eq. (4) using a measurement $\{M_i\}_{i\in\Lambda}$ where each measurement outcome from M_i means that the prepared state is guessed to be ρ_i. *ME of* $\mathcal{E}$ is to minimize the average probability of errors that occur in guessing the prepared state. Equivalently, ME of $\mathcal{E}$ is to maximize the average probability of correctly guessing the prepared state where the optimal success probability is defined as

$$p_{\mathrm{G}}(\mathcal{E})=\max_{\{M_i\}_{i\in\Lambda}}\sum_{i\in\Lambda}\eta_i\mathrm{Tr}(\rho_iM_i), \tag{6}$$

over all possible POVMs. The optimality of the POVMs in Eq. (6) can be confirmed by the following necessary and sufficient condition [21,22,27]:

$$\sum_{i\in\Lambda}\eta_i\rho_iM_i-\eta_j\rho_j\succeq 0\quad\forall\, j\in\Lambda. \tag{7}$$

When the available measurements are limited to LOCC measurements, we denote the maximum success probability by

$$p_{\mathrm{L}}(\mathcal{E})=\max_{\mathrm{LOCC}}\sum_{i\in\Lambda}\eta_i\mathrm{Tr}(\rho_iM_i). \tag{8}$$

Since the states of $\mathcal{E}$ are nonentangled, NLWE occurs in terms of ME if and only if ME of $\mathcal{E}$ cannot be achieved only by LOCC, that is,

$$p_{\mathrm{L}}(\mathcal{E})<p_{\mathrm{G}}(\mathcal{E}). \tag{9}$$

The following proposition provides an upper bound of $p_{\mathrm{L}}(\mathcal{E})$.

Proposition 1 ([28]). If H is a Hermitian operator with

$$H-\eta_i\rho_i\in\mathrm{SEP}^*\quad\forall\, i\in\Lambda, \tag{10}$$

then $\mathrm{Tr}\,H$ is an upper bound of $p_{\mathrm{L}}(\mathcal{E})$.

B. Quantum state discrimination with postmeasurement information

In the subsection, we consider ME of $\mathcal{E}$ in Eq. (4) when the classical information $b\in\{0,1\}$ about the prepared subensemble $\mathcal{E}_b$ defined in Eq. (5) is given after performing a measurement. In this situation, it is known that a measurement can be expressed by a POVM $\{M_{\vec{\omega}}\}_{\vec{\omega}\in\Omega}$ with the Cartesian product outcome space,

$$\Omega=\mathsf{A}_0\times\mathsf{A}_1, \tag{11}$$

where each $M_{(\omega_0,\omega_1)}$ indicates the detection of ρ_{ω_0} or ρ_{ω_1} according to PI $b=0$ or 1, respectively [17,18].

ME of $\mathcal{E}$ *with PI* is to minimize the average error probability. Equivalently, ME of $\mathcal{E}$ with PI is to maximize the average probability of correct guessing where the optimal success probability is defined as

$$p_{\mathrm{G}}^{\mathrm{PI}}(\mathcal{E})=\max_{\{M_{\vec{\omega}}\}_{\vec{\omega}\in\Omega}}\sum_{b\in\{0,1\}}\sum_{i\in\mathsf{A}_b}\eta_i\mathrm{Tr}\Bigg[\rho_i\sum_{\substack{\vec{\omega}\in\Omega\\ \omega_b=i}}M_{\vec{\omega}}\Bigg], \tag{12}$$

over all possible POVMs. Note that when ρ_i is prepared and PI $b\in\{0,1\}$ with $i\in\mathsf{A}_b$ is given, the prepared state is correctly guessed if we obtain a measurement outcome $\vec{\omega}\in\Omega$ with $\omega_b=i$; otherwise, errors occur in guessing the prepared state. Figure 1 illustrates ME of $\mathcal{E}$ with PI.

We note that for a given POVM $\{M_{\vec{\omega}}\}_{\vec{\omega}\in\Omega}$, the average probability of correct guessing, that is, the right-hand side of

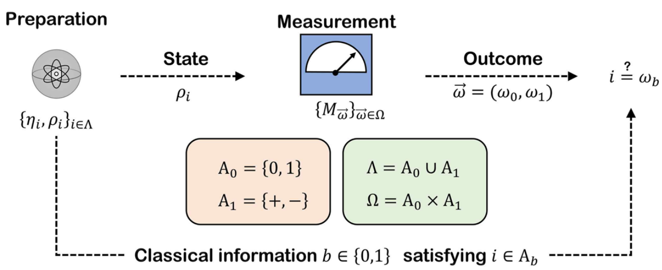

FIG. 1. ME of $\mathcal{E} = \{\eta_i, \rho_i\}_{i\in\Lambda}$ with PI. For each $i \in \Lambda$, state ρ_i is prepared with the probability η_i. After performing a measurement $\{M_{\vec{\omega}}\}_{\vec{\omega}\in\Omega}$, the classical information $b \in \{0, 1\}$ satisfying $i \in \mathsf{A}_b$ is given. For each measurement outcome $(\omega_0, \omega_1) = \vec{\omega} \in \Omega$, the prepared state is guessed to be ρ_{ω_0} or ρ_{ω_1} according to PI $b = 0$ or 1. When ρ_i is prepared, it is correctly guessed if a measurement outcome $\vec{\omega} \in \Omega$ with $\omega_b = i$ is obtained; otherwise, errors occur in guessing the prepared state.

Eq. (12) without maximization, can be rewritten as

$$\begin{aligned}
&\sum_{b\in\{0,1\}}\sum_{i\in\mathsf{A}_b}\eta_i \mathrm{Tr}\Bigg[\rho_i \sum_{\substack{\vec{\omega}\in\Omega\\ \omega_b=i}} M_{\vec{\omega}}\Bigg] \\
&= \sum_{b\in\{0,1\}}\sum_{i\in\mathsf{A}_b}\sum_{\substack{\vec{\omega}\in\Omega\\ \omega_b=i}} \mathrm{Tr}\big(\eta_{\omega_b}\rho_{\omega_b}M_{\vec{\omega}}\big) \\
&= \sum_{b\in\{0,1\}}\sum_{\vec{\omega}\in\Omega} \mathrm{Tr}\big(\eta_{\omega_b}\rho_{\omega_b}M_{\vec{\omega}}\big) \\
&= 2\sum_{\vec{\omega}\in\Omega}\frac{1}{2}\mathrm{Tr}\Bigg[\sum_{b\in\{0,1\}}\eta_{\omega_b}\rho_{\omega_b}M_{\vec{\omega}}\Bigg] \\
&= 2\sum_{\vec{\omega}\in\Omega}\tilde{\eta}_{\vec{\omega}}\mathrm{Tr}(\tilde{\rho}_{\vec{\omega}}M_{\vec{\omega}}),
\end{aligned} \tag{13}$$

where

$$\tilde{\eta}_{\vec{\omega}} = \frac{1}{2}\sum_{b\in\{0,1\}}\eta_{w_b}, \quad \tilde{\rho}_{\vec{\omega}} = \frac{\sum_{b\in\{0,1\}}\eta_{w_b}\rho_{\omega_b}}{\sum_{b'\in\{0,1\}}\eta_{w_{b'}}}. \tag{14}$$

When the available measurements are limited to LOCC measurements, we denote the maximum success probability by

$$p_{\mathrm{L}}^{\mathrm{PI}}(\mathcal{E}) = \max_{\mathrm{LOCC}}\sum_{b\in\{0,1\}}\sum_{i\in\mathsf{A}_b}\eta_i \mathrm{Tr}\Bigg[\rho_i \sum_{\substack{\vec{\omega}\in\Omega\\ \omega_b=i}} M_{\vec{\omega}}\Bigg]. \tag{15}$$

Because the states in $\mathcal{E}$ are nonentangled, NLWE occurs in terms of ME with PI if and only if ME of $\mathcal{E}$ with PI cannot be achieved only by LOCC, that is,

$$p_{\mathrm{L}}^{\mathrm{PI}}(\mathcal{E}) < p_{\mathrm{G}}^{\mathrm{PI}}(\mathcal{E}). \tag{16}$$

Here, we note that $\{\tilde{\eta}_{\vec{\omega}}\}_{\vec{\omega}\in\Omega}$ is a set of positive numbers satisfying $\sum_{\vec{\omega}\in\Omega}\tilde{\eta}_{\vec{\omega}} = 1$ and $\{\tilde{\rho}_{\vec{\omega}}\}_{\vec{\omega}\in\Omega}$ is a set of density operators. Thus, Eqs. (13) and (15) imply that $p_{\mathrm{L}}^{\mathrm{PI}}(\mathcal{E})$ is twice the maximum success probability for ME of $\tilde{\mathcal{E}}$,

$$p_{\mathrm{L}}^{\mathrm{PI}}(\mathcal{E}) = 2p_{\mathrm{L}}(\tilde{\mathcal{E}}), \tag{17}$$

where $\tilde{\mathcal{E}}$ is the ensemble consisting of the average states $\tilde{\rho}_{\vec{\omega}}$ prepared with the nonzero probabilities $\tilde{\eta}_{\vec{\omega}}$ in Eq. (14),

$$\tilde{\mathcal{E}} = \{\tilde{\eta}_{\vec{\omega}}, \tilde{\rho}_{\vec{\omega}}\}_{\vec{\omega}\in\Omega}. \tag{18}$$

In the following lemma, we provide an upper bound of $p_{\mathrm{L}}^{\mathrm{PI}}(\mathcal{E})$.

Lemma 1. If $\tilde{H}$ is a Hermitian operator satisfying

$$\tilde{H} - \tilde{\eta}_{\vec{\omega}}\tilde{\rho}_{\vec{\omega}} \in \mathrm{SEP}^* \quad \forall\, \vec{\omega} \in \Omega, \tag{19}$$

then $2\,\mathrm{Tr}\,\tilde{H}$ is an upper bound of $p_{\mathrm{L}}^{\mathrm{PI}}(\mathcal{E})$.

Proof. For the ensemble $\tilde{\mathcal{E}}$ in Eq. (18), Proposition 1 implies that $\mathrm{Tr}\,\tilde{H}$ is an upper bound of $p_{\mathrm{L}}(\tilde{\mathcal{E}})$. Thus, $2\,\mathrm{Tr}\,\tilde{H}$ is an upper bound of $p_{\mathrm{L}}^{\mathrm{PI}}(\mathcal{E})$ due to Eq. (17). ■

We close this section by providing the concept of *annihilating* and *creating* NLWE by PI.

Definition 1. For ME of an ensemble $\mathcal{E}$ in Eq. (4), we say that the PI $b \in \{0, 1\}$ about the prepared subensemble $\mathcal{E}_b$ in Eq. (5) annihilates NLWE if NLWE occurs in discriminating the states of $\mathcal{E}$ and the availability of PI b about the prepared subensemble vanishes the occurrence of NLWE, that is,

$$p_{\mathrm{L}}(\mathcal{E}) < p_{\mathrm{G}}(\mathcal{E}), \quad p_{\mathrm{L}}^{\mathrm{PI}}(\mathcal{E}) = p_{\mathrm{G}}^{\mathrm{PI}}(\mathcal{E}). \tag{20}$$

Also, we say that the PI b about the prepared subensemble $\mathcal{E}_b$ creates NLWE if NLWE does not occur in discriminating the states of $\mathcal{E}$ and the availability of PI b about the prepared subensemble releases the occurrence of NLWE, that is,

$$p_{\mathrm{L}}(\mathcal{E}) = p_{\mathrm{G}}(\mathcal{E}), \quad p_{\mathrm{L}}^{\mathrm{PI}}(\mathcal{E}) < p_{\mathrm{G}}^{\mathrm{PI}}(\mathcal{E}). \tag{21}$$

III. ANNIHILATING NLWE BY POSTMEASUREMENT INFORMATION

In this section, we consider a situation where the PI about the prepared subensemble $\mathcal{E}_b$ in Eq. (5) annihilates NLWE. We first provide a specific example of a state ensemble $\mathcal{E}$ in Eq. (4) and show that NLWE occurs in discriminating the states in the ensemble. With the same ensemble, we further show that the occurrence of NLWE in the state discrimination can be vanished if the PI about the prepared subensemble is available, thus, annihilating NLWE by PI.

Example 1. Let us consider the ensemble $\mathcal{E}$ in Eq. (4) with

$$\begin{aligned}
\eta_0 &= \frac{\gamma}{2(1+\gamma)}, \quad \rho_0 = |0\rangle\langle 0| \otimes |0\rangle\langle 0|, \\
\eta_1 &= \frac{\gamma}{2(1+\gamma)}, \quad \rho_1 = |0\rangle\langle 0| \otimes |1\rangle\langle 1|, \\
\eta_+ &= \frac{1}{2(1+\gamma)}, \quad \rho_+ = |+\rangle\langle +| \otimes |+\rangle\langle +|, \\
\eta_- &= \frac{1}{2(1+\gamma)}, \quad \rho_- = |-\rangle\langle -| \otimes |-\rangle\langle -|,
\end{aligned} \tag{22}$$

where $2 \leqslant \gamma < \infty$, $\{|0\rangle, |1\rangle\}$ is the standard basis in one-qubit system, and $|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$. In this case, the subensembles in Eq. (5) become

$$\begin{aligned}
\mathcal{E}_0 &= \left\{\tfrac{1}{2}, |0\rangle\langle 0| \otimes |0\rangle\langle 0|, \tfrac{1}{2}, |0\rangle\langle 0| \otimes |1\rangle\langle 1|\right\}, \\
\mathcal{E}_1 &= \left\{\tfrac{1}{2}, |+\rangle\langle +| \otimes |+\rangle\langle +|, \tfrac{1}{2}, |-\rangle\langle -| \otimes |-\rangle\langle -|\right\},
\end{aligned} \tag{23}$$

with the probabilities of preparation $\frac{\gamma}{1+\gamma}$ and $\frac{1}{1+\gamma}$, respectively.

To show the occurrence of NLWE in terms of ME about the state ensemble $\mathcal{E}$ in Example 1, we first evaluate the optimal success probability $p_{\mathrm{G}}(\mathcal{E})$ defined in Eq. (6). From the optimality condition in Eq. (7) together with a straightforward calculation, we can easily verify that the following POVM $\{M_i\}_{i\in\Lambda}$ is optimal for $p_{\mathrm{G}}(\mathcal{E})$:

$$\begin{aligned} M_0 &= |\Gamma_0\rangle\langle\Gamma_0|, \quad M_+ = |\mu_+\rangle\langle\mu_+|\otimes|+\rangle\langle+|,\\ M_1 &= |\Gamma_1\rangle\langle\Gamma_1|, \quad M_- = |\mu_-\rangle\langle\mu_-|\otimes|-\rangle\langle-|, \end{aligned} \tag{24}$$

where

$$\begin{aligned} |\mu_\pm\rangle &= \sqrt{\frac{1}{2}-\frac{\gamma}{2\sqrt{1+\gamma^2}}}|0\rangle \pm \sqrt{\frac{1}{2}+\frac{\gamma}{2\sqrt{1+\gamma^2}}}|1\rangle,\\ |\Gamma_0\rangle &= \sqrt{\frac{1}{2}+\frac{\gamma}{2\sqrt{1+\gamma^2}}}|00\rangle - \sqrt{\frac{1}{2}-\frac{\gamma}{2\sqrt{1+\gamma^2}}}|11\rangle,\\ |\Gamma_1\rangle &= \sqrt{\frac{1}{2}+\frac{\gamma}{2\sqrt{1+\gamma^2}}}|01\rangle - \sqrt{\frac{1}{2}-\frac{\gamma}{2\sqrt{1+\gamma^2}}}|10\rangle. \end{aligned} \tag{25}$$

Thus, the optimality of the POVM $\{M_i\}_{i\in\Lambda}$ in Eq. (24) and the definition of $p_{\mathrm{G}}(\mathcal{E})$ lead us to

$$p_{\mathrm{G}}(\mathcal{E}) = \frac{1}{2}\left(1+\frac{\sqrt{1+\gamma^2}}{1+\gamma}\right). \tag{26}$$

In order to obtain the maximum success probability $p_{\mathrm{L}}(\mathcal{E})$ in Eq. (8), we consider lower and upper bounds of $p_{\mathrm{L}}(\mathcal{E})$. A lower bound of $p_{\mathrm{L}}(\mathcal{E})$ can be obtained from the following POVM $\{M_i\}_{i\in\Lambda}$:

$$\begin{aligned} M_0 &= |0\rangle\langle 0|\otimes|0\rangle\langle 0|, \quad M_+ = |1\rangle\langle 1|\otimes|+\rangle\langle+|,\\ M_1 &= |0\rangle\langle 0|\otimes|1\rangle\langle 1|, \quad M_- = |1\rangle\langle 1|\otimes|-\rangle\langle-|, \end{aligned} \tag{27}$$

which gives $\frac{1}{2}(1+\frac{\gamma}{1+\gamma})$ as the success probability in discriminating the states of the ensemble $\mathcal{E}$ in Example 1. We also note that the measurement given in Eq. (27) can be achieved with finite-round LOCC: We perform a measurement $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ on the first subsystem and measure $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ or $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ on the second subsystem depending on the first measurement result $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$. Thus, the success probability for the LOCC measurement in Eq. (27) is a lower bound of $p_{\mathrm{L}}(\mathcal{E})$,

$$p_{\mathrm{L}}(\mathcal{E}) \geqslant \frac{1}{2}\left(1+\frac{\gamma}{1+\gamma}\right). \tag{28}$$

To obtain an upper bound of $p_{\mathrm{L}}(\mathcal{E})$, let us consider a Hermitian operator,

$$H = \frac{1}{4(1+\gamma)}(2\gamma|0\rangle\langle 0|\otimes\sigma_0 + |1\rangle\langle 1|\otimes\sigma_0 + \sigma_1\otimes\sigma_1), \tag{29}$$

where σ_0 and σ_1 are the Pauli operators,

$$\begin{aligned} \sigma_0 &= |0\rangle\langle 0| + |1\rangle\langle 1|,\\ \sigma_1 &= |0\rangle\langle 1| + |1\rangle\langle 0|. \end{aligned} \tag{30}$$

We will show that $H-\eta_i\rho_i \in \mathrm{SEP}^*$ for any $i\in\Lambda$, therefore, $\mathrm{Tr}\,H$ is an upper bound of $p_{\mathrm{L}}(\mathcal{E})$ by Proposition 1.

For each $i\in\Lambda$, $H-\eta_i\rho_i$ can be rewritten as

$$\begin{aligned} H-\eta_0\rho_0 &= \frac{1}{4(1+\gamma)}[T_0 + |11\rangle\langle 11| + \mathrm{PT}(T_0)],\\ H-\eta_1\rho_1 &= \frac{1}{4(1+\gamma)}[T_1 + |10\rangle\langle 10| + \mathrm{PT}(T_1)],\\ H-\eta_+\rho_+ &= \frac{2\gamma-1}{4(1+\gamma)}(\rho_0+\rho_1) + \frac{1}{2(1+\gamma)}\rho_-,\\ H-\eta_-\rho_- &= \frac{2\gamma-1}{4(1+\gamma)}(\rho_0+\rho_1) + \frac{1}{2(1+\gamma)}\rho_+, \end{aligned} \tag{31}$$

where T_0 and T_1 are positive-semidefinite operators,

$$\begin{aligned} T_0 &= \gamma|01\rangle\langle 01| + |01\rangle\langle 10| + |10\rangle\langle 01| + \tfrac{1}{2}|10\rangle\langle 10|,\\ T_1 &= \gamma|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + \tfrac{1}{2}|11\rangle\langle 11| \end{aligned} \tag{32}$$

for $2\leqslant\gamma<\infty$. In other words, each $H-\eta_i\rho_i$ in Eq. (31) is a sum of positive-semidefinite operators and PPT operators. From the argument after Eq. (3), $H-\eta_i\rho_i$ is in SEP^* for each $i\in\Lambda$, thus, Proposition 1 leads us to

$$p_{\mathrm{L}}(\mathcal{E}) \leqslant \mathrm{Tr}\,H = \frac{1}{2}\left(1+\frac{\gamma}{1+\gamma}\right). \tag{33}$$

Inequalities (28) and (33) imply

$$p_{\mathrm{L}}(\mathcal{E}) = \frac{1}{2}\left(1+\frac{\gamma}{1+\gamma}\right). \tag{34}$$

From Eqs. (26) and (34), we note that there exists a nonzero gap between $p_{\mathrm{G}}(\mathcal{E})$ and $p_{\mathrm{L}}(\mathcal{E})$,

$$p_{\mathrm{L}}(\mathcal{E}) = \frac{1}{2}\left(1+\frac{\gamma}{1+\gamma}\right) < \frac{1}{2}\left(1+\frac{\sqrt{1+\gamma^2}}{1+\gamma}\right) = p_{\mathrm{G}}(\mathcal{E}), \tag{35}$$

for $2\leqslant\gamma<\infty$, thus, NLWE occurs in terms of ME in discriminating the states of the ensemble $\mathcal{E}$ in Example 1.

Now, we show that the occurrence of NLWE in Inequality (35) can be vanished when the PI about the prepared subensemble is available. Let us consider the following POVM $\{M_{\vec{\omega}}\}_{\vec{\omega}\in\Omega}$:

$$\begin{aligned} M_{(0,+)} &= |+\rangle\langle+|\otimes|0\rangle\langle 0|, \quad M_{(1,+)} = |+\rangle\langle+|\otimes|1\rangle\langle 1|,\\ M_{(0,-)} &= |-\rangle\langle-|\otimes|0\rangle\langle 0|, \quad M_{(1,-)} = |-\rangle\langle-|\otimes|1\rangle\langle 1|, \end{aligned} \tag{36}$$

which can be performed using finite-round LOCC: Two local measurements $\{|+\rangle\langle+|, |-\rangle\langle-|\}$ and $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ are performed on first and second subsystems, respectively. Moreover, it is a straightforward calculation to show that the success probability for the LOCC measurement of Eq. (36) in discriminating the states in the ensemble $\mathcal{E}$ with PI is one. That is, the states in $\mathcal{E}$ can be perfectly discriminated when PI is available.

We note that the success probability obtained from the LOCC measurement in Eq. (36) is a lower bound of $p_{\mathrm{L}}^{\mathrm{PI}}(\mathcal{E})$ in Eq. (15), therefore,

$$p_{\mathrm{L}}^{\mathrm{PI}}(\mathcal{E}) \geqslant 1 \tag{37}$$

for the ensemble $\mathcal{E}$ in Example 1. Moreover, from the definitions of $p_{\mathrm{G}}^{\mathrm{PI}}(\mathcal{E})$ and $p_{\mathrm{L}}^{\mathrm{PI}}(\mathcal{E})$ in Eqs. (12) and (15), respectively,

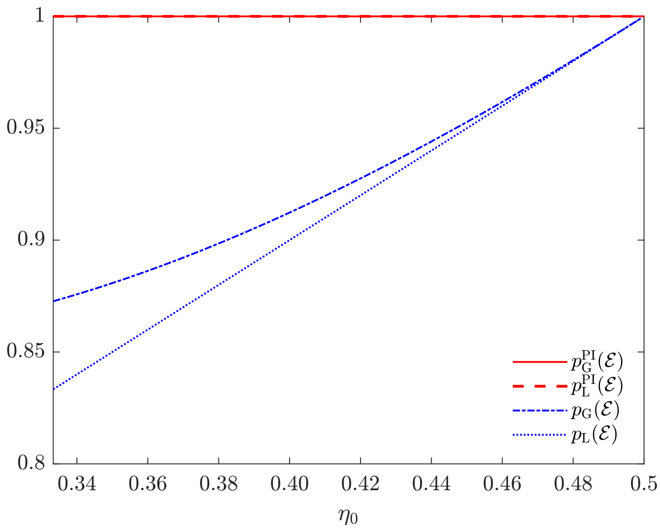

FIG. 2. Annihilating NLWE by PI in terms of ME. For all $\eta_0 \in [\frac{1}{3}, \frac{1}{2})$, $p_{\rm L}(\mathcal{E})$ (dotted blue line) is less than $p_{\rm G}(\mathcal{E})$ (dot-dashed blue line), but $p_{\rm L}^{\rm PI}(\mathcal{E})$ (dashed red line) is equal to $p_{\rm G}^{\rm PI}(\mathcal{E})$ (solid red line).

we have

$$p_{\rm G}^{\rm PI}(\mathcal{E}) \geqslant p_{\rm L}^{\rm PI}(\mathcal{E}). \tag{38}$$

As both $p_{\rm G}^{\rm PI}(\mathcal{E})$ and $p_{\rm L}^{\rm PI}(\mathcal{E})$ are bounded above by 1, we have

$$p_{\rm G}^{\rm PI}(\mathcal{E}) = p_{\rm L}^{\rm PI}(\mathcal{E}) = 1. \tag{39}$$

Thus, NLWE does not occur in terms ME in discriminating the states of the ensemble $\mathcal{E}$ in Example 1 when the PI about the prepared subensemble is available.

Inequality (35) shows that NLWE occurs in terms of ME about the ensemble $\mathcal{E}$ in Example 1, whereas Eq. (39) shows that NLWE does not occur when PI is available. Figure 2 illustrates the relative order of $p_{\rm G}(\mathcal{E})$, $p_{\rm L}(\mathcal{E})$, $p_{\rm G}^{\rm PI}(\mathcal{E})$, and $p_{\rm L}^{\rm PI}(\mathcal{E})$ for the range of $\frac{1}{3} \leqslant \eta_0 < \frac{1}{2}$.

Theorem 1. For ME of the ensemble in Example 1, the PI about the prepared subensemble annihilates NLWE.

IV. CREATING NLWE BY POSTMEASUREMENT INFORMATION

In this section, we consider the opposite situation to the previous section; the PI about the prepared subensemble $\mathcal{E}_b$ in Eq. (5) creates NLWE. After providing an example of a state ensemble $\mathcal{E}$ in Eq. (4), we first show that NLWE does not occur in discriminating the states of the ensemble. With the same ensemble, we further show the occurrence of NLWE in the state discrimination with the help of PI, thus, creating NLWE by PI.

Example 2. Let us consider the ensemble $\mathcal{E}$ in Eq. (4) with

$$\begin{aligned}
\eta_0 &= \frac{\gamma}{2(1+\gamma)}, \quad \rho_0 = |0\rangle\langle 0| \otimes |0\rangle\langle 0|, \\
\eta_1 &= \frac{\gamma}{2(1+\gamma)}, \quad \rho_1 = |0\rangle\langle 0| \otimes |1\rangle\langle 1|, \\
\eta_+ &= \frac{1}{2(1+\gamma)}, \quad \rho_+ = |+\rangle\langle +| \otimes |+\rangle\langle +|, \\
\eta_- &= \frac{1}{2(1+\gamma)}, \quad \rho_- = |+\rangle\langle +| \otimes |-\rangle\langle -|,
\end{aligned} \tag{40}$$

where $2 \leqslant \gamma < \infty$. In this case, the subensembles in Eq. (5) become

$$\begin{aligned}
\mathcal{E}_0 &= \left\{\tfrac{1}{2}, |0\rangle\langle 0| \otimes |0\rangle\langle 0|, \quad \tfrac{1}{2}, |0\rangle\langle 0| \otimes |1\rangle\langle 1|\right\}, \\
\mathcal{E}_1 &= \left\{\tfrac{1}{2}, |+\rangle\langle +| \otimes |+\rangle\langle +|, \ \tfrac{1}{2}, |+\rangle\langle +| \otimes |-\rangle\langle -|\right\},
\end{aligned} \tag{41}$$

with the probabilities of preparation $\frac{\gamma}{1+\gamma}$ and $\frac{1}{1+\gamma}$, respectively.

To show the nonoccurrence of NLWE in terms of ME about the ensemble $\mathcal{E}$ in Example 2, we first evaluate the optimal success probability $p_{\rm G}(\mathcal{E})$ defined in Eq. (6). From the optimality condition in Eq. (7) together with a straightforward calculation, we can easily verify that the following POVM $\{M_i\}_{i\in\Lambda}$ is optimal for $p_{\rm G}(\mathcal{E})$:

$$\begin{aligned}
M_0 &= |\nu_-\rangle\langle \nu_-| \otimes |0\rangle\langle 0|, \quad M_+ = |\nu_+\rangle\langle \nu_+| \otimes |+\rangle\langle +|, \\
M_1 &= |\nu_-\rangle\langle \nu_-| \otimes |1\rangle\langle 1|, \quad M_- = |\nu_+\rangle\langle \nu_+| \otimes |-\rangle\langle -|,
\end{aligned} \tag{42}$$

where

$$|\nu_\pm\rangle = \sqrt{\frac{1}{2} \mp \frac{\gamma}{2\sqrt{1+\gamma^2}}}|0\rangle \pm \sqrt{\frac{1}{2} \pm \frac{\gamma}{2\sqrt{1+\gamma^2}}}|1\rangle. \tag{43}$$

Thus, the optimality of the POVM $\{M_i\}_{i\in\Lambda}$ in Eq. (42) and the definition of $p_{\rm G}(\mathcal{E})$ lead us to

$$p_{\rm G}(\mathcal{E}) = \frac{1}{2}\left(1 + \frac{\sqrt{1+\gamma^2}}{1+\gamma}\right). \tag{44}$$

The measurement given in Eq. (42) can be achieved with finite-round LOCC: First, a local measurement $\{|\nu_+\rangle\langle \nu_+|, |\nu_-\rangle\langle \nu_-|\}$ is performed on the first subsystem, and then according to $|\nu_+\rangle\langle \nu_+|$ or $|\nu_-\rangle\langle \nu_-|$, a local measurement $\{|+\rangle\langle +|, |-\rangle\langle -|\}$ or $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ is performed on the second subsystem. Thus, the success probability for the LOCC measurement in Eq. (42) is a lower bound of $p_{\rm L}(\mathcal{E})$ in Eq. (8), therefore,

$$p_{\rm L}(\mathcal{E}) \geqslant \frac{1}{2}\left(1 + \frac{\sqrt{1+\gamma^2}}{1+\gamma}\right) \tag{45}$$

for the ensemble $\mathcal{E}$ in Example 2. Moreover, from the definitions of $p_{\rm G}(\mathcal{E})$ and $p_{\rm L}(\mathcal{E})$ in Eqs. (6) and (8), respectively, we have

$$p_{\rm G}(\mathcal{E}) \geqslant p_{\rm L}(\mathcal{E}). \tag{46}$$

Inequalities (45) and (46) lead us to

$$p_{\rm L}(\mathcal{E}) = p_{\rm G}(\mathcal{E}) = \frac{1}{2}\left(1 + \frac{\sqrt{1+\gamma^2}}{1+\gamma}\right). \tag{47}$$

Thus, NLWE does not occur in terms of ME in discriminating the states of the ensemble $\mathcal{E}$ in Example 2.

Now, we show that NLWE occurs when the PI about the prepared subensemble is available. Let us consider the following POVM $\{M_{\vec{\omega}}\}_{\vec{\omega}\in\Omega}$:

$$\begin{aligned}
M_{(0,+)} &= |\Phi_+\rangle\langle \Phi_+|, \quad M_{(0,-)} = |\Phi_-\rangle\langle \Phi_-|, \\
M_{(1,+)} &= |\Psi_+\rangle\langle \Psi_+|, \quad M_{(1,-)} = |\Psi_-\rangle\langle \Psi_-|,
\end{aligned} \tag{48}$$

where $|\Phi_\pm\rangle$ and $|\Psi_\pm\rangle$ are Bell states,

$$|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle),$$
$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}(|01\rangle \pm |10\rangle). \quad (49)$$

From a straightforward calculation, we can easily see that the success probability obtained from the measurement of Eq. (48) in discriminating the states in the ensemble $\mathcal{E}$ with PI is one,

$$p_{\rm G}^{\rm PI}(\mathcal{E}) = 1. \quad (50)$$

That is, the states of $\mathcal{E}$ can be perfectly discriminated when PI is available.

In order to obtain the maximum success probability $p_{\rm L}^{\rm PI}(\mathcal{E})$ in Eq. (15), we consider lower and upper bounds of $p_{\rm L}^{\rm PI}(\mathcal{E})$. For a lower bound of $p_{\rm L}^{\rm PI}(\mathcal{E})$, let us first consider the average state ensemble $\tilde{\mathcal{E}}$ defined in Eqs. (14) and (18) with respect to Example 2,

$$\tilde{\eta}_{\vec{\omega}} = \frac{1}{4} \quad \forall\,\vec{\omega}\in\Omega, \quad \tilde{\rho}_{(0,\pm)} = \frac{\eta_0}{\eta_0+\eta_\pm}\rho_0 + \frac{\eta_\pm}{\eta_0+\eta_\pm}\rho_\pm = \frac{\gamma}{1+\gamma}|0\rangle\langle 0|\otimes|0\rangle\langle 0| + \frac{1}{1+\gamma}|+\rangle\langle +|\otimes|\pm\rangle\langle\pm|,$$
$$\tilde{\rho}_{(1,\pm)} = \frac{\eta_1}{\eta_1+\eta_\pm}\rho_1 + \frac{\eta_\pm}{\eta_1+\eta_\pm}\rho_\pm = \frac{\gamma}{1+\gamma}|0\rangle\langle 0|\otimes|1\rangle\langle 1| + \frac{1}{1+\gamma}|+\rangle\langle +|\otimes|\pm\rangle\langle\pm|, \quad (51)$$

which satisfy

$$(\sigma_0\otimes\sigma_2)\tilde{\rho}_{(0,\pm)}(\sigma_0\otimes\sigma_2) = \tilde{\rho}_{(1,\mp)}, \quad (52a)$$
$$(\sigma_0\otimes\sigma_1)\tilde{\rho}_{(0,\pm)}(\sigma_0\otimes\sigma_1) = \tilde{\rho}_{(1,\pm)}, \quad (52b)$$

with the Pauli operators σ_0 and σ_1 in Eq. (30) and

$$\sigma_2 = -i|0\rangle\langle 1| + i|1\rangle\langle 0|. \quad (53)$$

We further consider the following Hermitian operators,

$$\tilde{\rho}_{(0,+)} - \tilde{\rho}_{(1,-)}, \quad \tilde{\rho}_{(1,+)} - \tilde{\rho}_{(0,-)}, \quad (54)$$

where both of them have the same four eigenvalues; two positive eigenvalues λ_+ and λ_-, and two negative eigenvalues $-\lambda_+$ and $-\lambda_-$ with

$$\lambda_\pm = \frac{\sqrt{1+\gamma+\gamma^2} \pm \sqrt{1-\gamma+\gamma^2}}{2(1+\gamma)} \quad (55)$$

for $2 \leqslant \gamma < \infty$. We denote $\Pi_{(0,+)}$ and $\Pi_{(1,-)}$ as the projection operators onto the positive and negative eigenspaces of $\tilde{\rho}_{(0,+)} - \tilde{\rho}_{(1,-)}$, respectively. Similarly, we denote $\Pi_{(1,+)}$ and $\Pi_{(0,-)}$ as the projection operators onto the positive and negative eigenspaces of $\tilde{\rho}_{(1,+)} - \tilde{\rho}_{(0,-)}$, respectively.

Now, we consider the following POVM $\{M_{\vec{\omega}}\}_{\vec{\omega}\in\Omega}$:

$$M_{(0,+)} = \tfrac{1}{2}\Pi_{(0,+)}, \quad M_{(0,-)} = \tfrac{1}{2}\Pi_{(0,-)},$$
$$M_{(1,+)} = \tfrac{1}{2}\Pi_{(1,+)}, \quad M_{(1,-)} = \tfrac{1}{2}\Pi_{(1,-)}. \quad (56)$$

From the property of (52a) and the definition of $\Pi_{\vec{\omega}}$, we can see that

$$(\sigma_0\otimes\sigma_2)\Pi_{(0,+)}(\sigma_0\otimes\sigma_2) = \Pi_{(1,-)},$$
$$(\sigma_0\otimes\sigma_2)\Pi_{(1,+)}(\sigma_0\otimes\sigma_2) = \Pi_{(0,-)},$$
$$\Pi_{(0,+)} + \Pi_{(1,-)} = \mathbb{1},$$
$$\Pi_{(1,+)} + \Pi_{(0,-)} = \mathbb{1}. \quad (57)$$

Here we note that for any Hermitian operator A satisfying,

$$A + (\sigma_0\otimes\sigma_2)A(\sigma_0\otimes\sigma_2) = \mathbb{1}, \quad (58)$$

it holds that

$$\langle i0|A|j1\rangle = \langle i1|A|j0\rangle \quad (59)$$

for any $i, j \in \{0, 1\}$. From Eqs. (57)–(59), we have

$$\mathrm{PT}(\Pi_{\vec{\omega}}) = \Pi_{\vec{\omega}} \quad \forall\,\vec{\omega}\in\Omega, \quad (60)$$

which implies that $\Pi_{\vec{\omega}}$ is in SEP for any $\vec{\omega}\in\Omega$. Thus, two POVMs $\{\Pi_{(0,+)}, \Pi_{(1,-)}\}$ and $\{\Pi_{(1,+)}, \Pi_{(0,-)}\}$ are separable. Moreover, both of them can be performed using finite-round LOCC because each of them consists of two orthogonal rank-2 projection operators [29]. The measurement given in Eq. (56) can be realized with finite-round LOCC by performing two LOCC measurements $\{\Pi_{(0,+)}, \Pi_{(1,-)}\}$ and $\{\Pi_{(1,+)}, \Pi_{(0,-)}\}$ with the equal probability $\frac{1}{2}$.

The success probability of the LOCC measurement in Eq. (56) for the average state ensemble $\tilde{\mathcal{E}}$ in Eq. (51) is

$$\sum_{\vec{\omega}\in\Omega}\tilde{\eta}_{\vec{\omega}}\mathrm{Tr}(\tilde{\rho}_{\vec{\omega}}M_{\vec{\omega}}) = \frac{1}{4}\left(1+\frac{\sqrt{1+\gamma+\gamma^2}}{1+\gamma}\right). \quad (61)$$

This probability is upper bounded by $p_{\rm L}(\tilde{\mathcal{E}})$ which is the maximum success probability for ME of $\tilde{\mathcal{E}}$ when the available measurements are limited to LOCC measurements,

$$p_{\rm L}(\tilde{\mathcal{E}}) \geqslant \frac{1}{4}\left(1+\frac{\sqrt{1+\gamma+\gamma^2}}{1+\gamma}\right). \quad (62)$$

Since any lower bound of $2p_{\rm L}(\tilde{\mathcal{E}})$ becomes a lower bound of $p_{\rm L}^{\rm PI}(\mathcal{E})$ due to Eq. (17), we have

$$p_{\rm L}^{\rm PI}(\mathcal{E}) \geqslant \frac{1}{2}\left(1+\frac{\sqrt{1+\gamma+\gamma^2}}{1+\gamma}\right). \quad (63)$$

To obtain an upper bound of $p_{\rm L}^{\rm PI}(\mathcal{E})$, let us first consider the following two operators:

$$K_0 = \tfrac{1}{2}\tilde{\rho}_{(0,+)}\Pi_{(0,+)} + \tfrac{1}{2}\tilde{\rho}_{(1,-)}\Pi_{(1,-)},$$
$$K_1 = \tfrac{1}{2}\tilde{\rho}_{(1,+)}\Pi_{(1,+)} + \tfrac{1}{2}\tilde{\rho}_{(0,-)}\Pi_{(0,-)}. \quad (64)$$

Since the projective measurement $\{\Pi_{(0,+)}, \Pi_{(1,-)}\}$ is optimal in ME between two states $\tilde{\rho}_{(0,+)}$ and $\tilde{\rho}_{(1,-)}$ with equal prior probability [20], it satisfies a necessary and sufficient condition for a measurement to be optimal in ME between two states $\tilde{\rho}_{(0,+)}$ and $\tilde{\rho}_{(1,-)}$ with equal prior probability $\frac{1}{2}$ [21,22,27],

$$K_0 - \tfrac{1}{2}\tilde{\rho}_{(0,+)} \succeq 0, \quad K_0 - \tfrac{1}{2}\tilde{\rho}_{(1,-)} \succeq 0. \quad (65)$$

Similarly, $\{\Pi_{(1,+)}, \Pi_{(0,-)}\}$ is the optimal measurement in ME between two states $\tilde{\rho}_{(1,+)}$ and $\tilde{\rho}_{(0,-)}$ with equal prior probability $\frac{1}{2}$, thus,

$$K_1 - \tfrac{1}{2}\tilde{\rho}_{(1,+)} \succeq 0, \quad K_1 - \tfrac{1}{2}\tilde{\rho}_{(0,-)} \succeq 0. \tag{66}$$

We further note that K_0 and K_1 are Hermitian operators due to the positive semidefiniteness of (65) and (66).

Now, we consider a Hermitian operator,

$$\tilde{H} = \tfrac{1}{4}K_0 + \tfrac{1}{4}K_1. \tag{67}$$

We will show that $\tilde{H} - \tilde{\eta}_{\vec{\omega}}\tilde{\rho}_{\vec{\omega}} \in \text{SEP}^*$ for all $\vec{\omega} \in \Omega$, therefore, $2\text{Tr}\tilde{H}$ is the upper bound of $p_{\rm L}^{\rm PI}(\mathcal{E})$ by Lemma 1.

From Eqs. (52) and (57), we can see that

$$\begin{aligned}(\sigma_0 \otimes \sigma_1)K_0(\sigma_0 \otimes \sigma_1) &= K_1,\\ (\sigma_0 \otimes \sigma_1)K_1(\sigma_0 \otimes \sigma_1) &= K_0,\\ (\sigma_0 \otimes \sigma_2)K_0(\sigma_0 \otimes \sigma_2) &= K_0,\\ (\sigma_0 \otimes \sigma_2)K_1(\sigma_0 \otimes \sigma_2) &= K_1.\end{aligned} \tag{68}$$

Moreover, for any Hermitian operator A with

$$(\sigma_0 \otimes \sigma_2)A(\sigma_0 \otimes \sigma_2) = A, \tag{69}$$

it holds that

$$\begin{aligned}\langle i0|A|j0\rangle &= \langle i1|A|j1\rangle,\\ \langle i0|A|j1\rangle &= -\langle i1|A|j0\rangle\end{aligned} \tag{70}$$

for any $i, j \in \{0, 1\}$. From Eqs. (68)–(70), we have

$$\begin{aligned}\text{PT}(K_0) &= (\sigma_0 \otimes \sigma_1)K_0(\sigma_0 \otimes \sigma_1) = K_1,\\ \text{PT}(K_1) &= (\sigma_0 \otimes \sigma_1)K_1(\sigma_0 \otimes \sigma_1) = K_0.\end{aligned} \tag{71}$$

Thus, for each $\vec{\omega} \in \Omega$, $\tilde{H} - \tilde{\eta}_{\vec{\omega}}\tilde{\rho}_{\vec{\omega}}$ can be rewritten as

$$\begin{aligned}\tilde{H} - \tilde{\eta}_{(0,+)}\tilde{\rho}_{(0,+)} &= \frac{1}{4}\left(K_0 - \frac{1}{2}\tilde{\rho}_{(0,+)}\right) + \frac{1}{4}\text{PT}\left(K_0 - \frac{1}{2}\tilde{\rho}_{(0,+)}\right),\\ \tilde{H} - \tilde{\eta}_{(1,-)}\tilde{\rho}_{(1,-)} &= \frac{1}{4}\left(K_0 - \frac{1}{2}\tilde{\rho}_{(1,-)}\right) + \frac{1}{4}\text{PT}\left(K_0 - \frac{1}{2}\tilde{\rho}_{(1,-)}\right),\\ \tilde{H} - \tilde{\eta}_{(1,+)}\tilde{\rho}_{(1,+)} &= \frac{1}{4}\text{PT}\left(K_1 - \frac{1}{2}\tilde{\rho}_{(1,+)}\right) + \frac{1}{4}\left(K_1 - \frac{1}{2}\tilde{\rho}_{(1,+)}\right),\\ \tilde{H} - \tilde{\eta}_{(0,-)}\tilde{\rho}_{(0,-)} &= \frac{1}{4}\text{PT}\left(K_1 - \frac{1}{2}\tilde{\rho}_{(0,-)}\right) + \frac{1}{4}\left(K_1 - \frac{1}{2}\tilde{\rho}_{(0,-)}\right).\end{aligned} \tag{72}$$

From the argument after Eq. (3) together with the positive semidefiniteness of (65) and (66), each $\tilde{H} - \tilde{\eta}_{\vec{\omega}}\tilde{\rho}_{\vec{\omega}}$ in Eq. (72) is in SEP*, therefore, Lemma 1 leads us to

$$p_{\rm L}^{\rm PI}(\mathcal{E}) \leqslant 2\,\text{Tr}\,\tilde{H} = \frac{1}{2}\left(1 + \frac{\sqrt{1+\gamma+\gamma^2}}{1+\gamma}\right). \tag{73}$$

Inequalities (63) and (73) imply

$$p_{\rm L}^{\rm PI}(\mathcal{E}) = \frac{1}{2}\left(1 + \frac{\sqrt{1+\gamma+\gamma^2}}{1+\gamma}\right). \tag{74}$$

From Eqs. (50) and (74), we note that there exists a nonzero gap between $p_{\rm G}^{\rm PI}(\mathcal{E})$ and $p_{\rm L}^{\rm PI}(\mathcal{E})$,

$$p_{\rm L}^{\rm PI}(\mathcal{E}) = \frac{1}{2}\left(1 + \frac{\sqrt{1+\gamma+\gamma^2}}{1+\gamma}\right) < 1 = p_{\rm G}^{\rm PI}(\mathcal{E}) \tag{75}$$

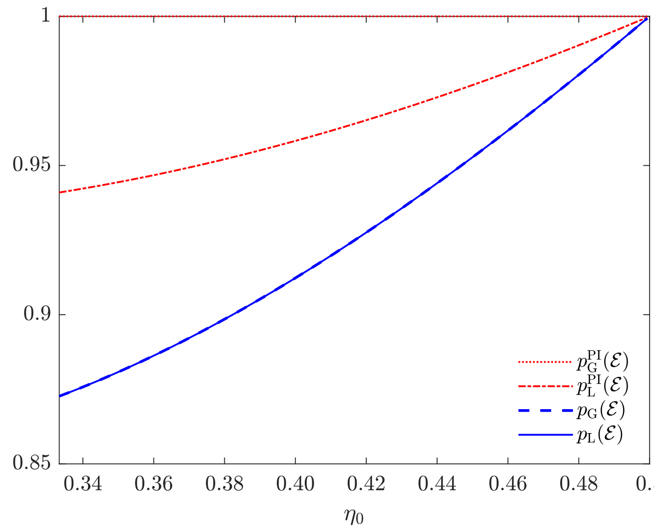

FIG. 3. Creating NLWE by PI in terms of ME. For all $\eta_0 \in [\frac{1}{3}, \frac{1}{2})$, $p_{\rm L}(\mathcal{E})$ (solid blue line) is equal to $p_{\rm G}(\mathcal{E})$ (dashed blue line), but $p_{\rm L}^{\rm PI}(\mathcal{E})$ (dot-dashed red line) is less than $p_{\rm G}^{\rm PI}(\mathcal{E})$ (dotted red line).

for $2 \leqslant \gamma < \infty$. Thus, NLWE occurs in terms of ME when the PI about the prepared subensemble is available.

Equation (47) shows that NLWE does not occur in terms of ME about the ensemble $\mathcal{E}$ in Example 2, whereas Inequality (75) shows that NLWE occurs when PI is available. Figure 3 illustrates the relative order of $p_{\rm G}(\mathcal{E})$, $p_{\rm L}(\mathcal{E})$, $p_{\rm G}^{\rm PI}(\mathcal{E})$, and $p_{\rm L}^{\rm PI}(\mathcal{E})$ for the range of $\frac{1}{3} \leqslant \eta_0 < \frac{1}{2}$.

Theorem 2. For ME of the ensemble $\mathcal{E}$ in Example 2, the PI about the prepared subensemble creates NLWE.

V. DISCUSSION

We have shown that the PI about the prepared subensemble can annihilate or create NLWE in discriminating multiparty nonorthogonal nonentangled quantum states. We have first provided a two-qubit state ensemble consisting of four nonorthogonal separable states (Example 1) and shown that NLWE occurs in discriminating the states in the ensemble. With the same ensemble, we have further shown that the occurrence of NLWE in the state discrimination can be vanished when the PI about the prepared subensemble is available, thus, annihilating NLWE by PI (Theorem 1). Moreover, we have provided another two-qubit state ensemble consisting of four nonorthogonal separable states (Example 2) and shown that NLWE does not occur in discriminating the states of the ensemble. With the same ensemble, we have further shown the occurrence of NLWE in the state discrimination with the PI about the prepared subensemble, thus, creating NLWE by PI (Theorem 2).

We note that in both Examples 1 and 2, the prepared state can be perfectly identified by a global measurement when the PI about the prepared subensemble is provided. In Example 1, the prepared state can be perfectly identified by a LOCC measurement when the PI about the prepared subensemble is available. However, in Example 2, the prepared state cannot be perfectly discriminated by a LOCC measurement even if the PI about the prepared subensemble is available. As far as we know, the latter is an example exhibiting NLWE in terms of perfect discrimination with the help of PI.

We remark that the phenomenon of creating NLWE by PI cannot arise in perfectly discriminating orthogonal separable states because there is no better state discrimination than perfect discrimination. On the other hand, the phenomenon of annihilating NLWE by PI can arise in perfectly discriminating orthogonal separable states with local indistinguishability, such as an *unextendible product basis* (UPB) [30].

For example, let us consider a two-qutrit state ensemble $\{\frac{1}{5}, \rho_i\}_{i=1}^5$ consisting of UPB states ρ_i with the equal prior probability $\frac{1}{5}$ [30],

$$\begin{aligned}\rho_1 &= |\phi_1\rangle\langle\phi_1| \otimes |2\rangle\langle 2|, \quad \rho_4 = |0\rangle\langle 0| \otimes |\phi_1\rangle\langle\phi_1|,\\ \rho_2 &= |\phi_2\rangle\langle\phi_2| \otimes |0\rangle\langle 0|, \quad \rho_5 = |2\rangle\langle 2| \otimes |\phi_2\rangle\langle\phi_2|,\\ \rho_3 &= |\phi_3\rangle\langle\phi_3| \otimes |\phi_3\rangle\langle\phi_3|, \end{aligned} \tag{76}$$

where $\{|0\rangle, |1\rangle, |2\rangle\}$ is the standard basis in one-qutrit system and

$$\begin{aligned}|\phi_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle), \quad |\phi_2\rangle = \frac{1}{\sqrt{2}}(|1\rangle - |2\rangle),\\ |\phi_3\rangle &= \frac{1}{\sqrt{3}}(|0\rangle + |1\rangle + |2\rangle).\end{aligned} \tag{77}$$

Since every UPB can be perfectly discriminated by global measurements but cannot be perfectly discriminated only by LOCC [8,31], NLWE occurs in terms of the perfect discrimination of $\{\frac{1}{5}, \rho_i\}_{i=1}^5$. However, the occurrence of NLWE can be vanished by PI because the prepared state can be perfectly identified in the following situation: The classical information on whether the prepared state belongs to $\{\rho_1, \rho_2, \rho_3\}$ or $\{\rho_4, \rho_5\}$ is provided after a LOCC measurement $\{M_{(i,j)}\}_{i,j=1}^3$,

$$M_{(i,j)} = |\phi_i\rangle\langle\phi_i| \otimes |\phi_j\rangle\langle\phi_j|, \quad i, j = 1, 2, 3, \tag{78}$$

where each $M_{(i,j)}$ indicates the detection of ρ_i or ρ_{3+j} depending on whether the set to which the prepared state belongs is $\{\rho_1, \rho_2, \rho_3\}$ or $\{\rho_4, \rho_5\}$. Thus, annihilating NLWE by PI.

Our result can provide a useful method to share or hide information using nonorthogonal separable states [32–37]. In Example 1, the PI about the prepared subensemble makes the information locally accessible, and the information can be locally shared between parties. On the other hand, in Example 2, the PI about the prepared subensemble makes the information globally accessible but not locally, and the globally accessible information can be locally hidden to some extent. Our results can also be applied to multiparty secret sharing, such as two-qubit nonlocal bases with multicopy adaptive local distinguishability [37]. We finally remark that it would be an interesting future task to investigate if the availability of PI affects the occurrence of NLWE in terms of other optimal discrimination strategies besides ME.

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[1] A. Chefles, Quantum state discrimination, Contemp. Phys. **41**, 401 (2000).
[2] S. M. Barnett and S. Croke, Quantum state discrimination, Adv. Opt. Photonics **1**, 238 (2009).
[3] J. A. Bergou, Discrimination of quantum states, J. Mod. Opt. **57**, 160 (2010).
[4] J. Bae and L.-C. Kwek, Quantum state discrimination and its applications, J. Phys. A: Math. Theor. **48**, 083001 (2015).
[5] C. H. Bennett, D. P. DiVincenzo, C. A. Fuchs, T. Mor, E. Rains, P. W. Shor, J. A. Smolin, and W. K. Wootters, Quantum nonlocality without entanglement, Phys. Rev. A **59**, 1070 (1999).
[6] A. Peres and W. K. Wootters, Optimal Detection of Quantum Information, Phys. Rev. Lett. **66**, 1119 (1991).
[7] E. Chitambar and M.-H. Hsieh, Revisiting the optimal detection of quantum information, Phys. Rev. A **88**, 020302(R) (2013).
[8] D. P. Divincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Unextendible product bases, uncompletable product bases and bound entanglement, Commun. Math. Phys. **238**, 379 (2003).
[9] J. Niset and N. J. Cerf, Multipartite nonlocality without entanglement in many dimensions, Phys. Rev. A **74**, 052103 (2006).
[10] G.-B. Xu, Q.-Y. Wen, S.-J. Qin, Y.-H. Yang, and F. Gao, Quantum nonlocality of multipartite orthogonal product states, Phys. Rev. A **93**, 032341 (2016).
[11] S. Halder, M. Banik, S. Agrawal, and S. Bandyopadhyay, Strong Quantum Nonlocality without Entanglement, Phys. Rev. Lett. **122**, 040403 (2019).
[12] S. S. Bhattacharya, S. Saha, T. Guha, and M. Banik, Nonlocality without entanglement: Quantum theory and beyond, Phys. Rev. Research **2**, 012068(R) (2020).
[13] R. Duan, Y. Feng, Z. Ji, and M. Ying, Distinguishing Arbitrary Multipartite Basis Unambiguously using Local Operations and Classical Communication, Phys. Rev. Lett. **98**, 230502 (2007).
[14] D. Ha and Y. Kwon, Quantum nonlocality without entanglement: explicit dependence on prior probabilities of nonorthogonal mirror-symmetric states, npj Quantum Inf. **7**, 81 (2021).
[15] D. Ha and J. S. Kim, Quantum nonlocality without entanglement depending on nonzero prior probabilities in optimal unambiguous discrimination, Sci. Rep. **11**, 17695 (2021).
[16] S. Akibue, G. Kato, and N. Marumo, Perfect discrimination of nonorthogonal quantum states with posterior classical partial information, Phys. Rev. A **99**, 020102(R) (2019).
[17] M. A. Ballester, S. Wehner, and A. Winter, State discrimination with post-measurement information, IEEE Trans. Inf. Theory **54**, 4183 (2008).
[18] D. Gopal and S. Wehner, Using postmeasurement information in state discrimination, Phys. Rev. A **82**, 022326 (2010).

[19] C. Carmeli, T. Heinosaari, and A. Toigo, State discrimination with postmeasurement information and incompatibility of quantum measurements, Phys. Rev. A **98**, 012126 (2018).

[20] C. W. Helstrom, *Quantum Detection and Estimation Theory* (Academic, New York, 1976).

[21] A. S. Holevo, *Probabilistic and Statistical Aspects of Quantum Theory* (North-Holland, Amsterdam, 1979).

[22] H. Yuen, R. Kennedy, and M. Lax, Optimum testing of multiple hypotheses in quantum detection theory, IEEE Trans. Inf. Theory **21**, 125 (1975).

[23] J. Bae, Structure of minimum-error quantum state discrimination, New J. Phys. **15**, 073037 (2013).

[24] E. Chitambar, D. Leung, L. Maninska, M. Ozols, and A. Winter, Everything you always wanted to know about LOCC (but were afraid to ask), Commun. Math. Phys. **328**, 303 (2014).

[25] A. Peres, Separability Criterion for Density Matrices, Phys. Rev. Lett. **77**, 1413 (1996).

[26] M. Horodecki, P. Horodecki, and R. Horodecki, Separability of mixed states: Necessary and sufficient conditions, Phys. Lett. A **223**, 1 (1996).

[27] S. M. Barnett and S. Croke, On the conditions for discrimination between quantum states with minimum error, J. Phys. A: Math. Theor. **42**, 062001 (2009).

[28] S. Bandyopadhyay, A. Cosentino, N. Johnston, V. Russo, J. Watrous, and N. Yu, Limitations on separable measurements by convex optimization, IEEE Trans. Inf. Theory **61**, 3593 (2015).

[29] E. Chitambar, R. Duan, and M.-H. Hsieh, When do local operations and classical communication suffice for two-qubit state discrimination?, IEEE Trans. Inf. Theory **60**, 1549 (2014).

[30] C. H. Bennett, D. P. DiVincenzo, T. Mor, P. W. Shor, J. A. Smolin, and B. M. Terhal, Unextendible Product Bases and Bound Entanglement, Phys. Rev. Lett. **82**, 5385 (1999).

[31] S. De Rinaldis, Distinguishability of complete and unextendible product bases, Phys. Rev. A **70**, 022309 (2004).

[32] B. M. Terhal, D. P. DiVincenzo, and D. W. Leung, Hiding Bits in Bell States, Phys. Rev. Lett. **86**, 5807 (2001).

[33] D. P. DiVincenzo, D. W. Leung, and B. M. Terhal, Quantum data hiding, IEEE Trans. Inf. Theory **48**, 580 (2002).

[34] T. Eggeling, and R. F. Werner, Hiding Classical Data in Multipartite Quantum States, Phys. Rev. Lett. **89**, 097905 (2002).

[35] R. Rahaman and M. G. Parker, Quantum scheme for secret sharing based on local distinguishability, Phys. Rev. A **91**, 022330 (2015).

[36] J. Wang, L. Li, H. Peng, and Y. Yang, Quantum-secret-sharing scheme based on local distinguishability of orthogonal multiqudit entangled states, Phys. Rev. A **95**, 022320 (2017).

[37] M. Banik, T. Guha, M. Alimuddin, G. Kar, S. Halder, and S. S. Bhattacharya, Multicopy Adaptive Local Discrimination: Strongest Possible Two-Qubit Nonlocal Bases, Phys. Rev. Lett. **126**, 210505 (2021).