Non-Gaussian entanglement swapping between three-mode spontaneous parametric down-conversion and three qubits

A. Agustí Casado [1,*] and C. Sabín [2]

[1]*Instituto de Física Fundamental, CSIC, Serrano 113-bis, 28006 Madrid, Spain*
[2]*Departamento de Física Teórica, Univerdad Autónoma de Madrid, 28049 Madrid, Spain*

In this work we study the production and swapping of non-Gaussian multipartite entanglement in a setup containing a parametric amplifier which generates three photons in different modes coupled to three qubits. We prove that the entanglement generated in this setup is of non-Gaussian nature. We introduce witnesses of genuine tripartite non-Gaussian entanglement, valid for both mode and qubit entanglement. Moreover, those witnesses show that the entanglement generated among the photons can be swapped to the qubits, and indeed the qubits display non-Gaussian genuine tripartite entanglement over a wider parameter regime, suggesting that our setup could be a useful tool to extract entanglement generated in higher-order parametric amplification for quantum metrology or quantum computing applications.

I. INTRODUCTION

Entanglement is the key ingredient to most quantum technologies being designed today, ranging from teleportation [1–3] to boson sampling [4] and in general any quantum computational scheme. Therefore, plenty of present-day literature deals with how to generate entanglement, and a very fruitful paradigm at that is parametric amplification. Take, for example, its role as a primitive ingredient in the recent claim on boson sampling quantum advantage [5].

The first instances of quantum parametric amplifiers date back to the 1980s [6,7] in the setting of outperforming quantum measurements with single-mode squeezing. Then, in that same decade, it was discovered that parametric amplification could pump energy in two modes at once, leading to the generation of two-mode squeezing [8], perhaps the simplest form of continuous-variable (CV) entanglement [9]. During the past five years, some of us have predicted that such two-mode squeezing can be used to entangle three modes in a genuinely tripartite way by applying the process to two pairs at once [10,11], a prediction that has been experimentally validated [12]. We denominate this process double two-mode spontaneous parametric down-conversion (2-2SPDC). In a recent work [13], we predicted that a similar process experimentally demonstrated in [14], capable of generating three photons on different modes at once, 3SPDC, produces genuine tripartite entanglement too. In order to experimentally detect 2-2SPDC entanglement, inspection of the covariances of the field quadratures was enough, whereas the 3SPDC entanglement requires inspecting higher statistical moments.

As entanglement generation has become a well-established technology, produced in countless laboratories around the globe, still interesting theoretical questions remain open.

Take, for example, the inequivalent entanglement of the three-qubit W and Greenberger-Horne-Zeilinger (GHZ) states [15]. Those states are entangled in a tripartite way and yet they cannot be converted into each other by means of stochastic local operations and classical communication (SLOCC). A generalization of this result to general discrete-variable (DV) d-level systems has been proposed recently [16], and the generalization to n qubits is still incomplete, although we know that there have to be infinitely many SLOCC classes for $N > 3$ [15], which therefore have to be gathered into some finite number of entanglement families (which proves to be a formidable task even for N=4 [17–20]) whose physical meaning is not always transparent. Furthermore, extensions of the above results to mixed states (even for three qubits) or to continuous variables beyond Gaussian states remain as open problems. A physically meaningful criterion to classify quantum entanglement, valid in principle both for CV and DV systems and for pure and mixed states, might be the distinction between gaussian and non-Gaussian entanglement. Besides the theoretical interest, non-Gaussian entanglement provides also technological advantages, for instance, in quantum metrology [21,22] or quantum computing applications [23].

In [13] we found that the states generated by 2-2SPDC and 3SPDC processes have different types of entanglement, suggesting some sort of continuous-variable equivalence with the three-qubit W and GHZ classes. In this work we formalize this insight as well as analyze the swapping of entanglement from 3SPDC to three qubits. In particular, we provide formal definitions to Gaussian and non-Gaussian entanglement and prove both the Gaussianity of 2-2SPDC entanglement and the non-Gaussianity of the 3SPDC entanglement, finding similarities and differences between GHZ and W classes.

Moreover, we propose an experimental setup in which 3SPDC non-Gaussian entanglement can be swapped to three qubits. An asymmetric superconducting quantum interference

---

*soyandres2@gmail.com

022401-1

device (SQUID) generating 3SPDC is coupled to three separate resonators, each containing a coupled superconducting qubit. We show that the entanglement generated among the qubits is also of non-Gaussian nature, by using a natural extension of our CV entanglement witness which accommodates DV systems. Interestingly, we detect non-Gaussian qubit entanglement in a wider parameter regime (as compared to mode entanglement) which suggests that the swapping to qubits could be an efficient way of extending the technological usefulness of 3SPDC entanglement.

The structure of this work is as follows. First we introduce the notions of Gaussian and non-Gaussian entanglement in such a way that they may be applied to both CV and DV systems and pure and mixed states. Then we relate these notions to the widely known W and GHZ states. After that, we present arguments that can be used to prove the non-Gaussianity of the entanglement contained in a state and we will apply them to our three-mode 3SPDC system in the presence of three qubits, each interacting with a bosonic mode. We will obtain proof of the tripartite non-Gaussianity of the field's state as well of the qubits'. Finally, some concluding remarks and future research directions will be presented.

II. NON-GAUSSIAN ENTANGLEMENT

We start with a description of non-Gaussian entanglement. The term is coined after the Gaussian states of quantum optics, those states represented by Wigner functions that happen to be Gaussians of the canonical variables.

Detecting entanglement in an experiment often involves measuring some witness, namely, a combination of expectation values of observables that is bounded by some constant for states that do not possess the kind of entanglement considered. An entanglement witness is Gaussian if its algebraic expression contains only linear and quadratic contributions of the canonical variables. That way, the witness is only sensitive to the means and (co)variances of a multipartite wave function or Wigner quasidistribution. If higher powers of the canonical variables appear in the witness or the witness cannot be brought into an algebraic formula of the canonical variables, then it is non-Gaussian.

The characterization of the entanglement of Gaussian states is well known [24]. Any entanglement in a Gaussian state will be detected by a Gaussian witness; thus a Gaussian state can only contain Gaussian entanglement. However, a non-Gaussian state might have the same mean and covariances of the canonical variables as some separable Gaussian state [13]. Then its entanglement would not be detected by a Gaussian witness and so it would be non-Gaussian entanglement. Finally, we can extend the concept of Gaussianity to DV systems, by replacing any reference to canonical variables with spin variables.

Interestingly, the concepts of Gaussian and non-Gaussian entanglement can be related with the two main representatives of tripartite qubit entanglement, the W and GHZ states. The W entanglement is Gaussian, since it can be detected by a Gaussian witness [25], while GHZ entanglement is non-Gaussian, since we can, for instance, find a state that contains no entanglement and yet has the same means and covariances

on the spin variables as the GHZ state:

$\rho_{\text{mimic GHZ}}$

$$= \tfrac{1}{12}(|0_1\rangle\langle 0_1| + |1_1\rangle\langle 1_1|) \otimes (|0_2 0_3\rangle\langle 0_2 0_3| + |1_2 1_3\rangle\langle 1_2 1_3|)$$

$$+ \tfrac{1}{12}(|0_2\rangle\langle 0_2| + |1_2\rangle\langle 1_2|) \otimes (|0_1 0_3\rangle\langle 0_1 0_3| + |1_1 1_3\rangle\langle 1_1 1_3|)$$

$$+ \tfrac{1}{12}(|0_3\rangle\langle 0_3| + |1_3\rangle\langle 1_3|) \otimes (|0_1 0_2\rangle\langle 0_1 0_2| + |1_2 1_2\rangle\langle 1_2 1_2|),$$

where $|0_i\rangle$ is the ground state of the ith qubit and $|1_i\rangle$ its excited state. Both the GHZ state and $\rho_{\text{mimic GHZ}}$ have the same first and second statistical moments of the spin variables

$$\langle S_x^i \rangle = 0, \quad \langle S_y^i \rangle = 0, \quad \langle S_z^i \rangle = 0,$$

$$\Delta^2 S_x^i S_x^j = 0, \quad \Delta^2 S_y^i S_y^j = 0, \quad \Delta^2 S_z^i S_z^j = \tfrac{1}{4},$$

where the spin variables are defined by $S_z^i|0_i\rangle = -1/2|0_i\rangle$ and $S_z^i|1_i\rangle = 1/2|1_i\rangle$ and the angular momentum algebra.

III. NON-GAUSSIANITY OF ENTANGLEMENT IN 3SPDC RADIATION

The 3SPDC process studied in [14] takes place in a system composed of three bosonic modes subject to time-dependent boundary conditions, implemented by means of a SQUID, which behaves as a tunable nonlinear inductor at the edge of a superconducting waveguide. The SQUIDs inductance is modulated with the sum of the characteristic frequencies of the three modes, producing an effective three-mode interaction described by

$$H_{\text{3SPDC RWA}} = \sum_{i=1}^{3} \omega_i a_i^\dagger a_i + g_0 \cos \omega_d t (a_1^\dagger a_2^\dagger a_3^\dagger + a_1 a_2 a_3),$$

where ω_1, ω_2, and ω_3 are the modes' characteristic frequencies, $a_i^\dagger$ and a_i are the creation and annihilation operators on the ith mode, respectively, g_0 is the intensity of the coupling between the modes, and ω_d is the driving to the SQUID, which is equal to $\sum_i \omega_i$. Note that the rotating-wave approximation (RWA) was performed in order to illustrate the main process induced by this Hamiltonian: parametric creation or destruction of triplets of photons, one on each mode. The Hamiltonian is actually an approximation of a more general Hamiltonian

$$H_{\text{3SPDC}} = \sum_{i=1}^{3} \omega_i a_i^\dagger a_i$$

$$+ g_0 \cos \omega_d t (a_1^\dagger + a_1)(a_2^\dagger + a_2)(a_3^\dagger + a_3),$$

which will be the one that we will study throughout the text. We use this Hamiltonian for the sake of completeness, although the RWA Hamiltonian above would suffice to obtain the main results of this work and is generally valid under experimental conditions. However, using the general Hamiltonian allows us not to be concerned with the regime of validity of the RWA. Before we begin proving the non-Gaussian nature of the entanglement produced among the three modes, we will extend the system with three qubits, each one interacting with one mode. This modification is of interest because it paves the way to experimental production of non-Gaussian entanglement in both CV systems (the reduced state of the three modes) and DV systems (the qubits).
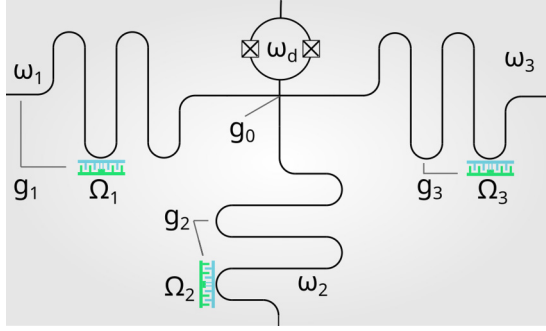
FIG. 1. Illustration of the system composed of three transmission lines (depicted as solid meandering lines) that meet at an asymmetric SQUID (loop with boxes, that is, Josephson junctions, at the sides). Each one of those transmission lines interacts with a transmon qubit (colored zippers, not to scale). Control lines have been omitted. If pumped with the appropriate tone, the asymmetric SQUID will drive three-mode spontaneous parametric down-conversion among the three fundamental modes of the transmission lines. Non-Gaussian tripartite entanglement will be produced between the modes as proved in Eq. (5) and Fig. 2 for some parameter regimes. Additionally, non-Gaussian tripartite entanglement will be swapped to the qubits, as proved in the text and Fig. 3. We show labels for the parameters that appear in the Hamiltonian (1) for reference.

Such a technological platform could ground our theory on experimental data and additionally find technical applications as the primitive for generation of tripartite entanglement between CV or DV systems.

When the three qubits are taken into account, the total Hamiltonian becomes

$$H_{3\text{SPDC}+3\text{qubits}} = \sum_{i=1}^{3} \omega_i a_i^\dagger a_i + \frac{\Omega_i}{2}\sigma_{z,i} + g_i \sigma_{x,i}(a_i^\dagger + a_i)$$
$$+ g_0 \cos\omega_d t (a_1^\dagger + a_1)(a_2^\dagger + a_2)(a_3^\dagger + a_3),$$
(1)

where $\sigma_{x,y,z,i}$ are the Pauli matrices for the ith qubit and g_i is the intensity of its coupling to the respective mode. Note that the qubit-mode interaction takes the form of the Rabi interaction. An experimental setup that could be effectively modeled with Eq. (1) is described in Fig. 1. It is composed of three superconducting cavities joined together from one of their edges [26–28]. At that meeting point lies an asymmetric SQUID driven with a single tone of frequency $\omega_d = \sum_i \omega_i$.

In order to prove the non-Gaussianity of the entanglement produced by the Hamiltonian in Eq. (1) when evolving the initial vacuum state $|0g0g0g\rangle$, where $|0\rangle$ is the mode vacuum state and $|g\rangle$ is the qubit ground state, we will examine the time derivatives of the quadratures and spin covariances, by making use of the condition

$$i\hbar \partial_t \Delta^2 O_i O_j = 0$$
$$\Leftrightarrow$$
$$\langle [O_i O_j, H] \rangle = \langle O_i \rangle \langle [O_j, H] \rangle + \langle [O_i, H] \rangle \langle O_j \rangle, \quad (2)$$

where O_i and O_j are canonical or spin variables, H is the Hamiltonian of the system and $\Delta^2 O_i O_j$ is the covariance between the measurements of O_i and O_j, that is, $\langle O_i O_j \rangle -$

$\langle O_i \rangle \langle O_j \rangle$. Equation (2) is easily derived from the Heisenberg equation of motion. See Appendix A for further notes on its derivation. Using the Hamiltonian in Eqs. (1) and (2), we have

$$\partial_t \Delta^2 x_i x_j = \left\langle \frac{x_i p_j}{m_j} + \frac{x_j p_i}{m_i} \right\rangle - \langle x_i \rangle \left\langle \frac{p_j}{m_j} \right\rangle - \left\langle \frac{p_i}{m_i} \right\rangle \langle x_j \rangle,$$

$$\partial_t \Delta^2 p_i p_j = -\langle m_j \omega_j^2 p_i x_j + m_i \omega_i^2 x_i p_j \rangle$$
$$- i\hbar \langle g_j \sigma_{xj} p_i + g_i \sigma_{xi} p_j \rangle$$
$$- \hat{g}(t) \langle p_i x_i x_k + x_j p_j x_k \rangle$$
$$+ \langle m_i \omega_i^2 x_i + g_i \sigma_{xi} + \hat{g}(t) x_j x_k \rangle \langle p_j \rangle$$
$$+ \langle m_j \omega_j^2 x_j + g_j \sigma_{xj} + \hat{g}(t) x_i x_k \rangle \langle p_i \rangle,$$

$$\partial_t \Delta^2 S_{xi} S_{xj} = \Omega_i \langle \sigma_x^i \sigma_y^j \rangle + \Omega_j \langle \sigma_y^i \sigma_x^j \rangle,$$

$$\partial_t \Delta^2 S_{yi} S_{yj} = \Omega_j \langle \sigma_y^i \sigma_x^j \rangle + \Omega_i \langle \sigma_x^i \sigma_y^j \rangle,$$

$$\partial_t \Delta^2 S_{zi} S_{zj} = \frac{g_j}{2} \langle \sigma_z^i x_j \sigma_y^j \rangle + \frac{g_i}{2} \langle x_i \sigma_y^i \sigma_z^j \rangle, \quad (3)$$

where x_i and p_i are the quadratures of the ith mode and $S_{x,y,z,i}$ are the analogous angular momentum operators along the x, y, and z axes for the ith qubit. For a detailed derivation of the covariances time derivatives see Appendix B. In order to tackle Eq. (3) we consider the projector

$$P = \sum_{\alpha,\beta=0}^{1} \bigotimes_{i=1}^{3} \sum_{n=0}^{\infty} P_i(2n+\alpha) \otimes P_{i,2\times2}(\beta), \quad (4)$$

where $P_i(n) = |n\rangle\langle n|$ is the projector onto the bosonic mode state with n photons or excitations and $P_{i,2\times2}(q)$ is $|g\rangle\langle g|$ if $q = 0$, the projector onto the qubit ground state, or $|e\rangle\langle e|$ if $q = 1$, the projector onto the qubit excited state. We find that this projector is a conserved quantity of the system. Consider the following motivation behind its definition: The Hamiltonian in Eq. (1) allows for some transitions between the stationary Hamiltonian eigenstates. In particular, it allows for transitions that change all three modes in one photon (via the 3SPDC process) as well as transitions changing a qubit-mode pair in one excitation (that is, any combination of creating or destroying a photon while exciting or relaxing the qubit). However, there are many other transitions that are not allowed: creating or destroying a pair of photons but not a third one, spontaneously exciting or relaxing a qubit without changing photon number, and so on. Then P is built to project onto all of the eigenstates the vacuum can transition to, while excluding those the vacuum cannot leak into. For further information about the derivation of P, as well as proof of how it commutes with the Hamiltonian, see Appendix C. The expectation value of P for the initial state $|0g0g0g\rangle$ is 1. Therefore, the time evolution of $|0g0g0g\rangle$ will never leave the subspace P projects onto, which we denote the dynamical subspace,

$$\psi(t) = \sum_{\alpha,\beta=0}^{1} \bigotimes_{i=1}^{3} \sum_{n=0}^{\infty} c_{\alpha,\beta,i,n}(t)|2n+\alpha\rangle \otimes |\beta\rangle.$$

With this we can evaluate many of the expectation values in the covariances time derivatives in Eq. (3). In particular, all time derivatives become zero, except for the $\Delta^2 S_{z,i} S_{z,j}$
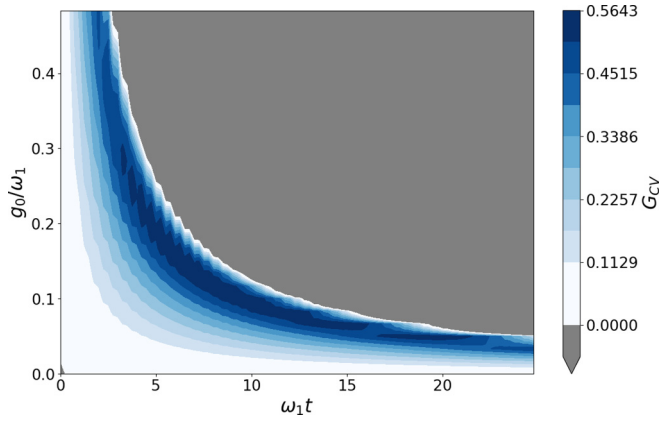
FIG. 2. Value of the witness G_{CV} defined in Eq. (6) as a function of time t and 3SPDC coupling g_0 in units of the lowest-frequency mode ω_1 when the initial state $|0g0g0g\rangle$ evolves under the Hamiltonian in Eq. (1). The other mode frequencies are $\omega_2 = 2\omega_1$ and $\omega_3 = \omega_1$. The qubits are resonant with their modes so that $\Omega_i = \omega_i$ and their couplings are all equal $g_i = 0.01\omega_1$. The witness reports non-Gaussian entanglement in the modes, that is, it is greater than zero, for short times. Note that entanglement is 0 at $t = 0$.

covariance

$$\partial_t \Delta^2 x_i x_j = 0,$$

$$\partial_t \Delta^2 p_i p_j = 0,$$

$$\partial_t \Delta^2 S_{xi} S_{xj} = 0,$$

$$\partial_t \Delta^2 S_{yi} S_{yj} = 0,$$

$$\partial_t \Delta^2 S_z^i S_z^j = \frac{g_j}{2} \langle \sigma_z^i x_j \sigma_y^j \rangle + \frac{g_i}{2} \langle x_i \sigma_y^i \sigma_z^j \rangle \neq 0. \quad (5)$$

Therefore, the reduced state of the three modes cannot contain Gaussian entanglement: It has the same covariances as a clearly separable state, the vacuum $|000\rangle$, but the state gets entangled with time, as we proved in [13] for the qubitless system. In that work we built a genuine tripartite entanglement witness defined as

$$G'_{\mathrm{CV}} = |\langle a_1 a_2 a_3 \rangle| - \sum_{\substack{i,j,k=1,2,3 \\ i \neq j \neq k \neq i}} \sqrt{\langle a_i^\dagger a_i \rangle \langle a_j^\dagger a_j a_k^\dagger a_k \rangle},$$

so that when $G'_{\mathrm{CV}} > 0$, genuine tripartite entanglement is detected. In fact, since the publication of [13] we have found an improved witness

$$G_{\mathrm{CV}} = |\langle a_1 a_2 a_3 \rangle| - \max_{\substack{i,j,k=1,2,3 \\ i \neq j \neq k \neq i}} \sqrt{\langle a_i^\dagger a_i \rangle \langle a_j^\dagger a_j a_k^\dagger a_k \rangle} \quad (6)$$

by following the derivation in [13] and making use of the fact that the expectation values of a mixed state cannot be larger than the largest of its pure components. Figure 2 shows the value of the genuine tripartite entanglement witness G_{CV} for different times and 3SPDC coupling strength. We conclude that the field contains non-Gaussian entanglement at times not much longer than $g_0 t = 1$. For longer times, all we know is that Gaussian witnesses will fail, but if there is any entanglement in the modes non-Gaussian witnesses might succeed.
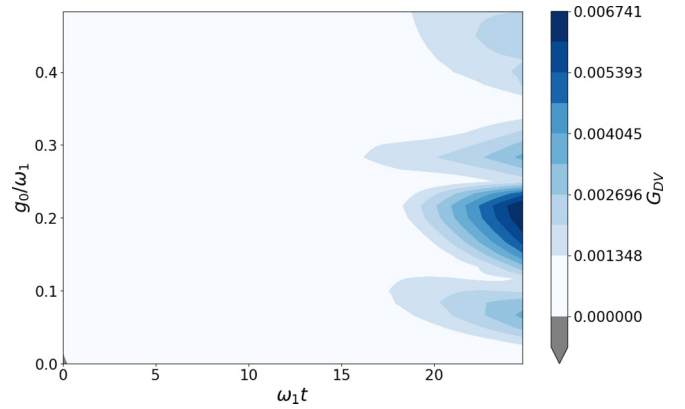

FIG. 3. Value of the witness G_{DV} defined in Eq. (7) as a function of time t and 3SPDC coupling g_0 in units of the lowest-frequency mode ω_1 in the same conditions as in Fig. 2. The witness reports non-Gaussian entanglement in the qubits, that is, it is greater than zero, for a broad parameter regime. Note that entanglement is 0 at $t = 0$.

IV. NON-GAUSSIAN THREE-QUBIT ENTANGLEMENT

The nature of the three-qubit entanglement is however more difficult to determine: Since the z covariances do change in time we need to answer the question of whether or not a Gaussian witness exists that uses only the z spin covariances. We find that the answer is no, and therefore the qubit entanglement, if there is any, is non-Gaussian too. See Appendix D for a proof.

In order to detect whether there is actually entanglement, we need a suitable non-Gaussian entanglement witness. The same proof [13] that leads to the construction of G_{CV} in CV systems can be extended to a DV witness by replacing the canonical variables with spin variables

$$G_{\mathrm{DV}} = |\langle \sigma_1^- \sigma_2^- \sigma_3^- \rangle| - \max_{\substack{i,j,k=1,2,3 \\ i \neq j \neq k \neq i}} \sqrt{\langle \sigma_i^+ \sigma_i^- \rangle \langle \sigma_j^+ \sigma_j^- \sigma_k^+ \sigma_k^- \rangle},$$

$$(7)$$

which works as G_{CV} but in DV systems; it reports genuine tripartite entanglement whenever $G_{\mathrm{DV}} > 0$. Figure 3 shows the value of G_{DV} for different times and 3SPDC coupling strengths. We conclude that the qubits are indeed entangled in a non-Gaussian way for a broad parameter regime. Indeed, it seems that the qubits are entangled in a wider regime of parameters, suggesting that swapping the entanglement from the photons to the qubits could be a way to exploit the multipartite entanglement generated in 3SPDC radiation. However, notice that there could be other witnesses detecting entanglement where ours fails. Note also that, as usual, an entanglement witness only tells us about the existence of entanglement, not necessarily its degree, which would require the use of an entanglement measure.

V. CONCLUSION AND OUTLOOK

In summary, we have presented a setup in which three qubits are coupled to a 3SPDC source. We have shown that there is genuine tripartite entanglement generated both among

the three modes of the electromagnetic field and among the qubits. Moreover, we have proved the non-Gaussian nature of this entanglement, as well as that of the GHZ state, suggesting that Gaussianity might be an extension to CV and mixed states of the W and GHZ classes. We have introduced witnesses of genuine tripartite entanglement for both the field and the qubits. Interestingly, in the case of the qubits, entanglement is detected for a wider regime of parameters, which suggests that our setup could provide an efficient way of exploiting the genuine non-Gaussian multipartite entanglement generated in 3SPDC interactions. In particular, qubits with non-Gaussian entanglement display useful properties for quantum metrology and quantum computing applications.

APPENDIX A: DYNAMICS OF STATISTICAL MOMENTS

In this Appendix we will derive the expression for the time derivatives of the canonical and spin variable covariances. We will be particularly interested in the cases when the moments are constant. If that is the case, Gaussian entanglement cannot be generated. We start with the Heisenberg equation of motion

$$i\hbar \partial_t O(t) = [O(t), H(t)], \tag{A1}$$

which immediately yields expressions for the time derivatives of the first-order statistical moments, the means

$$i\hbar \partial_t \langle O \rangle = \langle [O(t), H(t)] \rangle. \tag{A2}$$

In order to derive a similar expression for second-order statistical moments, that is, variances and covariances, we follow a similar approach. We recall the definition of the covariances of two observables O_i and O_j,

$$\Delta^2 O_i O_j = \langle O_i O_j \rangle - \langle O_i \rangle \langle O_j \rangle,$$

and by taking its time derivative we arrive at

$$i\hbar \partial_t \Delta^2 O_i O_j = \langle [O_i O_j, H] \rangle - \langle O_i \rangle \langle [O_j, H] \rangle - \langle O_j \rangle \langle [O_i, H] \rangle.$$

This equation gives us conditions systems must follow in order not to generate or destroy Gaussian entanglement

$$i\hbar \partial_t \Delta^2 O_i O_j = 0$$
$$\Leftrightarrow$$
$$\langle [O_i O_j, H] \rangle = \langle O_i \rangle \langle [O_j, H] \rangle + \langle [O_i, H] \rangle \langle O_j \rangle. \tag{A3}$$

Note that if the averages of O_i and O_j are zero, then the condition states that in order not to change the covariances, the operator $O_i O_j$ must be a conserved quantity in the subspace spanned by the state during all that time.

Summarizing, we have obtained expressions for the time derivatives of the means and covariances of general observables. Those equations led to Hamiltonian conditions in Eq. (2) that will tell when the covariances (and Gaussian

entanglement) are constant in a particular system. We will consider particular Hamiltonians in the calculations to come.

APPENDIX B: DERIVATION OF THE COVARIANCES' TIME DERIVATIVES

In this Appendix we will take Hamiltonian in Eq. (1) and compute the covariances' time derivatives as instructed by Eq. (2). Note that the Hamiltonian can be written in terms of the canonical and spin variables alone

$$H = \sum_{i=1}^{3} \left[\frac{p_i^2}{2m_i} + \frac{1}{2} m_i \omega_i^2 x_i^2 + \Omega S_z^i \right]$$
$$+ \hat{g}_0 \cos \left(\sum_i \omega_i t \right) x_1 x_2 x_3 + \sum_{i=1}^{3} g_i \sigma_x^i x_i.$$

Then the field's position covariances have the time derivatives

$$[x_i, H] = \frac{1}{2m_i} [x_i, p_i^2]$$
$$= \frac{1}{2m_i} ([x_i, p_i] p_i + p_i [x_i, p_i])$$
$$= \frac{i\hbar}{m_i} p_i,$$
$$[x_i x_j, H] = x_i [x_j, H] + [x_i, H] x_j$$
$$= i\hbar \left(\frac{x_i p_j}{m_j} + \frac{x_j p_i}{m_i} \right),$$
$$\partial_t \Delta^2 x_i x_j = \left\langle \frac{x_i p_j}{m_j} + \frac{x_j p_i}{m_i} \right\rangle - \langle x_i \rangle \left\langle \frac{p_j}{m_j} \right\rangle - \left\langle \frac{p_i}{m_i} \right\rangle \langle x_j \rangle. \tag{B1}$$

For the momentum's covariances

$$[p_i, H] = \frac{m_i \omega_i^2}{2} [p_i, x_i^2] + g_i \sigma_{xi} [p_i, x_i] + \hat{g}(t) [p_i, x_1 x_2 x_3]$$
$$= -i\hbar m_i \omega_i^2 x_i - i\hbar g_i \sigma_{xi}$$
$$- i\hbar \hat{g}(t) x_j x_k \quad \text{with } i \neq j \neq k \neq i,$$
$$[p_i p_j, H] = p_i [p_j, H] + [p_i, H] p_j$$
$$= -i\hbar (m_j \omega_j^2 p_i x_j + m_i \omega_i^2 x_i p_j)$$
$$- i\hbar (g_j \sigma_{xj} p_i + g_i \sigma_{xi} p_j)$$
$$- i\hbar \hat{g}(t) (p_i x_i x_k + x_j p_j x_k),$$

which results in a time derivative of the momenta's covariances

$$\partial_t \Delta^2 p_i p_j = -\langle m_j \omega_j^2 p_i x_j + m_i \omega_i^2 x_i p_j \rangle$$
$$- i\hbar \langle g_j \sigma_{xj} p_i + g_i \sigma_{xi} p_j \rangle$$
$$- \hat{g}(t) \langle p_i x_i x_k + x_j p_j x_k \rangle$$
$$+ \langle m_i \omega_i^2 x_i + g_i \sigma_{xi} + \hat{g}(t) x_j x_k \rangle \langle p_j \rangle$$
$$+ \langle m_j \omega_j^2 x_j + g_j \sigma_{xj} + \hat{g}(t) x_i x_k \rangle \langle p_i \rangle. \tag{B2}$$

The conditions derived in Eq. (2) not only apply to continuous-variable systems, but discrete ones as well. By plugging the spin variables S_x^i, S_y^i, and S_z^i and the Hamiltonian

in Eq. (1) we derive

$$\partial_t \Delta^2 S_x^i S_x^j = \Omega_i \langle \sigma_x^i \sigma_y^j \rangle + \Omega_j \langle \sigma_y^i \sigma_x^j \rangle,$$

$$\partial_t \Delta^2 S_y^i S_y^j = \Omega_j \langle \sigma_y^i \sigma_x^j \rangle + \Omega_i \langle \sigma_x^i \sigma_y^j \rangle$$
$$- 2g_j \langle x_j \sigma_y^i \sigma_z^j \rangle - 2g_i \langle x_i \sigma_z^i \sigma_y^j \rangle,$$

$$\partial_t \Delta^2 S_z^i S_z^j = \frac{g_j}{2} \langle \sigma_z^i x_j \sigma_y^j \rangle + \frac{g_i}{2} \langle x_i \sigma_y^i \sigma_z^j \rangle.$$

APPENDIX C: CONSERVED QUANTITIES

In this Appendix we provide proof of the conserved quantity P in Eq. (4). Note that P projects onto the subspace that contains every eigenstate with the same parity of qubit plus photon excitation on each pair of qubits and modes. That is, for every eigenstate in that subspace, the addition of the number of photons on the first mode plus the number of excitations on the first qubit (that is, zero for $|g\rangle$ or one for $|e\rangle$) will always be the same as the addition of the number of photons and qubit excitations in the second qubit-mode pair. The same happens with the third qubit-mode pair. In order to gain some insight into why that particular projector is a conserved quantity, we will first argue for its construction with perturbation theory. Then an actual proof calculating the commutator with the Hamiltonian is provided. Finally, we will compute some elementary expectation values within the image of P that happen to appear in the covariances' time derivatives.

1. Construction of a conserved quantity

We will begin with the first-order perturbative corrections to the time evolution of $H_{\text{3SPDC+3qubits}}$

$$\psi^{(1)}(t) = \frac{1}{i\hbar} \int_0^t dt' H_{\text{int}}(t') |000ggg\rangle$$
$$= \alpha |111ggg\rangle + \beta |100egg\rangle + \gamma |010geg\rangle + \delta |001gge\rangle,$$

where H_{int} is the Hamiltonian in the interaction picture. The important fact to note here is that all kets share some sort of parity. If we add together the number of photons in the first mode and the number of excitations in the first qubit, we obtain 2 or 0, even numbers. The same happens with every mode-qubit pair and for every ket.

The second-order correction takes the form

$$\psi^{(2)}(t) = \frac{1}{i\hbar} \int_0^t dt' H_{\text{int}}(t') \psi^{(1)}(t')$$

$$\in \text{span}(|000ggg\rangle, |002ggg\rangle, |020ggg\rangle, |022ggg\rangle,$$
$$|200ggg\rangle, |202ggg\rangle, |220ggg\rangle, |222ggg\rangle,$$
$$|110gge\rangle, |101geg\rangle, |011egg\rangle, |112gge\rangle,$$
$$|121geg\rangle, |211egg\rangle, |211egg\rangle, |011egg\rangle,$$
$$|000ggg\rangle, |200ggg\rangle, |110eeg\rangle, |101ege\rangle,$$
$$|121geg\rangle, |101geg\rangle, |110eeg\rangle, |000ggg\rangle,$$
$$|020ggg\rangle, |011gee\rangle, |112gge\rangle, |110gge\rangle,$$
$$|101ege\rangle, |011gee\rangle, |002ggg\rangle, |000ggg\rangle).$$

Again, all the kets involved in the second-order correction share a notion of parity, but it appears to be a different, or more general, parity than the first-order corrections. Some kets have an even number of photons plus qubit excitations (e.g., $|222ggg\rangle$) and other kets have an odd number of photons plus qubits excitations (e.g., $|110gge\rangle$), but there are no kets that mix odd and even numbers of photons plus qubit excitations (e.g., there is no $|211geg\rangle$).

The reader might have noticed that we are now in position to finish a proof by induction. We have proven that the first-order corrections are composed of kets with an even number of fields plus qubit excitations. We have proven that the second-order corrections are a superposition of ketrks with an odd *or* even number (but no mixtures) of fields plus qubit excitations. Now we will prove that if the nth-order correction is such a superposition, the $(n+1)$th correction has that same parity. In order to do so, we will study the effects each of the pieces of the Hamiltonian has on the parity of a ket.

First, the 3SPDC piece has the form $g(t)(a_1^\dagger + a_1)(a_2^\dagger + a_2)(a_3^\dagger + a_3)$. Note that the result of the application of this piece of the Hamiltonian on a vector with well-defined parity is to completely change the parity of each mode-qubit pair. That is, each mode has to change its number of photons in one unit, up or down, but their interacting qubit will remain the same. Therefore, the result is a superposition of vectors with the same parity on each qubit-mode pair.

Second, the Rabi piece has the form $g_i(t)\sigma_x^i(a_i^\dagger + a_i)$. The result of applying this piece of the Hamiltonian on a vector with well-defined parity is a superposition of vectors of the same parity. This is due to the fact that the ith qubit must change its quantum number and the same ith mode must change its number of photons in one unit. Therefore, the parity of that pair will be the same.

Because of these two facts, the parities of the kets forming the superposition that is the evolution of the vacuum will never mix. Therefore, the state must remain in the subspace of vectors with well-defined qubit plus mode excitation parity. The operator that projects onto the subspace of vectors with that well-defined excitation parity is

$$P = \bigotimes_{i=1}^3 \sum_{n=0}^\infty P_i(2n) \otimes P_{i,2\times2}(0)$$

$$+ \bigotimes_{i=1}^3 \sum_{n=0}^\infty P_i(2n+1) \otimes P_{i,2\times2}(0)$$

$$+ \bigotimes_{i=1}^3 \sum_{n=0}^\infty P_i(2n) \otimes P_{i,2\times2}(1)$$

$$+ \bigotimes_{i=1}^3 \sum_{n=0}^\infty P_i(2n+1) \otimes P_{i,2\times2}(1)$$

$$= \sum_{\alpha,\beta=0}^1 \bigotimes_{i=1}^3 \sum_{n=0}^\infty P_i(2n+\alpha) \otimes P_{i,2\times2}(\beta), \quad \text{(C1)}$$

where $P_i(n)$ is the Fock state projector $|n\rangle\langle n|$ and $P_{i,2\times2}(q)$ is

the projector onto the S_z lower eigenstate if $q = 0$ or onto the higher eigenstate if $q = 1$.

2. Proof that P is a conserved quantity

In this section we let the i indices drop as they are redundant notation The projector P clearly commutes with the Hamiltonian's stationary part. In order to prove that it commutes with the interacting pieces as well, we need to

introduce some notation

$$x_1 x_2 x_3 \rightarrow \bigotimes_{i=1}^{3} x \otimes \mathbb{I}_{2\times 2},$$

$$\sigma_x x \rightarrow \bigotimes_{j=1}^{3}[\delta_{ij} x \otimes \sigma_x + (1 - \delta_{ij})\mathbb{I} \otimes \mathbb{I}_{2\times 2}].$$

First, we will show that $x_1 x_2 x_3$ commutes with P,

$$x_1 x_2 x_3 P = \bigotimes_{i=1}^{3} x \otimes \mathbb{I}_{2\times 2} \sum_{\alpha,\beta=0}^{1} \bigotimes_{i=1}^{3} \sum_{n=0}^{\infty} P(2n + \alpha) \otimes P_{2\times 2}(\beta) = \sum_{\alpha,\beta=0}^{1} \bigotimes_{i=1}^{3} \sum_{n=0}^{\infty} xP(2n + \alpha) \otimes P_{2\times 2}(\beta)$$

$$= \sum_{\alpha,\beta=0}^{1} \bigotimes_{i=1}^{3} \sum_{n=0}^{\infty} (\sqrt{2n + \alpha}|2n + \alpha - 1\rangle\langle 2n + \alpha| + \sqrt{2n + \alpha + 1}|2n + \alpha + 1\rangle\langle 2n + \alpha|) \otimes P_{2\times 2}(\beta)$$

$$= \sum_{\alpha,\beta=0}^{1} \bigotimes_{i=1}^{3} \left[\sum_{n=0}^{\infty} \sqrt{2n + \alpha}|2n + \alpha - 1\rangle\langle 2n + \alpha| + \sum_{n=0}^{\infty} \sqrt{2n + \alpha + 1}|2n + \alpha + 1\rangle\langle 2n + \alpha| \right] \otimes P_{2\times 2}(\beta),$$

where we understand that if $2n + \alpha - 1 < 0$ then $|2n + \alpha - 1\rangle = 0$. We have split the summation on n into two different summations. We will perform a change of variables in the first one so that $n \rightarrow n + 1$. Note that in that case the summation index starts at -1,

$$x_1 x_2 x_3 P = \sum_{\alpha,\beta=0}^{1} \bigotimes_{i=1}^{3} \left(\left[\sum_{n=-1}^{\infty} \sqrt{2n + \alpha + 2}|2n + \alpha + 1\rangle\langle 2n + \alpha + 2| + \sum_{n=0}^{\infty} \sqrt{2n + \alpha + 1}|2n + \alpha + 1\rangle\langle 2n + \alpha| \right] \otimes P_{2\times 2}(\beta) \right)$$

Now compare both summations over n. They contain the same ket $|2n + \alpha + 1\rangle$ and have different coefficients and bras. Those coefficients and bras match the result of applying the x operator to the projector $P(2n + \alpha + 1)$ from the right. Therefore,

$$x_1 x_2 x_3 P = \sum_{\alpha,\beta=0}^{1} \bigotimes_{i=1}^{3} \left(\left[\sqrt{\alpha}|\alpha - 1\rangle\langle\alpha| + \sum_{n=0}^{\infty} P(2n + \alpha + 1)x \right] \otimes P_{2\times 2}(\beta) \right).$$

The term $\sqrt{\alpha}|\alpha - 1\rangle\langle\alpha|$ is due to one of the summations over n starting at $n = -1$. That term, however, is different from zero only when $\alpha = 1$. To regroup the term with the rest of the summations it is easier to study the cases $\alpha = 0$ and $\alpha = 1$ separately,

$$x_1 x_2 x_3 P = \sum_{\beta=0}^{1} \bigotimes_{i=1}^{3} \left(\left[\sum_{n=0}^{\infty} P(2n + 1)x \right] \otimes P_{2\times 2}(\beta) \right) + \sum_{\beta=0}^{1} \bigotimes_{i=1}^{3} \left(\left[|0\rangle\langle 1| + \sum_{n=0}^{\infty} P(2n + 2)x \right] \otimes P_{2\times 2}(\beta) \right).$$

The second line is the one representing the case $\alpha = 1$. Note that $|0\rangle\langle 1|$ is the result of applying x to the projector $|0\rangle\langle 0| = P(0)$ from the right. Additionally, we can change the variable in the summation on n so that $n \rightarrow n - 1$ and put $P(0)x$ together with the rest of the summation

$$x_1 x_2 x_3 P = \sum_{\beta=0}^{1} \bigotimes_{i=1}^{3} \left(\left[\sum_{n=0}^{\infty} P(2n + 1)x \right] \otimes P_{2\times 2}(\beta) \right) + \sum_{\beta=0}^{1} \bigotimes_{i=1}^{3} \left(\left[\sum_{n=0}^{\infty} P(2n)x \right] \otimes P_{2\times 2}(\beta) \right).$$

Finally, this expression can be formulated in terms of a new summation over α,

$$x_1 x_2 x_3 P = \sum_{\alpha,\beta=0}^{1} \bigotimes_{i=1}^{3} \left(\sum_{n=0}^{\infty} P(2n + \alpha)x \otimes P_{2\times 2}(\beta) \right) = Px_1 x_2 x_3.$$

Therefore, $[P, x_1 x_2 x_3] = 0$.

We are missing a second step to prove that P is a conserved quantity: It has to commute with the interaction Hamiltonians of the qubits and modes. In order to do so, we will prove that $x_i \sigma_{x,i} P = P x_i \sigma_{x,i}$,

$$x_i \sigma_{x,i} P = \left(\bigotimes_{j=1}^{3} \delta_{ij} x \otimes \sigma_x + (1 - \delta_{ij}) \mathbb{I} \otimes \mathbb{I}_{2 \times 2} \right) \times \left(\sum_{\alpha,\beta=0}^{1} \bigotimes_{j=1}^{3} \sum_{n=0}^{\infty} P(2n+\alpha) \otimes P_{2 \times 2}(\beta) \right)$$

$$= \sum_{\alpha,\beta=0}^{1} \bigotimes_{j=1}^{3} \sum_{n=0}^{\infty} [\delta_{ij} x P(2n+\alpha) \otimes \sigma_x P_{2 \times 2}(\beta) + (1 - \delta_{ij}) P(2n+\alpha) \otimes P_{2 \times 2}(\beta)].$$

Now we will study the action of x on $P(2n + \alpha)$,

$$x_i \sigma_{x,i} P = \sum_{\alpha,\beta=0}^{1} \bigotimes_{j=1}^{3} \sum_{n=0}^{\infty} [\delta_{ij} \sqrt{2n+\alpha} |2n+\alpha-1\rangle\langle 2n+\alpha| \otimes \sigma_x P_{2 \times 2}(\beta)$$

$$+ \delta_{ij} \sqrt{2n+\alpha+1} |2n+\alpha+1\rangle\langle 2n+\alpha| \otimes \sigma_x P_{2 \times 2}(\beta)(1 - \delta_{ij}) P(2n+\alpha) \otimes P_{2 \times 2}(\beta)].$$

As it happened with $x_1 x_2 x_3 P$, we will make a change in the variable n so that $n \to n+1$ only in the first line,

$$x_i \sigma_{x,i} P = \sum_{\alpha,\beta=0}^{1} \bigotimes_{j=1}^{3} \left(\delta_{ij} \sum_{n=-1}^{\infty} \sqrt{2n+\alpha+2} |2n+\alpha+1\rangle\langle 2n+\alpha+2| \otimes \sigma_x P_{2 \times 2}(\beta) \right.$$

$$\left. + \delta_{ij} \sum_{n=0}^{\infty} \sqrt{2n+\alpha+1} |2n+\alpha+1\rangle\langle 2n+\alpha| \otimes \sigma_x P_{2 \times 2}(\beta) + (1 - \delta_{ij}) \sum_{n=0}^{\infty} P(2n+\alpha) \otimes P_{2 \times 2}(\beta) \right).$$

The same way as before, the summation can be rewritten in terms of $P(2n + \alpha + 1)$ acting on x,

$$x_i \sigma_{x,i} P = \sum_{\alpha,\beta=0}^{1} \bigotimes_{j=1}^{3} \left(\delta_{ij} \sqrt{\alpha} |\alpha-1\rangle\langle\alpha| \otimes \sigma_x P_{2 \times 2}(\beta) + \delta_{ij} \sum_{n=0}^{\infty} P(2n+\alpha+1) x \otimes \sigma_x P_{2 \times 2}(\beta) \right.$$

$$\left. + (1 - \delta_{ij}) \sum_{n=0}^{\infty} P(2n+\alpha) \otimes P_{2 \times 2}(\beta) \right).$$

Now we will study the action of σ_x on $P_{2 \times 2}(\beta)$,

$$\sigma_x P_{2 \times 2}(\beta) = |\beta-1\rangle\langle\beta| + |\beta+1\rangle\langle\beta|,$$

where we understand that if $\beta - 1 < 0$ then $|\beta - 1\rangle = 0$ and if $\beta + 1 > 1$ then $|\beta + 1\rangle = 0$. Plugging this equation into the last expression for $x_i \sigma_{x,i} P$ results in

$$x_i \sigma_{x,i} P = \sum_{\alpha,\beta=0}^{1} \bigotimes_{j=1}^{3} \left[\delta_{ij} \left(\sqrt{\alpha} |\alpha-1\rangle\langle\alpha| + \sum_{n=0}^{\infty} P(2n+\alpha+1) x \right) \otimes [|\beta-1\rangle\langle\beta| + |\beta+1\rangle\langle\beta|] \right.$$

$$\left. + (1 - \delta_{ij}) \sum_{n=0}^{\infty} P(2n+\alpha) \otimes P_{2 \times 2}(\beta) \right].$$

By doing two different changes of variable in β for each of the terms $|\beta - 1\rangle\langle\beta|$ and $|\beta + 1\rangle\langle\beta|$ and realizing that only one of those is nonzero for a particular value of β, we conclude that

$$x_i \sigma_{x,i} P = \sum_{\alpha,\beta=0}^{1} \bigotimes_{j=1}^{3} \left(\delta_{ij} \sqrt{\alpha} |\alpha-1\rangle\langle\alpha| \otimes P_{2 \times 2}(\beta) \sigma_x + \delta_{ij} \sum_{n=0}^{\infty} P(2n+\alpha+1) x \otimes P_{2 \times 2}(\beta) \sigma_x \right.$$

$$\left. + (1 - \delta_{ij}) \sum_{n=0}^{\infty} P(2n+\alpha) \otimes P_{2 \times 2}(\beta) \right).$$

Finally, we study the cases $\alpha = 0$ and $\alpha = 1$ separately,

$$x_i \sigma_{x,i} P = \sum_{\beta=0}^{1} \bigotimes_{j=1}^{3} \left(\delta_{ij} \sum_{n=0}^{\infty} P(2n+1) x \otimes P_{2 \times 2}(\beta) \sigma_x + (1 - \delta_{ij}) \sum_{n=0}^{\infty} P(2n) \otimes P_{2 \times 2}(\beta) \right)$$

$$+ \sum_{\beta=0}^{1} \bigotimes_{j=1}^{3} \left(\delta_{ij} |0\rangle\langle 1| \otimes P_{2 \times 2}(\beta) \sigma_x + \delta_{ij} \sum_{n=0}^{\infty} P(2n+2) x \otimes P_{2 \times 2}(\beta) \sigma_x + (1 - \delta_{ij}) \sum_{n=0}^{\infty} P(2n+1) \otimes P_{2 \times 2}(\beta) \right).$$

Again, in the $\alpha = 1$ case we can regroup the matrix element $|0\rangle\langle 1|$ as $P(0)x$ and combine it with the summation on n,

$$x_i\sigma_{x,i}P = \sum_{\beta=0}^{1}\bigotimes_{j=1}^{3}\left(\delta_{ij}\sum_{n=0}^{\infty}P(2n+1)x\otimes P_{2\times2}(\beta)\sigma_x + (1-\delta_{ij})\sum_{n=0}^{\infty}P(2n)\otimes P_{2\times2}(\beta)\right)$$
$$+\sum_{\beta=0}^{1}\bigotimes_{j=1}^{3}\left(\delta_{ij}\delta_{ij}\sum_{n=0}^{\infty}P(2n)x\otimes P_{2\times2}(\beta)\sigma_x + (1-\delta_{ij})\sum_{n=0}^{\infty}P(2n+1)\otimes P_{2\times2}(\beta)\right).$$

This expression can be condensed again in a summation over α so that

$$x_i\sigma_{x,i}P = \sum_{\alpha,\beta=0}^{1}\bigotimes_{j=1}^{3}\sum_{n=0}^{\infty}[\delta_{ij}P(2n+\alpha)x\otimes P_{2\times2}(\beta)\sigma_x + (1-\delta_{ij})P(2n)\otimes P_{2\times2}(\beta)] = Px_i\sigma_{x,i}$$

Therefore, we have proven that $[x_i\sigma_{xi}, P] = 0$.

Summarizing, the projector P as defined in Eq. (4) commutes with each of the ingredients that compose the full 3SPDC + 3qubits Hamiltonian of Eq. (1). We conclude that P is a conserved quantity, and since the initial value of $\langle P\rangle$ for the initial state of vacuum $|0g0g0g\rangle$ is 1, it must remain one at all times. In other words, the state remains in the subspace that the projector P projects on at all times, regardless of whether or not the RWA is taken on any interaction:

$$\psi(t) = \sum_{\alpha,\beta=0}^{1}\bigotimes_{i=1}^{3}\sum_{n=0}^{\infty}c_{\alpha,\beta,i,n}(t)|2n+\alpha\rangle\otimes|\beta\rangle. \quad (C2)$$

We define the dynamical subspace as the subspace that contains ψ at all times, that is, the image of P.

3. Some expectation values in the dynamical subspace

With a closed expression of the dynamical subspace, that is, the subspace that contains the time evolution of vacuum under the Hamiltonian (prior to any RWA), it is possible to compute some expectation values, in particular single, pairs, and triplets of ladder operators, involving the fields or the qubits.

The expectation values of single creation operators on the modes are zero. In Eq. (C2) all eigenbras of the superposition $\psi(t)$ will be orthogonal to all eigenkets of that same superposition if a photon is added to each of them. That is, the $a_i^\dagger$ operator will produce kets with mixed parities and there are no bras at the other side of the expectation value with mixed parities. A similar argument holds for the annihilation operators on each mode,

$$\langle a_i\rangle = \langle a_i^\dagger\rangle = 0.$$

The expectation values of single creation operators on the qubits are zero too, because of the same argument

$$\langle\sigma_i^+\rangle = \langle\sigma_i^-\rangle = 0.$$

The expectation values of pairs of creation or annihilation operators on modes are zero only if they act on different modes. If that is the case, the result is zero because of the same argument as before. If the operators act on the same mode, we are talking about the expectation value of the number operator, which must not be zero, as there is photon generation and that

operator does not mix parities of the kets.

$$\langle a_i^\dagger a_j^\dagger\rangle = \langle a_i a_j^\dagger\rangle = \langle a_i^\dagger a_j\rangle = \langle a_i a_j\rangle = 0 \quad \text{provided } i \neq j.$$

The expectation values of pairs of ladder operators on the qubits are zero if and only if they act on different qubits, because of the same argument as for the modes,

$$\langle\sigma_i^+\sigma_j^+\rangle = \langle\sigma_i^+\sigma_j^-\rangle = \langle\sigma_i^-\sigma_j^+\rangle = \langle\sigma_i^-\sigma_j^-\rangle = 0 \quad \text{provided } i \neq j.$$

The expectation values of pairs of ladder operators on one mode and on one qubit are zero only if the former acts on a mode that does not interact with the qubit the latter acts on, that is,

$$\langle a_i^\dagger\sigma_j^+\rangle = \langle a_i^\dagger\sigma_j^-\rangle = \langle a_i\sigma_j^+\rangle = \langle a_i\sigma_j^-\rangle = 0 \quad \text{provided } i \neq j.$$

The reason is the same as before: Each operator will change the parity of two different pairs of modes and qubits, but will leave one pair with the previous parity.

The expectation values of triplets of ladder operators on the modes are zero as long as they act on two modes. If that is the case, one of the ladder operators acts on one mode, and by the same argument as before, that expectation value must be zero,

$$\langle a_i^\dagger a_i a_j\rangle = 0 \quad \text{provided } i \neq j$$

With these expressions we have enough information to prove that the covariances in the fields' canonical variables and qubits' x and y spin variables are constant in time.

APPENDIX D: THE z SPIN COVARIANCES ALONE ARE NOT GAUSSIAN ENTANGLEMENT

In this Appendix we will prove that any three-qubit mixed state that has the same x and y covariances as a separable state and only different z spin covariances has no Gaussian entanglement. The argument is very similar to those presented before: Separable states have access to a particular range of values of the z spin covariance. If general three-qubit states have access to a bigger range of the z spin covariances, then a Gaussian witness paying attention to only the z covariances could report entanglement; however, if the separable and general ranges are the same, then no witness can tell the difference between those states with only one covariance. Then a state that differs only in those z covariances from a separable state,

as is the case of the qubits state in the main text, cannot contain Gaussian entanglement.

For separable states the bound on the z spin covariances is given by classical probability theory, in particular, the Cauchy-Schwarz and Popoviciu inequalities

$$|\Delta^2 O_i O_j| \leqslant \sqrt{\Delta^2 O_i \Delta^2 O_j}$$
$$\leqslant \tfrac{1}{4}(\sup O_i - \inf O_i)(\sup O_j - \inf O_j),$$

where $\sup O$ and $\inf O$ are bounds to the values a measurement of the observable O may take. In particular, for spin variables we have

$$|\Delta^2 S_{zi} S_{zj}| \leqslant \tfrac{1}{4}.$$

The question remains whether this classical bound can be violated by some entangled state. The reader might suspect that the answer is negative, as in the many years of research on entanglement there are no Bell-like inequalities or witnesses built from covariances on only one axis. To prove that intuition, consider a pure two-qubit state

$\psi = \sum_{q_1=0}^{1} \sum_{q_2=0}^{1} c_{q_1,q_2} |q_1, q_2\rangle$ and the fact that the covariances of the spin variables can be expressed in terms of the covariance of the excitation projector's covariance

$$\Delta^2 S_{zi} S_{zj} = \Delta^2 P_{ei} P_{ej}$$
$$= (1 - |c_{10}|^2 - |c_{01}|^2)|c_{11}|^2 - |c_{11}|^4 - |c_{01}|^2|c_{10}|^2,$$

where P_{ei} is the projector onto the excited state of the ith qubit and $c_{q_1 q_2}$ are the coefficients of a two-qubit pure state in the computational basis. It is a simple exercise to find the pure two-qubit state that maximizes the covariance, which is a Bell state $\psi = \frac{1}{\sqrt{2}}[|00\rangle + |11\rangle]$, which yields a covariance $\Delta^2 S_{z1} S_{z2}$ of $\tfrac{1}{4}$. Two-qubit mixed states cannot violate this bound and the expectation value of a mixture is never larger than the largest of its pure components. General systems that contain two qubits cannot beat this bound either, as their expectation values will be the same as those of the reduced density matrix on the two qubits.

Therefore, we have proven that no witness will be able to report entanglement by inspecting the z covariances alone and a state that differs from a separable state only in those covariances will not contain Gaussian entanglement.

[1] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, Phys. Rev. Lett. **70**, 1895 (1993).

[2] D. Boschi, S. Branca, F. De Martini, L. Hardy, and S. Popescu, Phys. Rev. Lett. **80**, 1121 (1998).

[3] D. Bouwmeester, J.-W. Pan, K. Mattle, M. Eibl, H. Weinfurter, and A. Zeilinger, Nature (London) **390**, 575 (1997).

[4] S. Aaronson and A. Arkhipov, Theor. Comput. **9**, 143 (2013).

[5] H.-S. Zhong, H. Wang, Y.-H. Deng, M.-C. Chen, L.-C. Peng, Y.-H. Luo, J. Qin, D. Wu, X. Ding, Y. Hu, P. Hu, X.-Y. Yang, W.-J. Zhang, H. Li, Y. Li, X. Jiang, L. Gan, G. Yang, L. You, Z. Wang *et al.*, Science **370**, 1460 (2020).

[6] R. E. Slusher, L. W. Hollberg, B. Yurke, J. C. Mertz, and J. F. Valley, Phys. Rev. Lett. **55**, 2409 (1985).

[7] L.-A. Wu, H. J. Kimble, J. L. Hall, and H. Wu, Phys. Rev. Lett. **57**, 2520 (1986).

[8] A. Heidmann, R. J. Horowicz, S. Reynaud, E. Giacobino, C. Fabre, and G. Camy, Phys. Rev. Lett. **59**, 2555 (1987).

[9] Z. Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng, Phys. Rev. Lett. **68**, 3663 (1992).

[10] P. Lähteenmäki, G. S. Paraoanu, J. Hassel, and P. J. Hakonen, Nat. Commun. **7**, 12548 (2016).

[11] D. E. Bruschi, C. Sabín, and G. S. Paraoanu, Phys. Rev. A **95**, 062324 (2017).

[12] C. W. S. Chang, M. Simoen, J. Aumentado, C. Sabín, P. Forn-Díaz, A. M. Vadiraj, F. Quijandría, G. Johansson, I. Fuentes, and C. M. Wilson, Phys. Rev. Appl. **10**, 044019 (2018).

[13] A. Agustí, C. W. S. Chang, F. Quijandría, G. Johansson, C. M. Wilson, and C. Sabín, Phys. Rev. Lett. **125**, 020502 (2020).

[14] C. W. S. Chang, C. Sabín, P. Forn-Díaz, F. Quijandría, A. M. Vadiraj, I. Nsanzineza, G. Johansson, and C. M. Wilson, Phys. Rev. X **10**, 011011 (2020).

[15] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000).

[16] M. Gharahi and S. Mancini, Phys. Rev. A **104**, 042402 (2021).

[17] L. Lamata, J. León, D. Salgado, and E. Solano, Phys. Rev. A **74**, 052336 (2006).

[18] L. Lamata, J. León, D. Salgado, and E. Solano, Phys. Rev. A **75**, 022318 (2007).

[19] M. Sanz, D. Braak, E. Solano, and I. L. Egusquiza, J. Phys. A: Math. Theor. **50**, 195303 (2017).

[20] M. Gharahi, S. Mancini, and G. Ottaviani, Phys. Rev. Res. **2**, 043003 (2020).

[21] H. Strobel, W. Muessel, D. Linnemann, T. Zibold, D. B. Hume, L. Pezzè, A. Smerzi, and M. K. Oberthaler, Science **345**, 424 (2014).

[22] M. Gessner, A. Smerzi, and L. Pezzè, Phys. Rev. Lett. **122**, 090503 (2019).

[23] L. García-Álvarez, C. Calcluth, A. Ferraro, and G. Ferrini, Phys. Rev. Res. **2**, 043322 (2020).

[24] G. Adesso, Entanglement of Gaussian states, Ph.D. thesis, University of Salerno, 2007.

[25] R. Y. Teh and M. D. Reid, Phys. Rev. A **100**, 022126 (2019).

[26] J. Koch, A. A. Houck, K. L. Hur, and S. M. Girvin, Phys. Rev. A **82**, 043811 (2010).

[27] A. A. Houck, H. E. Türeci, and J. Koch, Nat. Phys. **8**, 292 (2012).

[28] S. Felicetti, M. Sanz, L. Lamata, G. Romero, G. Johansson, P. Delsing, and E. Solano, Phys. Rev. Lett. **113**, 093602 (2014).