

Conversion of Gaussian states under incoherent Gaussian operations

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The coherence resource theory needs to study the operational value and efficiency which can be broadly formulated as the question: when can one coherent state be converted into another under incoherent operations? We answer this question completely for one-mode continuous-variable systems by characterizing conversion of coherent Gaussian states under incoherent Gaussian operations in terms of their first and second moments. The no-go theorem of purification of coherent Gaussian states is also built. The structure of incoherent Gaussian operations of two-mode continuous-variable systems is discussed further and is applied to coherent conversion for pure Gaussian states with standard second moments. The standard second moments are images of all second moments under local linear unitary Bogoliubov operations. As concrete applications, we obtain some peculiarities of a Gaussian system: (1) there does not exist a maximally coherent Gaussian state which can generate all coherent Gaussian states; (2) the conversion between pure Gaussian states is reversible; (3) the coherence of input pure state and the coherence of output pure state are equal.

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I. INTRODUCTION

Manipulating physical systems always suffers from practical restrictions which limit the control we can exert. It is, e.g., extremely difficult to exchange quantum systems undisturbed over long distances [1]. In order to manipulate spatially separate subsystems effectively within the resource theoretic framework, this restricts us to local operation and classical communication (LOCC). Under these operations, we have to prepare a certain kind of states, i.e., separable states. The states which cannot be produced by LOCC are entangled. The entanglement is the key resource that allows one to implement operations such as quantum state teleportation to obtain perfect quantum state conversion by consuming entanglement [2]. The restrictions are vital in quantum communication and quantum technology and also drive a deep understanding of the fundamental laws of nature [1,3–5].

As entanglement of pure states is among the manifestations of the superposition principle, one can naturally see the phenomenon of coherent superposition as a valuable resource. Recently, the resource theory of quantum coherence has attracted much attention and various efforts are made to build the coherence resource theory [6–12]. In this frame, free operations corresponding to LOCC in entanglement theory are incoherent operations (IOs) that can be interpreted as a measurement which cannot create coherence even if one applies postselection on the measurement outcomes [7].

One of the central questions in coherence resource theory is conversion of coherent states. The aim is to study whether IOs can introduce an order on the set of coherent states, i.e., whether, given two coherent states ρ and σ , either ρ can be

transformed into σ or vice versa. The answer to this question determines the value of coherent states in technological applications. The question has been solved for pure state case [13–15] and for qubit state case [10,16,17]. More recently, the conversion between pure states and mixed states is characterized in [18–20].

All the above results for conversion of coherence states are implicitly assuming a discrete-variable system (finite-dimensional system). Note that the first framework for understanding quantum coherence is quantum optics which must require quantum states in a continuous-variable system (infinite-dimensional system), especially the Gaussian states which have arisen to a privileged position in continuous-variable quantum information [21–24]. The primary tool for analyzing Gaussian states is Gaussian operations. Indeed, Gaussian operations correspond exactly to those operations that can be implemented by means of optical elements such as beam splitters, phase shifts, and squeezers together with homodyne measurements [25–27]. Such operations are in principle experimentally accessible with present technology [28]. Especially, Gaussian unitary operations can be realized as a passive operation, a single-mode squeezing operation on each of the n modes, and a subsequent second passive operation [29]. In fact, phase rotation, the simplest and most common Gaussian unitary operation, is an optical implementation which preserves coherence in the process of conversion of coherent states [24]. For the process of evolution of optical cat states, coherence is consumed [30,31].

In the outlook of [7], Baumgratz, Cramer, and Plenio point out that coherence theory of Gaussian systems is needed. Closely mirroring the development of entanglement theory, mathematical problems concerning continuity that are inevitably emerging can be addressed by requiring energy constraints [32] or by considering special, experimentally relevant, subclasses such as Gaussian states [29].

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The study of coherence theory of Gaussian systems is moving ahead since the question is proposed [7]. Recently, coherence theory of Gaussian systems including incoherent Gaussian states, incoherent Gaussian operations, and coherence measures of Gaussian states is introduced [33–35]. The main contribution of our paper is to discuss conversion of coherence states in continuous-variable systems under incoherent Gaussian operations (IGOs).

The paper is organized as follows. In Sec. II, we review some background material and establish notations. In particular, we review the definitions and characterizations of incoherent Gaussian states and incoherent Gaussian operations obtained in [33]. In Sec. III, an explicit description of conversion of Gaussian states of one-mode continuous-variable systems is provided and the no-go theorem of purification for Gaussian states is built. In Sec. IV, we further study the structure of IGOs in two-mode continuous-variable systems and characterize conversion of one kind of important Gaussian pure states under IGOs. The last section is a summary of our findings.

II. BACKGROUND AND NOTATION

Let \mathcal{H} be an infinite-dimensional Hilbert space with fixed orthonormal basis $\{|n\rangle\}_{n=0}^{+\infty}$. When we consider the m -mode continuous-variable systems $\mathcal{H}^{\otimes m}$, we adopt $\{|n\rangle\}_{n=0}^{+\infty}$ as its reference basis. For a quantum state ρ on $\mathcal{H}^{\otimes m}$, the characteristic function of ρ is defined as

$$\begin{aligned}\mathcal{X}_\rho(\lambda) &= \text{tr}[\rho D(\lambda)], \\ D(\lambda) &= \otimes_{i=1}^m D(\lambda_i), \\ D(\lambda_i) &= e^{(\lambda_i \hat{a}_i^\dagger - \bar{\lambda}_i \hat{a}_i)}.\end{aligned}$$

Here \hat{a}_i and \hat{a}_i^\dagger are the annihilation and creation operator in mode i , $\lambda = (\lambda_1, \dots, \lambda_m)^t$, and $\bar{\lambda}_i$ denotes the complex conjugate of λ_i . A one-mode quantum state ρ is called Gaussian state if its characteristic function

$$\mathcal{X}_\rho(\lambda) = \exp^{-\frac{1}{4}(\lambda_x, \lambda_y) \Omega V \Omega^t (\lambda_x, \lambda_y)^t - i(\Omega d)^t (\lambda_x, \lambda_y)^t},$$

where $\lambda_x = \text{Re}(\lambda)$ and $\lambda_y = \text{Im}(\lambda)$ are the real and imaginary parts of λ , $\Omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $d = (d_1, d_2)^t \in \mathbb{R}^2$, and $V = \begin{pmatrix} v_{11} & v_{12} \\ v_{12} & v_{22} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$.

Recall that V is a positive-definite matrix satisfying $V + i\Omega \geq 0$ (all eigenvalues are non-negative) and $\det V \geq 1$, $\det V = 1$ if and only if ρ is pure [24]. d and V are called the first and second moment of ρ which can describe Gaussian state ρ completely. So ρ can be usually written in $\rho(V, d)$.

The quantification of coherence is fundamental in the resource theory of quantum coherence. For a given coherent Gaussian state, it is important to ask the amount of coherence it has. Inspired by the idea of discrete-variable systems [7], researchers have built the framework for quantifying coherence of Gaussian states [33–35]. For the convenience of reader, we give a brief overview of results in [33]. The incoherent Gaussian states are defined as diagonal Gaussian states. The set of incoherent Gaussian states will be labeled by \mathcal{I} . The nondiagonal Gaussian states are called coherent Gaussian states. In fact, \mathcal{I} consists of all thermal states [33]. A thermal

state has the form

$$\rho_{th}(\bar{n}) = \sum_{n=0}^{+\infty} \frac{\bar{n}^n}{(\bar{n} + 1)^{n+1}} |n\rangle\langle n|,$$

where $\bar{n} = \text{tr}[\hat{a}^\dagger \hat{a} \rho_{th}(\bar{n})]$ is the mean photon number. A Gaussian operation is incoherent if it maps incoherent Gaussian states into incoherent Gaussian states. A one-mode incoherent Gaussian operation is fully described by

$$(T, N), \quad T = tO, \quad N = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix},$$

where t is a real number, O is a 2×2 real orthogonal matrix ($OO^t = O^tO = I$), and $\omega \geq |t^2 \det O - 1|$ [33]. For a Gaussian state $\rho(V, d)$, it performs on $\rho(V, d)$ and obtain a Gaussian state with the first and second moment as follows:

$$d \mapsto Td, \quad V \mapsto TVT^t + N.$$

For m -mode case, every Gaussian state $\rho(V, d)$ is described by its first and second moment d and V , where d is a $2m$ -dimensional column vector and V is a $2m \times 2m$ real positive-definite matrix with $V + i\Omega \geq 0$, $\Omega = \oplus_m \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ [24]. Furthermore, $\det V \geq 1$ and $\det V = 1$ if and only if $\rho(V, d)$ is pure. In [33,35], Xu has given detailed characterizations of incoherent Gaussian states and incoherent Gaussian operations (IGOs). The incoherent Gaussian states have the form $\otimes_{j=1}^m \rho_{th}(v_j)$, where v_j is the symplectic eigenvalue of $\rho_{th}(v_j)$. A Gaussian operation $\Phi(T, N, d)$ is incoherent if and only if

$$\begin{aligned}d &= 0, \\ T &= \{t_j O_j\}_{j=1}^m \in \mathcal{T}_{2m}, \\ N &= \oplus_{j=1}^m \omega_j I_2, \\ \omega_j &\geq |1 - \sum_{k,r(k)=j} t_k^2 \det O_k|, \forall j,\end{aligned}$$

where $t_j, \omega_j \in \mathbb{R}$, O_j is a 2×2 real orthogonal matrix ($O_j O_j^t = I_2$), \mathcal{T}_{2m} denotes the set of $2m \times 2m$ real matrices such that, for any $T \in \mathcal{T}_{2m}$, the $(2j-1, 2j)$ two columns have just one 2×2 real matrix $t_j O_j$ located in $[2r(j)-1, 2r(j)]$ rows for $\forall j$, $r(j) \in \{k\}_{k=1}^m$, and other elements are all zero. For a Gaussian state $\rho(V, d)$, it performs on $\rho(V, d)$ and obtain a Gaussian state with the first and second moment as follows:

$$d \mapsto Td, \quad V \mapsto TVT^t + N.$$

Based on the definition of incoherent Gaussian states and incoherent Gaussian operations (IGOs), any proper coherence measure C is a non-negative function and must satisfy the following conditions.

(C1) $C(\rho) = 0$ for all $\rho \in \mathcal{I}$.

(C2) Monotonicity under all incoherent Gaussian operations (IGOs) Φ : $C[\Phi(\rho)] \leq C(\rho)$.

(C3) Nonincreasing under mixing of Gaussian states: $C(\sum_j p_j \rho_j) \leq \sum_j p_j C(\rho_j)$ for any set of Gaussian states $\{\rho_j\}$ and any $p_j \geq 0$ with $\sum_j p_j = 1$.

Furthermore, the relative entropy measure has been provided by

$$C_R(\rho) = \inf_{\delta \in \mathcal{I}} S(\rho||\delta),$$

where $S(\rho||\delta) = \text{tr}(\rho \log_2 \rho) - \text{tr}(\rho \log_2 \delta)$ is the relative entropy.

For Gaussian states $\rho(V_1, d_1), \sigma(V_2, d_2)$, if there exists an incoherent Gaussian operation Φ such that $\Phi(\rho) = \sigma$, we denote it by $\rho(V_1, d_1) \xrightarrow{\text{IGO}} \sigma(V_2, d_2)$ and call ρ and $\Phi(\rho)$ the input state and output state, respectively.

III. CONVERSION OF GAUSSIAN STATES OF ONE-MODE CONTINUOUS-VARIABLE SYSTEMS

Our first result provides a complete classification of conversion for pure Gaussian states. This offers an affirmative answer to the open question on coherence conversion in continuous-variable systems [7,13]. This question is to study how we can determine if there exists an IGO Φ such that $\Phi(\rho(V_1, d_1)) = \sigma(V_2, d_2)$ for pure Gaussian states $\rho(V_1, d_1)$ and $\sigma(V_2, d_2)$.

Theorem 3.1. For pure Gaussian states $\rho(V_1, d_1), \sigma(V_2, d_2)$, $\rho(V_1, d_1) \xrightarrow{\text{IGO}} \sigma(V_2, d_2)$ if and only if there exists a phase rotation operator $R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$ for some $\theta \in \mathbb{R}$ such that $V_2 = R(\theta)V_1R(\theta)^t, d_2 = R(\theta)d_1$ or $V_2 = I$ and $d_2 = 0$ ($\sigma(V_2, d_2) = |0\rangle \in \mathcal{I}$).

By Theorem 3.1, we obtain the following key peculiarities of Gaussian continuous-variable systems.

(i) $\rho(V_1, d_1) \xrightarrow{\text{IGO}} \sigma(V_2, d_2)$ iff $\sigma(V_2, d_2) \xrightarrow{\text{IGO}} \rho(V_1, d_1)$ for coherent pure Gaussian states $\rho(V_1, d_1), \sigma(V_2, d_2)$.

(ii) For any coherence measure C , $C(\rho(V_1, d_1)) = C(\sigma(V_2, d_2))$ if $\rho(V_1, d_1) \xrightarrow{\text{IGO}} \sigma(V_2, d_2)$. This implies that the coherence of input state is equal to the coherence of output state. It shows frozen behavior of coherence in Gaussian dynamical systems. Frozen coherence in discrete-variable systems studied in [36] is to discuss when $C(\rho) = C(\Phi(\rho))$ holds true. Frozen coherence in Gaussian dynamical systems is also listed as an open question in summary of [33].

(iii) There does not exist a maximally coherent Gaussian pure state $|\psi\rangle$ such that $|\psi\rangle \xrightarrow{\text{IGO}} \sigma$ for any Gaussian state σ .

Here we identify a maximally coherent state as a state that allows for the deterministic generation of all other Gaussian states by means of incoherent Gaussian operations. Note that maximally coherent states are independent of a specific coherence measure.

The above consequences demonstrate significant differences between discrete-variable systems and Gaussian systems. One key reason for these differences is the fact that Gaussian states are completely specified by their first and second moments. Intuitively, since determinant of the second moment for any pure Gaussian state is 1, conversion of pure Gaussian states by IGO can be realized by Gaussian unitary operations.

Theorem 3.1 is also a nice tool for conversion of pure Gaussian states because the first and second moments of pure Gaussian states have clear analytic formulas. In the following, we exhibit conditions for realizing conversion of pure states under IGOs.

For α state

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^{+\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

($\alpha \in \mathbb{C}$), the most important Gaussian states which are generated by the vacuum state $|0\rangle$ and Weyl displacement operator $\widehat{D}(\alpha) = e^{\alpha\widehat{a}^\dagger - \bar{\alpha}\widehat{a}}$, $|\alpha\rangle = \widehat{D}(\alpha)|0\rangle$. Note that

$$d = 2[\text{Re}(\alpha), \text{Im}(\alpha)]^t, \quad V = I \quad [25].$$

Obviously $|\alpha\rangle \notin \mathcal{I}$, i.e., $|\alpha\rangle$ is coherent. By Theorem 3.1, one can see that

$$|\alpha\rangle \xrightarrow{\text{IGO}} |\beta\rangle \Leftrightarrow |\alpha| = |\beta|.$$

The most general pure Gaussian state $|\psi\rangle$ of one mode is a displaced squeezed state obtained by the combined action of Weyl displacement operator $\widehat{D}(\alpha)$ and the squeezing operator

$$\widehat{S}(\beta) = e^{\frac{1}{2}[\beta\widehat{a}^{\dagger 2} - \bar{\beta}\widehat{a}^2]}, \quad \beta \in \mathbb{C},$$

on the vacuum state $|0\rangle$ [24]:

$$|\psi_{\alpha,\beta}\rangle = \widehat{D}(\alpha)\widehat{S}(\beta)|0\rangle.$$

The first and second moment of $|\psi_{\alpha,\beta}\rangle$ are [37]

$$\begin{pmatrix} 2[\text{Re}(\alpha), \text{Im}(\alpha)], \\ \left(\begin{array}{c} \cosh(2|\beta|) + \cos \theta \sinh(2|\beta|) \\ \sin \theta \sinh(2|\beta|) \end{array} \right), \\ \left(\begin{array}{c} \sin \theta \sinh(2|\beta|) \\ \cosh(2|\beta|) - \cos \theta \sinh(2|\beta|) \end{array} \right), \end{pmatrix}$$

where $\beta = |\beta|e^{i\theta}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$, and $\sinh(x) = \frac{e^x - e^{-x}}{2}$ are hyperbolic functions. By Theorem 3.1, one can deal with conversion of pure Gaussian states efficiently.

A particularly key conversion of coherent states in discrete-variable systems is purification which is the process that extracts pure coherent states from general states by IOs [8,20,38–40]. The importance of purification lies in that the quantum systems are rather susceptible to imperfect operations such as decoherence [41,42] which may jeopardize the reliability of quantum coherence and so one key question is

to extract coherent states with high quality for application. Especially, in [20], Fang and Liu show that it is impossible to exactly transform a full rank coherent mixed state to a pure output coherent state by IOs, even probabilistically. This builds no-go theorem for coherent mixed states with full rank. An interesting question is how about purification of continuous-variable systems?

Theorem 3.2. For a coherent pure Gaussian state $\sigma(V_2, d_2)$, if there exist an IGO Φ and a Gaussian state $\rho(V_1, d_1)$ such that $\Phi(\rho(V_1, d_1)) = \sigma(V_2, d_2)$, then $\rho(V_1, d_1)$ is a pure state.

Theorem 3.2 is a parallel result of no-go theorem of purification for coherent mixed states of discrete-variable systems [20]. It shows a strong limit on the efficiency of perfect coherent purification of Gaussian states.

In addition, by Theorem 3.2, there does not exist a maximally coherent mixed Gaussian state which can generate all Gaussian states. Combining this with Theorem 3.1, there is not a maximally coherent Gaussian state which can generate all other Gaussian states by means of IGOs. It is a peculiarity of Gaussian continuous-variable systems.

In practical applications such as evolution of quantum coherence of optical cat states, people need to deal with the mixed input and output states rather than pure ones [30,31]. We will provide structural characterization of conversion for mixed Gaussian states in the following. It is an answer to the question of characterizing mixed coherent state manipulation in infinite-dimensional systems [13]. That is, given two mixed Gaussian states $\rho(V_1, d_1)$ and $\sigma(V_2, d_2)$, when $\rho(V_1, d_1) \xrightarrow{\text{IGO}} \sigma(V_2, d_2)$ holds true. First, we need to classify IGOs for a clear presentation. By the definition of IGOs, one can see any IGO of one-mode has two kinds of type:

$$T = tO_1, \quad N = \omega I \quad (\text{type I}),$$

with $\omega \geq |1 - t^2|$, $\det O_1 = 1$ and O_1 is a 2×2 real orthogonal matrix,

$$T = tO_2, \quad N = \omega I \quad (\text{type II}),$$

with $\omega \geq 1 + t^2$, $\det O_2 = -1$ and O_2 is a 2×2 real orthogonal matrix. Secondly, for Gaussian states $\rho(V_1, d_1)$, $\sigma(V_2, d_2)$, there are real orthogonal matrices U and W with $\det U = \det W = 1$ such that

$$UV_1U^t = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad WV_2W^t = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

We also assume $\lambda_1 \neq \lambda_2$, $\|d_1\| \neq 0$ for generality; here $\|d_1\|$ is the Euclidean norm of d_1 .

Now, we are ready to give our results on conversion of mixed Gaussian states.

Theorem 3.3. For Gaussian states $\rho(V_1, d_1), \sigma(V_2, d_2)$, $\rho(V_1, d_1) \xrightarrow{\text{IGO}} \sigma(V_2, d_2)$ by type I IGO if and only if one of the following holds true.

- (i) $V_2 = \mu I \quad (\mu \geq 1), \quad d_2 = 0,$
- (ii) $\begin{cases} \frac{\|d_2\|^2}{\|d_1\|^2} = \frac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2}, \\ \frac{\|d_2\|^2}{\|d_1\|^2} \leq \min\left\{\frac{\mu_1}{\lambda_1}, \frac{1 + \mu_1}{1 + \lambda_1}\right\}, \\ 1 - \mu_1 \leq (1 - \lambda_1) \frac{\|d_2\|^2}{\|d_1\|^2}, \end{cases}$
- (iii) $\begin{cases} \frac{\|d_2\|^2}{\|d_1\|^2} = \frac{\mu_1 - \mu_2}{\lambda_2 - \lambda_1}, \\ \frac{\|d_2\|^2}{\|d_1\|^2} \leq \min\left\{\frac{\mu_1}{\lambda_2}, \frac{1 + \mu_1}{1 + \lambda_2}\right\}, \\ 1 - \mu_1 \leq (1 - \lambda_2) \frac{\|d_2\|^2}{\|d_1\|^2}. \end{cases}$

Theorem 3.4. For Gaussian states $\rho(V_1, d_1), \sigma(V_2, d_2)$, $\rho(V_1, d_1) \xrightarrow{\text{IGO}} \sigma(V_2, d_2)$ by type II IGO if and only if one of

the following holds true.

- (i) $V_2 = \mu I \quad (\mu \geq 1), \quad d_2 = 0,$
- (ii) $\begin{cases} \frac{\|d_2\|^2}{\|d_1\|^2} = \frac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2}, \\ \frac{\|d_2\|^2}{\|d_1\|^2} \leq \frac{\mu_1 - 1}{\lambda_1 + 1}, \end{cases}$
- (iii) $\begin{cases} \frac{\|d_2\|^2}{\|d_1\|^2} = \frac{\mu_1 - \mu_2}{\lambda_2 - \lambda_1}, \\ \frac{\|d_2\|^2}{\|d_1\|^2} \leq \frac{\mu_1 - 1}{\lambda_2 + 1}. \end{cases}$

Theorem 3.3 and Theorem 3.4 are very helpful to fulfill conversion of Gaussian states under IGOs. By [24], the most general one-mode Gaussian state has the second moment

$$V = (2\bar{n} + 1)R(\theta)S(2r)R(\theta)^t,$$

where $S(2r) = \begin{pmatrix} e^{-2r} & 0 \\ 0 & e^{2r} \end{pmatrix}$, $r \in \mathbb{R}$ is called the squeezing parameter. Note that $R(\theta)$ is real orthogonal; for Gaussian states $\rho(V_1, d_1), \sigma(V_2, d_2)$, one can decide whether $\rho(V_1, d_1)$ can be converted into $\sigma(V_2, d_2)$ or vice versa by parameters $\bar{n}_i, r_i, \|d\|_i, i = 1, 2$.

IV. CONVERSION OF PURE GAUSSIAN STATES OF TWO-MODE CONTINUOUS-VARIABLE SYSTEMS

Originating from Theorem 3.1, an interesting question is to describe conversion of pure Gaussian states of m -mode continuous-variable systems ($m \geq 2$) under IGOs. We will first attempt to discuss two-mode case. The exploratory study reveals a big task for conversion of pure Gaussian states of m -mode continuous-variable systems ($m \geq 2$). The difficulty lies in computational complexity of finding determinant relationship between second moments of Gaussian states and its blocks. In this section, we discuss the structure of IGOs of two-mode continuous-variable systems further. Based on this, conversion of one kind of important pure Gaussian states under IGOs is described.

For generality, we assume that the second moment of output state is not diagonal. By the definition of IGOs, it is easy to check that IGOs of two-mode have two kinds of type:

$$T = \begin{pmatrix} t_1 O_1 & 0 \\ 0 & t_2 O_2 \end{pmatrix}, \quad N = \begin{pmatrix} \omega_1 I & 0 \\ 0 & \omega_2 I \end{pmatrix} \quad (\text{type I}),$$

with

$$\omega_1 \geq |1 - t_1^2 \det O_1|, \quad \omega_2 \geq |1 - t_2^2 \det O_2|,$$

$$T = \begin{pmatrix} 0 & t_2 O_2 \\ t_1 O_1 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} \omega_1 I & 0 \\ 0 & \omega_2 I \end{pmatrix} \quad (\text{type II}),$$

with

$$\omega_1 \geq |1 - t_2^2 \det O_2|, \quad \omega_2 \geq |1 - t_1^2 \det O_1|,$$

where O_1, O_2 are 2×2 real orthogonal matrices with $\det O_i = \pm 1$ ($i = 1, 2$). We find that, if the above IGOs transform one pure state into the other pure state, then $\det O_i = 1$ ($i = 1, 2$) as follows.

Theorem 4.1. For pure Gaussian states $\rho(V_1, d_1), \sigma(V_2, d_2)$, writing V_1 and V_2 in their block form $V_1 = \begin{pmatrix} V_{11} & V_{12} \\ V_{12}^t & V_{11} \end{pmatrix}$, $V_2 = \begin{pmatrix} V'_{11} & V'_{12} \\ V'_{12} & V'_{11} \end{pmatrix}$, $V'_{12} \neq 0$, where V_{ij} and V'_{ij} ($i, j = 1, 2$) are 2×2 matrices, if there exists

some IGO Φ such that $\Phi(\rho(V_1, d_1)) = \sigma(V_2, d_2)$, then $\det O_1 = \det O_2 = 1$.

Theorem 4.1 is useful to realize the conversion of pure Gaussian states. An important class of two-mode Gaussian states has second moments in standard form

$$V = \begin{pmatrix} aI & C \\ C & bI \end{pmatrix}, \quad \text{with } C = \begin{pmatrix} c & 0 \\ 0 & d \end{pmatrix},$$

where $a \geq 1$, $b \geq 1$, and $c, d \in \mathbb{R}$ [24,43,44]. Any Gaussian state can be transformed to the Gaussian state with the second moment in standard form by local linear unitary Bogoliubov operations [43]. We will give a complete classification of conversion for such kind of pure states. One can check that such Gaussian states are pure if and only if

$$ab - c^2 > 0, \quad (ab - c^2)(ab - d^2) = 1, \quad a^2 + b^2 + 2cd \leq 2.$$

Let

$$V' = \begin{pmatrix} a'I & C' \\ C' & b'I \end{pmatrix}, \quad \text{with } C' = \begin{pmatrix} c' & 0 \\ 0 & d' \end{pmatrix}$$

be the second moment of pure target states. The key step for realizing $\rho(V, d) \xrightarrow{\text{IGO}} \sigma(V', d')$ is

$$V \xrightarrow{\text{IGO}} V'.$$

We will first classify the transformation on second moments. Based on this classification, conversion of pure Gaussian states can be investigated easily.

In the following, assuming $cd \neq 0$, $c'd' \neq 0$ and denoting $\alpha = \frac{c^2}{c^2}$ or $\alpha = \frac{c'^2}{d'^2}$, there are four important closed intervals which are needed to classify transformation between V and V' :

- (1) $[\frac{\alpha(b+1)}{b+1}, \min\{1, \frac{a'-1}{a-1}, \alpha\}]$,
- (2) $[\max\{\alpha, \frac{\alpha(b-1)}{b-1}\}, 1]$,
- (3) $[\max\{1, \frac{\alpha(b+1)}{b+1}\}, \min\{\alpha, \frac{a'+1}{a+1}\}]$,
- (4) $[\max\{\alpha, \frac{\alpha(b-1)}{b-1}\}, 1], \frac{a'+1}{a+1}]$.

Theorem 4.2.

$$V \xrightarrow{\text{IGO}} V' \Leftrightarrow \Omega \neq \emptyset.$$

Here $\Omega = (1) \cup (2) \cup (3) \cup (4)$.

For $\rho(V, d), \sigma(V', d')$, if $\Omega \neq \emptyset$ and the desired IGO is type I, from the proof of Theorem 4.2 in Appendix, then we choose arbitrarily $t_1 \in \Omega$. t_2, ω_1, ω_2 are decided by t_1 . One can check easily whether there exists suitable T such that $Td = d'$. Therefore the conversion between $\rho(V, d)$ and $\sigma(V', d')$ can be ascertained. If the IGO is type II, then we pick any $t_2 \in \Omega$. The existence of T satisfying $Td = d'$ can be checked directly.

V. SUMMARY

In this work, we have studied conversion of coherent Gaussian states under incoherent Gaussian operations. An explicit description on conversion of one-mode systems has been provided. Compared with the finite-dimensional results on conversion of coherent states [7,9,12,13], there are some peculiarities as follows: (1) there does not exist a maximally coherent Gaussian state which can generate all coherent Gaussian states; (2) the conversion between pure Gaussian

states is reversible; (3) the coherence of input pure state and the coherence of output pure state are equal. This implies frozen behavior [36] in Gaussian dynamical systems which is listed as an open question in [33]. Conversion of pure Gaussian states of two-mode systems under incoherent Gaussian operations is also discussed. We classify conversion for an important class of two-mode pure Gaussian states with second moments in the standard form [24,43,44].

Our results raise some interesting questions. It would be of great interest to classify conversion of pure Gaussian states or mixed Gaussian states for m -mode ($m \geq 2$) continuous-variable systems. This is very helpful for comprehending behaviors of coherence of Gaussian dynamical systems.

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APPENDIX: PROOF OF MAIN RESULTS

Proofs of all results in this paper are given in this Appendix.

The proofs of our theorems need structural classification of real orthogonal matrices and determinant formula of sum of two matrices borrowed from [45].

Proposition 1. Let $R(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}$, where θ is a real parameter. O is real orthogonal if and only if $O = R(\theta)$ or $O = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R(\theta)$.

Proposition 2. For $A, B \in \mathcal{M}_2(\mathbb{R})$,

$$\det(A + B) = \det(A) + \det(B) + \text{tr}(A^\sharp B),$$

where $(\cdot)^\sharp$ denotes the adjugate map given by

$$M^\sharp = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \text{where } M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The following characterization of Gaussian states can be found in [46,47].

Proposition 3. For any Gaussian state with second moment $V = \begin{pmatrix} A & C \\ C & B \end{pmatrix}$, where A, B, C are real 2×2 matrices. Denoting $\Delta = \det A + \det B + 2 \det C$, we have

$$V > 0, \quad \det V \geq 1, \quad \Delta \leq 1 + \det V.$$

Now we are in a position to give proofs of our theorems.

Proof of Theorem 3.1. “ \Rightarrow ” Assume that there exists an IGO Φ such that $\Phi(\rho(V_1, d_1)) = \sigma(V_2, d_2)$. By the definition of IGO, we can obtain

$$t^2 O V_1 O' + \omega I = V_2, \quad t O d_1 = d_2, \quad (\text{A1})$$

where O is a real orthogonal matrix, $\omega, t \in \mathbb{R}$, $\omega \geq |1 - t^2 \det O|$. We divide the proof into two cases by Proposition 1.

Case (i) $O = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R(\theta)$. It is easy to see that $\det O = -1$ and so $\omega \geq 1 + t^2$. Combining (1) with Proposition 2, one

can get

$$1 = \det V_2 = t^4 \det V_1 + \omega^2 + \omega t^2 \text{tr}(V_1) = t^4 + \omega^2 + \omega t^2 \text{tr}(V_1).$$

By $\omega \geq 1 + t^2$, we have $t = 0, \omega = 1$. Therefore, $V_2 = I, d_2 = 0$ and so $\sigma(V_2, d_2) = |0\rangle \in \mathcal{I}$.

Case (ii) $O = R(\theta)$ for some $\theta \in \mathbb{R}$. It is evident that $\det O = 1$ and so $\omega \geq |1 - t^2|$. Because $\det V_1 = 1$, we can assume eigenvalues of V_1 are λ_1 and $\frac{1}{\lambda_1} (\lambda_1 > 0)$. By (1) and spectral mapping theorem, we have

$$(t^2 \lambda_1 + \omega) \left(\frac{t^2}{\lambda_1} + \omega \right) = 1. \tag{A2}$$

It follows that

$$\omega^2 + \omega \left(\lambda_1 t^2 + \frac{t^2}{\lambda_1} \right) + t^4 - 1 = 0.$$

This implies that

$$|t| \leq 1.$$

From the relation of root and coefficient of quadratic equation, we have

$$\omega = \frac{-(t^2 \lambda_1 + \frac{t^2}{\lambda_1}) + \sqrt{\lambda_1^2 t^4 + \frac{t^4}{\lambda_1^2} - 2t^4 + 4}}{2}.$$

From

$$-\left(t^2 \lambda_1 + \frac{t^2}{\lambda_1}\right) + \sqrt{\lambda_1^2 t^4 + \frac{t^4}{\lambda_1^2} - 2t^4 + 4} \geq 2(1 - t^2),$$

it follows that

$$\lambda_1^2 t^4 + \frac{t^4}{\lambda_1^2} - 2t^4 + 4 \geq \left[2 + t^2 \left(\lambda_1 + \frac{1}{\lambda_1} - 2 \right) \right]^2.$$

A direct computation shows that

$$t^2 \left(2 - \lambda_1 - \frac{1}{\lambda_1} \right) \leq \left(2 - \lambda_1 - \frac{1}{\lambda_1} \right).$$

Note that $\lambda_1 + \frac{1}{\lambda_1} \geq 2$ and so

$$|t| \geq 1.$$

Hence $|t| = 1$. By (2), we have $\omega = 0$. From (1),

$$V_2 = R(\theta)V_1R(\theta)', \quad d_2 = R(\theta)d_1$$

if $t = 1$. If $t = -1$, replacing θ with $\theta + \pi$, we also get the desired.

“ \Leftarrow ” If

$$V_2 = I, \quad d_2 = 0,$$

taking $T = tR(\theta), t = 0, \omega = 1$, then the IGO induced by T and ω has desired property. If there exists $R(\theta)$ such that

$$V_2 = R(\theta)V_1R(\theta)', \quad d_2 = R(\theta)d_1,$$

then, choosing $T = tR(\theta), t = 1, \omega = 0$, we have the desired.

Proof of Theorem 3.2. For coherent Gaussian pure state $\sigma(V_2, d_2)$, suppose that there exists an IGO Φ with $\Phi(\rho(V_1, d_1)) = \sigma(V_2, d_2)$. Therefore,

$$t^2 OV_1 O' + \omega I = V_2, \quad t Od_1 = d_2, \tag{A3}$$

where O is a real orthogonal matrix, $\omega, t \in \mathbb{R}, \omega \geq |1 - t^2 \det O|$. We declare $O = R(\theta)$ for some $\theta \in \mathbb{R}$. Otherwise, $O = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R(\theta)$. This indicates $\det O = -1$ and so $\omega \geq 1 + t^2$. By (3) and Proposition 2, we get

$$1 = \det V_2 = t^4 \det V_1 + \omega^2 + \omega t^2 \text{tr}(V_1). \tag{A4}$$

Since $\omega \geq 1 + t^2$, we have $t = 0, \omega = 1$. Therefore, $V_2 = I, d_2 = 0$ and so $\sigma(V_2, d_2) = |0\rangle \in \mathcal{I}$, a contradiction. In (4), for conciseness, denote $\text{tr}(V_1) = a, \det V_1 = b$. It is evident that $a > 1, b \geq 1$. Hence

$$\omega^2 + at^2 \omega + t^4 b - 1 = 0.$$

This indicates $|t| \leq 1$ and so $\omega \geq 1 - t^2$. By the relation between root and coefficient of quadratic equation, we also obtain

$$\frac{-t^2 a + \sqrt{t^4 a^2 - 4t^4 b + 4}}{2} \geq 1 - t^2.$$

It is equivalent to

$$t^2(1 - a + b) \leq 2 - a.$$

The proof is divided into three cases in the following.

Case (i) $1 - a + b < 0$. In this case, we immediately have $\frac{2-a}{1-a+b} \leq t^2 \leq 1$. Thus $1 - a + b \leq 2 - a$ and so $b \leq 1$. Since $b = \det V_1 \geq 1, b = 1$ and $\rho(V_1, d_1)$ is pure.

Case (ii) $1 - a + b > 0$. This tells $\frac{2-a}{1-a+b} \geq t^2$ and so $a \leq 2$. From $a \geq 2\sqrt{b}$, it follows that $b \leq 1$. Therefore, $b = 1$ and $\rho(V_1, d_1)$ is pure.

Case (iii) $1 - a + b = 0$. By the assumption, we have $0 \leq 2 - a$. Using the proof of case (ii), we obtain $\rho(V_1, d_1)$ is pure.

Proof of Theorem 3.3. For Gaussian states $\rho(V_1, d_1), \sigma(V_2, d_2)$, let U and W be real orthogonal matrices with $\det U = \det W = 1$ satisfying

$$UV_1U' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad WV_2W' = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

Note that

$$\rho(V_1, d_1) \xrightarrow{\text{IGO}} \sigma(V_2, d_2)$$

if and only if

$$\rho'(UV_1U', Ud_1) \xrightarrow{\text{IGO}} \sigma'(WV_2W', Wd_2).$$

Hence, without loss of generality, we may assume

$$V_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

“ \Rightarrow ” Suppose there exists an IGO Φ of type I such that $\Phi(\rho(V_1, d_1)) = \sigma(V_2, d_2)$; then

$$tOd_1 = d_2, \quad t^2 OV_1 O' + \omega I = V_2.$$

Here O is a real orthogonal matrix with $\det O = 1, \omega \geq |1 - t^2|$ for some real number t . This implies that

$$\begin{cases} t^2 = \frac{\|d_2\|^2}{\|d_1\|^2}, \\ t^2(\lambda_1 - \lambda_2) \sin 2\theta = 0, \\ t^2(\lambda_1 \cos^2 \theta + \lambda_2 \sin^2 \theta) + \omega = \mu_1, \\ t^2(\lambda_1 \sin^2 \theta + \lambda_2 \cos^2 \theta) + \omega = \mu_2. \end{cases}$$

We divide the proof into two cases.

Case (i) $t = 0$. It is evident that $V_2 = \mu I$, $d_2 = 0$, and $\mu \geq 1$.

Case (ii) $t \neq 0$, $\sin 2\theta = 0$. We can assume that $\theta = 0$ or $\theta = \frac{\pi}{2}$. If $\theta = 0$, then

$$\begin{cases} t^2 \lambda_1 + \omega = \mu_1, \\ t^2 \lambda_2 + \omega = \mu_2. \end{cases}$$

Hence

$$\frac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2} = \frac{\|d_2\|^2}{\|d_1\|^2}, \quad \omega = \mu_1 - \frac{\|d_2\|^2}{\|d_1\|^2} \lambda_1.$$

From $\omega \geq 0$, we have

$$\frac{\|d_2\|^2}{\|d_1\|^2} \leq \frac{\mu_1}{\lambda_1}.$$

Using $\omega \geq |1 - t^2|$, we can obtain

$$\frac{\|d_2\|^2}{\|d_1\|^2} \leq \frac{1 + \mu_1}{1 + \lambda_1}, \quad 1 - \mu_1 \leq (1 - \lambda_1) \frac{\|d_2\|^2}{\|d_1\|^2}.$$

Therefore,

$$\begin{cases} \frac{\|d_2\|^2}{\|d_1\|^2} = \frac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2}, \\ \frac{\|d_2\|^2}{\|d_1\|^2} \leq \min\left\{\frac{\mu_1}{\lambda_1}, \frac{1 + \mu_1}{1 + \lambda_1}\right\}, \\ 1 - \mu_1 \leq (1 - \lambda_1) \frac{\|d_2\|^2}{\|d_1\|^2}. \end{cases}$$

If $\theta = \frac{\pi}{2}$, then

$$\begin{cases} t^2 \lambda_2 + \omega = \mu_1, \\ t^2 \lambda_1 + \omega = \mu_2. \end{cases}$$

Hence

$$t^2 = \frac{\mu_1 - \mu_2}{\lambda_2 - \lambda_1} = \frac{\|d_2\|^2}{\|d_1\|^2}.$$

Combining $\omega \geq |1 - t^2|$ with $\omega = \mu_1 - \frac{\|d_2\|^2}{\|d_1\|^2} \lambda_2$, we can get

$$\begin{cases} \frac{\|d_2\|^2}{\|d_1\|^2} \leq \frac{\mu_1}{\lambda_2}, \\ \frac{\|d_2\|^2}{\|d_1\|^2} \leq \frac{1 + \mu_1}{1 + \lambda_2}, \\ 1 - \mu_1 \leq (1 - \lambda_2) \frac{\|d_2\|^2}{\|d_1\|^2}. \end{cases}$$

Therefore,

$$\begin{cases} \frac{\|d_2\|^2}{\|d_1\|^2} = \frac{\mu_1 - \mu_2}{\lambda_2 - \lambda_1}, \\ \frac{\|d_2\|^2}{\|d_1\|^2} \leq \min\left\{\frac{\mu_1}{\lambda_2}, \frac{1 + \mu_1}{1 + \lambda_2}\right\}, \\ 1 - \mu_1 \leq (1 - \lambda_2) \frac{\|d_2\|^2}{\|d_1\|^2}. \end{cases}$$

“ \Leftarrow ” If $V_2 = \mu I$, $d_2 = 0$, $\mu \geq 1$, then $t = 0$, $\omega = \mu$ can induce the desired IGO. If

$$\begin{cases} \frac{\|d_2\|^2}{\|d_1\|^2} = \frac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2}, \\ \frac{\|d_2\|^2}{\|d_1\|^2} \leq \min\left\{\frac{\mu_1}{\lambda_1}, \frac{1 + \mu_1}{1 + \lambda_1}\right\}, \\ 1 - \mu_1 \leq (1 - \lambda_1) \frac{\|d_2\|^2}{\|d_1\|^2}, \end{cases}$$

then $t^2 = \frac{\|d_2\|^2}{\|d_1\|^2}$, $O = I$, $\omega = \mu_1 - \lambda_1 \frac{\|d_2\|^2}{\|d_1\|^2}$ can induce the desired. If

$$\begin{cases} \frac{\|d_2\|^2}{\|d_1\|^2} = \frac{\mu_1 - \mu_2}{\lambda_2 - \lambda_1}, \\ \frac{\|d_2\|^2}{\|d_1\|^2} \leq \min\left\{\frac{\mu_1}{\lambda_2}, \frac{1 + \mu_1}{1 + \lambda_2}\right\}, \\ 1 - \mu_1 \leq (1 - \lambda_2) \frac{\|d_2\|^2}{\|d_1\|^2}, \end{cases}$$

then one can choose $t^2 = \frac{\|d_2\|^2}{\|d_1\|^2}$, $O = \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$, and $\omega = \mu_1 - \lambda_2 \frac{\|d_2\|^2}{\|d_1\|^2}$.

Proof of Theorem 3.4. Using the same arguments as the start of proof of Theorem 3.3, we suppose

$$V_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad V_2 = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

“ \Rightarrow ” Assume there exists an IGO Φ of type II such that $\Phi(\rho(V_1, d_1)) = \sigma(V_2, d_2)$; then

$$t O d_1 = d_2, \quad t^2 O V_1 O^t + \omega I = V_2.$$

Here O is a real orthogonal matrix with $\det O = -1$; $\omega \geq 1 + t^2$ for some real number t . This implies that

$$\begin{cases} t^2 = \frac{\|d_2\|^2}{\|d_1\|^2}, \\ t^2(\lambda_1 - \lambda_2) \sin 2\theta = 0, \\ t^2(\lambda_1 \cos^2 \theta + \lambda_2 \sin^2 \theta) + \omega = \mu_1, \\ t^2(\lambda_1 \sin^2 \theta + \lambda_2 \cos^2 \theta) + \omega = \mu_2. \end{cases}$$

We divide the proof into two cases.

Case (i) $t = 0$. It is evident that $V_2 = \mu I$, $d_2 = 0$, $\mu \geq 1$.

Case (ii) $t \neq 0$, $\sin 2\theta = 0$. We assume $\theta = 0$ or $\theta = \frac{\pi}{2}$ and will treat them separately. If $\theta = 0$, then

$$\begin{cases} t^2 \lambda_1 + \omega = \mu_1, \\ t^2 \lambda_2 + \omega = \mu_2. \end{cases}$$

Therefore,

$$\frac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2} = \frac{\|d_2\|^2}{\|d_1\|^2}, \quad \omega = \mu_1 - \frac{\|d_2\|^2}{\|d_1\|^2} \lambda_1.$$

From $\omega \geq 1 + t^2$, we have $\frac{\|d_2\|^2}{\|d_1\|^2} \leq \frac{\mu_1 - 1}{\lambda_1 + 1}$, as desired. If $\theta = \frac{\pi}{2}$, then

$$\begin{cases} t^2 \lambda_2 + \omega = \mu_1, \\ t^2 \lambda_1 + \omega = \mu_2. \end{cases}$$

Hence

$$t^2 = \frac{\mu_1 - \mu_2}{\lambda_2 - \lambda_1} = \frac{\|d_2\|^2}{\|d_1\|^2}.$$

From $\omega = \mu_1 - \lambda_2 \frac{\|d_2\|^2}{\|d_1\|^2}$ and $\omega \geq 1 + t^2$, we obtain

$$\frac{\|d_2\|^2}{\|d_1\|^2} \leq \frac{\mu_1 - 1}{\lambda_2 + 1}.$$

“ \Leftarrow ” If $V_2 = \mu I$, $d_2 = 0$, $\mu \geq 1$, then $t = 0$, $\omega = \mu$ can induce desired IGO. If

$$\begin{cases} \frac{\|d_2\|^2}{\|d_1\|^2} = \frac{\mu_1 - \mu_2}{\lambda_1 - \lambda_2}, \\ \frac{\|d_2\|^2}{\|d_1\|^2} \leq \frac{\mu_1 - 1}{\lambda_1 + 1} \end{cases}$$

hold true, then $t^2 = \frac{\|d_2\|^2}{\|d_1\|^2}$, $O = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\omega = \mu_1 - \lambda_1 \frac{\|d_2\|^2}{\|d_1\|^2}$ can induce the desired. If

$$\begin{cases} \frac{\|d_2\|^2}{\|d_1\|^2} = \frac{\mu_1 - \mu_2}{\lambda_2 - \lambda_1}, \\ \frac{\|d_2\|^2}{\|d_1\|^2} \leq \frac{\mu_1 - 1}{\lambda_2 + 1} \end{cases}$$

are true, then we take $t^2 = \frac{\|d_2\|^2}{\|d_1\|^2}$, $O = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$, and $\omega = \mu_1 - \lambda_2 \frac{\|d_2\|^2}{\|d_1\|^2}$.

Proof of Theorem 4.1. Without loss of generality, we assume that Φ is type I; the type II case can be treated similarly. By a direct calculation one can obtain

$$V_2 = TV_1T' + N = \begin{pmatrix} t_1^2 O_1 V_{11} O_1' + \omega_1 I & t_1 t_2 O_1 V_{12} O_2' \\ t_1 t_2 O_2 V_{12}' O_1' & t_2^2 O_2 V_{22} O_2' + \omega_2 I \end{pmatrix}.$$

Combining Proposition 2 with Proposition 3, we have

$$\begin{aligned} \det(t_1^2 O_1 V_{11} O_1' + \omega_1 I) &\geq 1, \\ \det(t_2^2 O_2 V_{22} O_2' + \omega_2 I) &\geq 1, \\ \det(t_1 t_2 O_1 V_{12} O_2') &\leq 0, \\ \Delta &= t_1^4 \det V_{11} + \omega_1^2 + \omega_1 t_1^2 \text{tr}(V_{11}) \\ &\quad + t_2^4 \det V_{22} + \omega_2^2 + \omega_2 t_2^2 \text{tr}(V_{22}) \\ &\quad + 2t_1^2 t_2^2 \det O_1 \det O_2 \det V_{12} \\ &\leq 2. \end{aligned}$$

In the following, we divide the proof into two cases.

Case (i) $\det V_{12}' \neq 0$. Note that $\det V_{12} < 0$; from

$$\det(t_1 t_2 O_1 V_{12} O_2') = t_1^2 t_2^2 \det V_{12} \det O_1 \det O_2 \leq 0,$$

we can get $\det O_1 \det O_2 = 1$. Suppose $\det O_1 = -1$, $\det O_2 = -1$, then $\omega_1 \geq 1 + t_1^2$, $\omega_2 \geq 1 + t_2^2$. This deduces

$$\begin{aligned} \Delta &\geq t_1^4 \det V_{11} + (1 + t_1^2)^2 + (1 + t_1^2) t_1^2 \text{tr}(V_{11}) \\ &\quad + t_2^4 \det V_{22} + (1 + t_2^2)^2 + (1 + t_2^2) t_2^2 \text{tr}(V_{22}) \\ &\quad + 2t_1^2 t_2^2 \det V_{12} \\ &\geq 2t_1^2 t_2^2 (\sqrt{\det V_{11} \det V_{12}} + \det V_{12}) + 2 \\ &\quad + 2t_1^2 + t_1^4 + (1 + t_1^2) t_1^2 \text{tr}(V_{11}) \\ &\quad + 2t_2^2 + t_2^4 + (1 + t_2^2) t_2^2 \text{tr}(V_{22}). \end{aligned}$$

Since $V > 0$, we know that

$$\sqrt{\det V_{11} \det V_{22}} + \det V_{12} \geq 0.$$

This implies that $\Delta = 2$ and so $t_1 = t_2 = 0$, a contradiction. Hence $\det O_1 = \det O_2 = 1$, as desired.

Case (ii) $\det V_{12}' = 0$. By Proposition 3,

$$1 = \det(t_1^2 O_1 V_{11} O_1' + \omega_1 I) = \det(t_2^2 O_2 V_{22} O_2' + \omega_2 I).$$

Thus

$$t_1^4 \det V_{11} + \omega_1^2 + \omega_1 t_1^2 \text{tr}(V_{11}) = 1.$$

This tells $|1 - t_1^2 \det O_1| \leq \omega_1 \leq 1$. So $\det O_1 = 1$; otherwise, $t_1 = 0$ and hence $V_{12}' = 0$, a contradiction. Analogously, one can obtain $\det O_2 = 1$.

Proof of Theorem 4.2. “ \Rightarrow ” Assume the IGO is type I and

$$TVT' + N = V'. \tag{A5}$$

By Theorem 3.1 and Proposition 1, we suppose

$$O_1 = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}, \quad O_2 = \begin{pmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{pmatrix}$$

for some real numbers θ, ϕ . A direct computation of (5) shows that

$$\begin{cases} c \cos \theta \sin \phi - d \sin \theta \cos \phi = 0, \\ c \sin \theta \cos \phi - d \cos \theta \sin \phi = 0. \end{cases}$$

Therefore, $c^2 = d^2$ or $\cos \theta = \cos \phi = 0$ or $\sin \theta = \sin \phi = 0$. We divide the proof into three cases.

Case (i) $c^2 = d^2$. From (5) and Proposition 3, the following equations hold true:

$$\begin{aligned} c &= -d, \\ c \cos \theta \sin \phi + c \sin \theta \cos \phi &= 0, \\ ct_1 t_2 \cos \theta \cos \phi - ct_1 t_2 \sin \theta \sin \phi &= c', \end{aligned}$$

and

$$ct_1 t_2 \sin \theta \sin \phi - ct_1 t_2 \cos \theta \cos \phi = d'.$$

Thus $\sin(\theta + \phi) = 0$ and so $\pm ct_1 t_2 = c' = -d'$. Denoting $\alpha = \frac{c'^2}{c^2}$, comparing the entry of diagonal position of (5), one can obtain

$$\begin{cases} at_1^2 + \omega_1 = a', & \omega_1 \geq |1 - t_1^2|, \\ bt_2^2 + \omega_2 = b', & \omega_2 \geq |1 - t_2^2|, \\ t_1^2 t_2^2 = \alpha. \end{cases} \tag{A6}$$

If $t_1^2 \leq 1$, $t_1^2 \leq \alpha$, then

$$\begin{cases} \omega_1 = a' - at_1^2 \geq 1 - t_1^2, \\ \omega_2 = b' - b \frac{\alpha}{t_1^2} \geq \frac{\alpha}{t_1^2} - 1. \end{cases}$$

Hence

$$\frac{\alpha(b+1)}{b'+1} \leq t_1^2 \leq \min \left\{ 1, \frac{a'-1}{a-1}, \alpha \right\}.$$

If $t_1^2 \leq 1$, $t_1^2 \geq \alpha$, then $b't_1^2 - b\alpha \geq t_1^2 - \alpha$. Therefore,

$$\max \left\{ \alpha, \frac{\alpha(b-1)}{b'-1} \right\} \leq t_1^2 \leq \min \left\{ 1, \frac{a'-1}{a-1} \right\}.$$

If $t_1^2 > 1$, $t_1^2 \leq \alpha$, then

$$\begin{cases} \omega_1 = a' - at_1^2 \geq t_1^2 - 1, \\ \omega_2 = b' - b \frac{\alpha}{t_1^2} \geq \frac{\alpha}{t_1^2} - 1. \end{cases}$$

So

$$\max \left\{ 1, \frac{\alpha(b+1)}{b'+1} \right\} \leq t_1^2 \leq \min \left\{ \alpha, \frac{a'+1}{a+1} \right\}.$$

If $t_1^2 > 1$, $t_1^2 \geq \alpha$, then $b't_1^2 - b\alpha \geq t_1^2 - \alpha$. Thus

$$\max \left\{ \alpha, \frac{\alpha(b-1)}{b'-1}, 1 \right\} \leq t_1^2 \leq \frac{a'+1}{a+1}.$$

Case (ii) $\sin \theta = \sin \phi = 0$. By a direct computation, one can get the same intervals and α as case (i).

Case (iii) $\cos \theta = \cos \phi = 0$. In this case, one can get the same intervals as case (i) and $\alpha = \frac{c'^2}{d^2} = \frac{d'^2}{d^2}$ by similar calculation.

If the IGO is type II, a direct computation shows that we have the same intervals and α as type I. The only difference is that t_1 is replaced by t_2 .

“ \Leftarrow ” If the desired IGO is type I, then one can choose $t_1 \in \Omega$. t_2, ω_1, ω_2 are fixed by (6). Next, according to the interval that t_1 belongs, we pick suitable θ and ϕ to construct

some IGO for conversion of pure states. If the desired IGO is type II, then one can choose $t_2 \in \Omega$ and other parameters can be chosen analogously.

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- [1] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
- [2] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wootters, *Phys. Rev. Lett.* **70**, 1895 (1993).
- [3] A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.* **47**, 777 (1935).
- [4] V. Vedral, M. B. Plenio, M. A. Rippin, and P. L. Knight, *Phys. Rev. Lett.* **78**, 2275 (1997).
- [5] M. B. Plenio and S. Virmani, *Quantum Inf. Comput.* **7**, 1 (2007).
- [6] J. Åberg, [arXiv:quant-ph/0612146](https://arxiv.org/abs/quant-ph/0612146).
- [7] T. Baumgratz, M. Cramer, and M. B. Plenio, *Phys. Rev. Lett.* **113**, 140401 (2014).
- [8] A. Winter and D. Yang, *Phys. Rev. Lett.* **116**, 120404 (2016).
- [9] A. Streltsov, G. Adesso, and M. B. Plenio, *Rev. Mod. Phys.* **89**, 041003 (2017).
- [10] A. Streltsov, S. Rana, P. Boes, and J. Eisert, *Phys. Rev. Lett.* **119**, 140402 (2017).
- [11] L. Lami, B. Regula, and G. Adesso, *Phys. Rev. Lett.* **122**, 150402 (2019).
- [12] E. Chitambar and G. Gour, *Rev. Mod. Phys.* **91**, 025001 (2019).
- [13] S. Du, Z. Bai, and Y. Guo, *Phys. Rev. A* **91**, 052120 (2015).
- [14] G. Torun and A. Yildiz, *Phys. Rev. A* **97**, 052331 (2018).
- [15] G. Torun, H. T. Senyasa, and A. Yildiz, *Phys. Rev. A* **103**, 032416 (2021).
- [16] E. Chitambar and G. Gour, *Phys. Rev. A* **94**, 052336 (2016).
- [17] H. Shi *et al.*, *Sci. Rep.* **7**, 14806 (2017).
- [18] S. Du, Z. Bai, and X. Qi, *Phys. Rev. A* **100**, 032313 (2019).
- [19] C. L. Liu and D. L. Zhou, *Phys. Rev. Lett.* **123**, 070402 (2019).
- [20] K. Fang and Z. W. Liu, *Phys. Rev. Lett.* **125**, 060405 (2020).
- [21] R. J. Glauber, *Phys. Rev.* **131**, 2766 (1963).
- [22] E. C. G. Sudarshan, *Phys. Rev. Lett.* **10**, 277 (1963).
- [23] S. L. Braunstein and P. van Loock, *Rev. Mod. Phys.* **77**, 513 (2005).
- [24] C. Weedbrook, S. Pirandola, R. Garcia-Patron, N. J. Cerf, T. C. Ralph, J. H. Shapiro, and S. Lloyd, *Rev. Mod. Phys.* **84**, 621 (2012).
- [25] G. Giedke and J. I. Cirac, *Phys. Rev. A* **66**, 032316 (2002).
- [26] J. Eisert, S. Scheel, and M. B. Plenio, *Phys. Rev. Lett.* **89**, 137903 (2002).
- [27] J. Fiurášek, *Phys. Rev. Lett.* **89**, 137904 (2002).
- [28] U. Leonhardt, *Measuring the Quantum State of Light* (Cambridge University Press, Cambridge, UK, 1997).
- [29] J. Eisert and M. B. Plenio, *Int. J. Quantum Inf.* **01**, 479 (2003).
- [30] S. Glancy, H. M. Vasconcelos, and T. C. Ralph, *Phys. Rev. A* **70**, 022317 (2004).
- [31] M. Zhang, H. Kang, M. Wang, F. Xu, X. Su, and K. Peng, *Photon. Res.* **9**, 887 (2021).
- [32] J. Eisert, C. Simon, and M. B. Plenio, *J. Phys. A* **35**, 3911 (2002).
- [33] J. Xu, *Phys. Rev. A* **93**, 032111 (2016).
- [34] D. Buono *et al.*, [arXiv:1609.00913](https://arxiv.org/abs/1609.00913).
- [35] J. Xu, *Phys. Lett. A* **387**, 127028 (2021).
- [36] T. R. Bromley, M. Cianciaruso, and G. Adesso, *Phys. Rev. Lett.* **114**, 210401 (2015).
- [37] S. Olivares, *Eur. Phys. J. ST* **203**, 3 (2012).
- [38] K. Fang, X. Wang, L. Lami, B. Regula, and G. Adesso, *Phys. Rev. Lett.* **121**, 070404 (2018).
- [39] B. Regula, K. Fang, X. Wang, and G. Adesso, *Phys. Rev. Lett.* **121**, 010401 (2018).
- [40] G. Torun, L. Lami, G. Adesso, and A. Yildiz, *Phys. Rev. A* **99**, 012321 (2019).
- [41] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information: 10th Anniversary Edition* (Cambridge University Press, New York, 2011).
- [42] J. Preskill, *Quantum* **2**, 79 (2018).
- [43] L. M. Duan, G. Giedke, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **84**, 2722 (2000).
- [44] R. Simon, *Phys. Rev. Lett.* **84**, 2726 (2000).
- [45] R. Horn and C. Johnson, *Matrix Analysis* (Cambridge University Press, Cambridge, UK, 1986).
- [46] A. Serafini, *Phys. Rev. Lett.* **96**, 110402 (2006).
- [47] S. Pirandola, A. Serafini, and S. Lloyd, *Phys. Rev. A* **79**, 052327 (2009).