Wigner time delay of particles elastically scattered by a cluster of zero-range potentials

M. Ya. Amusia,[1] A. S. Baltenkov,[2] and I. Woiciechowski[3,*]

[1]*Racah Institute of Physics, the Hebrew University, 91904 Jerusalem, Israel and Ioffe Physical-Technical Institute, 194021 St. Petersburg, Russian Federation*

[2]*Arifov Institute of Ion-Plasma and Laser Technologies, 100125 Tashkent, Uzbekistan*

[3]*Alderson Broaddus University, 101 College Hill Drive, Philippi, West Virginia 26416, USA*

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The Wigner time delay of slow particles in the process of their elastic scattering by compound targets consisting of several centers modeled by zero-range potentials is studied. It is shown that at asymptotically large distances from the target, the Huygens-Fresnel interference pattern formed by the spherical waves emitted by each of the potentials is transformed into a system of spherical waves generated by the geometric center of the target. These wave functions determine the flow of particles inward and outward through the surface of a sphere surrounding the target. The energy derivatives of the phase shifts of these functions are the partial Wigner time delays. General formulas, which establish a relationship between the s-phase shifts in particle scattering from each of the zero-range potentials and the phase shift in scattering by a cluster of the potentials, have been obtained. The clusters consisting of two, three, and four centers were considered as model targets. The centers, equally distant from each other, were assumed to be described by the same delta-function potentials. The partial Wigner time delays of slow particles scattered by the considered model targets were evaluated. The derived general formulas were applied to both the processes of electron scattering by atomic clusters trapping electrons and the processes of π-meson scattering on few-nucleon systems.

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I. INTRODUCTION

It is known that the equations of multiple scattering of s waves from a set of fixed in space zero-range potentials reduce to a system of ordinary algebraic equations [1]. The problem of s waves scattering from two centers was first addressed in Ref. [2] to verify the correctness of the impulse approximation. A detailed analysis of this problem was given by Brueckner in [3] where exact solutions of the wave equation for s scattering of a particle from two-point scatterers have been obtained. The scattering wave function $\psi_{\mathbf{k}}(\mathbf{r})$ was presented in [3] as a combination of a plane wave and two spherical s waves generated by the scattering centers. The amplitudes of these spherical waves were determined by the boundary conditions imposed on $\psi_{\mathbf{k}}(\mathbf{r})$ at the points where the zero-range potentials of the target are centered. The asymptotic form of the wave function $\psi_{\mathbf{k}}(\mathbf{r} \to \infty)$ determines the exact elastic scattering amplitude $F(\mathbf{k}, \mathbf{k}')$ in closed form. This method of calculation of the scattering amplitude in closed form applies to targets with larger than two delta centers.

Due to the lack of spherical symmetry of a compound target, the wave function $\psi_{\mathbf{k}}(\mathbf{r})$ and the scattering amplitude $F(\mathbf{k}, \mathbf{k}')$ cannot be represented as an expansion into a series of spherical harmonics. However, at asymptotically large distances from the target, where the target size can be neglected in comparison with the distance to the observation point, the pair of Huygens-Fresnel spherical waves generated by the point potentials transforms into a system of partial spherical waves $\varphi_\lambda(\mathbf{r})$ with the center located at the geometrical center of the target. The phase shifts $\eta_\lambda(k)$ in the radial parts of these spherical waves define both the flow of particles inward and outward through the surface of a sphere surrounding the target and the particle capture time in the scattering process.

The concepts of time delay and capture time of elastically colliding particles as quantum dynamical observables known as EWS time delay were originally introduced by Eisenbud, Wigner, and Smith in [4–6], respectively. The application of this scattering characteristic ranges from atomic [7–9] to nuclear physics, where it is applied in studies of meson and baryon unstable states [10–12]. The EWS time delay $\tau(E)$ in particle collisions is defined by the energy E derivative of the scattering phase shift $\delta(E)$:

$$\tau(E) = 2\hbar \frac{d\delta(E)}{dE}. \tag{1}$$

Note that the scattering phase shifts $\delta(E)$ in Eq. (1) are real for elastically colliding particles.

To calculate the EWS time delay (1) for a particle colliding with a set of identical zero-range potentials it is necessary to establish a relationship between the phase shifts $\delta_0(E)$ for the particle scattered from an isolated delta potential and the phase shift $\eta(E)$ for scattering from a multicenter target. One way of doing it is by using the scattering amplitude function $F(\mathbf{k}, \mathbf{k}')$. This approach was implemented in [13], where the EWS time delay of slow electrons colliding with a pair of atoms in the model of nonoverlapping atomic spheres was calculated. Extension of the approach to three or more centers

*woiciechowskiia@ab.edu

in the target and obtaining the phase shifts $\eta(E)$ from the phase shift $\delta_0(E)$, making use of the scattering amplitude $F(\mathbf{k}, \mathbf{k}')$, becomes very complicated.

Demkov and Rudakov in [14] developed a partial wave method for a nonspherical scatterer that generalizes the conventional phase method for the spherically symmetric problem. They formulated variational principles that make it possible to calculate the above-mentioned partial waves $\varphi_\lambda(\mathbf{r})$ and their phase shifts $\eta_\lambda(k)$ by a direct approach without obtaining the scattering amplitude $F(\mathbf{k}, \mathbf{k}')$ in closed form. It was shown that for the case, when the scatterer can be represented as a set of N zero-range potentials, the problem reduces to a purely algebraic one, namely, to inversion of an Nth-order matrix.

In the present paper, making use of the general formulas developed in [13–15], we investigate particle scattering from targets consisting of two, three, and four centers modeled by zero-range potentials. The outline of our paper is as follows: In Sec. II, using a simple example of s-wave scattering by a pair of zero-range potentials, we look upon the transformation of the diffraction pattern at large distances from the target. In Sec. III, we derive the general formulas relating the scattering phase shifts for an isolated delta potential $\delta_0(E)$ to the scattering phase shifts $\eta_\lambda(k)$ for targets consisting of two, three, and four centers. The derived formulas are applied to both electron scattering by atomic clusters trapping an electron (Sec. IV) and calculation of the EWS time delays of π mesons scattered by few-nucleon systems (Sec. V). Section VI presents conclusions.

II. TRANSFORMATION OF THE DIFFRACTION PATTERN

Let us consider s-wave scattering from a pair of zero-radius potentials separated by distance R (Fig. 1). Suppose the scattering phase shift on each of them is $\delta_0(k)$. The total amplitude of the spherical Huygens waves formed in the process of incident wave scattering from this axially symmetric target at a point spaced at a distance r from its geometric center is

$$J(\theta) \propto \frac{1}{kr_1}\sin(kr_1+\delta_0) + \frac{1}{kr_2}\sin(kr_2+\delta_0). \quad (2)$$

Here the radii $\mathbf{r}_1$ and $\mathbf{r}_2$ are

$$r_1^2 = \frac{R^2}{4} + r^2 - Rr\cos\theta, \quad r_2^2 = \frac{R^2}{4} + r^2 + Rr\cos\theta,$$

where θ is the angle between vectors $\mathbf{R}/2$ and $\mathbf{r}$. The function $J(\theta)$ is shown in Fig. 2. The curves in Fig. 2 represent the diffraction pattern profile along a circle with radius r. The amplitude of the alternating function $J(\theta)$ rapidly decreases to zero with increasing r/R; that is, the crests of a pair of Huygens waves are transformed into crests of the spherical waves generated by the center of the target. The phase shifts of the radial parts of these waves $\eta(k)$ determine the cross section for elastic scattering of a particle from a target formed by a pair of delta potentials. The formulas connecting the phases $\delta_0(k)$ and $\eta(k)$ are obtained in the next section.

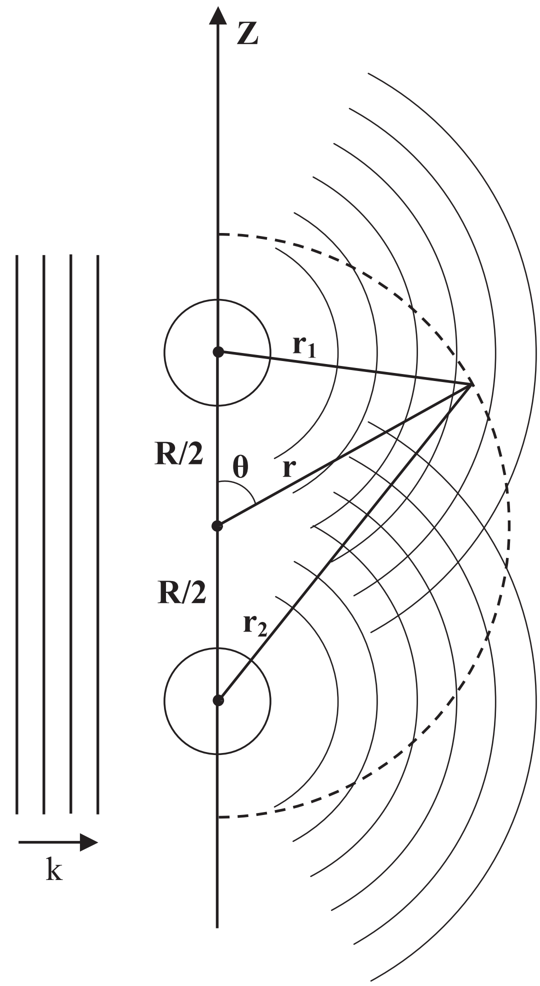

FIG. 1. The s-wave scattering by a pair of zero-range potentials separated by distance R; r is the position of an arbitrary observation point.

III. PHASE SHIFTS $\eta_\lambda(k)$ FOR PARTICLE SCATTERING ON COMPOUND TARGETS

According to [14], for the case when the scatterer can be represented as a superposition of N short-range potentials centered at points $\mathbf{R}_j$, the solution of the scattering problem is obtained by imposing boundary conditions at these points (the same as in [3]) on the following partial functions:

$$\varphi_\lambda(\mathbf{r}) = \sum_{j=1}^{N} D_j \frac{\sin(k|\mathbf{r}-\mathbf{R}_j| + \eta_\lambda)}{|\mathbf{r}-\mathbf{R}_j|}, \quad (3)$$

where k is the particle linear momentum relative to the target. As a result, we obtain a system of homogeneous equations, the solution of which determines the set of phase shifts $\eta_\lambda(k)$. Imposing the boundary conditions [15]

$$\varphi_\lambda(\mathbf{r})_{\mathbf{r}\to\mathbf{R}_j} \approx C_j\left[\frac{1}{|\mathbf{r}-\mathbf{R}_j|} + k\cot\delta_0\right] \quad (4)$$

at the positions $\mathbf{R}_j$ of N identical zero-range potentials, representing a target on the wave functions (3), leads to a system of N homogeneous linear equations for the unknown coefficients $D_1, D_2 \cdots D_N$.

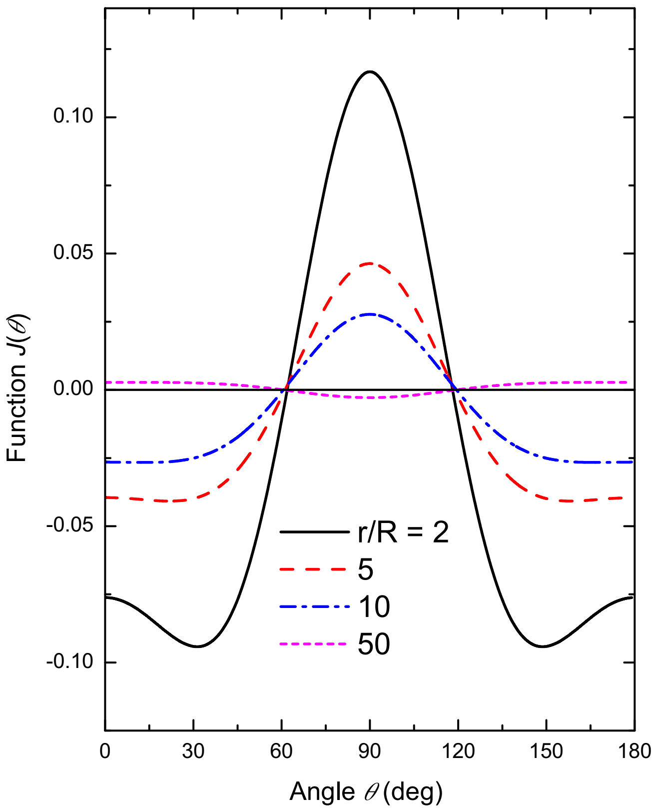

FIG. 2. The function $J(\theta)$ as the diffraction pattern profile along the circle with radius r (dashed curve in Fig. 1).

A. Scattering on a two-center target

The system of equations for a two-center target reads

$$\begin{aligned} D_1 k(\cot\eta - \cot\delta_0) + D_2[\mathrm{Im}a\cot\eta + \mathrm{Re}a] = 0, \\ D_1[\mathrm{Im}a\cot\eta + \mathrm{Re}a] + D_2 k(\cot\eta - \cot\delta_0) = 0, \end{aligned} \tag{5}$$

where $a = \exp(ikR)/R$ and R is the distance between the centers. In the system of equations (5) the number of equations is equal to the number of the unknowns. Therefore, this system has a nontrivial solution if and only if its determinant equals zero:

$$\begin{Vmatrix} B & A \\ A & B \end{Vmatrix} = (A^2 - B^2) = 0. \tag{6}$$

Here $A = \sin(kR + \eta)/R$ and $B = k(\cos\eta - \sin\eta\cot\delta_0)$. Solving (6) for $\eta_0(k)$ we obtain for the case $A + B = 0$, the phase shift for particle-target scattering:

$$\cot\eta_0(k) = \frac{kR\cot\delta_0(E) - \cos kR}{kR + \sin kR}. \tag{7}$$

For the case A–$B = 0$, the second phase shift reads

$$\cot\eta_1(k) = \frac{kR\cot\delta_0(E) + \cos kR}{kR - \sin kR}. \tag{8}$$

The phase shifts $\eta_\lambda(k)$ in (7) and (8) can be classified by considering their behavior at $k \to 0$ [14,15]. In this limit, the particle wavelength $\lambda\!\!\!^{-} = 1/k$ is much greater than the target size, and the scattering picture should approach a spherically symmetric one. From Eqs. (7) and (8), we obtain the asymptotic behavior $\eta_0(k \to 0) \sim k$ and $\eta_1(k \to 0) \sim k^3$, correspondingly. Thus, the phase shifts (7) and (8) at $k \to 0$ behave similarly to the s- and p-phase shifts in a spherically symmetric potential. This fact explains the choice of their indices.

Substituting the zero partial phase $\eta_0(k)$ into Eq. (5), we obtain the equality $D_1 - D_2 = 0$ for the coefficients at the two first wave functions given by Eq. (3). In the limit $r \to \infty$, we obtain the following expression for the partial wave $\varphi_0(\mathbf{r})$ in (3):

$$\begin{aligned} &\varphi_0(\mathbf{r} \to \infty) \\ &\quad \propto D_1\left[\frac{\sin(k|\mathbf{r} - \mathbf{R}_1| + \eta_0)}{|\mathbf{r} - \mathbf{R}_1|} + \frac{\sin(k|\mathbf{r} - \mathbf{R}_2| + \eta_0)}{|\mathbf{r} - \mathbf{R}_2|}\right]_{r\to\infty} \\ &\quad = 2D_1\cos(\mathbf{k}'\cdot\mathbf{R}/2)\left[\frac{1}{r}\sin(kr + \eta_0)\right]_{r\to\infty}, \end{aligned} \tag{9}$$

where $\mathbf{R}_1 = \mathbf{R}/2$ and $\mathbf{R}_2 = -\mathbf{R}/2$; $\mathbf{R}$ defines the position of the target axis in space and vector $\mathbf{k}'$ is particle linear momentum after scattering.

Repeating the same procedure for the phase shift $\eta_1(k)$, we obtain the equality $D_1 + D_2 = 0$ and the following asymptotic form for the second partial wave $\varphi_1(\mathbf{r})$:

$$\begin{aligned} &\varphi_1(\mathbf{r} \to \infty) \\ &\quad \propto D_1\left[\frac{\sin(k|\mathbf{r} - \mathbf{R}_1| + \eta_1)}{|\mathbf{r} - \mathbf{R}_1|} - \frac{\sin(k|\mathbf{r} - \mathbf{R}_2| + \eta_1)}{|\mathbf{r} - \mathbf{R}_2|}\right]_{r\to\infty} \\ &\quad = -2D_1\sin(\mathbf{k}'\cdot\mathbf{R}/2)\left[\frac{1}{r}\sin\left(kr + \frac{\pi}{2} + \eta_1\right)\right]_{r\to\infty}. \end{aligned} \tag{10}$$

The asymptotic expressions for the radial parts of these functions (in square brackets) coincide with the asymptotic behavior of the spherical s and p waves emitted from the geometrical center of the target, whereas the angle-dependent coefficients before the square brackets, functions $Z_0(\mathbf{k}')$ and $Z_1(\mathbf{k}')$, are analogs of the spherical harmonics. The wave function of a particle colliding with a nonspherical target is to be expanded into the Z_λ functions. For further details, see [13], where the explicit expressions for these functions were obtained. Normalized to the particle unit flow inward and outward through the surface of a sphere surrounding the target, the radial parts of the functions (9) and (10) determine (according to [6]) the EWS time delay of slow particles scattered by the target. In this case, the partial time delays of a particle scattered by a two-center target are determined by Eq. (1), in which the scattering phase shift on an individual center is replaced by the phase shifts $\eta_{0,1}$ in (9) and (10).

B. Scattering on a three-center target

Imposing the boundary conditions (4) on the wave functions (3) with $N = 3$ leads to the following homogeneous system of linear equations for the mixture coefficients D_1, D_2, and D_3:

$$\begin{aligned} &D_1 k(\cot\eta - \cot\delta_0) + D_2(\mathrm{Im}a_{12}\cot\eta + \mathrm{Re}a_{12}) \\ &\quad + D_3(\mathrm{Im}a_{13}\cot\eta + \mathrm{Re}a_{13}) = 0, \\ &D_1(\mathrm{Im}a_{21}\cot\eta + \mathrm{Re}a_{21}) + D_2 k(\cot\eta - \cot\delta_0) \\ &\quad + D_3(\mathrm{Im}a_{23}\cot\eta + \mathrm{Re}a_{23}) = 0, \end{aligned}$$

$$D_1(\mathrm{Im}a_{31}\cot\eta + \mathrm{Re}a_{31}) + D_2(\mathrm{Im}a_{32}\cot\eta + \mathrm{Re}a_{32}) + D_3k(\cot\eta - \cot\delta_0) = 0. \tag{11}$$

The following notation is introduced here: $a_{ij} = \exp(ikR_{ij})/R_{ij}$ and $R_{ij} = |\mathbf{R}_i - \mathbf{R}_j|$. Suppose the delta centers in the target are located at the vertices of an equilateral triangle. The side length of the triangle equals the distance between the centers in the two-center target $R = R_{ij}$. The system of equations (11) has a nontrivial solution if its determinant equals zero:

$$\begin{Vmatrix} B & A & A \\ A & B & A \\ A & A & B \end{Vmatrix} = (A-B)^2(2A+B) = 0. \tag{12}$$

From Eq. (12), we obtain three phase shifts of a particle scattered from the three-center target. The first of them reads

$$\cot\eta_0(k) = \frac{kR\cot\delta_0(E) - 2\cos kR}{kR + 2\sin kR}, \tag{13}$$

while the second and third are identical:

$$\cot\eta_1^{(1,2)}(k) = \frac{kR\cot\delta_0(E) + \cos kR}{kR - \sin kR}. \tag{14}$$

The last two phases (14) coincide with the phase (8) obtained above for a two-center target.

C. Scattering on a four-center target

Let us consider a four-center target, in which the zero-range potentials are centered at the vertices of a tetrahedron. The side length of the tetrahedron equals R as before. The model targets discussed in Sec. III exhaust all possible configurations, in which all center-to-center distances are the same. Imposing the boundary conditions (4) on the wave functions (3) with $N = 4$, we obtain the following equation for the phase shifts:

$$\begin{Vmatrix} B & A & A & A \\ A & B & A & A \\ A & A & B & A \\ A & A & A & B \end{Vmatrix} = (A-B)^3(3A+B) = 0. \tag{15}$$

Solving Eq. (15), we obtain four phase shifts for particles scattered by a four-center target. The first of them reads

$$\cot\eta_0(k) = \frac{kR\cot\delta_0(E) - 3\cos kR}{kR + 3\sin kR}, \tag{16}$$

and the other three are the same:

$$\cot\eta_1^{(1,2,3)}(k) = \frac{kR\cot\delta_0(E) + \cos kR}{kR - \sin kR}. \tag{17}$$

For all considered targets, only phase shifts $\eta_0(k)$ [Eqs. (7), (13), and (16)] are different, while the phase shifts with $\lambda = 1$ are all the same. At first glance, this is explained by the equidistant positions of the identical scattering centers, that is, by the same boundary conditions imposed on the wave function. Such an idealized design of the considered targets makes it possible to obtain compact solutions of the problem of particle scattering by these nonspherical targets.

In the following sections, we apply the derived formulas for phase shifts to calculate both the cross sections of elastic scattering and EWS time delays of particles scattered by model targets. The zero-range potential approximation is widely used in the description of scattering in both atomic and nuclear physics. According to [15], making use of a zero-range potential is the simplest and most natural way in studying the multiple scattering of an electron from a molecule. The next section generalizes the results obtained in [13] by considering quantum mechanical scattering of slow electrons by atomic clusters and EWS time delay in this process.

IV. ELECTRON ELASTICALLY SCATTERED BY A CLUSTER OF ATOMS

In this section, we use the atomic system of units (a.u.). The average effective cross section $\bar{\sigma}(k)$ of particles elastically scattered by a nonspherical target, that is, the total cross section integrated over all the angles between the vectors $\mathbf{k}'$ and $\mathbf{k}$ followed by averaging over all directions of the vector $\mathbf{k}$ in the target reference frame, is determined by the expression [14]

$$\bar{\sigma}(k) = \frac{4\pi}{k^2}\sum_{\lambda}\sin^2\eta_\lambda(k). \tag{18}$$

Here $\eta_\lambda(k)$ are the phase shifts obtained above. The index λ used to number the phase shifts for nonspherical targets, in the spherically symmetric case, should be replaced by two indices: l and m. Summation over the magnetic quantum numbers m leads to the appearance of the factor $(2l+1)$ in front of sine in Eq. (18). Then summation is conducted over the orbital angular momentum l only.

For a two-atom cluster, the average effective cross section $\bar{\sigma}(k)$ has the form

$$\bar{\sigma}_2(k) = \frac{4\pi}{k^2}[\sin^2\eta_0 + \sin^2\eta_1] = \frac{4\pi}{k^2}[(1+\cot^2\eta_0)^{-1} + (1+\cot^2\eta_1)^{-1}]. \tag{19}$$

The phase shifts here are determined by Eqs. (7) and (8).

For a three-atom cluster with the phase shifts (13) and (14), the average cross section is given by the expression

$$\bar{\sigma}_3(k) = \frac{4\pi}{k^2}\left[\sin^2\eta_0 + 2\sin^2\eta_1^{(1,2)}\right] = \frac{4\pi}{k^2}\left[(1+\cot^2\eta_0)^{-1} + 2\left(1+\cot^2\eta_1^{(1,2)}\right)^{-1}\right]. \tag{20}$$

The factor of 2 in front of the second term in Eq. (20) appears since two out of the three scattering phases in (14) are the same.

For a four-atom target with the phase shifts (16) and (17), the cross section becomes

$$\bar{\sigma}_4(k) = \frac{4\pi}{k^2}\left[\sin^2\eta_0 + 3\sin^2\eta_1^{(1,2)}\right] = \frac{4\pi}{k^2}\left[(1+\cot^2\eta_0)^{-1} + 3\left(1+\cot^2\eta_1^{(1,2)}\right)^{-1}\right]. \tag{21}$$

The factor of 3 in front of the second terms in Eq. (21) appears due to the multiplicity of 3 of the phase shifts with $\lambda = 1$.

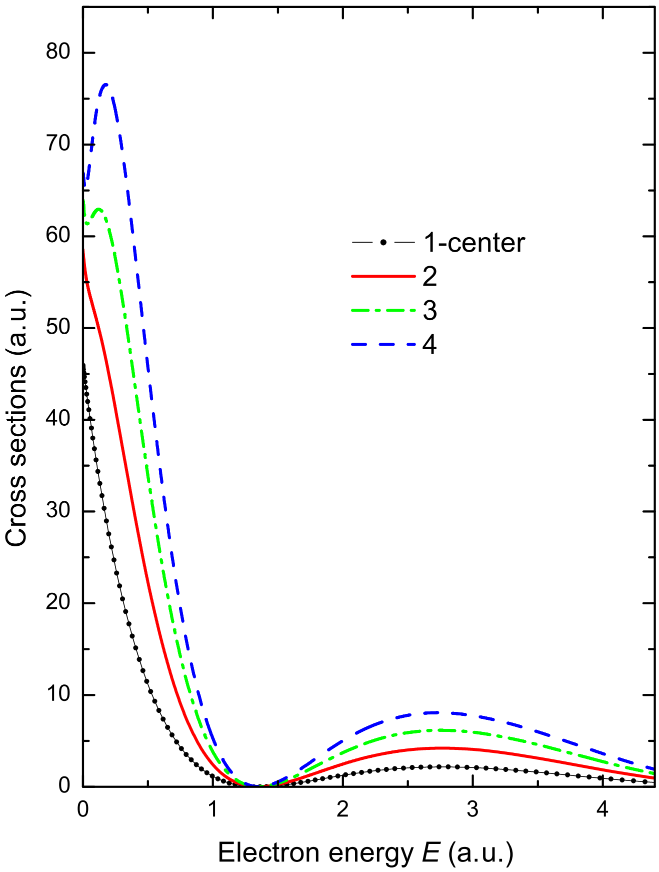

FIG. 3. The average effective cross section $\bar{\sigma}(E)$ for a particle elastically scattered by targets with one, two, three, and four delta centers.

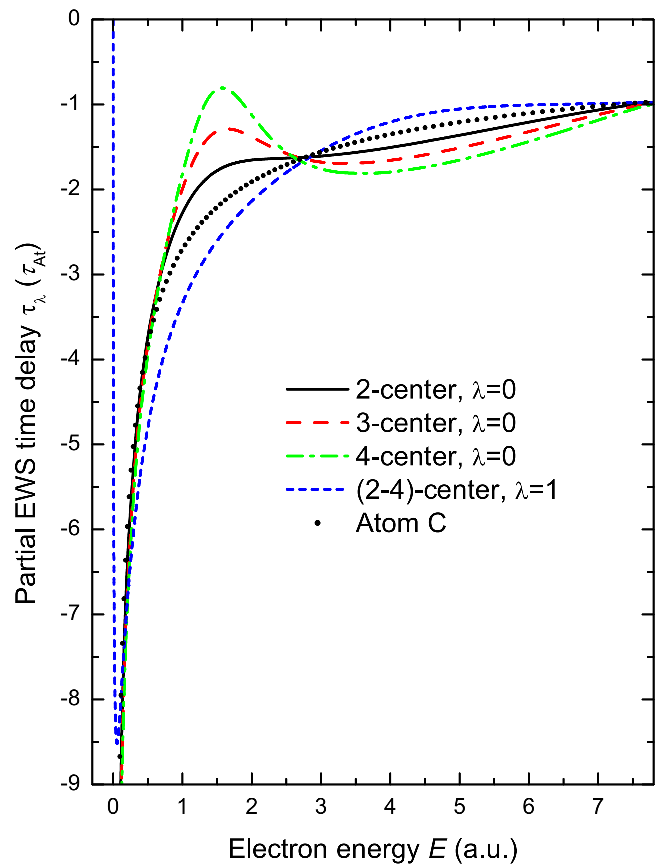

FIG. 4. The partial EWS time delays of electrons $\tau_\lambda(E)$ (in units of $\tau_{At} = 2.419 \times 10^{-17}$ s) in electron collisions with the compound targets.

Equations (20) and (21) generalize the results presented in [13], where the electron scattering cross sections for a cluster with two carbon atoms were calculated. We assume that the interatomic distances R in all targets are equal. As a specific value of R, we chose that to be the distance between carbon atoms in a molecule C_2, namely, $R = 2.479$ atomic units (a.u.) (as in [13]). In calculations of phase shifts $\eta_\lambda(k)$ (as in [13]), the phase shift $\delta_0(k)$ is evaluated as $\delta_0(k) = 2\pi - 1.912k$ [16]. The results of numerical calculations of the electron scattering cross sections by multicenter targets are shown in Fig. 3, where the electron s-scattering cross sections for a single carbon atom are also presented for comparison. As expected, the curve for a two-center target, calculated with Eq. (19), coincides with that obtained in [13] with the optical theorem [17].

The partial EWS time delay $\tau_\lambda(E)$ in electron-target collisions is expressed in terms of the energy derivative of the scattering phase shift $\eta_\lambda(E)$ [Eq. (1)]. According to the formulas derived above, there are three distinct scattering phase shifts with $\lambda = 0$ given by Eqs. (7), (13), and (16). Another scattering phase shift with $\lambda = 1$ is given by Eq. (8). Numerical results for partial EWS time delays with these phase shifts are given in Fig. 4. The curves corresponding to $\lambda = 0$ tend to $-\infty$ as $E \to 0$, while the curve for $\lambda = 1$ vanishes in this limit. All the curves oscillate around the curve that represents the EWS time delay for a single carbon atom. The curves form a node at electron energy $E \approx 2.75$ a.u. Analysis shows that $\eta_\lambda(E) = \pi/2$ in the vicinity of this energy. Oscillations of the curves are associated with the diffraction terms $\sin kR/kR$ and $\cos kR/kR$ in the formulas for the scattering phase shifts. All the time delays in Fig. 4 are negative. This is understandable since the considered scattering does not have resonance features. A big positive peak can be expected in the vicinity of a resonance. Indeed, let us assume that the phase $\delta_0(E)$ in the single-center case has an additional term:

$$\delta_{res}(E) \cong \tan^{-1}\left[\frac{\Gamma/2}{E_{res} - E}\right]. \qquad (22)$$

It gives a Breit-Wigner addition to the EWS time delay, namely,

$$\Delta\frac{d\delta}{dE} = \frac{\Gamma}{2}\frac{1}{(E_{res} - E)^2 + \Gamma^2/4}. \qquad (23)$$

We discuss the resonance behavior of the scattering phase shift in the next section.

V. MESON ELASTIC SCATTERING FROM A FEW-NUCLEON SYSTEM

The zero-range potential model is also widely used to describe scattering in nuclear physics. For example, it is applied for multiple scattering of π mesons by nucleons, and scat-

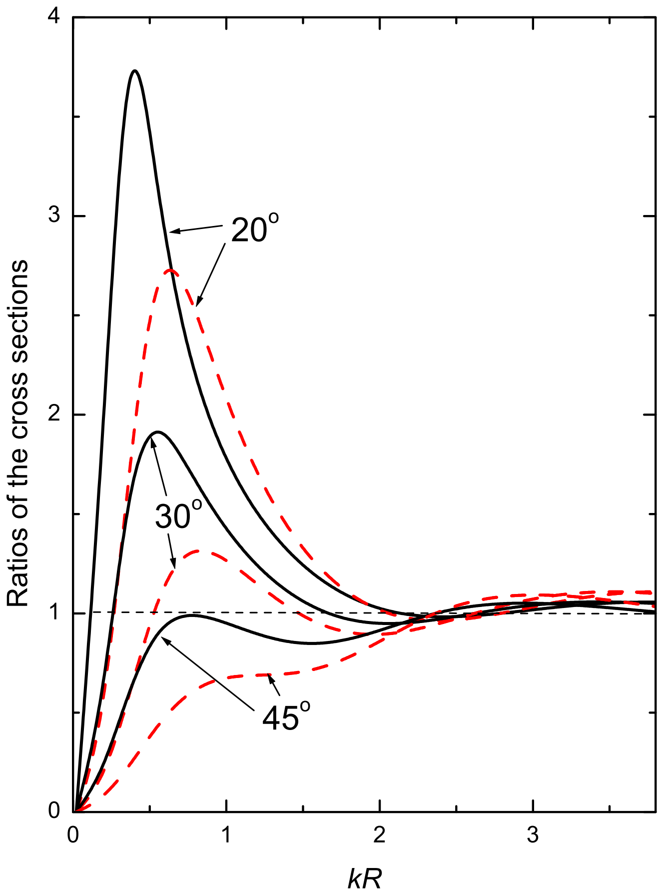

FIG. 5. The ratios of cross sections $\bar{\sigma}_D(k)/2\sigma_0(k)$ calculated with formula (19) (solid lines), and $\bar{\sigma}_T(k)/3\sigma_0(k)$ calculated with formula (20) (dashed lines). The meson-nucleon cross section $\sigma_0 = (4\pi/k^2)\sin^2\delta_0$.

tering of neutrons by nuclei in molecules or crystals [1–3]. We develop the results presented in [3] for the scattering of slow mesons by two-, three-, and four-nucleon systems and calculate the corresponding EWS time delays. In this section, the meson energy is measured in MeV and the time delay in units of $\tau_{\mathrm{Nuc}} = \hbar/\mathrm{MeV} = 6.582 \times 10^{-22}$ s.

The internucleon distances in all considered targets are assumed to be equal to the mean internucleon distance in a deuteron, $R = 2.142$ fm [18]. The scattering phase shifts of a meson on an isolated proton and neutron are assumed to be the same and equal, $\delta_0(E)$. Let us apply Eqs. (19)–(21) for calculation of the cross sections for elastic scattering of mesons by targets consisting of two, three, and four nucleons in the zero-range potential model. We assume (as in [3]) the phase shift $\delta_0(E)$ for the case of meson s scattering on an isolated nucleon to be fixed in Eqs. (7), (8), (13), and (16). Calculations of the cross sections $\bar{\sigma}_D(k)$ [using Eq. (19)] and $\bar{\sigma}_T(k)$ [using Eq. (20)] are conducted for $\delta_0(E) = 20^\circ$, 30°, or 45°. The ratios of these cross sections to $2\sigma_0(k)$ and $3\sigma_0(k)$, respectively [$\sigma_0 = (4\pi/k^2)\sin^2\delta_0$ is the meson-nucleon cross section] are depicted in Fig. 5. As in the case of the deuteron [3], the processes of multiple scattering of mesons on a triton essentially contribute to the total scattering cross section at $\delta_0(k)$ less than 45°. After elementary manipulations, one can show that Eq. (19) coincides with Eq. (5) in [3], obtained using the optical theorem [17]. The curves for the deuteron in Fig. 5, therefore, coincide with the curves in Fig. 1 in [3]. Numerical calculations of the effective cross sections for a four-nucleon target (21) lead to curves similar to those shown in Fig. 5 for a deuteron and triton. They are not shown in Fig. 5 to avoid overloading the picture.

The EWS time delay in the processes of elastic scattering of slow mesons from the considered targets is determined by the derivatives of the scattering phases $\eta_\lambda(k)$ with respect to the kinetic energy E of the meson, i.e., by Eq. (1). When calculating the curves in Fig. 5, we assume the phase shifts for s scattering from a single nucleon $\delta_0(E)$ to be fixed. For calculation of the partial time delays $\tau_\lambda(k)$ using Eq. (1), one needs a relationship between meson wavelength $1/k$ and the phase shift $\delta_0(E)$. Numerous experimental data for phase shifts as functions of meson linear momentum k are available in the literature in the energy range E between tens and hundreds of MeV. We have chosen the empirical dependence obtained in [19–21]. The nuclear phase shifts $\delta_0(E)$ are fitted with an analytical function, which incorporates the threshold behavior (three first terms in the formula below) and a term that represents the nearest π-nucleon resonance,

$$\frac{\tan\delta_0(E)}{q} = b + fq^2 + dq^4 + \frac{x\Gamma_0\omega_0 q_0^{-1}}{\omega_0^2 - \omega^2}, \tag{24}$$

where q is the center-of-mass linear momentum and ω is the center-of-mass-energy of meson-nucleon collision. Following [3], we consider the target to be infinitely heavy. The wave number $q = c\hbar k$ in (24) is expressed in units of MeV/c; k is the meson wave number in fm^{-1}. The constants b, f, and d measured in the corresponding powers of MeV/c were obtained in [21] (first row of Table II) to achieve the best fit of the expression (24) to the large set of experimental data. We also used the resonance parameters x, ω_0, q_0, and Γ_0, given in the first row of Table 1 in [19].

The results of the calculations of the phase shifts $\delta_0(E)$ (24) for mesons scattered on the compound targets are presented in Fig. 6. The scattering phases $\delta_0(E)$ are represented by a monotonically increasing smooth curve, whereas the scattering phase shifts $\eta_\lambda(E)$ oscillate around this curve. Moreover, the curves for the phase shifts with $\lambda = 0$ and $\lambda = 1$ oscillate in the antiphase. The appearance of these oscillations in the phases is associated with the diffraction of the meson waves on the scattering centers of the targets.

The phase shift $\eta_\lambda(k)$ is a function of meson momentum k, which is related to its kinetic energy E by the following expression [22]:

$$k = \frac{\sqrt{E(E + 2m_0c^2)}}{\hbar c} = \frac{1}{\lambda\!\!\!^{-}}, \tag{25}$$

where $c\hbar = 197.326$ MeV fm and the π-meson rest mass is $m_0 = 139.57\,\mathrm{MeV}/\mathrm{c}^2$ [22].

The results of the calculation of the partial EWS time delays of mesons are shown in Fig. 7. It is seen that the partial EWS time delays as functions of the meson energy E are oscillating curves. The large portions of the curves are located in the positive half plane of the coordinate system. The dominant peaks of the curves correspond to the meson

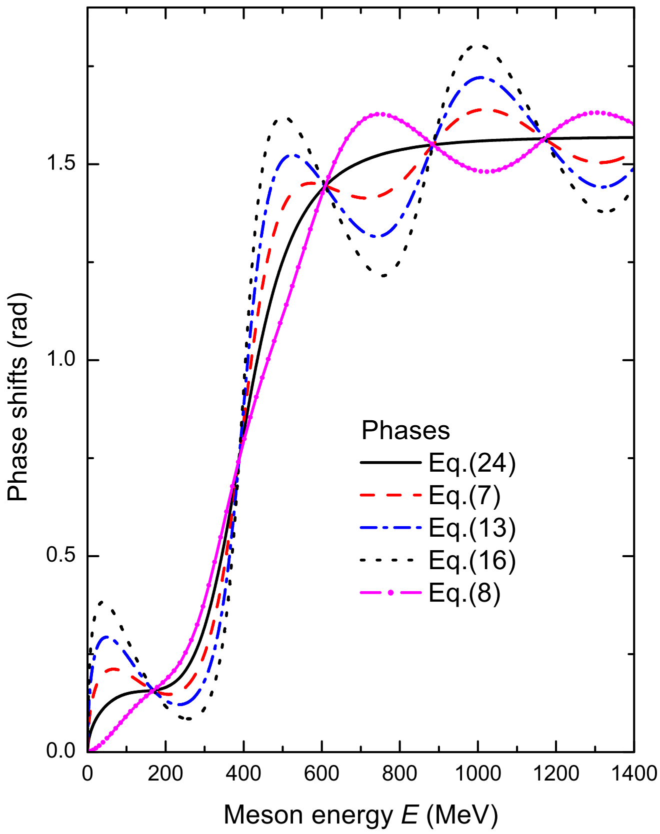

FIG. 6. The scattering phase shifts of mesons for all considered targets.

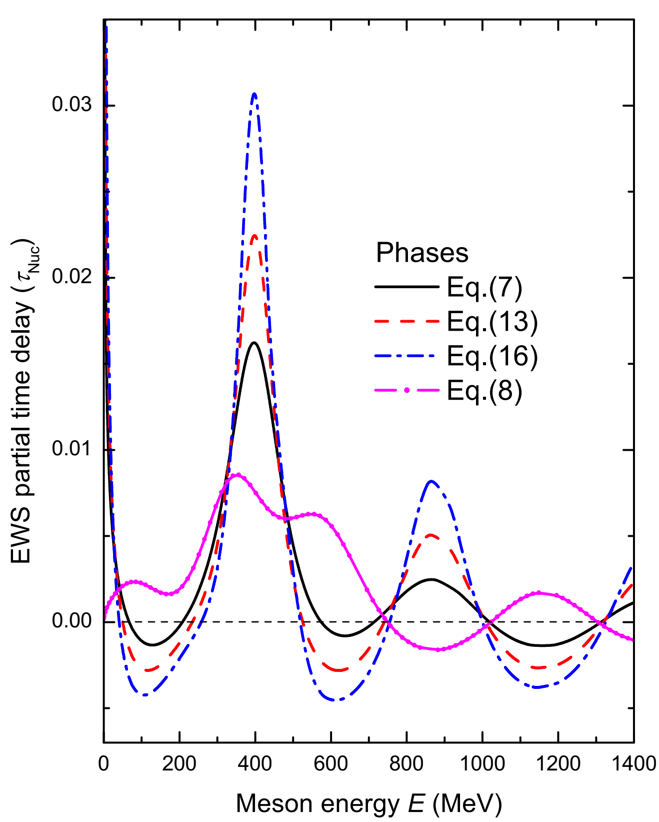

FIG. 7. The partial EWS time delays of mesons (in units of $\tau_{\mathrm{Nuc}} = \hbar/\mathrm{MeV} = 6.582 \times 10^{-22}$ s) as functions of meson energy E.

energy of about 400 MeV. The dependence for $\lambda = 1$ turns out to be in antiphase with the dependencies for $\lambda = 0$ at $E > 600$ MeV.

VI. CONCLUSIONS

In the present work, we discuss the method of calculation of the phase shifts and partial EWS time delays for particles elastically scattered by compound targets modeled by a set of zero-range potentials. The multicenter target does not have spherical symmetry. Therefore, the wave function of a scattered particle cannot be represented as an expansion into a series of spherical harmonics in all space. However, the target can be considered as a point source of spherical waves at asymptotically large distances from the scatterer. The phase shifts of these waves define the scattering cross section for a nonspherical target. The application of variational principles, as shown in [14,15], makes it possible to calculate the scattering phases in a system of zero-range potentials from the partial wave functions (3). Imposing boundary conditions on these functions results in algebraic equations for the scattering phase shifts. By analogy with the spherically symmetric case, there is every reason to believe that the partial wave method is the most convenient for studying and calculating particle scattering by nonspherical systems.

The set of model targets considered here makes it possible to complete the solution of the particle scattering problem. The scattering phases of particles for all zero-range potentials are considered the same for all the compound targets. The targets at the moment of collision are assumed to be motionless. The distances between centers in all the targets are to be equal. The used approximations maximally simplify Eqs. (6), (12), and (15), and, therefore, the calculation of the scattering phases shift becomes simple.

It is obvious that the replacement of identical phases $\delta_0(E)$ by different ones, or the introduction of different distances R between scattering centers, transforms the determinants. For example, along with the considered triangular target configuration, one can consider a linear configuration of the scatterers with the distance $R/2$ between the closest centers. For this target, the equation for the matrix of system (11) reads

$$\begin{Vmatrix} B & C & A \\ C & B & C \\ A & C & B \end{Vmatrix} = (A - B)(B^2 + AB - 2C^2) = 0. \tag{26}$$

The first phase from this equation coincides with the phase obtained from Eq. (13). The second and third phases are obtained from the quadratic equation,

$$(B^2 + AB - 2C^2) = 0, \tag{27}$$

containing another parameter $C = \sin(kR/2 + \eta)/R/2$. Analytical expressions for the phases could be easily obtained from this equation but the formulas are too cumbersome. In the general case of all different intercenter distances, we have a cubic equation with roots having a very complicated form. This is the reason why we do not present the general case here and limit ourselves by the equidistant configurations allowing us to obtain compact equations for the phases.

The cross sections of elastic scattering and the EWS time delays as functions of the intercenter distances have to be averaged over the system of the wave functions. However, we omitted such computations since our goal was to estimate in the first approximation the magnitude of the EWS time delay for target systems with few identical centers. It turned out that the time delays obtained for mesons are of the same order of magnitude as times considered, for example, in Refs. [11,23]. The existence of the positive time delay peaks in Fig. 7 is one of the main observations of the present work because a positive maximum in time delay is the necessary condition for the existence of resonances. The resonance appearance for meson scattering on compound targets, as shown in this paper, is associated with the resonance in meson-nucleon scattering. Experimental search for resonances and their study is, in our opinion, an interesting problem.

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