Editors' Suggestion

Resonant generation of electromagnetic modes in nonlinear electrodynamics: Classical approach

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The paper explores a theoretical possibility of resonant amplification of electromagnetic modes generated by a nonlinear effect in Euler-Heisenberg electrodynamics. Precisely, we examine the possibility of the amplification for the third harmonics induced by a single electromagnetic mode in a radio-frequency cavity as well as the generation of the signal mode of combined frequencies induced by two pump modes (ω_1 and ω_2) in the cavity. Solving inhomogeneous wave equations for the signal mode, we formulate two resonant conditions for a cavity of arbitrary shape and apply the obtained formalism to linear and rectangular cavities. We explicitly show that the third harmonics as well as the mode of combined frequency $2\omega_1 + \omega_2$ are not resonantly amplified whereas the signal mode with frequency $2\omega_1 - \omega_2$ is amplified for a certain cavity geometry.

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I. INTRODUCTION

The self-interaction of an electromagnetic field, being absent in classical theory, appears in quantum theory due to radiative corrections which include the contribution of virtual electrons. At low frequency of the electromagnetic field the quantum effect is described in terms of the effective Euler-Heisenberg Lagrangian [1,2] (the detailed historical review is presented in Ref. [3]). The most distinctive effect of Euler-Heisenberg nonlinear electrodynamics is the process of photon-photon scattering [1]. Other nonlinear electrodynamics effects include vacuum birefringence and dichroism for an electromagnetic wave in a classical intense electromagnetic background [4,5]. Besides the Euler-Heisenberg contribution, effective nonlinear interactions in electrodynamics appear if the full theory contains scalar or pseudoscalar particles interacting with the electromagnetic field [6–8].

No effect, predicted in nonlinear electrodynamics, has been experimentally observed to this moment. The reason is the extreme smallness of the self-coupling for the electromagnetic field. Nevertheless, several experimental attempts to probe it with high-intensity electromagnetic fields have been performed. The experiment which comes closest to the Euler-Heisenberg limit is the polarization experiment with intensive laser fields polarization of vacuum with a laser (PVLAS) [9,10]. The final PVLAS experimental sensitivity to photon self-coupling is one order of magnitude weaker than the prediction of Euler-Heisenberg [10].

Another experimental proposal referred to high-intensity electromagnetic modes in cavities instead of laser fields. The idea of such an experiment was suggested in the early 2000s [11,12]. The proposed setup consists of a single superconducting cavity filled with two nonequal "pump" modes. In the presence of self-interaction, one expects an excitation of a third "signal" mode whose frequency is a linear combination of the pump modes' frequencies. Due to the smallness of the nonlinear effect, the signal mode can be detected only if it is resonantly amplified. The application of a single cavity setup to the searching for pseudoscalar axionlike particles was proposed in Ref. [13]:[1] If the particle is heavy (the mass is much greater than the frequencies of the pump modes),[2] the problem is reduced to the aforementioned nonlinear electrodynamics.

In papers [11–13], the solutions of nonlinear wave equations describing the resonant growth of a signal mode have not been provided explicitly. In a recent paper [18], that nonlinear wave equation was exactly solved in a simplified one-dimensional (1D) model. It was shown that, contrary to the naive estimates, signal mode with triple frequency is not resonantly generated in a one-dimensional "cavity;" the only resonant amplification is observed at the pump mode's frequency. The goal of the current article is to generalize these calculations to realistic three-dimensional cavities.

The paper is organized as follows. Section II is devoted to nonlinear Maxwell and wave equations. In Sec. III we introduce our general formalism of searching for resonant modes in an arbitrary cavity. In Sec. IV, we apply the formalism to a one-dimensional cavity filled by one or two pump modes. In Sec. V, we generalize the results of the previous section to the three-dimensional rectangular cavity. In Sec. VI, we discuss obtained results.

II. NONLINEAR MAXWELL AND WAVE EQUATIONS

In this section, we briefly review the field equations appeared in nonlinear electrodynamics. The Euler-Heisenberg

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[1] The generalization to scalar particles and the CP-violating term was considered in Ref. [14].

[2] In the case of small mass of an axionlike particle, the similar experiments with two cavities have been proposed [15–17].

Lagrangian in the limit of weak electric and magnetic-field $[E, cB \ll m_e^2 c^3/(e\hbar) \sim 10^{18}\ \mathrm{V/m}]$ takes the form

$$\mathcal{L} = -\frac{\varepsilon_0}{4}\mathcal{F} + \kappa\varepsilon_0^2(\mathcal{F}^2 + \beta\mathcal{G}^2), \quad \kappa = \frac{\alpha_e^2\hbar^3}{90 m_e^4 c^5}, \quad \beta = \frac{7}{4}, \tag{1}$$

where $\alpha_e \approx 1/137$ is the fine-structure constant, m_e and $-e$ are the electron's mass and charge, ε_0, and μ_0 are the electric and magnetic constants, $\hbar$ is the reduced Planck's constant, and c is the speed of light in vacuum. In the presence of hypothetical scalar or pseudoscalar particles in theory, the coefficients κ and β are modified [8,14]. The electromagnetic-field invariants have the standard form;

$$\mathcal{F} \equiv F_{\mu\nu}F^{\mu\nu} = -2(\boldsymbol{E}^2 - c^2\boldsymbol{B}^2),$$
$$\mathcal{G} \equiv F_{\mu\nu}\widetilde{F}^{\mu\nu} = -4(\boldsymbol{E}\cdot c\boldsymbol{B}). \tag{2}$$

The electromagnetic-field equations obtained from the Lagrangian (1) have the form analogous to Maxwell equations in a medium [2],

$$\mathrm{rot}\,\boldsymbol{B} = \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t} + \mu_0\left[\frac{\partial \boldsymbol{P}}{\partial t} - \mathrm{rot}\,\boldsymbol{M}\right], \quad \mathrm{div}\,\boldsymbol{B} = 0,$$
$$\mathrm{rot}\,\boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \quad \mathrm{div}\,\boldsymbol{E} = \frac{1}{\varepsilon_0}[-\,\mathrm{div}\,\boldsymbol{P}], \tag{3}$$

where $\boldsymbol{P}$ and $\boldsymbol{M}$ denote vacuum polarization and magnetization, respectively,

$$\boldsymbol{P}(\boldsymbol{x}, t) \equiv 16\kappa\varepsilon_0^2[(\boldsymbol{E}^2 - c^2\boldsymbol{B}^2)\boldsymbol{E} + 2\beta(\boldsymbol{E}\cdot c\boldsymbol{B})c\boldsymbol{B}],$$
$$\boldsymbol{M}(\boldsymbol{x}, t) \equiv 16\kappa\varepsilon_0^2 c[(\boldsymbol{E}^2 - c^2\boldsymbol{B}^2)c\boldsymbol{B} - 2\beta(\boldsymbol{E}\cdot c\boldsymbol{B})\boldsymbol{E}]. \tag{4}$$

The field equations (3) yield modified wave equations both for amplitudes for electric and magnetic fields,

$$\Box\boldsymbol{E} = \mu_0\left[\frac{\partial}{\partial t}\,\mathrm{rot}\,\boldsymbol{M} + c^2\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{P} - \frac{\partial^2\boldsymbol{P}}{\partial t^2}\right],$$
$$\Box\boldsymbol{B} = \mu_0\left[\frac{\partial}{\partial t}\,\mathrm{rot}\,\boldsymbol{P} - \,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{M} + \Delta\boldsymbol{M}\right]. \tag{5}$$

Note that the plane electromagnetic wave is a solution of modified wave equations (5) since both electromagnetic invariants vanish at the plane-wave configuration $\mathcal{F} = \mathcal{G} = 0$. However, a linear combination of plane waves is no longer a solution of Eqs. (5) and so becomes unstable, leading to the production of new modes.

III. GENERAL FORMALISM OF SEARCHING FOR RESONANT MODES

One of the interesting features of modified wave equations (5) is the possibility for generation of higher-order harmonics by one or two initial electromagnetic modes in vacuum. Traditionally [11–13], we consider the following setup devoted to the search for higher-order harmonics. We take a superconducting radio-frequency cavity filled with one or two pump modes of very high amplitudes $\boldsymbol{E}^{\mathrm{pump}}, \boldsymbol{B}^{\mathrm{pump}}$, and look for generation of a signal mode of different frequencies with amplitudes $\boldsymbol{E}^{\mathrm{sig}}, \boldsymbol{B}^{\mathrm{sig}}$ which are expected to be small due the smallness of nonlinear coupling coefficient κ. Treating the signal mode as a small perturbation in Eq. (5) and assuming the hierarchy of scales $|\boldsymbol{E}^{\mathrm{sig}}| \sim \kappa\varepsilon_0(|\boldsymbol{E}^{\mathrm{pump}}|)^3 \ll |\boldsymbol{E}^{\mathrm{pump}}|$, one obtains in the zeroth-order trivial wave equations for the pump modes $\Box\boldsymbol{E}^{\mathrm{pump}} = 0$, $\Box\boldsymbol{B}^{\mathrm{pump}} = 0$, and in the first order,

$$\Box\boldsymbol{E}^{\mathrm{sig}} = \mu_0\left[\frac{\partial}{\partial t}\,\mathrm{rot}\,\boldsymbol{M}(\boldsymbol{E}^{\mathrm{pump}}, \boldsymbol{B}^{\mathrm{pump}}) + c^2\,\mathrm{grad}\,\mathrm{div}\boldsymbol{P}(\boldsymbol{E}^{\mathrm{pump}}, \boldsymbol{B}^{\mathrm{pump}}) - \frac{\partial^2\boldsymbol{P}(\boldsymbol{E}^{\mathrm{pump}}, \boldsymbol{B}^{\mathrm{pump}})}{\partial t^2}\right],$$
$$\Box\boldsymbol{B}^{\mathrm{sig}} = \mu_0\left[\frac{\partial}{\partial t}\,\mathrm{rot}\,\boldsymbol{P}(\boldsymbol{E}^{\mathrm{pump}}, \boldsymbol{B}^{\mathrm{pump}}) - \,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{M}(\boldsymbol{E}^{\mathrm{pump}}, \boldsymbol{B}^{\mathrm{pump}}) + \Delta\boldsymbol{M}(\boldsymbol{E}^{\mathrm{pump}}, \boldsymbol{B}^{\mathrm{pump}})\right]. \tag{6}$$

Here the polarization and magnetization vectors (4) are computed on the pump mode configuration. Instead of nonlinear Eq. (5), Eq. (6) is a linear wave equation on the signal mode amplitudes with the nonzero right-hand side. The solution of Eq. (6) determines the evolution of the signal mode at the classical level.

Equations (6) are to be solved in a given cavity D of finite volume. Furthermore, in order to take into account small dissipation, we introduce the dissipative term which includes the first-order time derivative and the damping coefficient Γ,

$$\left(\Box - \frac{1}{c^2}\Gamma\partial_t\right)\boldsymbol{E}^{\mathrm{sig}}(\boldsymbol{x}, t) = \boldsymbol{F}(\boldsymbol{x}, t), \quad \boldsymbol{x} \in D, \quad t > 0,$$
$$\boldsymbol{E}^{\mathrm{sig}}(\boldsymbol{x}, 0) = 0, \quad \boldsymbol{x} \in D,$$
$$\boldsymbol{n} \times \boldsymbol{E}^{\mathrm{sig}}(\boldsymbol{x}, t) = 0, \quad \boldsymbol{x} \in S. \tag{7}$$

Here S denotes the surface of the cavity D, $\boldsymbol{n}$ is the normal to the surface S. $\boldsymbol{F}(\boldsymbol{x}, t)$ denotes the right-hand side of the electric equation in (6). The boundary conditions refer to an ideal conducting surface. The similar system should be written for magnetic component of the signal mode $\boldsymbol{B}^{\mathrm{sig}}$, however, we will further skip it for the sake of shortness.

The signal field $\boldsymbol{E}^{\mathrm{sig}}(\boldsymbol{x}, t)$ can be expanded into the cavity eigenmodes,

$$\boldsymbol{E}^{\mathrm{sig}}(\boldsymbol{x}, t) = \sum_k E_k^{\mathrm{sig}}(t)\,\boldsymbol{\mathcal{E}}_k(\boldsymbol{x}). \tag{8}$$

Here $\boldsymbol{\mathcal{E}}(\boldsymbol{x})$ comprise the full system of eigenfunctions with eigenvalues ω_k, satisfying the equation $(\Delta + \frac{\omega_k^2}{c^2})\boldsymbol{\mathcal{E}}_k(\boldsymbol{x}) = 0$ and boundary conditions given in the last line of Eq. (7).

TABLE I. Examination of the resonance criterion for a single pump mode in the 1D cavity.

	n	$3n$
$\boldsymbol{F}^{\mathrm{el}}$	$\omega_n,\ \omega_{3n}$	ω_n
$\boldsymbol{F}^{\mathrm{mg}}$	$\omega_n,\ \omega_{3n}$	ω_n

Substituting expansion (8) to the first equation of (7) and integrating over the whole cavity with a mode $\boldsymbol{\mathcal{E}}_n(\boldsymbol{x})$, one obtains

$$\ddot{E}_n^{\rm sig}(t) + \Gamma \dot{E}_n^{\rm sig}(t) + \omega_n^2 E_n^{\rm sig}(t) = F_n(t),$$
$$F_n(t) \equiv c^2 \frac{\int_D dV\, \boldsymbol{F}(\boldsymbol{x},t)\cdot \boldsymbol{\mathcal{E}}_n(\boldsymbol{x})}{\int_D dV\, \boldsymbol{\mathcal{E}}_n^2(\boldsymbol{x})} \equiv c^2 \frac{(\boldsymbol{F}, \boldsymbol{\mathcal{E}}_n)}{\|\boldsymbol{\mathcal{E}}_n\|^2}. \quad (9)$$

Here we made the following notations: $\int_D dV$ for the integration over the volume of the cavity D, $(\boldsymbol{F}, \boldsymbol{\mathcal{E}})$ for the inner product of two vector functions $\boldsymbol{F}$ and $\boldsymbol{\mathcal{E}}$, and $\|\mathcal{E}\| = \sqrt{(\mathcal{E},\mathcal{E})}$ for the norm of function $\mathcal{E}$. All frequency components of $F_n(t)$ lead to the generation of a signal mode with a certain amplitude. If $F_n(t)$ includes the component $F_n(t) \supseteq \mathrm{Re}(F_n^0 e^{-i\omega t})$, the signal field of the amplitude,

$$E_n^{\rm sig}(t) = \mathrm{Re}\, \frac{F_n^0 e^{-i\omega t}}{-\omega^2 - i\omega\Gamma + \omega_n^2} \quad (10)$$

is generated in the steady regime. If the frequency ω coincides with one of the cavity eigenfrequencies, $\omega = \omega_n$, the first and the third terms in the denominator of (10) cancel each other, and so the amplitude of such a signal mode is resonantly enhanced $E_n^{\rm sig}(t) = \mathrm{Re}[iF_n^0 e^{-i\omega t}/(\Gamma\omega)]$.

Let us summarize aforementioned expressions in a more strict way as a criterion for resonant amplification of the signal mode. Assume that the right-hand side of one of the six scalar wave equations (6) contains the frequency component $\omega_{\rm sig}$. The signal mode with the frequency $\omega_{\rm sig}$ is resonantly amplified if both of the following conditions hold simultaneously:

(1) Frequency $\omega_{\rm sig}$ belongs to the cavity spectrum ($\exists n \in \mathbb{N}: \omega_{\rm sig} = \omega_n$,

(2) The scalar product $F_n(t)$ of the right-hand side of the considered scalar wave equation from (6) with the nth cavity eigenmode contains the frequency component $\omega_{\rm sig} = \omega_n$.

Note that even if $\boldsymbol{F}(\boldsymbol{x},t)$ contains a frequency component ω_n, it may disappear from $F_n(t)$ due to the integration with orthogonal cavity mode. An example supporting this statement will be provided in the next section.

IV. ONE-DIMENSIONAL CAVITY

In this section, we consider a model of one-dimensional cavity directed along the Ox axis, $D = (0, a)$. The y and z dimensions of the cavity are assumed to be significantly larger than the x-dimension $L_y, L_z \gg L_x \equiv a$. The cavity system[3] of eigenfunctions assuming ideal conducting walls takes a simple form

$$\begin{matrix} \boldsymbol{\mathcal{E}}_n(x) = \sin(k_n x) e^{-i\frac{\pi}{2}}\, \boldsymbol{e}_y \\ \boldsymbol{\mathcal{M}}_n(x) = \cos(k_n x)\, \boldsymbol{e}_z \end{matrix}, \quad k_n = \frac{\pi n}{a}, \quad \|\boldsymbol{\mathcal{E}}_n\|^2 = \|\boldsymbol{\mathcal{M}}_n\|^2 = \frac{a}{2}, \quad n \in \mathbb{N}. \quad (11)$$

The dynamics of a cavity mode with wave-number k_n and magnetic amplitude B_0 is just an oscillation with frequency $\omega_n = k_n c$,

$$\boldsymbol{E}^{\rm pump}(x,t) = cB_0\, \mathrm{Re}\left[\boldsymbol{\mathcal{E}}_n(x) e^{i\omega_n t}\right] = cB_0 \sin(k_n x)\sin(\omega_n t)\, \boldsymbol{e}_y,$$
$$\boldsymbol{B}^{\rm pump}(x,t) = B_0\, \mathrm{Re}\left[\boldsymbol{\mathcal{M}}_n(x) e^{i\omega_n t}\right] = B_0 \cos(k_n y)\cos(\omega_n t)\, \boldsymbol{e}_z. \quad (12)$$

A. Single pump mode

First, we consider an excitation of the one-dimensional cavity with a single pump mode of frequency ω_n, see (12). At the pump mode configuration (12) the invariant $\mathcal{F} \neq 0$ whereas the second invariant $\mathcal{G}$ vanishes. Substituting the pump mode fields (12) to the expression for the inhomogeneities of nonlinear wave equation (6) and performing a simple but cumbersome trigonometric calculation,[4] we obtain inhomogeneous wave equations for signal modes (see Ref. [18]),

$$\left(\Box - \frac{1}{c^2}\Gamma\partial_t\right)\boldsymbol{E}^{\rm sig} = 8\kappa\varepsilon_0 c^3 B_0^3 k_n^2 [2\, \sin(k_n x)\sin(\omega_n t) + \sin(3k_n x)\sin(\omega_n t) - 3\, \sin(k_n x)\sin(3k_n t)]\, \boldsymbol{e}_y = \boldsymbol{F}^{\rm el}(x,t),$$
$$\left(\Box - \frac{1}{c^2}\Gamma\partial_t\right)\boldsymbol{B}^{\rm sig} = 8\kappa\varepsilon_0 c^2 B_0^3 k_n^2 [2\, \cos(k_n x)\cos(\omega_n t) + 3\, \cos(3k_n x)\cos(\omega_n t) - \cos(k_n x)\cos(3k_n t)]\, \boldsymbol{e}_z = \boldsymbol{F}^{\rm mg}(x,t). \quad (13)$$

Note that both Eqs. (13) contain terms $\sin(k_n x)\sin(\omega_n t)$ or $\cos(k_n x)\cos(\omega_n t)$, which obviously result in a resonant enhancement of the signal mode with frequency ω_n. However, the right-hand side of (13) do not contain terms, such as $\sin(3k_n x)\sin(3\omega_n t)$, which would produce a signal mode of triple frequency. Formally, let us use the resonance criterion formulated in the previous section. The projections of the right-hand side of (13) on the cavity eigenfunctions are as follows:

$$F_n^{\rm el}(t) \equiv c^2 \frac{(\boldsymbol{F}^{\rm el}, \boldsymbol{\mathcal{E}}_n)}{\|\boldsymbol{\mathcal{E}}_n\|^2} = c^2 \frac{2}{a}\int_0^a F^{\rm el}(x,t)\sin(k_n x)dx = 8\kappa\varepsilon_0 c^3 B_0^3 \omega_n^2 [2\, \sin(\omega_n t) - 3\, \sin(3\omega_n t)],$$
$$F_n^{\rm mg}(t) \equiv c^2 \frac{(\boldsymbol{F}^{\rm mg}, \boldsymbol{\mathcal{M}}_n)}{\|\boldsymbol{\mathcal{M}}_n\|^2} = c^2 \frac{2}{a}\int_0^a F^{\rm mg}(x,t)\cos(k_n x)dx = 8\kappa\varepsilon_0 c^2 B_0^3 \omega_n^2 [2\, \cos(\omega_n t) - \cos(3\omega_n t)],$$

[3] For mathematical completeness, one has to add a constant mode (with zero frequency) to the system (11).

[4] The calculations were additionally verified in the computer algebra system WXMAXIMA 21.02.0, see Ref. [19].

$$F^{\mathrm{el}}_{3n}(t) \equiv c^2 \frac{(\boldsymbol{F}^{\mathrm{el}}, \boldsymbol{\mathcal{E}}_{3n})}{\|\boldsymbol{\mathcal{E}}_{3n}\|^2} = c^2 \frac{2}{a}\int_0^a F^{\mathrm{el}}(x,t)\sin(3k_n x)dx = 8\kappa\varepsilon_0 c^3 B_0^3\omega_n^2[\sin(\omega_n t)],$$

$$F^{\mathrm{mg}}_{3n}(t) \equiv c^2 \frac{(\boldsymbol{F}^{\mathrm{mg}}, \boldsymbol{\mathcal{M}}_{3n})}{\|\boldsymbol{\mathcal{M}}_{3n}\|^2} = c^2 \frac{2}{a}\int_0^a F^{\mathrm{mg}}(x,t)\cos(3k_n x)dx = 8\kappa\varepsilon_0 c^2 B_0^3\omega_n^2[3\cos(\omega_n t)]. \quad (14)$$

As the amplitudes are on the same order [as small as $F^{\mathrm{el}}, cF^{\mathrm{mg}} \sim \kappa\varepsilon_0(cB_0)^3\omega_n^2$], for the sake of shortness, we organize the computed projections into a table.

Table I is to be interpreted as follows: The upper row contains mode numbers (on which projections were computed), the leftmost column enumerates the right-hand side of nonhomogeneous wave equations for signal modes. Then, every cell contains those frequencies, which have been found in a projection of the right-hand side from the corresponding row onto an eigenmode with the number from the corresponding column. In the case of the one-dimensional cavity being excited with a single pump mode, one can easily correlate the four projections computed right above with the four cells in the table. The understanding of how the examination of the resonance criterion is presented in the table will be important later for more complex configurations, where the direct calculations result in formulas too long to be listed entirely.

As we see from Table I, the new triple frequency does not belong to a spectrum of any projection onto the $3n$ eigenmode. Thus, condition 2 of the resonance criterion is not satisfied, and, therefore, the signal mode with triple frequency is not resonantly enhanced.

B. Two pump modes

The next configuration we consider is the excitation of the one-dimensional cavity with two pump modes of frequencies ω_n and ω_p. Since the linear cavity exposes rotational symmetry along axis Ox, we introduce an arbitrary angle α between the polarization planes of the pump modes,

$$\boldsymbol{E}^{\mathrm{pump}}(x,t) = cB_0\,\mathrm{Re}\left[\boldsymbol{\mathcal{E}}_n(x)e^{i\omega_n t} + \hat{\boldsymbol{R}}_x(\alpha)\boldsymbol{\mathcal{E}}_p(x)e^{i\omega_p t}\right],$$
$$\boldsymbol{B}^{\mathrm{pump}}(x,t) = B_0\,\mathrm{Re}\left[\boldsymbol{\mathcal{M}}_n(x)e^{i\omega_n t} + \hat{\boldsymbol{R}}_x(\alpha)\boldsymbol{\mathcal{M}}_p(x)e^{i\omega_p t}\right], \qquad \hat{\boldsymbol{R}}_x(\alpha) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{pmatrix}. \quad (15)$$

The eigenmodes $\boldsymbol{\mathcal{E}}_n(x)$ and $\boldsymbol{\mathcal{M}}_n(x)$ are given by (11). In contrast to the case of a single pump mode, both electromagnetic invariants (2) are nonzero at the current configuration. The inhomogeneous wave equations for the signal mode read

$$\left(\Box - \frac{1}{c^2}\Gamma\partial_t\right)\boldsymbol{E}^{\mathrm{sig}}(x,t) = \boldsymbol{F}^{\mathrm{el}}(x,t) = \begin{pmatrix} 0 \\ F^{\mathrm{el}}_y \\ F^{\mathrm{el}}_z \end{pmatrix}, \quad \left(\Box - \frac{1}{c^2}\Gamma\partial_t\right)\boldsymbol{B}^{\mathrm{sig}}(x,t) = \boldsymbol{F}^{\mathrm{mg}}(x,t) = \begin{pmatrix} 0 \\ F^{\mathrm{mg}}_y \\ F^{\mathrm{mg}}_z \end{pmatrix}, \quad (16)$$

Here the inhomogeneities $\boldsymbol{F}^{\mathrm{el}}(x,t)$ and $\boldsymbol{F}^{\mathrm{mg}}(x,t)$ are calculated by the substitution of the field configuration (15) to the general expression (6). This calculation was performed in the computer algebra system WXMAXIMA; the resulting expressions are too long to be listed here explicitly, see Ref. [19].

Nevertheless, the restrictions on the set of signal modes can be obtained taking a look at the structure of the inhomogeneities $\boldsymbol{F}^{\mathrm{el}}(x,t)$ and $\boldsymbol{F}^{\mathrm{mg}}(x,t)$. Since the expressions are cubic [see Eq. (4)] relative to the pump modes which are simple trigonometric functions, one can expect at most the following frequencies for signal modes in (16):

$$\omega_{\mathrm{sig}} \in \{\omega_n, \omega_p, 3\omega_n, 3\omega_p, 2\omega_n \pm \omega_p, 2\omega_p \pm \omega_n\}. \quad (17)$$

The possible wave numbers for the signal mode belong to the similar set,

$$k_{\mathrm{sig}} \in \{k_n, k_p, 3k_n, 3k_p, 2k_n \pm k_p, 2k_p \pm k_n\}. \quad (18)$$

It follows from condition 1 of the criterion that the wave-numbers (18) have to match with the corresponding frequencies (17).

At the following step, we examine condition 2: We project the inhomogeneities of Eqs. (16) onto cavity eigenmodes, whose frequencies may hypothetically appear due to cubic nonlinearities (17). This stage of calculations was also carried out in THE WXMAXIMA system (see Ref. [19]); the results are presented in Table II which is constructed analogously to that in the previous section.

Condition 2 of the resonance criterion is satisfied only for the signal frequencies ω_n and ω_p, which are shadowed by the pump modes. Thus, in the case of two pump modes in the one-dimensional cavity, resonant amplification of signal modes with mixed frequencies does not occur.

TABLE II. Examination of the resonance criterion for two pump modes in the 1D cavity.

	n	$3n$	$2n-p$	$2n+p$	p	$3p$	$2p-n$	$2p+n$
$\boldsymbol{F}^{\mathrm{el}}$ $\boldsymbol{F}^{\mathrm{mg}}$	$\omega_n,\ \omega_{2p\pm n},\ \omega_{3n}$	ω_n	$\omega_p,\ \omega_{2n+p}$	$\omega_p,\ \omega_{2n-p}$	$\omega_p,\ \omega_{2n\pm p},\ \omega_{3p}$	ω_p	$\omega_n,\ \omega_{2p+n}$	$\omega_n,\ \omega_{2p-n}$

TABLE III. Examination of the resonance criterion for a single pump mode in the rectangular cavity.

<table>
<tr><td>TM(⋯)</td><td>n, p, q</td><td>$3n, p, q$</td><td>$n, 3p, q$</td><td>$n, p, 3q$</td><td>$n, 3p, 3q$</td><td>$3n, p, 3q$</td><td>$3n, 3p, q$</td><td>$3n, 3p, 3q$</td></tr>
<tr><td>$\boldsymbol{F}^{\text{el}}$
$\boldsymbol{F}^{\text{mg}}$</td><td colspan="7">$\omega_{npq}$, $3\omega_{npq}$</td><td>ω_{npq}</td></tr>
<tr><td>TE(⋯)</td><td>n, p, q</td><td>$3n, p, q$</td><td>$n, 3p, q$</td><td>$n, p, 3q$</td><td>$n, 3p, 3q$</td><td>$3n, p, 3q$</td><td>$3n, 3p, q$</td><td>$3n, 3p, 3q$</td></tr>
<tr><td>$\boldsymbol{F}^{\text{el}}$
$\boldsymbol{F}^{\text{mg}}$</td><td colspan="6">$\omega_{npq}$, $3\omega_{npq}$</td><td colspan="2">ω_{npq}</td></tr>
</table>

V. RECTANGULAR CAVITY

In this section, we proceed with a rectangular cavity $D = (0, L_x)(0, L_y)(0, L_z)$. Within the approximation of perfectly conducting walls the system of eigenfunctions separates into two subsets (relative to the Oz axis)—TE modes and TM modes [[20], pp. 25–28],

$$\begin{matrix} \boldsymbol{\mathcal{E}}^{\text{TM}}_{npq}(\boldsymbol{x}), \quad \boldsymbol{\mathcal{M}}^{\text{TM}}_{npq}(\boldsymbol{x}) \perp \boldsymbol{e}_z, \quad n, p \in \mathbb{N},\ q \in \mathbb{N}_0 \\ \boldsymbol{\mathcal{E}}^{\text{TE}}_{npq}(\boldsymbol{x}) \perp \boldsymbol{e}_z, \quad \boldsymbol{\mathcal{M}}^{\text{TE}}_{npq}(\boldsymbol{x}), \quad n, p \in \mathbb{N}_0,\ q \in \mathbb{N} \end{matrix}, \quad \boldsymbol{k}_{npq} = \left(\frac{\pi n}{L_x}, \frac{\pi p}{L_y}, \frac{\pi q}{L_z}\right), \quad \|\mathcal{E}^i\|^2 = \|\mathcal{M}^i\|^2 = \frac{L_x L_y L_z}{8}. \tag{19}$$

Time evolution of npq modes is an oscillation with the frequency $\omega_{npq} = c|\boldsymbol{k}_{npq}| = \pi c\sqrt{\frac{n^2}{L_x^2} + \frac{p^2}{L_y 2} + \frac{q^2}{L_z^2}}$.

A. Single pump mode

Let us consider a single pump mode with an eigenfrequency ω_{npq}. Since the division into TE and TM modes is purely artificial in the case of the rectangular cavity, we arbitrarily choose the TMnpq pump mode,

$$\begin{aligned} \boldsymbol{E}^{\text{pump}}(\boldsymbol{x}, t) &= cB_0 \operatorname{Re}\left[\boldsymbol{\mathcal{E}}^{\text{TM}}_{npq}(\boldsymbol{x}) e^{i\omega_{npq}t}\right], \\ \boldsymbol{B}^{\text{pump}}(\boldsymbol{x}, t) &= B_0 \operatorname{Re}\left[\boldsymbol{\mathcal{M}}^{\text{TM}}_{npq}(\boldsymbol{x}) e^{i\omega_{npq}t}\right]. \end{aligned} \tag{20}$$

At the single mode configuration (20), the electromagnetic invariant $\mathcal{G}$ vanishes, similar to the one-dimensional case; the invariant $\mathcal{F}$ is still nonzero. As previously, the right-hand side parts of the linearized wave equations are calculated using Eqs. (4) and (5),

$$\left(\Box + \frac{1}{c^2}\Gamma\partial_t\right)\boldsymbol{E}(\boldsymbol{x}, t) = \boldsymbol{F}^{\text{el}}(\boldsymbol{x}, t), \quad \left(\Box + \frac{1}{c^2}\Gamma\partial_t\right)\boldsymbol{B}(\boldsymbol{x}, t) = \boldsymbol{F}^{\text{mg}}(\boldsymbol{x}, t). \tag{21}$$

For arbitrary integers (n, p, q), all components of the obtained inhomogeneities are nonzero. In order to examine the resonance criterion, their spatial projections on corresponding eigenfunctions are calculated (for instance, $\boldsymbol{F}^{\text{el}}$ is to be projected on $\boldsymbol{\mathcal{E}}^{\text{TM}}$ and $\boldsymbol{\mathcal{E}}^{\text{TE}}$). The temporal spectra of all nonvanishing projections are listed in Table III.

From Table III, one concludes that only the lowest-frequency ω_{npq} is amplified, whereas the higher-order harmonics remain suppressed. Note that, in contrast to the case of the one-dimensional cavity, the modes with intermediate sets of wave numbers [e.g.. $(n, 3p, q)$] do appear in the rectangular cavity. However, neither the pump mode frequency nor the triple frequency fits these sets of wave numbers (condition 1 of resonance criterion fails), so these modes are not resonantly amplified.

B. Two pump modes

The last configuration being considered includes two pump modes (for certainty, one TM and one TE mode) excited in a rectangular cavity. The electric and magnetic fields of this configuration read

$$\begin{aligned} \boldsymbol{E}^{\text{pump}}(\boldsymbol{x}, t) &= cB_0 \operatorname{Re}\left[\boldsymbol{\mathcal{E}}^{\text{TM}}_{n_1,p_1,q_1}(\boldsymbol{x}) e^{i\omega_1 t} + \boldsymbol{\mathcal{E}}^{\text{TE}}_{n_2,p_2,q_2}(\boldsymbol{x}) e^{i\omega_2 t}\right], \\ \boldsymbol{B}^{\text{pump}}(\boldsymbol{x}, t) &= B_0 \operatorname{Re}\left[\boldsymbol{\mathcal{M}}^{\text{TM}}_{n_1,p_1,q_1}(\boldsymbol{x}) e^{i\omega_1 t} + \boldsymbol{\mathcal{M}}^{\text{TE}}_{n_2,p_2,q_2}(\boldsymbol{x}) e^{i\omega_2 t}\right]. \end{aligned} \tag{22}$$

Here the subscript 1(2) refers to the TM (TE) mode, $\omega_1 = \omega_{n_1,p_1,q_1}$ and $\omega_2 = \omega_{n_2,p_2,q_2}$. Now, Eqs. (21) are to be solved, where the right-hand side is computed at the pump field (22).

Since the right-hand side. of the wave equation for the signal modes is cubic relative to the pump modes [see (4)] and the latter are simple trigonometric functions, the right-hand side of (21) may contain terms only of the following form:

$$A\, h(\omega_{\text{sig}}t)\, h(k_{\text{sig},x}x)\, h(k_{\text{sig},y}y)\, h(k_{\text{sig},z}z),$$

TABLE IV. Examination of the resonance criterion for two arbitrary pump modes in the rectangular cavity.

Modes:	n_1 p_1 q_1	$3n_1$ p_1 q_1	n_1 $3p_1$ q_1	n_1 p_1 $3q_1$	n_1 $3p_1$ $3q_1$	$3n_1$ p_1 $3q_1$	$3n_1$ $3p_1$ q_1	$3n_1$ $3p_1$ $3q_1$
F_z^{el} on TM F_z^{mg} on TE	$\omega_1,\ 3\omega_1,\ 2\omega_2+\omega_1,\ 2\omega_2-\omega_1$	$\omega_1,\ 3\omega_1$		$\omega_1,\ 3\omega_1$	ω_1			ω_1
Modes:	$2n_2 \pm n_1$ p_1 q_1	n_1 $2p_2 \pm p_1$ q_1	n_1 p_1 $2q_2 \pm q_1$	$2n_2 \pm n_1$ $2p_2 \pm p_1$ q_1	$2n_2 \pm n_1$ p_1 $2q_2 \pm q_1$	n_1 $2p_2 \pm p_1$ $2q_2 \pm q_1$	$2n_2 \pm n_1$ $2p_2 \pm p_1$ $2q_2 \pm q_1$	
F_z^{el} on TM F_z^{mg} on TE				$\omega_1,\ 2\omega_2+\omega_1,\ 2\omega_2-\omega_1$				

where the notation $h(\cdot)$ stands just for trigonometrical functions $\sin(\cdot)$ and $\cos(\cdot)$, whereas the signal mode frequency ω_{sig} and wave-vector components $k_{\mathrm{sig},x}$, $k_{\mathrm{sig},y}$, and $k_{\mathrm{sig},z}$ can take arbitrary values at most from the following sets:

$$\begin{array}{llllll} \omega_{\mathrm{sig}} \in \{\omega_1, & \omega_2, & 2\omega_1 \pm \omega_2, & 2\omega_2 \pm \omega_1, & 3\omega_1, & 3\omega_2\}, \\ k_{\mathrm{sig},x} \in \{k_{1x}, & k_{2x}, & 2k_{1x} \pm k_{2x}, & 2k_{2x} \pm k_{1x}, & 3k_{1x}, & 3k_{2x}\}, \\ k_{\mathrm{sig},y} \in \{k_{1y}, & k_{2y}, & 2k_{1y} \pm k_{2y}, & 2k_{2y} \pm k_{1y}, & 3k_{1y}, & 3k_{2y}\}, \\ k_{\mathrm{sig},z} \in \{k_{1z}, & k_{2z}, & 2k_{1z} \pm k_{2z}, & 2k_{2z} \pm k_{1z}, & 3k_{1z}, & 3k_{2z}\}. \end{array} \tag{23}$$

For instance, some mixed combinations, such as $\sin(3\omega_1 t)\cos[(2k_{1x}-k_{2x})x]\cos(k_{1y}y)\sin(3k_{2z}z)$ might hypothetically appear within the right-hand side of wave equations (21). However, to search reliably for the resonant components, we have to check the conditions of the criterion. The first condition reads

$$\omega_{\mathrm{sig}}^2 = \left(k_{\mathrm{sig},x}^2 + k_{\mathrm{sig},y}^2 + k_{\mathrm{sig},z}^2\right)c^2. \tag{24}$$

Before the direct test of the second condition in order to simplify computer algebra computations, we make some additional theoretical statements concerning possible generation of a signal mode with the frequency $\omega_{\mathrm{sig}} = 2\omega_1 + \omega_2$. First, one can write the triangle inequality for the wave vectors of the pump modes,

$$\omega_{\mathrm{sig}} = 2\omega_1 + \omega_2 = c(|2\boldsymbol{k}_1| + |\boldsymbol{k}_2|) \geqslant c|2\boldsymbol{k}_1 + \boldsymbol{k}_2| = c\sqrt{(2k_{1x}+k_{2x})^2 + (2k_{1y}+k_{2y})^2 + (2k_{1z}+k_{2z})^2}. \tag{25}$$

The equality holds if the two pump mode wave vectors are parallel $\boldsymbol{k}_1 \parallel \boldsymbol{k}_2$. Condition (24) for this case is satisfied automatically. In the case of nonparallel wave vectors, the triangle inequality implies that, at least, one of the components of $\boldsymbol{k}_{\mathrm{sig}}$ (say, $k_{\mathrm{sig},x}$) should be larger than $2k_{1x}+k_{2x}$. The only case is the triple maximal projection of wave-numbers $k_{\mathrm{sig},x} = 3\ \max(k_{1x}, k_{2x})$.

Now, we are to check the second condition of the resonance criterion. This involves, following the algorithm from Sec. III, the calculation of $(\boldsymbol{F}, \boldsymbol{\mathcal{E}}_{npq}) = \int_D F_x \mathcal{E}_{npq,x} dV + \int_D F_y \mathcal{E}_{npq,y} dV + \int_D F_z \mathcal{E}_{npq,z} dV$ where $\boldsymbol{F} \in \{\boldsymbol{F}^{\mathrm{el}}, \boldsymbol{F}^{\mathrm{mg}}\}$ and (npq) takes, at least, $6^3 = 216$ possible combinations from (23). Full symbolic evaluation in WXMAXIMA requires too much computational resources. Since we are going to prove the absence of resonant generation of the$(2\omega_1 + \omega_2)$ signal mode, we slightly simplify the algorithm described in Sec. III.

Instead of solving full vector system (7), we focus only on the z-component $E_z^{\mathrm{sig}}(\boldsymbol{x}, t)$ of the signal mode. We expand $E_z^{\mathrm{sig}}(\boldsymbol{x}, t)$ over the z components of cavity eigenmodes,

$$E_z^{\mathrm{sig}}(\boldsymbol{x}, t) = \sum_{npq} E_{z,k}^{\mathrm{sig}}(t)\, \mathcal{E}_{npq,z}(\boldsymbol{x}). \tag{26}$$

Substituting expansion (26) into the z projection of Eq. (7), we obtain a second-order differential equation for $E_{z,k}^{\mathrm{sig}}(t)$ which is completely similar to (9) with the only difference: its right-hand side contains only the z component of the scalar product $F_k(t) \propto \int_D F_z(\boldsymbol{x}, t)\mathcal{E}_{npq,z}(\boldsymbol{x}) dV$; the related calculations become ~3 times simpler than before. Nevertheless, this partial treatment is still sufficient to prove the *absence* of resonant amplification for a certain signal mode: The z component of the signal mode induced by arbitrary pump modes should not be zero if such a signal mode is indeed resonantly amplified.

The result of the computer algebra calculations for the temporal spectra of nonzero projections is presented in the simplified form in the Table IV. Note that these spectra relate only to the testing of condition 2, condition 1 remains still to be examined. Here the signs "±" are independent from each other so that the table is compressed due to the lack of space. Table IV presents "an upper limit" on possible spectra, in specific cases the coefficients before some harmonics vanish. Particularly, these include the case of pump modes with parallel wave-vectors $\boldsymbol{k}_1 \parallel \boldsymbol{k}_2$. Thus, it turns out that the corresponding signal mode is not resonantly amplified.

In addition, it is shown in Table IV that the frequency $2\omega_2 + \omega_1$ does not appear in the spectra of projections on modes with, at least, one triple wave number. Therefore, the only remaining case of nonparallel wave vectors which require, at least, one triple wave number [see the paragraph after Eq. (25)] is ruled out by condition 2.

TABLE V. Examination of the resonance criterion for two pump modes in the rectangular cavity.

TM(⋯)	110	130	310	330	112	132	211	231
$\boldsymbol{F}^{\mathrm{el}}$ $\boldsymbol{F}^{\mathrm{mg}}$	$\omega_{110},\ 3\omega_{110},\ 2\omega_{011}\pm\omega_{110}$		$\omega_{110},\ 3\omega_{110}$	ω_{110}	$\omega_{110},\ 2\omega_{011}\pm\omega_{110}$		$\omega_{011},\ 2\omega_{110}\pm\omega_{011}$	
TE(⋯)	011	031	013	033	112	132	211	231
$\boldsymbol{F}^{\mathrm{el}}$ $\boldsymbol{F}^{\mathrm{mg}}$	$\omega_{011},\ 3\omega_{011},\ 2\omega_{110}\pm\omega_{011}$		$\omega_{011},\ 3\omega_{011}$	ω_{011}	$\omega_{110},\ 2\omega_{011}\pm\omega_{110}$		$\omega_{011},\ 2\omega_{110}\pm\omega_{011}$	

As a result, we have shown by scanning all possible combinations that the signal mode with the frequency $2\omega_2+\omega_1$ is not resonantly amplified.

However, as we demonstrate in the following subsection, resonant enhancement does occur for the signal mode of frequency $2\omega_2-\omega_1$.

Resonant solution for the $2\omega_2-\omega_1$ signal mode

Now, we return to the original algorithm of Sec. III and consider two certain pump modes TM110+TE011 in order to demonstrate explicitly the resonant amplification of a signal mode with mixed frequency $2\omega_2-\omega_1$ (where $\omega_1\equiv\omega_{110}$ and $\omega_2\equiv\omega_{011}$).

The lowest and the pure triple frequencies trivially match cavity eigenvalues. However, some of $2\omega_{011}\pm\omega_{110}$ and $2\omega_{110}\pm\omega_{011}$ might coincide with intermediate eigenfrequencies too (e.g., with ω_{130}) if one carefully adjusts the cavity sides' lengths. Thus, we first project the right-hand side of (21) on every possible eigenmode from the lowest through the triple ones [even higher frequencies are guaranteed to be absent in spectra of (21)]. Temporal spectra of all nonvanishing projections are presented in Table V.

As usual, only the lowest harmonics ω_{011} and ω_{110} resonate unconditionally. Following the results of the previous subsection, the third harmonics as well as the plus combined modes do not resonate. The two remaining options (up to the permutation of indices) are $2\omega_{011}-\omega_{110}=\omega_{130}$ and $2\omega_{011}-\omega_{110}=\omega_{132}$. Substituting the expression via wave numbers [see Eq. (24)] to the frequency matching conditions, one obtains the conditions to the cavity dimensions L_x, L_y, L_z. On one hand, no choice of the cavity dimensions satisfy the second condition $2\omega_{011}-\omega_{110}=\omega_{132}$. On the other hand, the first condition can be satisfied

$$2\omega_{011}-\omega_{110}=\omega_{130}\quad\Leftrightarrow\quad\left(\frac{L_z}{L_x}\right)^2\left(\frac{L_z}{L_y}\right)^2+\left(\frac{L_z}{L_x}\right)^2+3\left(\frac{L_z}{L_y}\right)^2=1. \tag{27}$$

Assuming for simplicity the square section of the cavity $L_x=L_y$, we obtain from Eq. (27) the condition for the cavity length c,

$$\frac{L_z}{L_x}=\frac{L_z}{L_y}=\xi=\sqrt{\sqrt{5}-2}\approx 0.486. \tag{28}$$

The resonantly enhanced signal mode ω_{130} (for shortness, the z component of the electric field) in the cavity satisfying Eq. (28) reads

$$E_z^{\mathrm{sig}}(\boldsymbol{x},t)=G\frac{\kappa\varepsilon_0(cB_0)^3\pi^2Q}{(L_z\omega_{130}/c)^2}\sin(\omega_{130}t)\sin\left(\frac{\pi x}{L_x}\right)\sin\left(\frac{3\pi y}{L_y}\right), \tag{29}$$

where $Q=\omega_{130}/\Gamma$ is the cavity quality factor related to mode ω_{130}, and the geometric factor G reads

$$G=\frac{1}{\xi^2}\big(\xi\sqrt{2(1+\xi^2)}+4+\xi^2\big)-\frac{\beta}{\xi^2}\big(\xi\sqrt{2(1+\xi^2)^3}+1-\xi^2\big)\;\approx\;8.517\quad\text{for }\beta=\frac{7}{4}. \tag{30}$$

Note that for a certain critical $\beta\approx 2.92$ the signal mode vanishes even for the resonant cavity geometry. However, for other choice of pump and signal modes (say, $\omega_{\mathrm{sig}}=\omega_{150}=2\omega_{130}-\omega_{011}$), the resonant geometry configuration would be changed, and the numerical value of critical β would be different.

It seems to be a counterintuitive result that one can resonantly enhance signal mode $2\omega_2-\omega_1$ (after certain adjustment of cavity geometry), whereas mode $2\omega_2+\omega_1$ remains always suppressed.

VI. DISCUSSION

In the current paper, we have formulated the conditions for the resonant amplification of a signal mode in a cavity of arbitrary shape and applied them to the analysis of linear and rectangular cavities. We have demonstrated that two pump modes with frequencies ω_1 and ω_2 in a rectangular cavity resonantly produce a signal mode with frequency $2\omega_2-\omega_1$ ($2\omega_1-\omega_2$) for a certain cavity geometry. On the other side, we have proved that the signal modes with frequencies $2\omega_1+\omega_2$ ($2\omega_2+\omega_1$) as well as the third harmonics $3\omega_1$ ($3\omega_2$) are not resonantly amplified. Recall that the resonant amplification means an enhancement in Q times, where Q is the cavity quality factor which can achieve a numerical value up to 10^{12} [21].

The crucial point of our proof for the absence of resonance in rectangular cavity is that the cavity eigenmodes include only trigonometric functions which allows us to make analytic calculations for arbitrary cavity modes. This stops working for a cavity of arbitrary shape, in that case the numerical calculations for specific cavity modes become necessary.

The absence of resonant amplification of the third harmonics and combined plus modes seems to be connected with the polarization structure of the vector gauge field. Considering the similar problem of higher-order harmonics generation for massless scalar field with $\lambda\varphi^4$ interaction instead of the electromagnetic one, one first obtains the analog of the inhomogeneous wave equation (5) which reads $\Box\varphi = \lambda\varphi^3$. Decomposing φ into the pump mode $\varphi_{\rm pump}$ which is a cavity eigenmode, and the signal mode $\varphi_{\rm sig}$ of small amplitude, one can easily check that the third harmonic does generate for the scalar field. We hope that the reason for this difference between scalar and electromagnetic field would become more clear after considering aforementioned processes on a quantum level.

To conclude, we have considered the problem of vacuum generation of higher-order harmonics only from the theoretical side. Although the scheme of such an experiment for the case of cylindrical cavity was, in general, studied in Ref. [13], there are still several unsolved issues, including proper treatment of nonlinearities from the cavity walls, etc.

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