Designing robust quantum refrigerators in disordered spin models

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We explore a small quantum refrigerator in which the working substance is made of paradigmatic nearest-neighbor quantum spin models, the XYZ and the XY model with Dzyaloshinskii-Moriya interactions, consisting of two and three spins, each of which is in contact with a bosonic bath. We identify a specific range of interaction strengths which can be tuned appropriately to ensure a cooling of the selected spin in terms of its local temperature in the weak-coupling limit. Moreover, we report that in this domain, when one of the interaction strengths is disordered, the performance of the thermal machine operating as a refrigerator remains *almost* unchanged instead of degrading, thereby establishing the flexibility of this device. However, to obtain a significant amount of cooling via ordered as well as disordered spin models, we observe that one has to go beyond the weak-coupling limit and compute the figures of merit by using global master equations.

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I. INTRODUCTION

The quest for small quantum thermal machines [1] that can supersede their classical counterparts in performance [2] has been an important and vibrant component in the field of quantum thermodynamics [3,4]. These machines are expected to not only provide a better understanding of the interplay between the concepts from quantum information theory and thermodynamics [4–6], but also lead to building efficient quantum technologies [7]. Moreover, the interdisciplinary nature of the designs and working principles of these machines has also attracted attention from researchers in statistical [8] and quantum many-body physics [9,10]. To verify the theoretical proposals on these machines, several experiments have been performed by using trapped ions [11], mesoscopic systems [12], nuclear magnetic resonance [13], and superconducting materials [14].

Among the wide variety of small quantum thermal machines, quantum refrigerators made of quantum systems with Hilbert spaces of small dimension have gained a lot of interest [15–21]. Special attention has recently been given to three-spin quantum refrigerators, where a local cooling of one of the spins is achieved by connecting each of the spins in the system with a local Markovian thermal bath. Depending on the choice of the system parameters, the refrigerator may operate in either the absorption region where energy is conserved or in an external energy-driven region, where a channel exists between the refrigerator and an external energy source or sink. The performance of the refrigerator and its type are assessed in terms of the heat currents between the spins and their respective baths, and a lowering of temperature either in the steady state or during the transient dynamics can be observed via an increase in the ground-state population of the spin undergoing local cooling [15–19]. Along with theoretical proposals to implement these machines in various substrates such as quantum dots [22], circuit QED architectures [23], and atom-cavity systems [24], three-spin quantum refrigerators have recently been implemented in laboratories using trapped ions [25].

While the original model for the three-spin refrigerator exploits a three-body interaction among the spins constituting the working substance [15], it has been shown that one can construct a three-spin refrigerator with two-body interactions also [21], where the spin-spin interactions constitute the well-known XXZ model [26], thereby highlighting the possibility of building small quantum thermal machines using paradigmatic low-dimensional quantum spin models [27–31] of few spins. On one hand, it allows one to control the performance of these machines by appropriately tuning the parameters of the quantum spin Hamiltonian, which is now possible in experiments using the same substrates used for realizing thermal machines [32–39]. On the other hand, existing studies on the interface of the quantum information theory and quantum spin models [40–42] may prove useful in establishing the connection between quantum thermodynamics and quantum information theory. However, identifying appropriate spin Hamiltonian among numerous low-dimensional quantum spin models available in literature [30,31,43,44] to implement a quantum refrigerator remains a demanding task.

Another challenge in implementing a working quantum refrigerator using a quantum spin model in the laboratory would be disorder, since imperfections are inevitably present in the system [45–49]. A disordered system has two fundamental timescales—the observation time, τ, over which the system undergoes a dynamics and subsequent observation via a measurement, and the time τ' taken by the disordered parameter to attain its equilibrium. When $\tau' \gg \tau$, an effectively frozen disorder configuration during the observation time happens which can be incorporated by performing the average

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over configurations after computing the physical quantity of interest, known as quenched disordered averaging [50–53]. The realization of quantum spin models with disordered parameters being now possible in laboratories [54–57], it is natural to ask how the performance of quantum refrigerators, built out of quantum spin models, can alter in the presence of disorder in the system, which is a focus of the current paper.

In the present paper, we construct quantum refrigerators using a one-dimensional quantum spin chain consisting of two or three spin-$\frac{1}{2}$ particles, each of which is connected to a local Markovian bosonic thermal bath. We consider nearest-neighbor interactions among the spins, and examine a number of paradigmatic quantum spin Hamiltonians, namely, quantum XYZ [31,43,44] and quantum XY models with Dzyaloshinskii-Moriya (DM) interaction [58–61], as possible system Hamiltonians for a thermal machine to operate as a refrigerator where the latter model is chosen to introduce asymmetry in the system. More specifically, we focus on two main questions as to (1) whether a small quantum refrigerator built out of quantum spin systems always provides a significant cooling to a selected spin in terms of the population-dependent definition of local temperature (if the answer is positive, we focus on the identification of the parameter regimes to be tuned) and (2) whether the performance of the quantum thermal machine as a refrigerator remains unaffected in the presence of quenched disorder.

We answer both questions affirmatively in terms of heat current and local temperature of the selected spins, by considering the local as well as the global master equation. For the local master equation, we first notice that since the magnetic fields of the initial states are aligned to the z directions, the interaction strength in the z plane of the XYZ model has negligible effect on the refrigeration. We observe that when the couplings are weaker than the strengths of the magnetic fields, the refrigerator based on the XY model with DM interactions performs better than that of the XYZ model. Moreover, numerical simulations reveal a small subspace of the entire parameter space in which cooling of a selected spin can take place. Such a hierarchy remains unaltered when either the interaction strengths in the xy plane or the DM ones are chosen randomly from the Gaussian distribution. Notice that although they are demonstrated by fixing the strengths of the magnetic fields, the results remains true even for the large range of parameters. However, in this domain, the refrigerator described by a quantum spin Hamiltonian, ordered as well as disordered, does not ensure a significant cooling for a selected spin in terms of the local temperature of the spin. To overcome this, we go beyond the local master equation, and by employing the global master equation we illustrate that the local cooling provided by the ordered as well as disordered spin models can substantially be improved.

The rest of the paper is organized as follows. In Sec. II, we briefly introduce the construction of the three-spin quantum refrigerator by discussing the system Hamiltonians, the evolution of the system due to the interaction between the spins and the local Markovian bosonic baths, and the idea of local refrigeration of a selected spin during the dynamics of the system. In Sec. III, we present our results on the two-spin refrigerator using ordered as well as disordered systems while we demonstrate the results for the three-spin refrigerator in Sec. IV. Section V offers concluding remarks.

II. QUANTUM REFRIGERATOR: MODEL AND DYNAMICS

In this section, we briefly describe the quantum spin Hamiltonians used to implement a two-spin and a three-spin quantum refrigerator. The setup of the local thermal baths in contact with the individual spins, and the quantities that we have used for assessing the performance of the machine, are also discussed.

A. Interacting quantum spin models

We model the refrigerator as a one-dimensional quantum spin chain with N spin-1/2 particles, governed by a Hamiltonian, $H_S = H_F + H_I$. Here H_F and $H_I = H_{xy} + H_z + H_{\rm DM}$ correspond to the components of the system Hamiltonian H_S due to the local external magnetic fields acting on each spin, and the spin-exchange interactions between the spins, respectively. They are given by

$$H_F = \sum_{i=1}^{N} h_i \sigma_z^i, \quad (1)$$

$$H_{xy} = \sum_{i=1}^{N} J_{i,i+1}^{xy}\big[(1+\gamma)\sigma_x^i\sigma_x^{i+1} + (1-\gamma)\sigma_y^i\sigma_y^{i+1}\big], \quad (2)$$

$$H_z = \sum_{i=1}^{N} J_{i,i+1}^{z}\sigma_z^i\sigma_z^{i+1}, \quad (3)$$

$$H_{\rm DM} = \sum_{i=1}^{N} J_{i,i+1}^{\rm DM}\big(\sigma_x^i\sigma_y^{i+1} - \sigma_y^i\sigma_x^{i+1}\big). \quad (4)$$

Here γ is the xy anisotropy parameter, h_i is the strength of the local magnetic field acting on the spin i, σ_p^i ($p = x, y, z$) are Pauli matrices, $J_{i,i+1}^{xy}$ and $J_{i,i+1}^{z}$ respectively represent the xy and the zz nearest-neighbor antiferromagnetic interaction strengths, and $J_{i,i+1}^{\rm DM}$ denotes the strength of the Dzyaloshinskii-Moriya interaction [58–61]. Moreover, we consider interaction strengths to be site independent as well as site dependent, leading to the ordered and disordered spin systems, respectively. A number of paradigmatic quantum spin Hamiltonians emerged from H_S for different values of these system parameters as follows: (1) $J_{i,i+1}^{xy}, J_{i,i+1}^{\rm DM} = 0$-the classical Ising model in a parallel magnetic field; (2) $\gamma = 1$, $J_{i,i+1}^{z} = 0$, $J_{i,i+1}^{\rm DM} = 0$-the transverse-field Ising model; (3) $0 < \gamma < 1$, $J_{i,i+1}^{z} = 0$, $J_{i,i+1}^{\rm DM} = 0$-the anisotropic XY model in a transverse field; (4) $\gamma = 0$, $J_{i,i+1}^{z} = 0$, $J_{i,i+1}^{\rm DM} = 0$-the XX model in a transverse magnetic field; (5) $\gamma = 0$, $J_{i,i+1}^{\rm DM} = 0$-the XXZ model with magnetic field; and (6) $\gamma = 0$, $J_{i,i+1}^{z} = 0$-the XX model in a transverse magnetic field with DM interaction.

In this paper, we focus on small quantum refrigerators, where the size is justified by the low dimension of the Hilbert space of the system. More specifically, we consider a two- and a three-spin refrigerator ($N = 2, 3$) for demonstrating the results in the subsequent sections.

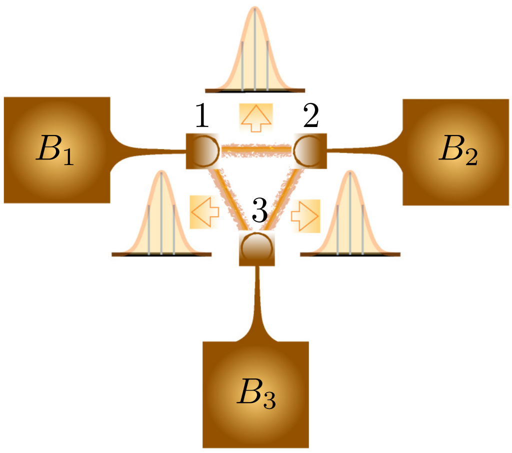


FIG. 1. A three-spin refrigerator in the presence of disorder. Three spin-1/2 particles are interacting with each other via spin-exchange interactions, while individually interacting with a local thermal heat bath. The spin-exchange interactions can be disordered, where the values of their strengths can be chosen from Gaussian distributions of fixed mean and standard deviations.

B. Local environments and the open quantum dynamics

We now describe the system-environment setup for implementing the quantum refrigerator. We consider N local heat baths, $B_1, B_2, \cdots, B_N$, each of which is connected to a spin in the N-spin system (see Fig. 1 for the $N = 3$ case), such that any spin is completely insulated from the effect of the $N-1$ baths, except the one connected to it. We assume that at $t = 0$, the spin-exchange interactions are absent, i.e., $H_S = H_F$, and each of the spins is at thermal equilibrium with its respective environment, so that the temperature $T_i(0)$ of the spin i at $t = 0$ is T_i^0, with T_i^0 being the absolute temperature of the bath i. The initial state of the system, therefore, is given by $\rho_s^0 = \bigotimes_{i=1}^N \rho_i^0$, where $\rho_i^0 = \exp(-\beta_i^0 h_i \sigma_z^i)/\text{Tr}[\exp(-\beta_i^0 h_i \sigma_z^i)]$, with $\beta_i^0 = (k_B T_i^0)^{-1}$, where k_B is the Boltzmann constant. At $t > 0$, all of the spin-exchange interactions or a subset of them are turned on, so that the system is taken out of equilibrium, and it undergoes an open system dynamics. The evolution of the state of the system, ρ_s, during this dynamics is described by a quantum master equation (QME) of the form

$$\dot{\rho}_s = -\frac{i}{\hbar}[H_S, \rho] + \mathcal{D}(\rho), \tag{5}$$

where $\mathcal{D}(.)$ represents the dissipator, emerging due to the spin-bath interaction. The state of the system, $\rho_s(t)$, as a function of t is obtained as the solution of the QME.

We consider each of the local thermal baths B_i to be a collection of harmonic modes with a Hamiltonian $H_b = \int_0^{\omega_m} d\omega a_\omega^\dagger a_\omega$, where a_ω ($a_\omega^\dagger$) is the annihilation (creation) operator corresponding to the harmonic mode of energy ω, obeying $[a_\omega, a_{\omega'}^\dagger] = \delta(\omega - \omega')$, and ω_m is the maximum ω. The total interaction between the spins and their corresponding baths is represented by the Hamiltonian $H_{sb} = \sum_{i=1}^N \sum_\omega (\sigma_i^+ \otimes a_\omega + \sigma_i^- \otimes a_\omega^\dagger)$, where σ_i^+ and σ_i^- are the raising and lowering operators of the ith spin, respectively. The dynamical term in the QME [Eq. (5)] takes the form [62] $\mathcal{D}(\rho) = \sum_{i=1}^N \mathcal{D}_i(\rho)$, with

$$\mathcal{D}_i(\rho) = \Gamma_i\Big[\big(n_\omega^i + 1\big)\big(\sigma_i^- \rho \sigma_i^+ - \tfrac{1}{2}\{\sigma_i^+\sigma_i^-, \rho\}\big) + n_\omega^i\big(\sigma_i^+ \rho \sigma_i^- - \tfrac{1}{2}\{\sigma_i^-\sigma_i^+, \rho\}\big)\Big], \tag{6}$$

in the case of the Markovian spin-bath interactions at the strict weak-coupling limit given by $h_i, \Gamma_i \gg \max\{J_{i,i+1}^{xy}, J_{i,i+1}^z, J_{i,i+1}^{\text{DM}}\}$. In Eq. (6), n_ω^i is the occupation number of the Bose-Einstein distribution corresponding to bath B_i given by $n_\omega^i = (e^{\hbar\omega/k_B T_i^0} - 1)^{-1}$, with $\omega = 2\hbar h_i$, and Γ_i being a constant. Note that the Lindblad operators represented by $\sigma_i^\pm$ here signify local transitions among the eigenstates of the subsystem i, and the QME in such situations belongs to the class of local master equations. It is also important to note that in such scenarios, a violation of the second law of thermodynamics may take place, implying that a local quantum master equation may not always be appropriate to describe the stationary nonequilibrium properties of the system (see Refs. [63–65]). Therefore, in the case of the local quantum master equation, the results should be interpreted carefully, and there have been proposals for rectifying this issue by constructing the master equation in a different fashion [66].

On the other hand, in the strong-coupling limit, the spin-interaction strengths are comparable to the strengths of the local magnetic fields, and the dynamical term corresponding to spin i in Eq. (5) takes the form [20]

$$\mathcal{D}_i(\rho) = \sum_{\omega>0} \gamma_i^\omega \Bigg[\Bigg(A_\omega^i \rho A_\omega^{i\dagger} - \frac{1}{2}\{A_\omega^{i\dagger} A_\omega^i, \rho\}\Bigg) + \Bigg(A_\omega^{i\dagger} \rho A_\omega^i - \frac{1}{2}\{A_\omega^i A_\omega^{i\dagger}, \rho\}\Bigg)\Bigg], \tag{7}$$

where the operator A_ω^i, given by

$$e^{iH_S t}(\sigma_i^+ + \sigma_i^-)e^{-iH_S t} = 2\sum_\omega A_\omega^i e^{-i\omega t}, \tag{8}$$

is the Lindblad operator on the spin i corresponding to the transition of energy ω among the energy levels of the system, and is derived by decomposing the spin part of H_{sb} in the eigenbasis of H_S. Note that in contrast to the previous case of the local master equation, the Lindblad operators here correspond to the transitions among the eigenstates of the entire system, and the QME in this situation is a global one. The coefficient γ_i^ω is the transition rate corresponding to the energy gap ω for the spin i, where

$$\gamma_i^\omega = f_i(\omega)[1 + \kappa_i(\omega)], \quad \text{for} \quad \omega \geqslant 0,$$
$$\gamma_i^\omega = f_i(|\omega|)\kappa_i(|\omega|), \quad \text{for} \quad \omega < 0, \tag{9}$$

with $f_i(\omega) = \alpha_i \omega e^{-\frac{\omega}{\Omega}}$, with Ω being the cutoff frequency and $\kappa_i(\omega) = (e^{\hbar\beta_i\omega} - 1)^{-1}$ representing the Ohmic spectral function and the Bose-Einstein distribution, respectively. Here, α_i is a constant for the bath, i, quantifying the strength of the spin-bath interaction strength. In order for the Markovian approximation to be valid, we restrict the values of α_i such that $\max\{\alpha_i\} \ll 1$. Here, the second law of thermodynamics is always valid. However, care must be taken while constructing quantities that are local to a subsystem of the quantum spin model. We shall elaborate on this in Sec. IV C.

C. Local refrigeration

If the N-spin system operates as a refrigerator for the spin i, then the heat current,

$$\dot{Q}_i = \mathrm{Tr}[H_S \mathcal{D}_i(\rho_s)], \tag{10}$$

corresponding to the spin i in the steady state is positive [1,20,21]. This represents a situation where heat flows from the bath B_i to the spin i, which is at a lower temperature than T_i^0 in the steady state. This can also be visualized by defining a local temperature for the spin i [15] as follows. At $t = 0$, the initial state of the ith spin is a diagonal state, which can be written in the eigenbasis of σ_z, $\{|0\rangle, |1\rangle\}$, having eigenvalues 1 and -1 respectively, as $\rho_i^0 = \tau_i^0|0\rangle\langle 0| + (1-\tau_i^0)|1\rangle\langle 1|$, where $\tau_i^0 = \exp(-2\beta_i^0 h_i)/[1+\exp(-2\beta_i^0 h_i)]$. During the dynamics, the forms of the Lindblad operators (see Sec. II B) ensure that the single-spin density matrix

$$\rho_i(t) = \mathrm{Tr}_{\substack{j,k(\neq i)\\ j,k=1,2,3}}[\rho_s(t)], \tag{11}$$

at every time instant t, remains diagonal, i.e., $\rho_i(t) = \tau_i(t)|0\rangle\langle 0| + [1-\tau_i(t)]|1\rangle\langle 1|$, while $\tau_i(t)$ varies with time starting from $\tau_i(0) = \tau_i^0$. It allows us to define a local temperature of the spin i as

$$T_i(t) = \frac{2h_i}{\ln[\tau_i(t)^{-1} - 1]} \tag{12}$$

at every time t, which is in agreement with the initial temperature $T_i(0)$ of the spin i to be equal to T_i^0.

A local steady-state cooling of the spin i is achieved if

$$T_i^s = T_i(t \to \infty) < T_i^0 \tag{13}$$

at any $t > 0$. Note, however, that as of now, no specific correlation between the values of $\dot{Q}_i$ and T_i^s exists as we will also show here. In the subsequent sections, we demonstrate the status of the local refrigeration of a spin in the (two-) three-spin system via the heat current as well as the local temperature corresponding to the chosen spin, by appropriately tuning the system as well as the spin-bath interaction parameters.

III. TWO-SPIN QUANTUM REFRIGERATOR: ORDER VS DISORDER

We begin our discussion with a two-spin refrigerator model (see Fig. 1 where the third spin and its corresponding bath, B_3, are absent), where we focus on the local refrigeration of a chosen spin in the system. For the purpose of demonstration, we choose spin 1 to be cooled, although the system as well as the environment parameters can be chosen appropriately to locally cool any one of the spins. To ensure that the two-spin thermal machine operates as a refrigerator for the spin 1, we exhibit $\dot{Q}_1 > 0$ as well as $T_1^s < T_1^0$ by properly tuning the parameter values. Note that maintaining $\dot{Q}_1 > 0$ alone describes a situation that includes all the operating regimes (see Ref. [21] for the three-spin refrigerator) corresponding to the two-spin thermal machine that refrigerates spin 1.

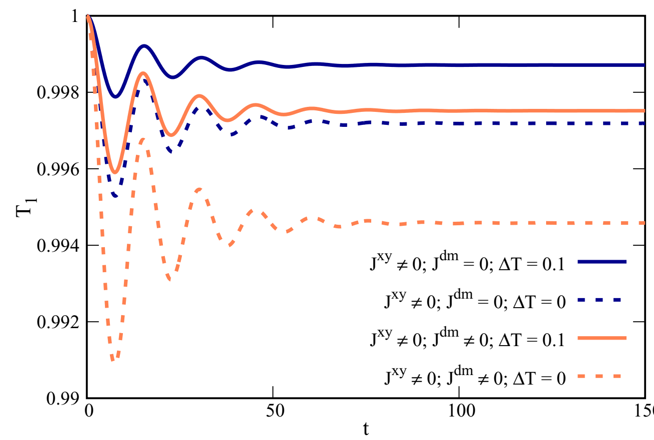


FIG. 2. Temperature dynamics for spin 1 of a two-spin refrigerator in the weak-coupling limit: variation of T_1 (ordinate) vs t (abscissa). The initial temperatures of the two spins are $T_1(0) = 1$, $T_2(0) = 1.1$ (solid lines), and $T_1(0) = T_2(0) = 1$ (dashed line). Dark (red) lines represent the XX model with $J^{xx} = 0.02$ while light (orange) lines are for the XX model with DM interactions where $J^{xx} = J^{\mathrm{DM}} = 0.02$. In both cases, we fix $h_1 = 1.1$, $h_2 = 1.3$, $\Gamma = 0.05$, and $\gamma = 0$. Both the axes are dimensionless.

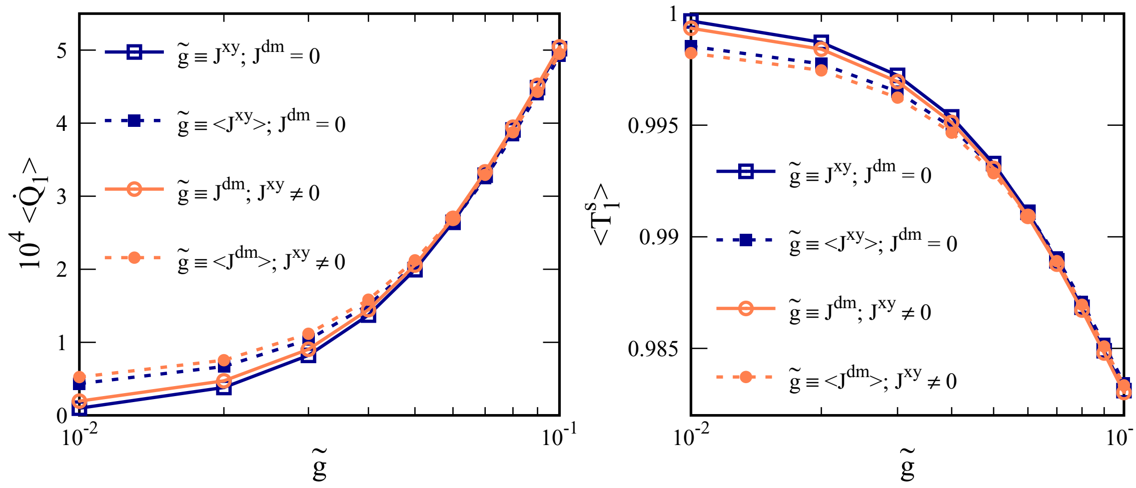
A. Ordered spin models as refrigerators

1. Transverse XY model

Let us first consider XY-type spin-exchange interaction between the spins so that $H_S = H_F + H_{xy}$ for $N = 2$ [see Eqs. (1) and (2)], where we set $\gamma = 0$ for demonstration. Solving Eq. (5) for the two-spin refrigerator model via the local master equation, followed by the calculation of the local density matrix for spin 1, leads to the local temperature of spin 1 as $T_1(t) = 2h_1/\ln[\sigma_{11}(t)^{-1} - 1]$ (see the Appendix). Notice that when H_S represents a classical Ising model in a parallel magnetic field and the initial state of the system is a diagonal one, the system does not evolve under the local master equation, implying that a local refrigeration of spin 1 is absent. Note also that under the strict weak-coupling limit (see Sec. II B) where the spin interactions are negligible compared to both the local magnetic fields and the dissipation rates, our numerical analysis does not find any point in the parameter space for which a local cooling for spin 1 can take place. This motivates us to relax the weak-coupling condition as $h_i > J^{xy} \sim \Gamma_i$ (see Ref. [18]), where significant subspace in the parameter space of the system is found where the designed refrigerator demonstrates cooling in spin 1. This is a feature valid for both two- and three-spin refrigerators, and from now onward, unless otherwise mentioned, we use the relaxed weak-coupling condition in terms of appropriate spin-interaction strengths (i.e, a subset of $\{J^z, J^{xy}, J^{\mathrm{DM}}\}$) to investigate the performance of refrigerators.

The observations obtained for the two-spin refrigerator modeled via a spin system other than the classical Ising model are the following.

(1) A nonzero XY interaction strength, J^{xy}, results in an evolution of the system, leading to a local cooling of spin 1, irrespective of the value of J^z. In Fig. 2, the dynamics of the local temperature of spin 1 in a two-spin refrigerator is depicted, thereby demonstrating a local steady-state cooling.



(a) (b)

FIG. 3. Variation of heat current and steady-state temperature (vertical axis) as functions of the strength of the spin-exchange interactions (horizontal axis). (a, b) Heat current and temperature of spin 1 with increasing XX interactions (squares) where $J^{\mathrm{DM}} = 0$ and with the increase of DM interactions, J^{DM} (circles) having $J^{xy} = 0.02 \neq 0$. Hollow and solid symbols (squares as well as circles) represent ordered and disordered spin models, respectively. Other parameter of the systems, namely, magnetic-field strengths and the spin-bath interactions, are chosen as $h_1 = 1.1$, $h_2 = 1.3$, and $\Gamma = 0.05$, and the initial temperature of each spin is $T_1(0) = 1$ and $T_2(0) = 1.1$ respectively. Here $\gamma = 0$. All the axes are dimensionless.

(2) Interestingly, we find that even when $\Delta T = T_2^0 - T_1^0 = 0$, a steady-state cooling occurs where an energy bias is given to the system in terms of two unequal strengths of the magnetic field to the individual spins. More importantly, we report that vanishing ΔT proves to be more advantageous with respect to cooling than nonvanishing ΔT (see Ref. [67]) if we suitably adjust the parameters of H_s and the spin-bath interaction strength (comparing solid and dashed lines of Fig. 2).

(3) The heat current (the steady-state temperature) remains almost constant when the strength of the spin-exchange interaction is $\leqslant 10^{-2}$, and increases with an increase in the value of J^{xy} within the weak-coupling limit ($\leqslant 10^{-1}$), irrespective of the presence of the interactions in the z plane, i.e., independent of the values of J^z.

The variation of the heat current and the steady-state temperature of spin 1 against the strength of the spin-exchange interaction J^{xy} is depicted in Figs. 3(a) and 3(b).

Remark 1. The amount of steady-state cooling achieved in the two-spin refrigerator is very small in magnitude, and it possibly indicates that one has to go beyond the local master equation to achieve a significant steady-state cooling of spin 1.

Remark 2. The trend remains unchanged for $\gamma \approx 0$, with negligible effect on the amount of steady-state cooling attained during the refrigeration of spin 1. On the other hand, when $\gamma \to 1$, the performance of the refrigerator diminishes. Hence the entire analysis in the rest of the paper is performed for the spin model with $\gamma = 0$.

2. *Transverse XY model with DM interaction*

To answer the question as to whether a change in the type of the spin-exchange interaction between the two spins affects the performance of the two-spin refrigerator, we add an asymmetric spin-spin interaction, specifically, the DM interaction in the system Hamiltonian, i.e., $H_s = H_{xy} + H_{\mathrm{DM}}$. We explore the behaviors of $\dot{Q}_1$ and T_1^s as functions of J^{DM}, where J^{xy} is kept fixed.

Our analysis clearly indicates that the qualitative behaviors of both the quantities, the heat current as well as the steady-state temperature observed in the XX model, remain the same even in the presence of DM interactions although the slight improvement in terms of cooling can be seen in the presence of asymmetric DM interactions, especially when the coupling constant is weak (of the order of 10^{-2}) (see Fig. 3). The local temperature dynamics of spin 1 is shown in Fig. 2, while the variation of the heat current and the steady-state temperature of spin 1 with increasing J^{DM} is plotted in Fig. 3.

B. Robustness in a disordered two-spin refrigerator

Let us now determine the response of the performance of the machine against disorder in the two-spin refrigerator model. As mentioned in Sec. II A, impurities are introduced in this model by choosing random spin-exchange interaction strengths, g, from a Gaussian distribution with a mean $\langle g \rangle$ and standard deviation σ_g, keeping the values of the local magnetic fields fixed. In this paper, either J^{xy} or J^{DM} is chosen to be random, by keeping the other coupling constants ordered. Notice that a vanishing standard deviation reduces to a perfectly ordered system discussed above.

For each random parameter configuration constituted of a random value of the spin-exchange interaction strength corresponding to a random realization of the system, one can compute the quantities of interest, and subsequently take an average of the quantity over a statistically large number of parameter configurations, known as *quenched* averaging of the physical quantity. Mathematically, the quenched averaging

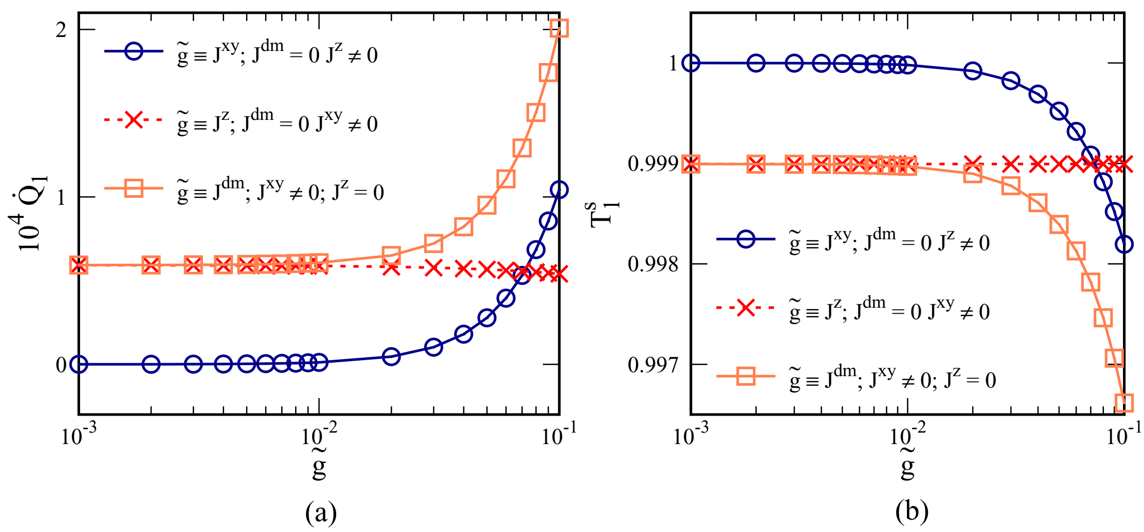


FIG. 4. Three-spin refrigerator: (a) $\dot{Q}_1$ and (b) T_1^s as functions of different spin-exchange interaction strengths where $g_{i,i+1} = g\ \forall i \in [1, 2, 3]$. The other relevant parameters, which are kept constant, are chosen as follows. For $\tilde{g} \equiv J^{xy}$, $J^z = 0.019$ and $J^{\rm DM} = 0$ (circles). When $\tilde{g} \equiv J^z$, $J^{xy} = 0.073$ and $J^{\rm DM} = 0$ (crosses), while, for $\tilde{g} \equiv J^{\rm DM}$, $J^{xy} = 0.073$ and $J^z = 0$ (squares). In all these cases, the local magnetic fields corresponding to the individual spins are fixed to $h_1 = 1.11$, $h_2 = 2.82$, and $h_3 = 3.65$, and the values of the spin-bath interaction parameters are $\Gamma_1 = 0.0639$, $\Gamma_2 = 0.0984$, and $\Gamma_3 = 0.0673$. All the axes are dimensionless.

of a physical quantity, $\mathcal{Q}$, can be represented as

$$\langle \mathcal{Q}(\langle g\rangle, \sigma_g)\rangle = \int \mathcal{P}(g)\mathcal{Q}(g)d(g), \tag{14}$$

where g is the parameter the values of which are chosen from a Gaussian distribution $[\mathcal{P}(g)]$ of mean $\langle g\rangle$ and standard deviation σ_g quantifying the strength of the disorder. Note that no restrictions on the possible values of the exchange interactions are imposed in order to keep the two-spin thermal machine operating in a specific working region, and a change in the values of the system parameters may, in principle, shift the two-spin thermal machine from one working region like an absorption refrigerator to another such as an external source driven thermal machine.

We investigate the patterns of quenched averaged heat current, $\langle \dot{Q}_1\rangle$, and steady-state temperature, $\langle T_1^s\rangle$, with the increase of $\langle J^{xy}\rangle$ or $\langle J^{\rm DM}\rangle$ where the averaging is performed over 2×10^3 realizations by keeping the value of the strength of disorder fixed at 2×10^{-2}. As shown in Fig. 3, we demonstrate that for small $\langle J^{xy}\rangle$ ($\langle J^{\rm DM}\rangle$), the quenched steady-state temperature (the quenched heat current) is smaller (higher) than that obtained via an ordered spin model as a refrigerator. It is also clear from the figure that the overall performance of the refrigerator remains qualitatively as well quantitatively similar in the presence of any amount of disorder in exchange interactions, thereby establishing a *robustness* of the refrigerator model against impurities.

These results provide a certain insight into how a small quantum refrigerator may behave when designed using a low-dimensional quantum spin Hamiltonian, and when disorder is present in the system. However, it is not clear whether these trends remain the same when one considers the traditional three-spin refrigerator. We explore this in the next section.

IV. THREE-SPIN REFRIGERATOR BASED ON THE QUANTUM SPIN MODEL

In order to check whether the results of the two-spin refrigerator remain qualitatively valid also for the widely studied three-spin refrigerator, we first explore the case of identical spin-exchange interactions between all spins, i.e., $g_{i,i+1} = g\ \forall i \in [1, 2, 3]$, where g stands for different types of spin-exchange interactions (see Secs. II A and III B). For brevity, we denote $J_{i,i+1}^{xy} = J^{xy}$, $J_{i,i+1}^{z} = J^z$, and $J_{i,i+1}^{\rm DM} = J^{\rm DM}$ for all i.

Unless otherwise stated, we assume the constraint $T_1^0 \leqslant T_2^0 \leqslant T_3^0$ for the bath temperatures, and always choose their values as $T_1^0 = 1$, $T_2^0 = 2$, $T_3^0 = 3$ for demonstration. By fixing the strengths of the magnetic fields, we study the response of the machine on the local cooling phenomena, specifically in terms of $\dot{Q}_1$ as well as T_1^s, when interaction strengths J^{xy}, J^z, and $J^{\rm DM}$ are varied in the range $[10^{-3}, 10^{-1}]$ (see Fig. 4). Notice that a stark difference between the two- and the three-spin refrigerators is that for the latter, there are possibilities to choose different interaction strengths between spins, i and $i+1$, $i = 1, 2, 3$. In this paper, we take them to be site independent although site dependence does not substantially effect the cooling procedure as we will see in the succeeding subsection.

A. Role of interaction strength on refrigeration

The observations for the three-spin refrigerators are quite similar to the two-spin ones and can be divided into three categories: (1) increase of J_z while $J^{xy} \neq 0$, $J^{\rm DM} = 0$; (2) variations of J^{xy} with fixed J^z and $J^{\rm DM} = 0$, leading to the XYZ refrigerator; and (3) change of $J^{\rm DM}$ by fixing J^{xy} with $J^z = 0$ which can be referred as the XY DM refrigerator. In the first case, the presence of a nonzero xy interaction in the system results in a slow variation of $\dot{Q}_1$ with J^z, while the corresponding change in the steady-state temperature T_1^s

of spin 1 is vanishing [see Fig. 4(b) for the behavior of T_1^s corresponding to the data presented in Fig. 4(a)]. The increase (decrease) of $\dot{Q}_1$ (T_1^s) becomes more prominent in the second and the third scenarios. As pointed out in the case of the two-spin refrigerator, the refrigeration can be improved by varying DM interaction strength compared to the XXZ refrigerator as depicted in Fig. 4. In all these calculations, we fix $\gamma = 0$ in H_{xy} (i.e., the XX model) since our data suggest that a nonzero value of γ in the neighborhood of the XX model has no significant effect on the refrigeration of spin 1 and the performance of the refrigerator degrades with the increase of γ.

As it is clear from Figs. 4(a) and 4(b), there is little or no variation of $\dot{Q}_1$ and T_1^s as a function of the spin-exchange interactions, when the interaction strength is $\leqslant 10^{-2}$. Beyond 10^{-2}, the variations of $\dot{Q}_1$ and T_1^s increase with increasing the spin-exchange interaction strength. Also, it is important to note that in the strictly weak-coupling regime, the local refrigeration obtained in spin 1 is negligible, although the three-spin machine operates in the refrigerator region for spin 1. These findings suggest that in order to obtain a significant cooling in terms of the temperature of spin 1, one needs to explore beyond the local master equation, as was also indicated by the results on the two-spin refrigerator. To investigate whether significant cooling can be found beyond this local master equation domain, we relax the weak-coupling condition to $h_i > \max\{J^{xy}, J^z, J^{\rm DM}\}$, and find that a considerable steady-state cooling may indeed be present in such situations. See Fig. 7 for a typical example, where we have set $J^{xy}, J^z \neq 0$, and $J^{\rm DM} = 0$.

B. Connecting heat current with local temperature in a three-spin model based refrigerator

Let us here address the question of whether a high positive value of $\dot{Q}_1$ always implies a low value of steady-state temperature in a specific spin. To demonstrate it, we choose 10^4 random parameter configurations of the three-spin refrigerator, where the system Hamiltonian is represented by $H_S = H_F + H_{xy} + H_z$, and we assume $g_{i,i+1} = g \ \forall i \in [1, 2, 3]$, where $g \equiv J^{xy}, J^z$. The random values of the spin-exchange interaction strengths, and the spin-bath coupling strengths Γ_i, $\forall i \in [1, 2, 3]$, are chosen from a uniform distribution within $[0, 10^{-1}]$. In the scatter diagram presented in Fig. 5, each point represents a three-spin thermal machine performing local refrigeration for spin 1, which is indicated by $T_1^0 - T_1^s > 0$ and $\dot{Q}_1 > 0$. It is clear from the corresponding amounts of the steady-state cooling that no specific correlation exists between $T_1^0 - T_1^s$ and $\dot{Q}_1$. Specifically, a very low value of heat current can lead to a substantially low steady-state temperature and vice versa. Note also that only about 4.11% of the 10^4 randomly chosen points result in $\dot{Q}_1 > 0$, which remains almost unchanged even in the presence of an additional DM term in H_S (in this case, the percentage is 3.25%). It again indicates the scarcity of a working three-spin refrigerator providing a significant amount of cooling by considering the local master equation, which indicates the importance of identifying the subspace in the entire parameter space for designing a small quantum refrigerator using the chosen quantum spin models.

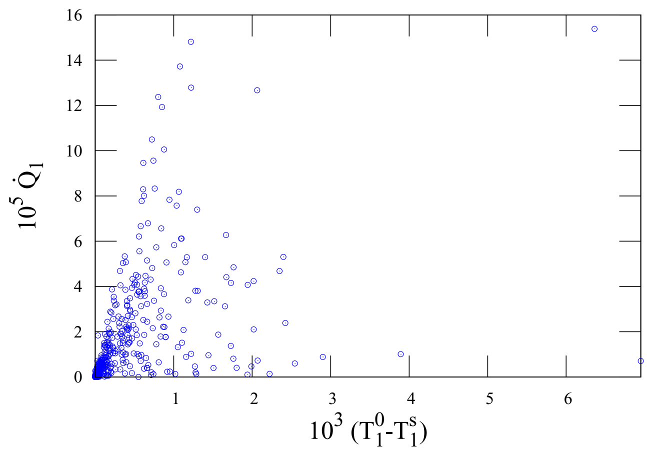


FIG. 5. Scattered plot of $\dot{Q}_1$ (ordinate) against $T_1^0 - T_1^s$ (abscissa) of the three-spin XXZ refrigerator. The values of the local magnetic fields, $\{h_1, h_2, h_3\}$, corresponding to the individual spins are chosen uniformly from [1.1,5] while the values of the spin-bath interaction parameters $\{\Gamma_1, \Gamma_2, \Gamma_3\}$ as well as the spin-exchange interaction strengths $\{J^{xy}, J^z\}$ are chosen from a uniform distribution of range $[0, 10^{-1}]$. Here $T_1^0 = 1$, $T_2^0 = 2$, and $T_3^0 = 3$. Among 10^4 choices of parameters, only 4.11% points are displayed for which local temperature of the first spin is lower than unity. Results indicate that there is no monotonic relation between them. Both the axes are dimensionless.

C. Disorder-enhanced refrigeration in three-spin systems

We will now examine how impurities arising naturally in the spin model affect the refrigeration. To incorporate impurities in this three-spin refrigerator model, interaction strengths, i.e., $J^{xy}_{i,i+1}$ and $J^{\rm DM}_{i,i+1}$, are taken to be site dependent and are chosen randomly from the Gaussian distribution with mean, $\langle J^{xy}\rangle$ and $\langle J^{\rm DM}\rangle$, having standard deviation $\sigma_{J^{xy}}$ and $\sigma_{J^{\rm DM}}$, respectively. The magnetic fields are fixed to the same value mentioned in the ordered case (see Fig. 4). Finally we compute the quenched averaged heat current, $\langle\dot{Q}_1\rangle$, and quenched steady-state temperature, $\langle T_1^s\rangle$, of spin 1 by averaging over 2×10^3 random configurations for a given strength of the disorder. Both with the random XY as well as DM interaction strength, i.e., for a given $\langle J^{xy}\rangle$ or $\langle J^{\rm DM}\rangle$ and their corresponding $\sigma_{J^{xy}}$ or $\sigma_{J^{\rm DM}}$, we report that

$$\langle\dot{Q}_1\rangle > \dot{Q}_1 \quad \text{and} \quad \langle T_1^s\rangle < T_1^s, \tag{15}$$

which establishes the *disorder-induced thermal device* although the increase (decrease) of heat current (temperature of the first spin) is small. It should be noted that although in Figs. 6(a) and 6(b), we depict the enhancement of the cooling feature by using the disordered three-spin refrigerator over its ordered counterparts by choosing exemplary values of magnetic fields and other interaction strengths, the characteristics remain the same even for another range of parameters in the local master equation. Therefore, as argued in the case of the two-spin refrigerator, our analysis clearly indicates that the spin model as a thermal machine is robust against impurities.

A comment on the significance of the enhancement of the cooling phenomena in the disordered refrigerator is in

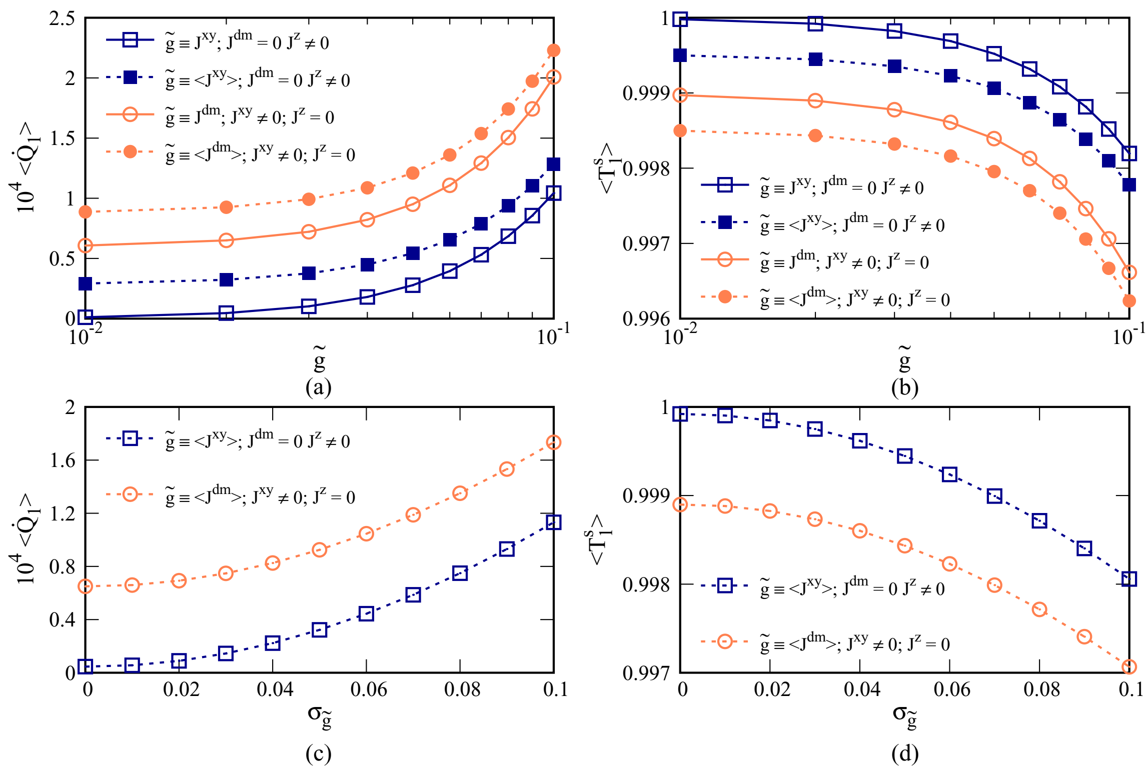


FIG. 6. Ordered vs disordered spin models as refrigerators. (a, b) For $\tilde{g} \equiv \langle J^{xy} \rangle$, $J^z = 0.019$ and $J^{\rm DM} = 0$ with $\sigma_{J^{xy}} = 5 \times 10^{-2}$ (dashed line with solid squares). When $\tilde{g} \equiv \langle J^{\rm DM} \rangle$, $J^{xy} = 0.073$ and $J^z = 0$ while $\sigma_{J^{\rm DM}} = 5 \times 10^{-2}$ (dashed line with solid circles). The quenched averaging is performed over 2×10^3 random configurations, chosen from Gaussian distribution with mean $\tilde{g}$ and standard deviation, $\sigma_{\tilde{g}}$. A similar set of parameters is also used for the ordered system (hollow circles and squares). All other specifications are the same as in Fig. 4. (c, d) $\langle \dot{Q}_1 \rangle$ and $\langle T_1^s \rangle$ with varying strength of disorder, $\sigma_{\tilde{g}}$ and $J^{\rm DM} = 0.02$. Other specifications are similar to (a) and (b). All the axes are dimensionless.

order here. For brevity of the notation, let us again denote the disordered spin-interaction strength by g, where in the present paper we choose g to be either J^z or J^{xy} (see also Sec. III B, and Figs. 3, 4, 6, and 7). Let us denote by g_0 the value of g for which

$$\dot{Q}_1(g_0) = \max \dot{Q}(g),$$
$$T_1^s(g_0) = \min T_1^s(g),$$

where the maximization and minimization are performed over the entire range of g satisfying the weak-coupling constraint, and by definition, $\langle \dot{Q}_1 \rangle \leqslant \dot{Q}_1(g_0)$ and $\langle T_1^s \rangle \geqslant T_1^s(g_0)$. This can interpret the results reported in Figs. 6(c) and 6(d) as being far from the optimal value g_0 of g. Note, however, that under the local master equation, $\dot{Q}_1$ (T_1^s) increases (decreases) monotonically with g, and g_0 is the point $g_0 = 10^{-1}$ in the chosen range of g. While finding $\langle \dot{Q}_1 \rangle \leqslant \dot{Q}_1(\langle g \rangle)$ [$\langle T_1^s \rangle \geqslant T_1^s(\langle g \rangle)$] is likely for such monotonically increasing (decreasing) behavior of $\dot{Q}_1$ (T_1^s) when $\langle g \rangle$ is far from g_0, such straightforward predictions cannot be made for quantities that vary nonmonotonically with g. This highlights the importance of investigating the possibility of enhancement (decrease) in the value of $\dot{Q}_1$ (T_1^s).

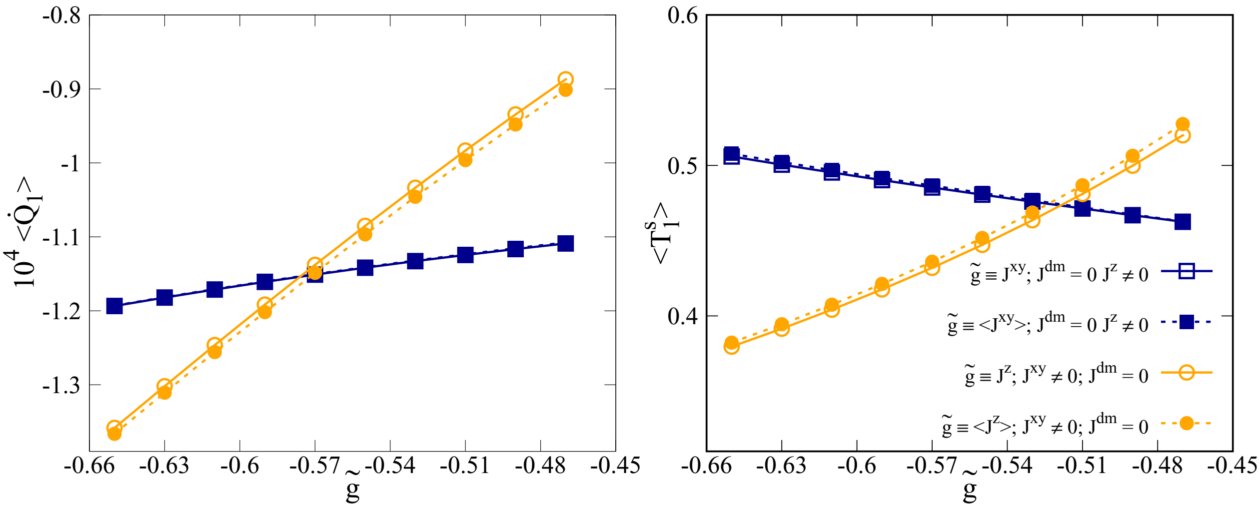
1. Effects of strength of disorder on refrigeration

To probe further, let us check the role of the magnitude of the disorder on the observed robustness. We systematically increase the value of the disorder strength up to 10^{-1}, and observe that with increasing strength of the disorder, the average value of the heat current of the first spin attains a more positive value, while the steady-state temperature becomes lower [see Figs. 6(c) and 6(d)] than that of the model with low disorder strength. It clearly exhibits an advantage to attain a lower steady-state temperature of the refrigerated spin in the presence of disorder where one is forced to operate a small quantum thermal machine made of three spins as a refrigerator.

2. Beyond the weak-coupling limit

All the results obtained until now strongly pinpoint that spin-exchange interaction strength beyond the weak-coupling limit aids in attaining a lower steady-state temperature of the refrigerated spin. This poses the natural question as to whether a quantum refrigerator in the strong-coupling domain performs advantageously to obtain an even lower steady-state temperature. It is also logical to ask whether the robustness of the three-spin refrigerator against disorder remains unaltered in the strong-coupling regime. Our numerical study of the three-spin refrigerator in the strong-coupling limit using the global master equation, as described in Sec. II B, answers both the questions positively.

Both in ordered as well as disordered scenarios, we find that the steady-state temperature and the corresponding quenched averaged temperature of the first spin can substantially be decreased in the strong-coupling domain compared to



(a) (b)

FIG. 7. Study of refrigeration with a global master equation. (a) $\dot{Q}_1$ for the ordered spin model and $\langle \dot{Q}_1 \rangle$ for the disordered ones vs $\tilde{g}$. (b) Steady-state temperature and its quenched averaged one with varying interaction strengths. Both for disordered and ordered situations, when $\tilde{g} \equiv J^{xy}$ or $\langle J^{xy} \rangle$, $J^z = -0.55$ and $J^{\mathrm{DM}} = 0$ (solid squares for disordered and hollow squares for ordered), while, when $\tilde{g} \equiv J^z$ or $\langle J^z \rangle$, $J^{xy} = -0.4$ and $J^{\mathrm{DM}} = 0$ (solid circles and hollow circles for disordered and ordered, respectively). Initial temperatures are the same as in other three-spin refrigerators. Here $h_1 = 0.1$, $h_2 = 1.5$, and $h_3 = 1.4$, and $\alpha_1 = 10^{-4}$, $\alpha_2 = 10^{-3}$, and $\alpha_3 = 10^{-2}$. In the disordered case, averaging is performed over 5×10^2 configurations. All the axes are dimensionless.

that obtained in the weak-coupling limit. In Fig. 7, the patterns of the steady-state temperature T_1^s as well as $\langle T_1^s \rangle$ by varying the corresponding interaction strengths, J^{xy} or J^z, are depicted by fixing local magnetic fields of all the spins comparable to the coupling constants. Note here that due to numerical limitations, we perform here quenched averaging over 5×10^2 configurations. In this regime also, we exhibit that effects of randomness in interaction strengths on the physical quantities quantifying the performance of the thermal machine are not significant, thereby supporting our claim of the robustness of the quantum refrigerator against quenched disorder.

While the robustness of local cooling in the disordered refrigerator is a common feature in both local and global master equations, an interesting difference between these two situations emerges from Fig. 7. Note that in the ordered case, a lower steady-state temperature for spin 1 can be obtained by varying J^z for a fixed value of J^{xy}, compared to the situation when J^{xy} is varied keeping J^z fixed. The situation alters after a certain threshold value of the varying parameter.

A higher enhancement of cooling, in terms of both heat current as well as local temperature of spin 1, is also obtained when disorder is present in J^z, compared to when J^{xy} is disordered. These observations indicate that J^z occasionally outperforms J^{xy} in enhancing the performance of the refrigerator. In the same context, note that the results reported on the weak-coupling range of the spin-interaction strengths remain invariant under changing the value of J^z from a zero to a nonzero value. However, under the global master equation, the performance of the refrigerator depends qualitatively (i.e., in terms of presence or absence of cooling) as well as quantitatively (i.e., in terms of the amount of cooling obtained) on the value of J^z. This is justified by the result that for a fixed nonzero value of J^{xy} (for instance, when $-0.65 \leqslant J^{xy} \leqslant -0.45$), the system may also exhibit a steady-state heating of spin 1 at $J^z = 0$, and a local cooling of spin 1 starts to appear only when $J^z \leqslant J_c^z$, where J_c^z is a critical value of J^z that depends on the chosen value of J^{xy}.

Before concluding, let us point out that the heat current for spin 1 in the strong-coupling scenario is negative, which is in contrast to a positive heat current expected for a spin, undergoing a local cooling. Note that the strong-coupling scenario corresponds to a global approach of constructing the quantum master equation (see Sec. II B). In view of this, one needs to be careful in defining the heat current, since a definition in terms of the local Hamiltonian, given by $\dot{Q}_i = \mathrm{Tr}[H_F^i \mathcal{L}_i(\rho)]$, where H_F^i and $\mathcal{L}_i(\rho)$ are, respectively, the local Hamiltonian and the dissipating term corresponding to the subsystem i, may not be appropriate for the validity of the balance equation given by

$$\Delta = \frac{dS}{dt} - \sum_i \frac{Q_i}{k_B T_i}, \tag{16}$$

which, in turn, ensures the validity of the second law of thermodynamics [63–65,67]. Here, Δ and S, respectively, are the entropy production rate and the entropy of the system, Q_i is the heat flow from the system to the ith bath, k_B is the Boltzmann constant, and T_i is the absolute temperature of the bath i. This implies that the determination of $\dot{Q}_i$ requires a careful analysis (see, for example, Ref. [68]), and in an effort to avoid the inconsistency arising from defining the heat currents using the local Hamiltonian, we have used the full system Hamiltonian H_S, including both the local and the interaction parts, to define the heat current as $\dot{Q}_i = \mathrm{Tr}[H_S \mathcal{L}_i(\rho)]$. It is important to stress here that although one is interested in the local properties of the refrigerator, in a global approach, the dynamics of the system is determined as a whole, and extracting information about a specific subsystem is nontrivial due to the strong interactions between individual subsystems.

However, this does not affect the main thesis of this paper, since local cooling of spin 1 is seen in both cases of the local and global master equation approach.

V. CONCLUSION

A potential method to build small scale quantum thermal machines is via quantum spin models which can be implemented by using physical substrates like trapped ions and neutral atoms in optical lattices. We chose this avenue to design quantum refrigerators consisting of two and three spins based on the nearest-neighbor quantum XYZ model as well as the quantum XY model with DM interactions. The initial state of the device is prepared in the thermal equilibrium states of the individual spins which are attached with their respective local baths, and their interactions are turned on during the dynamics, which is the refrigeration process. In this paper, the interaction strength is considered to be both ordered as well as disordered. Our aim is to show the reduction of local temperature in one of spins at the steady state, thereby exhibiting the refrigeration. We call this device a refrigerator when the temperature of that spin is lower than the minimum of the initial temperatures of all the spins.

By considering the local master equation, we found that the cooling of one of the spins occurs when the parameters of the ordered spin models are appropriately tuned. Specifically, we observed that DM interactions help to reach lower temperature than that of the XYZ model while interactions in the z plane of the XYZ model do not help at all. During the preparation procedure of the spin model, it is quite natural to have impurities in the system and hence refrigeration should be affected by the disorder. We observed that both in two- and three-spin refrigerator models, instead of decreasing the performance, disorder in the interaction strength can help to increase the figures of merit for refrigeration, although the advantage is not significant. It clearly illustrates that the spin model based quantum thermal machines are robust against impurities. We finally showed that the robustness against disorder can also be confirmed beyond the weak-coupling limit by investigating the global master equation. In future, it will be interesting to study whether the robustness observed against disorder on quantum spin model based thermal devices remains valid for other spin models having different intricacies.

ACKNOWLEDGMENTS

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APPENDIX: QUANTUM MASTER EQUATION FOR THE TWO-SPIN MODEL

For a two-spin model, let us consider the general form of the density matrix at time t, given by

$$\rho(t)=\begin{bmatrix}\rho_{11}(t) & \rho_{12}(t) & \rho_{13}(t) & \rho_{14}(t)\\ \rho_{21}(t) & \rho_{22}(t) & \rho_{23}(t) & \rho_{24}(t)\\ \rho_{31}(t) & \rho_{32}(t) & \rho_{33}(t) & \rho_{34}(t)\\ \rho_{41}(t) & \rho_{42}(t) & \rho_{43}(t)) & \rho_{44}(t)\end{bmatrix}, \tag{A1}$$

where $\rho_{ij}(t)=a_{ij}(t)+ib_{ij}(t)$, $\forall\ i\neq j$ and $\rho_{ii}(t)=a_{ii}(t)$, $\forall\ i=j$, both $a_{ij}(t)$ and $b_{ij}(t)$ being real. Consider the initial state of the system to be $\rho^0=\rho_1^0\otimes\rho_2^0$, where $\rho_i^0=\tau_i^0|0\rangle\langle 0|+(1-\tau_i^0)|1\rangle\langle 1|$ with $\tau_i^0=\exp(-2\beta_i^0h_i)/[1+\exp(-2\beta_i^0h_i)]$, $i=1,2$. Time evolution of this state, according to Eqs. (5) and (6), with $H_S=H_F+H_{xy}$ ($\gamma=0$), can be determined by solving the 16 coupled differential equations, given by

$$\begin{aligned}
\dot a_{11}&=\Gamma\big[a_{33}n^1_{2h_1}-a_{11}\big(2+n^1_{2h_1}+n^2_{2h_2}\big)+a_{22}n^2_{2h_2}\big],\quad \dot a_{12}=\Gamma\big[-a_{12}\big(1.5+n^1_{2h_1}+n^2_{2h_2}\big)+a_{34}n^1_{2h_1}\big]-2b_{13}J+2b_{12}h_2,\\
\dot b_{12}&=\Gamma\big[-b_{12}\big(1.5+n^1_{2h_1}+n^2_{2h_2}\big)+b_{34}n^1_{2h_1}\big]+2a_{13}J-2a_{12}h_2,\ \dot a_{13}=\Gamma\big[-a_{13}\big(1.5+n^1_{2h_1}+n^2_{2h_2}\big)+a_{24}n^2_{2h_2}\big]-2b_{12}J+2b_{13}h_1,\\
\dot b_{13}&=\Gamma\big[-b_{13}\big(1.5+n^1_{2h_1}+n^2_{2h_2}\big)+b_{24}n^2_{2h_2}\big]+2a_{12}J-2a_{13}h_1,\quad \dot a_{14}=-\Gamma a_{14}\big(1+n^1_{2h_1}+n^2_{2h_2}\big)+2b_{14}(h_1+h_2),\\
\dot b_{14}&=-\Gamma b_{14}\big(1+n^1_{2h_1}+n^2_{2h_2}\big)+2a_{14}(h_1+h_2),\quad \dot a_{22}=\Gamma\big[a_{11}\big(1+n^2_{2h_2}\big)-a_{22}\big(1+n^1_{2h_1}-n^2_{2h_2}\big)+a_{44}n^1_{2h_1}\big]-4b_{23}J,\\
\dot a_{23}&=-\Gamma a_{23}\big(1+n^1_{2h_1}+n^2_{2h_2}\big)+2b_{23}(h_1-h_2),\quad \dot b_{23}=-\Gamma b_{23}\big(1+n^1_{2h_1}+n^2_{2h_2}\big)+2J(a_{22}-a_{33})-2a_{23}(h_1-h_2),\\
\dot a_{24}&=\Gamma\big[a_{13}\big(1+n^2_{2h_2}\big)-a_{24}\big(0.5+n^1_{2h_1}+n^2_{2h_2}\big)\big]+2b_{34}J+2b_{24}h_1,\quad \dot b_{24}=\Gamma\big[b_{13}\big(1+n^2_{2h_2}\big)-b_{24}\big(0.5+n^1_{2h_1}+n^2_{2h_2}\big)\big]\\
&\quad-2a_{34}J-2a_{24}h_1,\\
\dot a_{33}&=\Gamma\big[a_{11}\big(1+n^1_{2h_1}\big)-a_{33}\big(1+n^1_{2h_1}+n^2_{2h_2}\big)a_{44}n^2_{2h_2}\big]+4b_{23}J,\quad \dot a_{34}=\Gamma\big[a_{12}\big(1+n^1_{2h_1}\big)-a_{34}\big(0.5+n^1_{2h_1}+n^2_{2h_2}\big)\big]\\
&\quad+2b_{24}J+2b_{34}h_2,\\
\dot b_{34}&=\Gamma\big[b_{12}\big(1+n^1_{2h_1}\big)-b_{34}\big(0.5+n^1_{2h_1}+n^2_{2h_2}\big)\big]-2a_{24}J-2a_{34}h_2,\quad \dot a_{44}=\Gamma\big[a_{22}\big(1+n^1_{2h_1}\big)+a_{33}\big(1+n^2_{2h_2}\big)-a_{44}\big(n^1_{2h_1}+n^2_{2h_2}\big)\big],
\end{aligned}$$

with $n^1_{2h_1}=1/[\exp(2\beta_1^0h_1)-1]$ and $n^2_{2h_2}=1/[\exp(2\beta_2^0h_2)-1]$ [see Eq. (6) and the following discussion]. Notice that the above coupled differential equations will be changed when $H_s=H_F+H_{xy}+H_{\rm DM}$. The time-dependent density matrix

$\rho(t)$ of the two-spin system reads as

$$\rho_s(t) = \begin{bmatrix} \rho_{11}(t) & 0 & 0 & 0 \\ 0 & \rho_{22}(t) & \rho_{23}(t) & 0 \\ 0 & \rho_{32}(t) & \rho_{33}(t) & 0 \\ 0 & 0 & 0 & \rho_{44}(t) \end{bmatrix}. \quad \text{(A2)}$$

Tracing out spin 2, the local density matrix of spin 1 takes the form

$$\rho_1(t) = \begin{bmatrix} \sigma_{11}(t) & 0 \\ 0 & \sigma_{22}(t) \end{bmatrix}, \quad \text{(A3)}$$

where $\sigma_{11}(t) = \rho_{11}(t) + \rho_{22}(t)$ and $\sigma_{22}(t) = \rho_{33}(t) + \rho_{44}(t)$.

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