


Symmetry-protected non-Abelian geometric phases in optical waveguides with nonorthogonal modes

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(Received 13 May 2021; accepted 20 December 2021; published 5 January 2022)

The generation of non-Abelian geometric phases from a system of evanescently coupled waveguides is considered within the framework of nonorthogonal coupled-mode theory. Here, we study an experimentally feasible tripod arrangement of waveguides that contain dark states from which a nontrivial $U(2)$ mixing can be obtained by means of an adiabatic parameter variation. We investigate the influence of higher-order contributions beyond nearest-neighbor coupling as well as self-coupling on the stability of a non-Abelian $U(3)$ phase generated from an optical tetrapod setup. Our results indicate that, despite the mode nonorthogonality, the symmetry of dark states protects the geometric evolution of light from distortion.

DOI: [10.1103/PhysRevA.105.013507](https://doi.org/10.1103/PhysRevA.105.013507)

I. INTRODUCTION

Abelian as well as non-Abelian gauge theories feature prominently in modern theories of fundamental interactions. They also occur frequently in a variety of geometric settings and are therefore central to much of modern mathematics. Their formulation in terms of gauge fields is deeply connected to the notion of a geometric phase. These phase factors arise naturally when considering the adiabatic evolution of a state vector in Hilbert space [1–3]. Besides this deeper insight into geometric and topological notions at an experimentally feasible scale, non-Abelian phases (holonomies) are important for holonomic and topological quantum computation, where they offer parametric robustness to increase stability of the computational process [4,5]. Consequently, there has been increased interest in the study and implementation of artificial gauge fields and symmetry groups [6]. Successful implementations ranged from single artificial gauge fields in photonic [7], superconducting [8], and atomic systems [9] to experimental simulation of lattice gauge theories [10].

In recent years, a novel approach was put forward to realize geometric phases in terms of integrated photonic structures such as laser-written waveguides in fused silica. In such structures, adiabatic population transfer [11] can be simulated by coherent tunneling between slowly varying waveguide geometries [12]. These optical structures have been proven to be a versatile tool box that combines the proven capabilities of modern optics with a high degree of interferometric stability [13]. For instance, the emergence of a non-Abelian Berry phase was observed when injecting coherent states of light into topologically guided modes [14,15], whereas other implementations made use of a tripod arrangement of evanescently coupled waveguides [16,17]. In Ref. [18], a proposal based on a photonic bus mode was studied with the view to implement a controlled-NOT gate on single photons. An

extension to an N -pod scheme with arbitrary photon-number states injected was proposed in Ref. [19]. However, all the above-mentioned proposals rely on power orthogonality of transverse modes. In general, the deviations from mode orthogonality can significantly distort the dynamics of light in coupled-mode systems.

In our present paper, we overcome the limitations of the weak-coupling regime by investigating the properties of geometric phases within the framework of nonorthogonal coupled-mode theory (NOCMT). In this setting, the nonorthogonality of transverse modes comes to light when decreasing the separation between adjacent waveguides. In this regime, the energy splitting between an adiabatic subspace and excited states may be large, thus favoring the generation of an adiabatic quantum holonomy. We present an optical setup for the generation of $U(2)$ -valued and $U(3)$ -valued geometric phases arising from tripod and tetrapod arrangements of waveguides, respectively, into which coherent light is being injected. After an expansion of the system in terms of normal modes, an analytical computation of the geometric phase reveals that the symmetry of dark states protects the state of light from any distortion.

The paper is structured as follows. In Sec. II, we review the NOCMT for a network of weak-index contrast waveguides. Section III is dedicated to the study of a tripod arrangement of waveguides in an experimentally feasible setting. We compute its geometric phase from a normal-mode expansion of the dark states and compare it to a numerical propagation of the longitudinal fields, thus quantifying the diabatic error. In Sec. IV, the influence of higher-order coupling and self-coupling is illuminated using the example of a $U(3)$ -valued geometric phase generated from an optical tetrapod. Finally, Sec. V contains a summary as well as some concluding remarks.

II. NONORTHOGONAL COUPLED-MODE THEORY

Since the inception of the concept of coupled modes in electromagnetic systems [20,21], coupled-mode theory has

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become a well-established tool in the description of parametric nonlinear devices, waveguide structures, optical fiber networks, and various other optoelectronic structures [22–24]. The starting point of our investigation is the propagation of coherent light in a linear, lossless, and isotropic medium patterned with an inhomogeneous lattice of N waveguides with refractive-index profile $n(\mathbf{r}) = n_0 + \sum_j \Delta n_j(\mathbf{r})$, where n_0 is the refractive index of the host material and Δn_j is the refractive-index contrast of the j th waveguide. The dynamics of an electromagnetic wave is determined by the inhomogeneous wave equation for the electric field $\mathbf{E}(\mathbf{r}, t)$:

$$\nabla^2 \mathbf{E} - \frac{n^2}{c^2} \partial_t^2 \mathbf{E} = -\nabla(\mathbf{E} \cdot \nabla \ln n^2), \quad (1)$$

where c denotes speed of light in vacuum. Under the assumption of a weak refractive-index contrast, that is, $\nabla \ln n^2 \approx \mathbf{0}$, Eq. (1) turns into a homogeneous wave equation. It follows that the Helmholtz equation for the paraxial propagation (slowly varying envelope) of an electromagnetic plane wave $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})e^{i(\omega t - \beta z)}$ reads

$$[\nabla_{\perp}^2 + \tilde{\lambda}^{-2} n^2(\mathbf{r}) - 2i\beta \partial_z - \beta^2] \mathbf{E}(\mathbf{r}) = \mathbf{0} \quad (2)$$

where $\tilde{\lambda} = c/\omega$, z denotes the propagation coordinate, β is the propagation constant, and ∇_{\perp}^2 denotes the Laplacian with respect to the transverse coordinates \mathbf{r}_{\perp} . In a weak index-contrast lattice we have $\Delta n_j \ll n_0$, hence the vector character of the Helmholtz equation can be neglected by factoring out a constant unit vector \mathbf{v} , because polarization effects at the interface of a waveguide become unimportant, and scalar wave theory can be applied [25]. For the treatment of coupled-mode theory that includes high-index contrast waveguides we refer the reader to Ref. [26].

In the light of the assumption of weak index contrast, we expand the wave packet in terms of tight-binding modes as

$$\mathbf{E}(\mathbf{r}) = \sum_{j=1}^N a_j(z) w_j(\mathbf{r}) \mathbf{v}, \quad (3)$$

where w_j is the transverse field mode that is localized around the waveguide Δn_j , and a_j is the corresponding longitudinal field amplitude determining the dynamics of the light field. The former constitute a quasicomplete (complete for divergence free fields) basis in which fields with compact support over the transverse plane can be expanded. Here we assume that the shape of the transverse field w_j remains constant throughout the propagation. However, because we consider waveguides that vary their position in the transverse plane along the propagation direction, the field $w_j(x - x_j(z), y - y_j(z))$ still depends on z , where (x_j, y_j) marks the center of the j th waveguide. On the other hand, in any typical setup the waveguide position changes only very slowly over the propagation length (paraxial approximation), i.e., $\partial_z x_j \approx \partial_z y_j \approx 0$, so that we can safely assume $w_j(\mathbf{r}) \approx w_j(\mathbf{r}_{\perp})e^{i\beta_j z}$ with an individual propagation constant β_j of the j th waveguide. The latter ones can be neglected throughout this paper by noting that they act merely as offsets when considering an identical writing process for each waveguide.

We assume the transverse fields to satisfy their own Helmholtz equation in the (x, y) plane in the absent of the

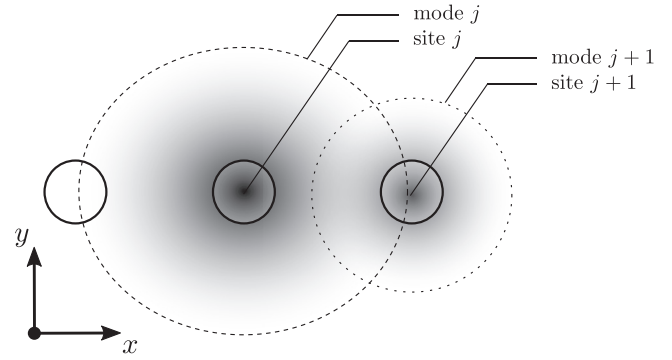


FIG. 1. Schematic front view of evanescently coupled waveguides (black circles). Each site j supports a fundamental transverse mode w_j (shaded areas). The modes extend towards the nearest-neighbor sites.

other waveguides, i.e.,

$$\{\nabla_{\perp}^2 + \tilde{\lambda}^{-2} [n_0 + \Delta n_j(\mathbf{r})]^2 - \beta^2\} w_j(\mathbf{r}_{\perp}) = 0. \quad (4)$$

This puts the expansion (3) in the light of a weighted eigenmode expansion. We can simplify the above expression by noting that for a weak index contrast we have $[n_0 + \Delta n_j]^2 \approx n_0^2 + 2n_0 \Delta n_j$ and $\beta^2 \approx n_0^2 / \tilde{\lambda}^2$. It thus follows that Eq. (4) takes the form

$$\left(\frac{\tilde{\lambda}^2}{2n_0} \nabla_{\perp}^2 + \Delta n_j(\mathbf{r}) \right) w_j(\mathbf{r}_{\perp}) = 0. \quad (5)$$

Inserting the ansatz (3) into Eq. (2), making use of Eq. (5), and employing the same assumptions as before, we obtain

$$\sum_{j=1}^N \left(i\tilde{\lambda} \partial_z a_j w_j - \sum_{\substack{m=1 \\ m \neq j}}^N \Delta n_m a_j w_j \right) = 0. \quad (6)$$

When contracting Eq. (6) with w_k^* , we observe that the transverse fields overlap with the surrounding waveguides (sites) and their modes gives rise to transverse interactions (see Fig. 1). Here, we distinguish between two different contributions that occur in Eq. (6) after contraction, namely,

$$\begin{aligned} \kappa_{jk} &= \frac{1}{\tilde{\lambda}} \sum_{\substack{m=1 \\ m \neq j}}^N \int_{S_{\infty}} \Delta n_m w_k^* w_j d^2 \mathbf{r}_{\perp}, \\ \sigma_{jk} &= \int_{S_{\infty}} w_k^* w_j d^2 \mathbf{r}_{\perp}, \end{aligned} \quad (7)$$

where we integrate over the entire (x, y) -plane S_{∞} . In Eq. (7), $\kappa_{jk} \approx \frac{1}{\tilde{\lambda}} \int_{S_{\infty}} \Delta n_k w_k^* w_j d^2 \mathbf{r}_{\perp}$ is the evanescent coupling between waveguides j and k , while σ_{jk} describes the overlap of their respective transverse modes. In particular, there is also a self-coupling $v_j = \kappa_{jj}$ due to the presence of other waveguides around the j th mode. The latter is usually the smallest contribution to the dynamical propagation (6). This can be seen from Fig. 2 where these quantities were computed for two adjacent (identically written) cylindrical waveguides as a function of their separation. We considered a scenario in which each waveguide supports only its first transverse

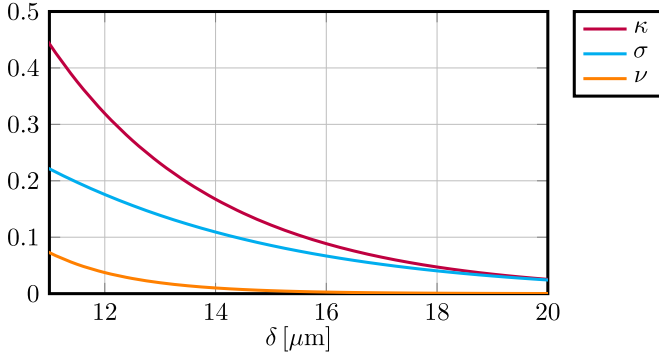


FIG. 2. Coupling κ (upper line) in mm^{-1} , overlap σ (middle line), and self-coupling ν (lowest line) in mm^{-1} between two adjacent waveguides as a function of their separation δ (measured from the center). The results are shown for cylindrical fibers of radius $R = 4.8 \mu\text{m}$ with bulk index $n_0 = 1.452$ and weak contrast $n_1 = 6.53 \times 10^{-4}$ for the inside of each waveguide. The wavelength of the injected light beam considered is $\lambda = 633 \text{ nm}$. The interpolating functions were obtained by evaluating Eq. (7) for several values of δ .

mode $w_j(\mathbf{r}_\perp)$, given in terms of Bessel functions [27]. These assumptions are not too restrictive, as monomode operation and mode synchronism are often required as operating conditions. Furthermore, due to mode overlaps, the transverse modes form a nonorthogonal basis for describing the light propagation. The nonorthogonality can be neglected if the distance between adjacent waveguides is large. Then one works in the regime of orthogonal coupled-mode theory (OCMT), i.e., $\sigma_{jk} \approx 0$ for $j \neq k$ [13,28,29].

With the definitions from Eqs. (7) at hand, the coupled-mode equations become

$$i \sum_{j=1}^N \sigma_{jk} \partial_z a_j = \sum_{j=1}^N \kappa_{jk} a_j. \quad (8)$$

This form of the paraxial Helmholtz equation resembles a discrete Schrödinger equation. Equation (8) can be reformulated as a first-order matrix differential equation in z , with $(\Sigma)_{jk} = \sigma_{jk}$ and $(\mathbf{K})_{jk} = \kappa_{jk}$ being the power and coupling matrix, respectively. Because Σ and \mathbf{K} do not commute in general, the generator $\Sigma^{-1}\mathbf{K}$ of the dynamics is not necessarily Hermitian, and $\mathbf{a}^\dagger \mathbf{a}$ would not be conserved throughout the propagation. Particularly, we observe that these generators of the NOCMT lie within the class of real-valued nonsymmetric matrices. Nevertheless, the modified intensity distribution $\mathbf{a}^\dagger \Sigma \mathbf{a}$ (positive operator-valued measurement) remains constant as long as \mathbf{K} is Hermitian, that is, dissipative effects are negligible as assumed.

The non-Hermitian nature of Eq. (8) can even be lifted completely by transforming to a set of longitudinal normal modes (also referred to as supermodes in Refs. [16,26]) $\{b_j(z)\}_{j=1}^N$ such that the overall propagation is unitary [28]. Even though we did not start our derivation from power-orthogonal modes, conservation of energy demands that such normal modes always exist. Obviously, in contrast to the waveguide mode a_j , the corresponding normal mode b_j will contain contributions from adjacent waveguides. To be precise, their relation is given by the conserved quantity $\mathbf{b}^\dagger \mathbf{b} =$

$\mathbf{a}^\dagger \Sigma \mathbf{a}$. Moreover, because the power matrix Σ is positive definite, there exists a (nonunique) matrix \mathbf{Q} such that $\Sigma = \mathbf{Q}^\dagger \mathbf{Q}$. It follows that the normal modes are given by the transformation $\mathbf{b} = \mathbf{Q} \mathbf{a}$. With these preparations, Eq. (8) can be rewritten as

$$i \partial_z \mathbf{b} = \mathbf{H} \mathbf{b}, \quad (9)$$

where \mathbf{H} is given by [28]

$$\mathbf{H} = (\mathbf{Q}^{-1})^\dagger \mathbf{K} \mathbf{Q}^{-1}, \quad (10)$$

which is Hermitian as long as $\mathbf{K} = \mathbf{K}^\dagger$. From similarity with Eq. (10) it follows that $\Sigma^{-1}\mathbf{K}$ is always diagonalizable with a real spectrum. Even though the matrix \mathbf{H} in Eq. (10) can be viewed as a generalized coupling matrix mediating population transfer between the normal modes $\{b_j\}_j$, its off-diagonal entries cannot be viewed as the usual evanescent couplings as nonorthogonality of the waveguide modes $\{a_j\}_j$ has a fundamentally different distance behavior with decreasing waveguide separation. In particular, the ansatz of exponentially decaying coupling behavior as in Ref. [12] becomes inappropriate. This distortion due to nonorthogonality was observed in Ref. [30] by means of a fluorescence measurement.

Adiabatic propagation

While a general state vector $\Psi(z)$ evolves according to Eq. (9), under the adiabatic assumption any initial preparation $\Psi(z_0)$ in a dark subspace (zero-eigenvalue eigenspace) will be mapped to a state $\Psi(z_f) = \mathbf{U}(z_0, z_f) \Psi(z_0)$ that is also in the dark subspace. Then, we can expand the state Ψ at every instance z in terms of the dark states, i.e., $\Psi(z) = \sum_a U_{ab}(z) \mathbf{d}_a(z)$, with initial condition $\Psi(z_0) = \mathbf{d}_a(z_0)$. Inserting this ansatz into Eq. (9) and following Ref. [3], we obtain $(\mathbf{U}^{-1} \partial_z \mathbf{U})_{ba} = (\mathbf{A}_z)_{ab}$ where we used $\mathbf{H} \Psi = \mathbf{0}$ for states Ψ in the dark subspace. Here we defined the adiabatic connection [31] as $(\mathbf{A}_z)_{ab} = \mathbf{d}_b^\dagger \partial_z \mathbf{d}_a$. A formal solution for \mathbf{U} is then given in terms of the matrix exponential

$$\mathbf{U}(z_0, z_f) = \mathcal{T} \exp \int_{z_0}^{z_f} \mathbf{A}_z dz, \quad (11)$$

where \mathcal{T} denotes z ordering.

For a collection of coupled waveguides, the z dependence of the system is functionally connected to the distances between the waveguides $\{\delta_\mu(z)\}_\mu$ that form local coordinates of an abstract parameter space \mathcal{M} (control manifold). Then, the propagation along the z direction can equivalently be viewed as a parameter variation along a curve $\mathcal{C} : [z_0, z_f] \rightarrow \mathcal{M}$. In the following sections we are not interested in arbitrary paths, but only those that (approximately) form loops in \mathcal{M} , i.e., $\mathcal{C}(z_0) = \mathcal{C}(z_f)$. Population transfer between waveguides associated with such a loop appears to be robust against stochastic fluctuations of the control parameter, as the transformation $\mathbf{U}(z_0, z_f)$ only depends on the area enclosed by the loop \mathcal{C} [32].

III. TRIPOD ARRANGEMENT OF WAVEGUIDES

In order to examine the properties of geometric phases in relation to nonorthogonal modes more closely, we shall focus on a benchmark example that can be implemented using the

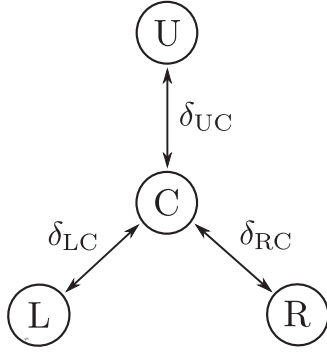


FIG. 3. Front view of the tripod arrangement; arrows indicate interactions between the outer waveguides and the central waveguide. Within NOCMT, this includes evanescent coupling as well as the overlap of transverse modes. In OCMT, this overlap is neglected. The interaction strength depends on the separation δ_{ic} between the corresponding waveguides.

current state of technology. A schematic representation of the waveguide network under investigation is shown in Fig. 3. There, the outer waveguides L , R , and U interact only implicitly via the central waveguide C (nearest-neighbor coupling). To be more precise, we consider a situation in which the transverse fields of the outer waveguides have couplings and overlaps that can be safely disregarded. Then, the coupling matrix of the system is

$$\mathbf{K} = \sum_{j \neq C} \kappa_{jC} (\mathbf{w}_j^\dagger \mathbf{w}_C + \mathbf{w}_C^\dagger \mathbf{w}_j) + \sum_j \nu_j \mathbf{w}_j^\dagger \mathbf{w}_j,$$

where j enumerates the waveguides $\{L, R, U, C\}$. Here, $\{\mathbf{w}_j\}_j$ denotes the standard basis in \mathbb{C}^4 that forms an orthonormal representation of the transverse eigenmodes of the j th waveguide. For the moment, we neglect self-coupling on the matrix diagonal, as it appears to be the smallest contribution to the overall propagation. The corresponding power matrix of the system then reads

$$\mathbf{\Sigma} = \sum_{j \neq C} \sigma_{jC} (\mathbf{w}_j^\dagger \mathbf{w}_C + \mathbf{w}_C^\dagger \mathbf{w}_j) + \sum_j \mathbf{w}_j^\dagger \mathbf{w}_j, \quad (12)$$

where we assumed normalized transverse modes such that $\sigma_{jj} = 1$. The setup depicted in Fig. 3 is known as a tripod system. The tripod scheme, arising in different physical settings, is often used as a starting point for the generation of $U(2)$ -valued phases (see, e.g., Refs. [11,33,34]).

One possible transformation between waveguide and normal modes can be obtained by a Cholesky factorization of the matrix (12), that reads

$$\mathbf{Q} = \sum_{j \neq C} (\sigma_{jC} \mathbf{w}_j^\dagger \mathbf{w}_C + \mathbf{w}_j^\dagger \mathbf{w}_j) + \sqrt{1 - \sigma^2} \mathbf{w}_C^\dagger \mathbf{w}_C, \quad (13)$$

where we introduced the vector $\boldsymbol{\sigma} = (\sigma_{LC}, \sigma_{RC}, \sigma_{UC})$. From the matrix (13) we can compute the Hermitian generator for the evolution of normal modes according to Eq. (10), and we find

$$\mathbf{H} = \sum_{j \neq C} s \kappa_{jC} (\mathbf{w}_j^\dagger \mathbf{w}_C + \mathbf{w}_C^\dagger \mathbf{w}_j) - 2s^2 \boldsymbol{\sigma} \cdot \boldsymbol{\kappa} \mathbf{w}_C^\dagger \mathbf{w}_C,$$

with $\boldsymbol{\kappa} = (\kappa_{LC}, \kappa_{RC}, \kappa_{UC})$ and $s = 1/\sqrt{1 - \sigma^2}$. The system's dark states

$$\begin{aligned} \mathbf{d}_1 &= \sin \theta \mathbf{w}_L - \cos \theta \mathbf{w}_R, \\ \mathbf{d}_2 &= \cos \theta \sin \varphi \mathbf{w}_L + \sin \theta \sin \varphi \mathbf{w}_R - \cos \varphi \mathbf{w}_U \end{aligned} \quad (14)$$

satisfy $\mathbf{H} \mathbf{d}_a = \mathbf{0}$, where we defined $\theta = \arctan(\kappa_{RC}/\kappa_{LC})$ and $\varphi = \arctan(\kappa_{UC}/\sqrt{\kappa_{LC}^2 + \kappa_{RC}^2})$. Quite remarkably, as long as the second-order (next-nearest-neighbor) contributions to the coupling [35] as well as self-coupling are negligible, the introduction of mode overlaps does not break the degeneracy of the system, and still gives rise to a twofold degenerate dark subspace similar to its counterpart in OCMT [17]. In particular, the dark states of \mathbf{H} have the same structure as the ones found in OCMT but with their coefficients now giving the population of the associated normal mode that differs from the individual modes of each waveguide.

With the dark states known, computing the connection \mathbf{A}_z becomes a straightforward task and, thus, the matrix exponential (11) can be evaluated analytically to

$$\mathbf{U}(z_0, z_f) = \begin{pmatrix} \cos \phi & \sin \phi \\ -\sin \phi & \cos \phi \end{pmatrix}, \quad (15)$$

written in the basis $\{\mathbf{d}_a(z)\}_a$. Here

$$\phi(z_0, z_f) = \int_{z_0}^{z_f} \frac{\kappa_{UC} (\kappa_{LC} \partial_z \kappa_{RC} - \kappa_{RC} \partial_z \kappa_{LC})}{\sqrt{\kappa_{LC}^2 + \kappa_{RC}^2 + \kappa_{UC}^2 (\kappa_{LC}^2 + \kappa_{RC}^2)}} dz \quad (16)$$

is a phase factor depending only on the geometry of the associated parameter variation \mathcal{C} . The holonomy matrix (15) is the same unitary transformation as for the case of orthogonal transverse modes [17]. This implies that the results of the NOCMT and the OCMT predict the same output state as long as the evolution takes place solely in the dark subspace. This means that the geometric phase generated from the tripod is robust against the nonorthogonal nature of transverse modes. This is a remarkable result as, in general, the deviations from mode orthogonality can significantly distort the dynamics of light in a coupled-mode system. In contrast to Ref. [16] where a normal-mode expansion for the tripod-well structure was proposed, here we showed that the interaction of normal modes is independent of the mode overlaps σ_{iC} as long as the propagation appears to be adiabatic.

Propagation with Gaussian-shaped geometries

Here, we present an experimentally feasible parameter loop \mathcal{C} that can be realized by current laser-writing techniques [13]. Suppose one implements the geometry

$$\begin{aligned} \delta_{LC}(z) &= \Delta - \Omega \exp\left(-\frac{(z - \bar{z} - \tau)^2}{T^2}\right), \\ \delta_{RC}(z) &= \Delta - \Omega \exp\left(-\frac{(z - \bar{z} + \tau)^2}{T^2}\right), \\ \delta_{UC}(z) &= \Delta_U, \end{aligned} \quad (17)$$

in which the central waveguide is assumed to remain at the origin of the transverse plane, while the separation to the L and R waveguide changes according to a Gaussian function. In Eq. (17), \bar{z} denotes half the total propagation length, and

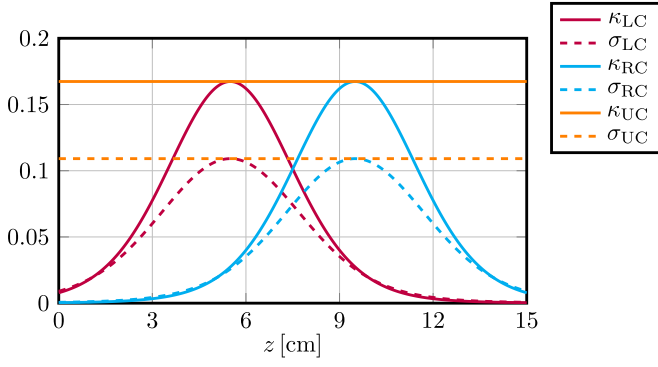


FIG. 4. Couplings κ_{iC} (in mm^{-1}) and overlaps σ_{iC} between the i th outer waveguide and the central waveguide as a function of the propagation distance z for the case of the z -dependent waveguide position (17). The first interaction with waveguide L is increased (red left peaks) and then the interaction with waveguide R is increased (blue right peaks) while the waveguide U is kept at constant distance. The chosen parameters are $\Delta = 40 \mu\text{m}$, $\Delta_U = 14 \mu\text{m}$, $\Omega = 26 \mu\text{m}$, $T = 8 \text{ cm}$, and $\tau = 2 \text{ cm}$.

$\Delta - \Omega$ is the minimal separation between waveguide L (R) and the central waveguide C , with waveguide U having constant distance Δ_U . Furthermore, T is the width parameter, and τ is the separation of the two Gaussian peaks from the center at \bar{z} . The relevant coupling strengths and overlaps of the respective waveguides can be determined from their mutual separation, shown in Fig. 2. Given the variation (17), the respective coupling constants and overlaps are shown in Fig. 4.

A parameter loop starting at the initial point $\delta_0 = (\Delta, \Delta, \Delta_U)$ will induce a nontrivial mixing of states

$$\mathbf{d}_1(\delta_0) \approx \mathbf{w}_L, \quad \mathbf{d}_2(\delta_0) \approx \mathbf{w}_R.$$

After integration of the phase factor (16) and subsequent evaluation of the transformation (15), this mixing can be given explicitly. The adiabatic limit is applicable if the loop \mathcal{C} given by Eq. (17) is traversed slowly enough compared to the energy splitting between the dark subspace and the excited bright subspaces [11,12]. We found that the separation between dark and bright states is slightly shifted when compared to the eigenvalue splitting known from OCMT, due to the contribution from mode overlaps. In order to evaluate to which extent the adiabatic approximation is justified in NOCMT, we compute the (expected) gate fidelity $0 \leq \bar{F} \leq 1$ in terms of input state fidelities

$$F_{\Psi(z_0)} = |\tilde{\Psi}(z_f)^\dagger \mathbf{Q}^{-1}(z_f) \mathbf{U}(z_f) \Psi(z_0)|^2, \quad (18)$$

with $\mathbf{U}(z_f)\Psi(z_0)$ being the ideal output state predicted by the holonomic theory (15) and $\tilde{\Psi}(z_f)$ obtained from a numerical propagation subject to the paraxial Helmholtz equation (8). For the scenario under study, the gate fidelity of the dark state ensemble $\{\{1/2, \mathbf{w}_L\}, \{1/2, \mathbf{w}_R\}\}$ amounts to approximately $\bar{F} \approx 98.6\%$.

We can conclude that robust adiabatic parallel transport of an initial wave packet can be achieved within the range of experimentally feasible setups, despite the influence of mode nonorthogonality. In fact, the expected distortion from close-coupling dynamics comes to light when one of the bright states \mathbf{w}_U or \mathbf{w}_C is excited. In Fig. 5 we compare the popula-

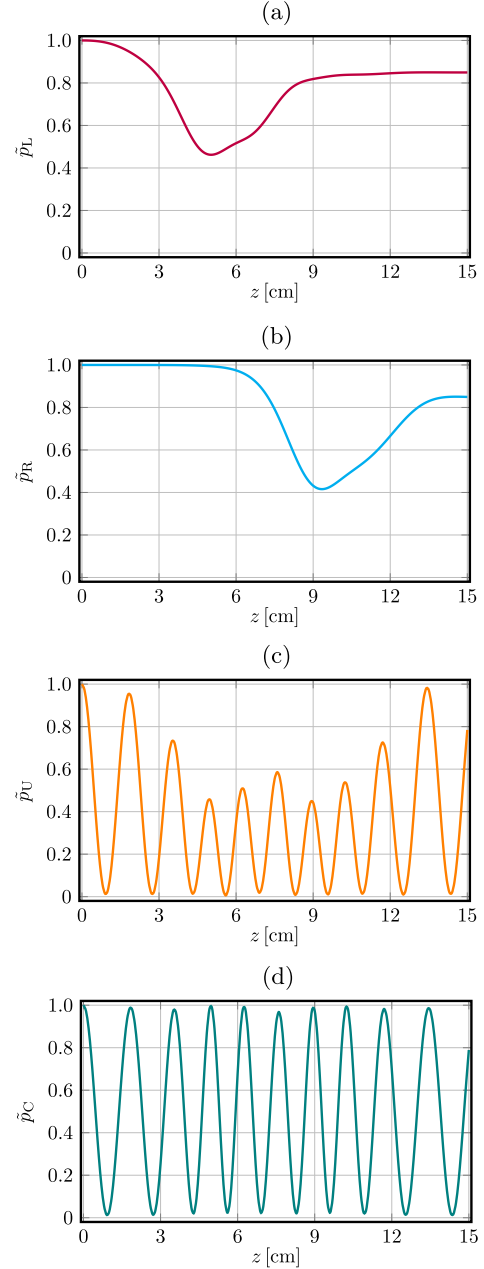


FIG. 5. Simulated propagation of $\Psi(z)$ in terms of waveguide modes [subject to Eq. (8)] for the waveguide position (17) under investigation. The four plots show the change of intensity throughout propagation for different initial input states $\{\mathbf{w}_j\}_j$. It can be seen that the bright states (U and C) are subject to rapid intensity fluctuations due to deviations from mode orthogonality. The parameter values are $\Delta = 40 \mu\text{m}$, $\Delta_U = 14 \mu\text{m}$, $\Omega = 26 \mu\text{m}$, $T = 8 \text{ cm}$, and $\tau = 2 \text{ cm}$. (a) Intensity in site L for $\Psi(z_0) = \mathbf{w}_L$. (b) Intensity in site R for $\Psi(z_0) = \mathbf{w}_R$. (c) Intensity in site U for $\Psi(z_0) = \mathbf{w}_U$. (d) Intensity in site C for $\Psi(z_0) = \mathbf{w}_C$.

tion transfer between the waveguides for different initial input states. One can clearly see the robustness of the dark states, while the excited states are exposed to more rapid population transfer. In fact, when adding the bright states to the input ensemble, a gate error emerges that is not of diabatic nature, but has its origin in the fact that the parameter variation (17)

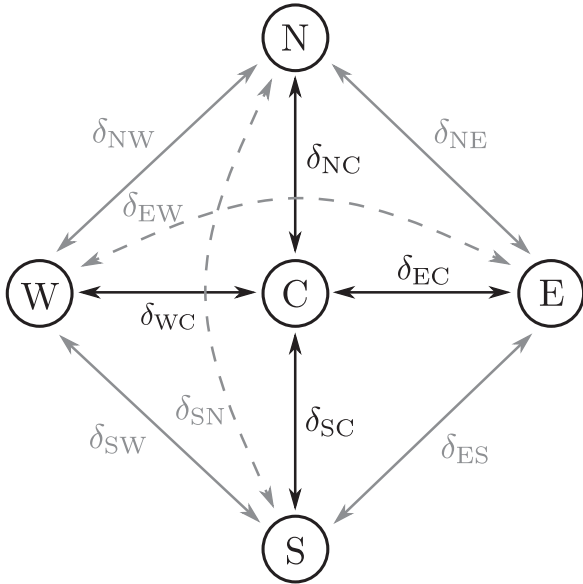


FIG. 6. Front view of the tetrapod arrangement; arrows indicate interaction between the respective waveguides. Within the NOCMT this includes evanescent coupling as well as the overlap of transverse modes. In the case of OCMT the overlap is neglected. The strength of interaction depends on the separation δ_{iC} between the corresponding waveguides. Black arrows were chosen for nearest-neighbor coupling (first order). Gray solid lines are reserved for next-nearest-neighbor coupling (second order), while gray dotted lines depict coupling between opposite placed waveguides (third order).

does not form a perfect loop. This matching error of input and output basis is rather small for the dark states but gets amplified for the bright states due to rapid intensity oscillations, thus leading to a substantial decrease in gate fidelity, $\bar{F} \approx 88.9\%$. The relation between mode nonorthogonality and rapid intensity oscillations was experimentally verified in Ref. [30]. In general, the robustness of dark states can be attributed to their form given by Eq. (14) which is independent of σ_{iC} and therefore identical to their counterparts in OCMT [17]. Hence, despite the overlaps, absence of light in waveguide C can still be used as a measure of adiabaticity.

IV. HIGHER-ORDER COUPLING IN THE TETRAPOD ARRANGEMENT

We now turn to the influence of self-coupling and higher-order contributions in a waveguide network which had been neglected in our calculations thus far. These effects potentially break the degeneracy structure of the system under investigation. We found numerically that, in the tripod arrangement, these contributions are completely negligible for the propagation lengths investigated ($|z_f - z_0| \leq 15$ cm). However, this result depends crucially on the distance behavior shown in Fig. 2, and does not need to be valid beyond step-index waveguides.

In order to show that coupling beyond nearest-neighbor hopping as well as self-coupling can become relevant when the distance between waveguides is substantially decreased, we now turn to a tetrapod arrangement depicted in Fig. 6.

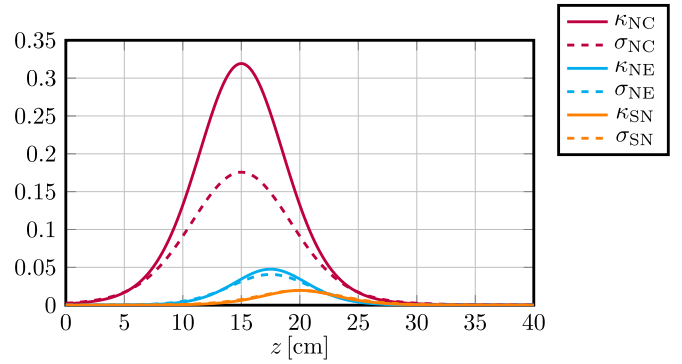


FIG. 7. Couplings (in mm^{-1}) and overlaps between the northern (N) waveguide and j th waveguide ($j \in \{C, E, S\}$) of the tetrapod arrangement as a function of the propagation distance z for the waveguide geometry (21). Coupling to the central (C) waveguide is of first order (upper lines), coupling to the eastern (E) waveguide is of second order (middle lines), and coupling to the south (S) waveguide is of third order (lowest lines). The relevant parameters are given by $\Delta = 38 \mu\text{m}$, $\Delta_w = 12 \mu\text{m}$, $\Omega = 26 \mu\text{m}$, $T = 15$ cm, and $\tau = 5$ cm.

Placing more waveguides around the central site will inevitably result in a system having no degeneracy at all. The coupling and power matrices of the (ideal) tetrapod system without self-coupling and higher-order effects are given by

$$\mathbf{K} = \sum_{j \neq C} \kappa_{jC} (\mathbf{w}_j^\dagger \mathbf{w}_C + \mathbf{w}_C^\dagger \mathbf{w}_j) \quad (19)$$

and

$$\mathbf{\Sigma} = \sum_{j \neq C} \sigma_{jC} (\mathbf{w}_j^\dagger \mathbf{w}_C + \mathbf{w}_C^\dagger \mathbf{w}_j) + \sum_j \mathbf{w}_j^\dagger \mathbf{w}_j, \quad (20)$$

where j runs through the alphabet $\{N, E, S, W, C\}$, and hence $\{\mathbf{w}_j\}_j$ might be thought of as the standard basis in \mathbb{C}^5 . For this system, the adiabatic propagation of normal modes will generate a $U(3)$ -valued mixing within the threefold degenerate dark subspace. A geometric phase designed via the tetrapod will

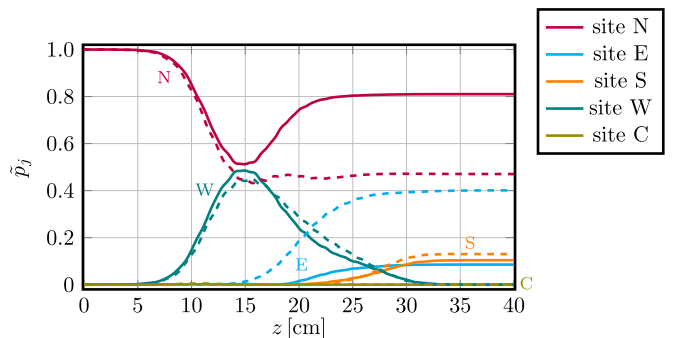


FIG. 8. Intensity distribution for the tetrapod arrangement subject to the geometry (21). The solid lines belong to the ideal tetrapod [Eqs. (19) and (20)], while the dashed lines describe propagation including higher-order couplings as well as self-coupling. The results were computed via a numerical simulation of Eq. (8) for an initial wave package \mathbf{w}_N , with parameters $\Delta = 38 \mu\text{m}$, $\Delta_w = 12 \mu\text{m}$, $\Omega = 26 \mu\text{m}$, $T = 15$ cm, and $\tau = 5$ cm for a network of 40-cm total length.

be not only matrix valued as the holonomy in Eq. (15), but truly noncommutative in its nature due to the richer parameter space. In a tripod arrangement, complex-valued couplings are necessary in order to create non-Abelian characteristics [16].

Propagation with Gaussian-shaped geometries

Higher-order coupling becomes relevant as soon as the propagation length of the evolution is substantially increased. In this regime, we can assume adiabaticity to apply. Then, the matrices in Eqs. (19) and (20) are no longer sparse, but have only nonvanishing entries. The diagonal entries of \mathbf{K} now contain the self-couplings ν_j . In the following, we will study the influence of such contributions on a benchmark geometry:

$$\begin{aligned}\delta_{NC}(z) &= \Delta - \Omega \exp\left(-\frac{(z - \bar{z} - \tau)^2}{T^2}\right), \\ \delta_{EC}(z) &= \Delta - \Omega \exp\left(-\frac{(z - \bar{z})^2}{T^2}\right), \\ \delta_{SC}(z) &= \Delta - \Omega \exp\left(-\frac{(z - \bar{z} + \tau)^2}{T^2}\right), \\ \delta_{WC}(z) &= \Delta_W.\end{aligned}\quad (21)$$

In the chosen waveguide geometry, the evolution of coherent light approximately starts and ends at the point $\delta_0 = (\Delta, \Delta, \Delta, \Delta_W)$, where the holonomy will generate a mixing of dark states

$$\mathbf{d}_1(\delta_0) \approx \mathbf{w}_N, \quad \mathbf{d}_2(\delta_0) \approx \mathbf{w}_E, \quad \mathbf{d}_3(\delta_0) \approx \mathbf{w}_S.$$

Given the geometry (21), we can derive the z -dependent form of the relevant parameters from Fig. 2. Knowing these parameters allows us, by simple geometric considerations (see Fig. 6), to construct higher-order couplings and overlaps as well. In Fig. 7, the coupling strengths of these different contributions are shown for the waveguide N . One observes clearly how the higher-order couplings have a substantially smaller influence compared to the nearest-neighbor coupling. Nevertheless, for longer propagation lengths (e.g., $|z_f - z_0| = 40$ cm) these contributions will lead to a deviation in the intensity distribution from the evolution predicted by the propagation through the ideal tetrapod system.

We also investigated the change of self-coupling $\nu_j(z)$ of each respective waveguide. There we observed that the order of magnitude of the self-coupling in the outer waveguides lies between those of the second-order and third-order contributions. However, the self-coupling ν_C is substantially larger, as the central waveguide mode is closely surrounded by the other waveguides.

In Fig. 8, the propagation through the ideal tetrapod (solid line) is compared to a propagation including higher-order couplings and self-coupling (dashed line). The state was propagated numerically for an initial input \mathbf{w}_N . Our results show that these additional effects (mostly second-order coupling and self-coupling ν_C) lead to increased population transfer between the waveguides. Due to symmetry breaking, this additional population transfer will not be robust against parametric fluctuations and mode nonorthogonalities as it is no longer described by a parallel transport in the degenerate dark subspace of the tetrapod.

V. CONCLUSION

In this paper, we considered the generation of geometric phases in integrated photonic waveguide structures within the framework of nonorthogonal coupled-mode theory. General arguments show that including the nonorthogonality of transverse modes leads to a set of tight-binding equations governing an evolution in which the conventional intensity distribution is not necessarily conserved. This issue was lifted by means of a normal-mode expansion. An analytical computation of the geometric phase and a subsequent evaluation of the intensity distribution for the tripod system showed a robustness of the adiabatic parallel transport against deviations from mode orthogonalities as long as adiabaticity holds and higher-order couplings are negligible. A subsequent study of the tetrapod arrangement showed that higher-order coupling as well as self-coupling can become relevant for longer propagation lengths, eventually breaking the degeneracy of the system, thus perturbing the generation of a $U(3)$ -valued geometric phase. However, for appropriate propagation lengths our numerical simulations of the population transfer between the waveguides showed that high fidelity transformations can be generated when working in the close-coupling regime.

Our paper paves the way for the study of adiabatic parallel transport by photonic structures within the close-coupling regime. The symmetry of the dark subspace leads to an inherent robustness of geometric phases towards the distortion of light by mode nonorthogonalities. Together with their parametric stability, geometric phases can be considered a powerful tool for the generation of stable population transfer of light even within the realm of the nonorthogonal coupled-mode theory.

ACKNOWLEDGMENT

Financial support by Deutsche Forschungsgemeinschaft Grant No. SCHE 612/6-1 is gratefully acknowledged.

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