Recovery of a generic local Hamiltonian from a steady state

Jing Zhou[1,2] and D. L. Zhou[1,2,3,4,*]

[1]*Institute of Physics, Beijing National Laboratory for Condensed Matter Physics, Chinese Academy of Sciences, Beijing 100190, China*
[2]*School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China*
[3]*Collaborative Innovation Center of Quantum Matter, Beijing 100190, China*
[4]*Songshan Lake Materials Laboratory, Dongguan, Guangdong 523808, China*

(Received 30 September 2021; accepted 11 January 2022; published 31 January 2022)

With the development of the quantum many-body simulator, Hamiltonian tomography has become an increasingly important technique for verification of quantum devices. Here we investigate recovering the generic Hamiltonians of two spin chains with two-local interactions and three-local interactions by measuring local observables. For these two models, we show that, when the chain length reaches a certain critical number, we can recover the generic local Hamiltonian from its one steady state by solving the homogeneous operator equation (HOE) developed by Bairey, Arad, and Lindner [Phys. Rev. Lett. **122**, 020504 (2019)]. To explain the existence of such a critical chain length, we develop an alternative method to recover the Hamiltonian by solving the energy eigenvalue equations (EEE). By using the EEE method, we completely recovered the numerical results from the HOE method. Then we theoretically prove the equivalence between the HOE method and the EEE method. In particular, we obtain the analytical expression of the rank of the constraint matrix in the HOE method by using the EEE method, which can be used to determine the correct critical chain length in all the cases.

DOI: 10.1103/PhysRevA.105.012615

I. INTRODUCTION

In quantum mechanics, all the information on a quantum system is contained in its Hamiltonian [1,2]. For example, all the energy eigenvalues and eigenstates can be obtained by solving the eigenvalue problem of the Hamiltonian. For a condensed-matter system, however, its (effective) Hamiltonian is unknown when prepared. Then it is crucial to determine the Hamiltonian by making some quantum measurements, which is called Hamiltonian tomography [3–5]. In a Hamiltonian tomography, the quantum measurements made must provide sufficient information such that the Hamiltonian can be specified uniquely. For a generic Hamiltonian, a successful Hamiltonian tomography needs the information on all the energy eigenvalues and eigenstates. This implies that the number of independent quantum measurements increases exponentially with the number of particles in a Hamiltonian tomography.

Fortunately, the Hamiltonian of a physically realizable system is usually not generic but local, which means that interactions arise only between (or among) local particles [6,7]. This information on the local interaction pattern of the Hamiltonian is extremely useful to reduce the necessary information from the quantum measurements in a Hamiltonian tomography. For example, it has been shown that the local Hamiltonian can be reconstructed uniquely by the information on one eigenstate when the particle number becomes large in most cases [8,9].

One of the major challenges in Hamiltonian tomography is to develop an algorithm to recover the Hamiltonian from numbers of measurements which is in accord with demands of resource limitation with high accuracy [10]. The measurement resource in the state-of-the-art algorithm for recovering a generic local Hamiltonian scales polynomial to the system size [11,12]. Many algorithms have been proposed to recover the Hamiltonian by making quantum measurements on its eigenstate [13–18], dynamics [19–22], and quantum quench process [23]. Several algorithms have been employed to successfully recover some local Hamiltonians with a specific pattern [24–26].

Recent years have witnessed the rapid development of quantum simulators and computation devices, such as controlling trapped ions [27–30] and superconducting circuits [31,32]. To verify the above devices, it is necessary to recover its Hamiltonian from the measured observables, which makes the Hamiltonian tomography become increasingly important in condensed-matter physics and quantum computing. Given the practical value of Hamiltonian tomography and significant development of numerical methods of this task, several Hamiltonian tomography algorithms have been implemented on real physical systems [33–35].

However, there is still a fundamental problem in Hamiltonian tomography that does not have a satisfactory answer: when can a generic local Hamiltonian be uniquely recovered from a steady state? Note that this problem has been solved partially. For example, the authors in Ref. [13] found that, when the rank of the constraint matrix equals the number of independent parameters minus one, the generic Hamiltonian can be uniquely recovered. However, we do not know

*zhoudl72@iphy.ac.cn

what factors determine the rank of the constraint matrix for a given generic local Hamiltonian. Here we aim to present an analytical answer to this fundamental problem. Based on our analytical results, in particular, we can predict the critical chain length for a random local Hamiltonian, i.e., the generic Hamiltonian with the chain length beyond which can be uniquely recovered.

The paper is organized as follows. In Sec. II, after reviewing the HOE method derived from a steady state [13], we implement the HOE to reconstruct two local Hamiltonians in a spin chain from a steady state, where we find that HOE fails to recover the Hamiltonian when the chain length is smaller than the critical chain length. In Sec. III, we develop the EEE method to give the analytical expressions for the critical chain lengths for any local Hamiltonian tomography. Section III contains three subsections. In Sec. III A, we employ the EEE method to recover the same two local Hamiltonians as studied in Sec. II. In Sec. III B, we prove the equivalence of the HOE and EEE in a Hamiltonian tomography. In Sec. III C, we determine when the local Hamiltonian can be recovered from a steady state. In Sec. IV, we give a brief summary.

II. RECONSTRUCTING THE HAMILTONIAN BY HOMOGENEOUS OPERATOR EQUATIONS

In this section, we review and apply the method developed in Ref. [13] to solve the Hamiltonian tomography problem whose goal is to recover the Hamiltonian of a quantum system by measuring some observables when the system stays in a steady state. According to quantum mechanics, the steady state may be an eigenstate of the Hamiltonian or a mixed state of several such eigenstates.

In general, the Hamiltonian to be recovered is decomposed as the sum

$$H = \sum_{n=1}^{N} a_n h_n, \tag{1}$$

where each h_n is a known Hermitian operator, a_n is an unknown real parameter, and N is the number of terms in the Hamiltonian. The task of the Hamiltonian tomography is to determine the vector $\vec{a} = (a_1, a_2, \ldots, a_N)$, which is formed by all the unknown parameters in the Hamiltonian. Suppose that our system stays in the steady state ρ. Thus the expectation value of any observable K is invariant under the quantum dynamic evolution, which is expressed in the Heisenberg picture as

$$\partial_t \langle K \rangle = -\langle i[K, H] \rangle = 0, \tag{2}$$

where $\langle O \rangle = \mathrm{Tr}[O\rho]$ denotes the expectation value of the observable O in the steady state ρ. Inserting Eq. (1) into Eq. (2), we conclude a homogeneous linear equation for the vector $\vec{a}$,

$$\sum_{n=1}^{N} a_n \langle i[K, h_n] \rangle = 0. \tag{3}$$

Since Eq. (3) works for any observable K, we can choose a set of observables $\{K_m\}_{m=1}^{M}$ and obtain M linear constraints on the vector $\vec{a}$

$$\forall m : \sum_{n=1}^{N} a_n \langle i[K_m, h_n] \rangle = 0, \tag{4}$$

which can be briefly written in the matrix form as

$$G\vec{a} = 0, \quad G_{mn} = \langle i[K_m, h_n] \rangle. \tag{5}$$

Equations (4) or (5) are called the linear homogeneous operator equations (HOE) [36], which are the basic equations to recover the Hamiltonian developed in Ref. [13].

The number of degrees of freedom of the vector $\vec{a}$ satisfying Eqs. (5) is determined by the rank of the constraint matrix G, denoted as $\mathrm{Rank}G = r$. In the Hamiltonian tomography, we assume that there always exists a nonzero solution of $\vec{a}_{\mathrm{true}}$, which implies that the rank $r < N$. The larger the rank of G, the more determined the solutions of $\vec{a}$. However, even when the rank of G arrives at its maximum $r = N - 1$, there still are an infinite number of solutions in the form of $\alpha\vec{a}_{\mathrm{true}}$, with α being any real number. To remove the trivial ambiguity of the solutions, we reconstruct the task into a convex optimization with constraint

$$\min_{\vec{a}} ||G\vec{a}||, \quad \text{such that } ||\vec{a}|| = 1. \tag{6}$$

The solution to Eq. (6) is the lowest right-singular vector of the constraint matrix G, i.e., the row vector of V^T that corresponds to the lowest singular value of G in the singular value decomposition $G = U\Sigma V^T$. The error of the reconstructing task is defined as the distance between the normalized true vector $\vec{a}_{\mathrm{true}}$ and the recovered vector $\vec{a}_{\mathrm{recovered}}$:

$$\Delta = \left\| \frac{\vec{a}_{\mathrm{true}}}{||\vec{a}_{\mathrm{true}}||} - \frac{\vec{a}_{\mathrm{recovered}}}{||\vec{a}_{\mathrm{recovered}}||} \right\|. \tag{7}$$

In Ref. [13], the HOE method has been applied to recover the local Hamiltonian from local measurements. More precisely, the local Hamiltonians of six middle qubits in a one-dimensional 12-qubit chain with random two-local interactions are successfully recovered by measuring the middle qubits.

Here, we apply the HOE to study how to recover the local Hamiltonian from one single steady state. For comparison, we study recovering the Hamiltonians of two forms of the spin-1/2 chain. The first spin chain consists of local terms and nearest-neighbor interactions, whose Hamiltonian

$$H_2 = \sum_{l=1}^{L} \sum_{\eta} a_{l\eta} \sigma_l^{\eta} + \sum_{l=1}^{L-1} \sum_{\eta} \sum_{\theta} a_{l\eta\theta} \sigma_l^{\eta} \sigma_{l+1}^{\theta}, \tag{8}$$

where L is the spin chain length, η and θ take values in the set $\{x, y, z\}$, σ_l^{η} is the η component of the Pauli matrix of the lth spin, and all $a_{l\eta}$ and $a_{l\eta\theta}$ are the unknown parameters to be recovered.

The second spin chain consists of all three-neighbor interactions besides the terms appearing in the Hamiltonian H_2, i.e., its Hamiltonian

$$H_3 = H_2 + \sum_{l=1}^{L-2} \sum_{\eta} \sum_{\theta} \sum_{\delta} a_{l\eta\theta\delta} \sigma_l^{\eta} \sigma_{l+1}^{\theta} \sigma_{l+2}^{\delta}. \tag{9}$$

The state prepared to be measured is the mixed state which is a mixture of q eigenstates of H, the Hamiltonian to be recovered.

The corresponding single steady state is assumed to be

$$\rho = \sum_{\mu=1}^{q} p_\mu |\lambda_\mu\rangle\langle\lambda_\mu|, \tag{10}$$

where $|\lambda_\mu\rangle$ is the μth eigenstate of the Hamiltonian (H_2 or H_3) with nonzero probability p_μ, where p_μ is sampled from a normalized uniform distribution. q is the rank of the state ρ. Intuitively, the Hamiltonian H_3 contains more unknown parameters than the Hamiltonian H_2, and we expect that H_3 is more difficult to recover from the information in a steady state.

Now we apply the HOE method reviewed in Sec. II to recover the above Hamiltonians, whose procedure is given as follows. First, for each given chain length L we prepare the Hamiltonians to be recovered by generating 200 random vectors $\{\vec{a}_{\text{true}}\}$ of the Gaussian distribution with zero mean and unit standard deviation. Second, for each random vector $\vec{a}_{\text{true}}$ we numerically calculate the eigenstates of the prepared Hamiltonian and construct three mixed states ρ with $q = 1, 2, 3$ as given in Eq. (10). Third, we choose the terms $\{h_i\}_{i=1}^{N}$ in the prepared Hamiltonian as the observables $\{K_m\}$ and calculate the constraint matrix G. Note that such a choice makes the number of equations equal to the number of unknown parameters in the prepared Hamiltonian. Fourth, we calculate the recovered vector $\vec{a}_{\text{recovered}}$ by the singular decomposition of G, along with the error as given in Eq. (7).

Following the above HOE procedure, we numerically obtain the reconstructing errors of the Hamiltonians H_2 and H_3 from mixed states ρ with the chain length L from 1 to 9 shown in Fig. 1. We find that the HOE method successfully recovers the Hamiltonian H_2 when $L \geqslant L_c$, where (1) $L_c = 5$ when $q = 1$, (2) $L_c = 3$ when $q = 2$, and (3) $L_c = 3$ when $q = 3$, and it successfully recovers the Hamiltonian H_3 when $L \geqslant L_c$, where (1) $L_c = 7$ when $q = 1$, (2) $L_c = 6$ when $q = 2$, and (3) $L_c = 5$ when $q = 3$. Here the reconstructing error $\Delta \simeq 1$ implies the failure of the HOE method and $\Delta < 10^{-6}$ implies the success of the HOE method.

Note that for a given type of Hamiltonian, the more eigenstates are contained in mixed state ρ, i.e., q is larger, the more Hamiltonians can be uniquely recovered. In addition, it is easier to recover H_2 than to recover H_3 from a steady state with the same q.

As discussed in Sec. II, the condition for the Hamiltonian to be successfully recovered by the HOE method is $\delta = N - (r + 1) = 0$, where N is the number of unknown parameters in the Hamiltonian, and r is the rank of the constraint matrix G. Here we numerically verify that $\delta = 0$ only when $L \geqslant L_c$ for the Hamiltonians H_2 and H_3, which are shown in Tables II and III, respectively. In fact, the number of unknown parameters in the Hamiltonians can be directly counted. For the Hamiltonian H_2, $N = 12L - 9$; for the Hamiltonian H_3, $N = 39L - 63$. From the numerical results, we observe that r depends not only on the Hamiltonian (including the length L) but also on the rank q of the steady state. However, we have no idea of how to directly determine the analytical relation between r and the variables L and q for a given Hamiltonian from the HOE method.

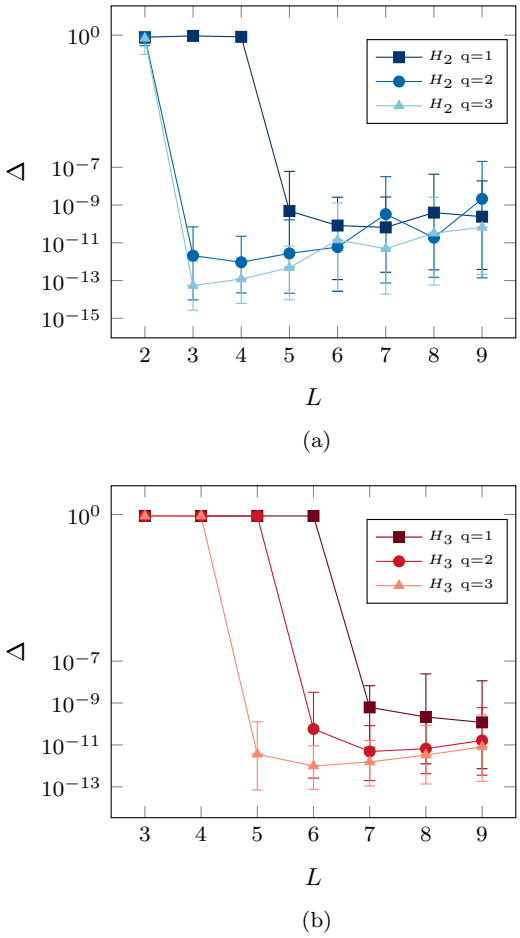

FIG. 1. We reconstruct (a) H_2 and (b) H_3 by means of HOE from steady state. Simulations are executed over 200 random Hamiltonians with three different states for each chain length L. The squares, circles, and triangles represent states of $q = 1, 2$, and 3, respectively.

III. DETERMINING THE RANK OF CONSTRAINT MATRIX WITH ENERGY EIGENVALUE EQUATIONS

To determine the value of Rank$G$, it is instructive to study the energy eigenvalue equations (EEE). In this section, we first apply EEE to recover Hamiltonians H_2 and H_3 from the mixed state with different rank. Then, we prove the equivalence of HOE and EEE. Finally, we determine the value of Rank$G$ using the characteristics of EEE.

A. Reconstructing Hamiltonians by the energy eigenvalue equation

When our system stays in the steady state ρ in Eq. (10), the most complete information about the state ρ can be obtained through quantum tomography. In general, we assume the spectrum of ρ is not degenerate. Then we can explicitly obtain every eigenstate $|\lambda_\mu\rangle$ and its probability p_μ. Since generally the probability p_μ contains no information of the Hamiltonian, all the information of the Hamiltonian is contained in the eigenstates $\{|\lambda_\mu\rangle\}$. Based on this consideration, we develop

the following approach to recover the Hamiltonian directly based on the energy eigenvalue equation, which is briefly called the EEE approach.

The energy eigenvalue equation of generic local Hamiltonian $H = \sum_{n=1}^{N} a_n h_n$ can be written as

$$\sum_{n=1}^{N} a_n h_n |\lambda_\mu\rangle = \lambda_\mu |\lambda_\mu\rangle, \tag{11}$$

where $|\lambda_\mu\rangle$ is the eigenstate with eigenvalue λ_μ appearing in Eq. (10). In a specific basis $\{|i\rangle\}$ Eq. (11) becomes

$$\sum_{n=1}^{N} a_n \langle i|h_n|\lambda_\mu\rangle = \lambda_\mu \langle i|\lambda_\mu\rangle. \tag{12}$$

Splitting Eq. (12) into the real and the imaginary part

$$\sum_{n=1}^{N} a_n \mathrm{Re}\langle i|h_n|\lambda_\mu\rangle - \lambda_\mu \mathrm{Re}\langle i|\lambda_\mu\rangle = 0, \tag{13a}$$

$$\sum_{n=1}^{N} a_n \mathrm{Im}\langle i|h_n|\lambda_\mu\rangle - \lambda_\mu \mathrm{Im}\langle i|\lambda_\mu\rangle = 0, \tag{13b}$$

where $\mathrm{Re}z$ and $\mathrm{Im}z$ denote the real and the imaginary part of complex number z, respectively. Denoting L as the chain length, we can get $2^{L+1}q$ homogeneous linear equations with the unknowns $\vec{x} = (a_1, \ldots, a_N, \lambda_1, \ldots, \lambda_q)$, which can be written in the matrix form as

$$Q\vec{x} = 0, \tag{14}$$

where the constraint matrix Q is a $2^{L+1}q \times (N+q)$ matrix:

$$Q = \begin{pmatrix} \mathrm{Re}A_1 & \mathrm{Re}B_1 \\ \mathrm{Im}A_1 & \mathrm{Im}B_1 \\ \mathrm{Re}A_2 & \mathrm{Re}B_2 \\ \mathrm{Im}A_2 & \mathrm{Im}B_2 \\ \vdots & \vdots \\ \mathrm{Re}A_q & \mathrm{Re}B_q \\ \mathrm{Im}A_q & \mathrm{Im}B_q \end{pmatrix}, \tag{15}$$

with A_μ being a $2^L \times N$ matrix:

$$A_\mu = \begin{pmatrix} \langle 1|h_1|\lambda_\mu\rangle & \cdots & \langle 1|h_N|\lambda_\mu\rangle \\ \vdots & \vdots & \vdots \\ \langle 2^L|h_1|\lambda_\mu\rangle & \cdots & \langle 2^L|h_N|\lambda_\mu\rangle \end{pmatrix} \tag{16}$$

and B_μ being a $2^L \times q$ matrix with nonzero elements in its μth column:

$$B_\mu = \begin{pmatrix} 0 & \cdots & 0 & -\langle 1|\lambda_\mu\rangle & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & -\langle 2^L|\lambda_\mu\rangle & 0 & \cdots & 0 \end{pmatrix}. \tag{17}$$

Here Eq. (14) plays the same role as Eq. (5) in the HOE method. Similarly, we can solve the parameter vector $\vec{a}$ by the following constraint optimization problem:

$$\min_{\vec{a}} ||Q\vec{x}||, \quad \text{such that } ||\vec{a}|| = 1. \tag{18}$$

The degree of freedom of the vector $\vec{x}$ is determined by rank of constraint matrix Q, which is denoted as $\mathrm{Rank}Q = r'$. The number of unknowns of linear equations (14) is represented as $N' = N + q$.

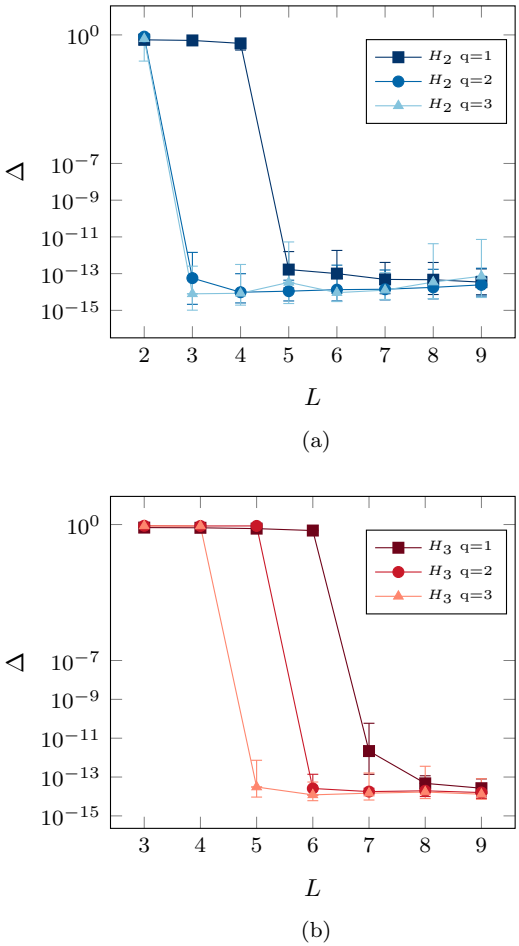

FIG. 2. We reconstruct (a) H_2 and (b) H_3 by means of EEE from steady state. Simulations are executed over 200 random Hamiltonians with three different states for each chain length L. The squares, circles, and triangles represent states of $q = 1, 2$, and 3, respectively.

The procedure to apply the EEE to recover the local Hamiltonians is given as follows. First, for each given chain length L we prepare the Hamiltonians to be recovered by generating 200 random vectors $\{\vec{a}_{\text{true}}\}$ of the Gaussian distribution with zero mean and unit standard deviation. Second, for each random vector $\vec{a}_{\text{true}}$ we numerically calculate the eigenstates of the prepared Hamiltonian and construct three mixed states ρ with $q = 1, 2, 3$ as given in Eq. (10). Third, we extract the eigenstates by eigendecomposition of density matrix ρ. Then we construct constraint matrix Q by Eq. (15) and calculate the Rank$Q$. Fourth, we solve Eq. (18) using the least-squares method by NumPy function numpy.linalg.svd and calculate the reconstructing errors.

Following the above EEE procedure, we obtain the reconstructing errors of the Hamiltonians H_2 and H_3 from mixed states with chain length L from 1 to 9 shown in Fig. 2. We present the accurate value of the number of unknowns N', Rank$Q$, and difference $\delta' = N' - (r' + 1)$ as the function of L with $q = 1, 2, 3$ for H_2 and H_3 in Tables IV and V, respec-

tively. For all the cases with the same N and q, we observe that

$$\delta' = \delta, \tag{19}$$

$$r' = r + q. \tag{20}$$

Equation (19) implies that the EEE method and the HOE method has the same power to recover the Hamiltonians in all the cases, which are numerically verified by the results shown in Fig. 1 and Fig. 2. Equation (20) shows that the rank of G can be obtained by calculating the rank of Q.

Up to now, we tackled the Hamiltonian tomography problem of H_2 and H_3 by both HOE and EEE methods. It turns out that the local Hamiltonian space that HOE and EEE can successfully recover contains the area that satisfies $\delta = \delta' = 0$. In other words, it gives the same critical chain length L_c in all the cases. Here, we emphasize that, in the HOE procedure, when all the Hamiltonian terms are used as observables $\{K_m\}$, adding new observables to matrix G will not increase the value of Rank$G$. Subsequently, r in Tables II and III is the maximum value of Rank$G$ to the corresponding type of Hamiltonians. We can infer that the number of linearly independent functions in HOE can be no more than the number of independent functions in EEE.

B. Equivalence between HOE and EEE

In this subsection, we prove the equivalence of HOE and EEE. We first derive the HOE from the EEE.

In the EEE method, the complex conjugation of Eq. (12) gives

$$\sum_{n=1}^{N} a_n \langle \lambda_\mu | h_n | j \rangle = \lambda_\mu \langle \lambda_\mu | j \rangle, \tag{21}$$

where $|j\rangle$ is any basis vector. Combining Eq. (12) and Eq. (21), for any two basis vectors $|i\rangle$ and $|j\rangle$ we obtain

$$\begin{aligned} \langle \lambda_\mu | [|j\rangle\langle i|, H] | \lambda_\mu \rangle &= \langle \lambda_\mu | j \rangle \langle i | H | \lambda_\mu \rangle - \langle \lambda_\mu | H | j \rangle \langle i | \lambda_\mu \rangle \\ &= \lambda_\mu \langle \lambda_\mu | j \rangle \langle i | \lambda_\mu \rangle - \lambda_\mu \langle \lambda_\mu | j \rangle \langle i | \lambda_\mu \rangle = 0, \end{aligned} \tag{22}$$

which immediately leads to the basic equations of the HOE:

$$\sum_{n=1}^{N} a_n \langle i [K_m, h_n] \rangle = \sum_{\mu,j,i} i p_\mu \langle j | K_m | i \rangle \langle \lambda_\mu | [|j\rangle\langle i|, H] | \lambda_\mu \rangle = 0, \tag{23}$$

where K_m is any linear operator on the Hilbert space.

Now we derive the EEE from the HOE. We start from the basic equations of the HOE, Eq. (23). Because K_m is an arbitrary operator, we can always make the coefficients $\langle j | K_m | i \rangle$ linearly independent. Thus we can obtain

$$\sum_{\mu} p_\mu \langle \lambda_\mu | [|j\rangle\langle i|, H] | \lambda_\mu \rangle = 0. \tag{24}$$

Note that it can be written as

$$\mathrm{Tr}\left(|j\rangle\langle i| \left[H, \sum_{\mu} p_\mu |\lambda_\mu\rangle\langle\lambda_\mu| \right] \right) = 0. \tag{25}$$

Because $\{|j\rangle\langle i|\}$ constructs a basis of the operator space, Eq. (25) gives

$$\left[H, \sum_{\mu} p_\mu |\lambda_\mu\rangle\langle\lambda_\mu| \right] = [H, \rho] = 0. \tag{26}$$

Equation (26) implies that H has the same eigenstates with ρ when $\{p_\mu, \mu = 1, \ldots, N\}$ are nondegenerate, i.e., that it satisfies the eigenvalue equation (11). This completes our proof of the equivalence of the HOE and the EEE. Consequently, Eq. (19) and Eq. (20) directly follow from the above equivalence.

C. Determining the rank of the constraint matrix

We are now in a position to determine Rank$G$ from Rank$Q$ by Eq. (20). As mentioned above, the matrix Q satisfies Eq. (14), which contains $2^{L+1}q$ homogeneous linear equations with the unknowns $\vec{x} = (a_1, \ldots, a_N, \lambda_1, \ldots, \lambda_q)$. However, the Hamiltonian is diagonal in its own eigenbasis and Eq. (12) gives

$$\sum_{n} a_n \langle \lambda_\nu | h_n | \lambda_\mu \rangle = \lambda_\mu \delta_{\mu\nu}, \quad \mu, \nu = 1, \ldots, q. \tag{27}$$

Since $\langle \lambda_\nu | h_n | \lambda_\mu \rangle$ is complex in general, the above equations give $2q^2$ real constraint linear equations. Because every h_n is Hermitian, Eq. (27) is real valued; then it implies

$$\sum_{n} a_n \langle \lambda_\mu | h_n | \lambda_\nu \rangle = \lambda_\mu \delta_{\mu\nu}, \quad \mu, \nu = 1, \ldots, q. \tag{28}$$

Then there are q^2 constraint independent homogeneous linear equations with the unknowns $\vec{x}$ in Eq. (27). Thus there are at most $2^{L+1}q - q^2$ independent linear equations in Eq. (14), i..e., $r' \leqslant 2^{L+1}q - q^2$. In addition, there are always nonzero solutions of Eq. (14), which implies that $r' \leqslant N + q - 1$. Therefore, we obtain

$$r' = \min\{2^{L+1}q - q^2, N + q - 1\}. \tag{29}$$

By using Eq. (20), we arrive at

$$r = \min\{2^{L+1}q - q^2 - q, N - 1\}. \tag{30}$$

The above analytical expressions of the rank of G in Eq. (30) and the rank of Q in Eq. (29) are numerically verified in Tables II, III, IV, and V.

The critical chain length is denoted as L_c. When chain length $L \geqslant L_c$, we can uniquely recover the corresponding Hamiltonian. Now, we determine the L_c from Eq. (30). To uniquely recover the Hamiltonian, Rank$G$ should equal the number of unknowns minus 1, which leads to

$$2^{L+1}q - q^2 - q \geqslant N - 1. \tag{31}$$

For the two-local Hamiltonian H_2, $N = 12L - 9$. From Eq. (31), the critical chain length

$$L_c(H_2, \rho) = \min_{L} 2^{L+1}q - q^2 - q \geqslant 12L - 10, \tag{32}$$

where $L \geqslant 2$ and $1 \leqslant q \leqslant 2^L$.

Similarly, for the three-local Hamiltonian H_3, $N = 39L - 63$. From Eq. (31), the critical chain length

$$L_c(H_3, \rho) = \min_{L} 2^{L+1}q - q^2 - q \geqslant 39L - 64, \tag{33}$$

where $L \geqslant 3$ and $1 \leqslant q \leqslant 2^L$.

TABLE I. Critical chain length L_c for H_2, H_2', and H_3 with $q = 1, \ldots, 6$.

$H \backslash L_c \backslash q$	1	2	3	4	5	6
H_2	5	3	3	3	3	3
H_2'	6	4	3	3	3	3
H_3	7	6	5	4	4	3

We point out that our method not only works for local Hamiltonians H_2 and H_3, it can also be used to predict L_c for any one-dimensional spin-1/2 chain with local Hamiltonians. To demonstrate its effectiveness, we calculate the L_c of H_2', which contains the nearest- and the next-nearest-neighbor interaction,

$$H_2' = H_2 + \sum_{l=1}^{L-2} \sum_{\eta} \sum_{\theta} a_{l\eta\theta} \sigma_l^{\eta} \sigma_{l+2}^{\theta}. \tag{34}$$

For the Hamiltonian H_2', $N = 21L - 27$. The critical chain length can be calculated by

$$L_c(H_2', \rho) = \min_L 2^{L+1} q - q^2 - q \geqslant 21L - 28. \tag{35}$$

The critical chain lengths for H_2, H_2', and H_3 with $q = 1, \ldots, 6$ are shown in Table I.

IV. CONCLUSION

We revisit the problem of reconstructing a generic local Hamiltonian when the system stays in a steady state by measuring a collection of observables. Applying the HOE method to the two spin chains with two-local interactions and three-local interactions, we numerically find that only when the chain length L is not less than some critical chain length L_c can we uniquely recover the corresponding local Hamiltonian. The critical chain length L_c depends not only on the spin chain model, but also on the rank q of the steady state.

To explain the underlying mechanism for the existence of the critical chain length L_c, we observe that, when the rank r of the constraint matrix G is not less than the number of unknown parameters in the recovered Hamiltonian minus 1, the Hamiltonian can be uniquely recovered. To further determine the rank r, we develop an alternative method called the EEE method, which is used to recover all the results from the HOE method. Furthermore, we proved the equivalence between the HOE method and the EEE method. Especially, we obtain the analytical expression of the rank r by using the EEE method, which can be used to determine the critical chain length L_c analytically.

Note that our method can be applied to generic local Hamiltonians, but it fails for some local Hamiltonians whose probability measure is zero, for example, any local Hamiltonian that can be written as the sum of two terms that commute with each other, $H = H_1 + H_2$ with $[H_1, H_2] = 0$. Then the eigenstate space of H is also the eigenstate space of H_1 and H_2, which leads to the fact that Hamiltonians $H' = \alpha H_1 + \beta H_2$ have the same eigenstate space for any different real numbers α and β. Obviously it is impossible to recover a unique Hamiltonian from any number of eigenstates in this case.

Our work studies the condition for a generic local Hamiltonian that can be recovered from its one steady state. For the two spin chain models with two-local interactions and three-local interactions, we show the generic Hamiltonians can be reconstructed uniquely only when the chain length is not less than the critical chain length. Furthermore, our quantitative result in Eq. (31) for determining the critical chain length L_c can be used on any one-dimensional spin-1/2 chain with generic local Hamiltonians. In principle, we can extend our analytical result on the critical length to the critical system size for two-dimensional and three-dimensional generic local Hamiltonians. We hope that our work will shed light on the Hamiltonian tomography problem.

ACKNOWLEDGMENTS

This work is supported by the National Key Research and Development Program of China (Grants No. 2021YFA0718302 and No. 2016YFA0300603), the National Natural Science Foundation of China (NSFC) of China (Grants No. 11775300 and No. 12075310), and the Strategic Priority Research Program of Chinese Academy of Sciences (CN) (Grant No. XDB28000000).

APPENDIX

In this Appendix, we list the numerical results on our Hamiltonian reconstructions in four tables as follows. Tables II and III show the numerical results of reconstructing Hamiltonians H_2 and H_3 by means of HOE, respectively. For Hamiltonian H_2 (or H_3), the number of unknowns (N) in Eq. (5) is a function of the chain length L, and the rank of the constraint matrix G (r) is the function of the chain length L and the rank of the steady state ρ (q). Only when the difference $\delta = N - 1 - r$ is zero can we recover the generic Hamiltonian H_2 (or H_3). All the numerical results of N, r, and δ for $L \leqslant 9$ and $q = 1, 2, 3$ are presented in Tables II and III.

Tables IV and V show the numerical results of reconstructing Hamiltonians H_2 and H_3 by means of EEE, respectively. For Hamiltonian H_2 (or H_3), the number of unknowns (N') in Eq. (14) and the rank of the constraint matrix Q (r') is the function of the chain length L and the rank of the steady state ρ (q). Only when the difference $\delta' = N' - 1 - r'$ is zero can we recover the generic Hamiltonian H_2 (or H_3). All the numerical results of N', r', and δ' for $L \leqslant 9$ and $q = 1, 2, 3$ are presented in Tables IV and V.

TABLE II. N, r, and δ as the function of L and q when reconstructing H_2 by means of HOE from the steady state.

		$q=1$		$q=2$		$q=3$	
L	N	r	δ	r	δ	r	δ
2	15	6	8	10	4	12	2
3	27	14	12	26	0	26	0
4	39	30	8	38	0	38	0
5	51	50	0	51	0	51	0
6	63	62	0	62	0	62	0
7	75	74	0	74	0	74	0
8	87	86	0	86	0	86	0
9	99	98	0	98	0	98	0

TABLE III. N, r, and δ as the function of L and q when reconstructing H_3 by means of HOE from the steady state.

		$q=1$		$q=2$		$q=3$	
L	N	r	δ	r	δ	r	δ
3	63	14	48	26	36	36	26
4	111	30	80	58	52	84	26
5	159	62	96	122	36	158	0
6	207	126	80	206	0	206	0
7	255	254	0	254	0	254	0
8	303	302	0	302	0	302	0
9	351	350	0	350	0	350	0

TABLE IV. N', r', and δ' as the function of L and q when reconstructing H_2 by means of EEE from the steady state.

	$q=1$			$q=2$			$q=3$		
L	N'	r'	δ'	N'	r'	δ'	N'	r'	δ'
2	16	7	8	17	12	4	18	15	2
3	28	15	12	29	28	0	30	29	0
4	40	31	8	41	40	0	42	41	0
5	52	51	0	53	52	0	54	53	0
6	64	63	0	65	64	0	66	65	0
7	76	75	0	77	76	0	78	77	0
8	88	87	0	89	88	0	90	89	0
9	100	99	0	101	100	0	102	101	0

TABLE V. N', r', and δ' as the function of L and q when reconstructing H_3 by means of EEE from the steady state.

	$q=1$			$q=2$			$q=3$		
L	N'	r'	δ'	N'	r'	δ'	N'	r'	δ'
3	64	15	48	65	28	36	66	39	26
4	112	31	80	113	60	52	114	87	26
5	160	63	96	161	124	36	162	161	0
6	208	127	80	209	208	0	210	209	0
7	256	255	0	257	256	0	258	257	0
8	304	303	0	305	304	0	306	305	0
9	352	351	0	353	352	0	354	353	0

[1] A. Paramekanti, M. Randeria, and N. Trivedi, Phys. Rev. Lett. **87**, 217002 (2001).

[2] P. Anderson, Mater. Res. Bull. **8**, 153 (1973).

[3] C. Di Franco, M. Paternostro, and M. S. Kim, Phys. Rev. Lett. **102**, 187203 (2009).

[4] J. S. Lundeen, A. Feito, H. Coldenstrodt-Ronge, K. L. Pregnell, C. Silberhorn, T. C. Ralph, J. Eisert, M. B. Plenio, and I. A. Walmsley, Nat. Phys. **5**, 27 (2009).

[5] S. G. Schirmer and D. K. L. Oi, Phys. Rev. A **80**, 022333 (2009).

[6] H. L. Haselgrove, M. A. Nielsen, and T. J. Osborne, Phys. Rev. Lett. **91**, 210401 (2003).

[7] A. Molnar, N. Schuch, F. Verstraete, and J. I. Cirac, Phys. Rev. B **91**, 045138 (2015).

[8] J. R. Garrison and T. Grover, Phys. Rev. X **8**, 021026 (2018).

[9] X.-L. Qi and D. Ranard, Quantum **3**, 159 (2019).

[10] A. Anshu, S. Arunachalam, T. Kuwahara, and M. Soleimanifar, Nat. Phys. **17**, 931 (2021).

[11] G. Bentsen, I.-D. Potirniche, V. B. Bulchandani, T. Scaffidi, X. Cao, X.-L. Qi, M. Schleier-Smith, and E. Altman, Phys. Rev. X **9**, 041011 (2019).

[12] A. Zubida, E. Yitzhaki, N. H. Lindner, and E. Bairey, arXiv:2108.08824.

[13] E. Bairey, I. Arad, and N. H. Lindner, Phys. Rev. Lett. **122**, 020504 (2019).

[14] S.-Y. Hou, N. Cao, S. Lu, Y. Shen, Y.-T. Poon, and B. Zeng, New J. Phys. **22**, 083088 (2020).

[15] C. Cao, S.-Y. Hou, N. Cao, and B. Zeng, J. Phys.: Condens. Matter **33**, 064002 (2020).

[16] W. Zhu, Z. Huang, and Y.-C. He, Phys. Rev. B **99**, 235109 (2019).

[17] X. Turkeshi, T. Mendes-Santos, G. Giudici, and M. Dalmonte, Phys. Rev. Lett. **122**, 150606 (2019).

[18] J. Chen, Z. Ji, B. Zeng, and D. L. Zhou, Phys. Rev. A **86**, 022339 (2012).

[19] J. Zhang and M. Sarovar, Phys. Rev. Lett. **113**, 080401 (2014).

[20] Y. Wang, D. Dong, B. Qi, J. Zhang, I. R. Petersen, and H. Yonezawa, IEEE Trans. Autom. Control **63**, 1388 (2018).

[21] N. Wiebe, C. Granade, C. Ferrie, and D. G. Cory, Phys. Rev. Lett. **112**, 190501 (2014).

[22] D. Burgarth, K. Maruyama, and F. Nori, Phys. Rev. A **79**, 020305(R) (2009).

[23] Z. Li, L. Zou, and T. H. Hsieh, Phys. Rev. Lett. **124**, 160502 (2020).

[24] E. Bairey, C. Guo, D. Poletti, N. H. Lindner, and I. Arad, New J. Phys. **22**, 032001 (2020).

[25] M. Dupont, N. Macé, and N. Laflorencie, Phys. Rev. B **100**, 134201 (2019).

[26] C. Kokail, R. van Bijnen, A. Elben, B. Vermersch, and P. Zoller, Nat. Phys. **17**, 936 (2021).
[27] R. Blatt and C. Roos, Nat. Phys. **8**, 277 (2011).
[28] A. E. Webb, S. C. Webster, S. Collingbourne, D. Bretaud, A. M. Lawrence, S. Weidt, F. Mintert, and W. K. Hensinger, Phys. Rev. Lett. **121**, 180501 (2018).
[29] C. J. Ballance, T. P. Harty, N. M. Linke, M. A. Sepiol, and D. M. Lucas, Phys. Rev. Lett. **117**, 060504 (2016).
[30] J. I. Cirac and P. Zoller, Phys. Rev. Lett. **74**, 4091 (1995).
[31] R. Barends, J. Kelly, A. Megrant, D. Sank, E. Jeffrey, Y. Chen, Y. Yin, B. Chiaro, J. Mutus, C. Neill, P. O'Malley, P. Roushan, J. Wenner, T. C. White, A. N. Cleland, and J. M. Martinis, Phys. Rev. Lett. **111**, 080502 (2013).
[32] R. Barends, J. Kelly, A. Megrant, A. Veitia, D. Sank, E. Jeffrey, T. C. White, J. Mutus, A. G. Fowler, B. Campbell, Y. Chen, Z. Chen, B. Chiaro, A. Dunsworth, C. Neill, P. O'Malley, P. Roushan, A. Vainsencher, J. Wenner, A. N. Korotkov *et al.*, Nature (London) **508**, 500 (2014).
[33] S.-Y. Hou, H. Li, and G.-L. Long, Sci. Bull. **62**, 863 (2017).
[34] X. Chen, Y. Li, Z. Wu, R. Liu, Z. Li, and H. Zhou, Phys. Rev. A **103**, 042429 (2021).
[35] J. Wang, S. Paesani, R. Santagati, S. Knauer, A. A. Gentile, N. Wiebe, M. Petruzzella, J. O'Brien, J. G. Rarity, and A. Laing, Nat. Phys. **13**, 551 (2017).
[36] E. F. Dumitrescu and P. Lougovski, Phys. Rev. Research **2**, 033251 (2020).