

Characterizing entanglement using quantum discord over state extensions

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We propose a framework to characterize entanglement with quantum discord, both asymmetric and symmetric, over state extensions. In particular, we show that the minimal Bures distance of discord over state extensions is equivalent to the Bures distance of entanglement. This equivalence places quantum discord in a more primitive position than entanglement conceptually in the sense that entanglement can be interpreted as an irreducible part of discord over all state extensions. Based on this equivalence, we also offer an operational meaning of the Bures distance of entanglement by connecting it to quantum state discrimination. Moreover, for the relative entropy part, we prove that the entanglement measure introduced by Devi and Rajagopal [A. R. U. Devi and A. K. Rajagopal, *Phys. Rev. Lett.* **100**, 140502 (2008)] is actually equivalent to the relative entropy of entanglement. We also provide several quantifications of entanglement based on discord measures.

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I. INTRODUCTION

Quantum correlations [1,2] are defined from different viewpoints and they in turn are expected to offer different advantages. Hence, the characterization and quantification of quantum correlations are instrumental in exploring and exploiting the quantum phenomena. The remarkable advantages that quantum correlations offer make quantum information theory more powerful than classical theory. Entanglement [1] is an important quantum resource which plays a crucial role in quantum information processing, quantum algorithms, quantum computation, and cryptography [3]. The notion of quantum correlations and resources beyond entanglement [2,4–6] such as quantum discord [2,7–10] and quantum coherence [6,11–13] are also very prominent and useful in quantum information theory. For instance, quantum discord is a genuine resource in the deterministic quantum computation with one quantum bit algorithm [14,15].

Among several quantum correlations, entanglement and quantum discord are significant and are usually regarded as very distinct in nature. While entanglement belongs to the entanglement-separability paradigm, quantum discord

belongs to the information-theoretic paradigm. Nevertheless, entanglement and quantum discord have both essential similarities and significant differences. Several remarkable investigations [16–28] relating to entanglement and discord have been studied in recent years. In particular, the correspondence between classical states and separable states [29], the characterization and quantification of entanglement with the generalized information-theoretic measure [30], the minimal quantum discord of bipartite state over state extensions [31], and the minimal correlated coherence over symmetric state extensions [32,33] are of special interest.

It should be noted that the framework that quantifies entanglement with quantum discord or coherence over state extensions is quite different from the existing entanglement measures such as entanglement of formation and entanglement cost [34], distillable entanglement [35], relative entropy of entanglement and Bures distance of entanglement [36,37], robustness of entanglement [38], and squashed entanglement [39,40], which are mostly based on operational meaning, information principles, and mathematics. Thus far, the interrelationship between these two different categories of entanglement measures is not clear.

In this paper we study characterization of entanglement using quantum discord over state extensions. For the meaning of “over state extensions” see [41]. Our work is different from and a generalization of previous works [30–32]. In Ref. [30] the authors introduced quantumness of correlations as an entanglement measure. They proved that for a bipartite state ρ_{ab} , the relative entropy of asymmetric discord over all extended

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states is a candidate of entanglement measure, that is, this quantification is faithful for separable states and nonincreasing under local operations. It also turns out to be an upper bound to the relative entropy of entanglement. On the other hand, it was shown in Ref. [31] that the minimal discord over all extended states is also a valid entanglement measure. These two investigations revealed the potential relation between entanglement and quantum discord over state extensions. Also, the conservation law for distributed entanglement and quantum discord [19] hints at some kind of fundamental relation between them. However, a general formalism relating entanglement and discord (quantum correlations regarded as very distinctive in nature) has been elusive. Our study provides a natural connection between the two via state extensions. The main findings of our work can be summarized as follows.

(i) For the Bures distance [42,43], we prove that the minimal Bures distance of discord over state extensions is equivalent to the Bures distance of entanglement [37,44,45]. Moreover, the Bures distance of entanglement was proved to be equal to its corresponding convex roof [45], which plays a key role in our study. In fact, we prove the equivalence by showing that the minimal Bures distance of discord over state extensions is bounded by the Bures distance of entanglement and its convex roof.

The Bures distance of entanglement is defined from a geometric viewpoint whose operational meaning is not very clear. A correspondence, however, between the Bures distance of discord and quantum state discriminations has been established in Ref. [46]. Based on this correspondence and the equivalence between the Bures distance of entanglement and discord over state extensions, we offer an operational meaning of the Bures distance of entanglement by linking it to quantum state discriminations.

(ii) We propose a framework to characterize entanglement using quantum discord over state extensions. Actually, for a generalized discord measure including entropic discord [7,8], geometric discord [47], measurement-induced geometric discord [48], correlated coherence [32], and geometric correlated partial coherence [49], we prove that the minimal discord over state extensions is a candidate for an entanglement measure. This provides an alternative perspective to understand entanglement from the viewpoint of quantum discord. Results in previous studies [30–32] are special cases of our framework.

(iii) We show that the quantification of entanglement proposed in Ref. [30] is actually equivalent to the relative entropy of entanglement [37], which is a well-studied entanglement measure.

This paper is structured as follows. In Sec. II we recall various concepts prerequisite for our study. We discuss characterization of entanglement using asymmetric and symmetric quantum discord in Secs. III and IV, respectively. We give an operational meaning of the Bures distance of entanglement in Sec. III C. We summarize our findings in Sec. V. Appendixes A and B present proofs of Theorems 3 and 4, respectively.

II. PRELIMINARIES

In this section we briefly recall various concepts which are prerequisite for our study.

A. Separable states

Let $\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$ be the composite Hilbert space of a bipartite system and $D(\mathcal{H})$ be the set of density matrices on \mathcal{H} . A quantum state $\rho_{ab} \in D(\mathcal{H})$ shared between two parties a and b is called separable if it can be represented as

$$\rho_{ab} = \sum_i p_i \rho_a^i \otimes \rho_b^i, \quad (1)$$

where $p_i \geq 0$, $\sum_i p_i = 1$, and ρ_a^i and ρ_b^i are local states for parties a and b , respectively. Otherwise, it is called entangled. We denote the set of separable states by \mathcal{S} . Moreover, a bipartite separable state ρ_{ab} is called classical quantum (CQ) if it can be written as $\rho_{ab} = \sum_i p_i |i\rangle_a \langle i| \otimes \rho_b^i$ and classical classical (CC) if $\rho_{ab} = \sum_{i,j} p_{ij} |i\rangle_a \langle i| \otimes |j\rangle_b \langle j|$, where $\{|i\rangle\}$, $\{|j\rangle\}$ are two set of orthogonal pure states.

B. Relative entropy

Relative entropy is defined as $S(\rho||\sigma) := \text{tr}[\rho(\log_2 \rho - \log_2 \sigma)]$ [3]. For any classical-quantum state $\sigma_{ab}^{(cq)} = \sum_j p_j |j\rangle_a \langle j| \otimes \sigma_b^j$, it is easy to verify that $\text{tr}(\rho_{ab} \log_2 \sigma_{ab}^{(cq)}) = \text{tr}[\Pi_a(\rho_{ab}) \log_2 \sigma_{ab}^{(cq)}]$, where $\Pi_a(\rho_{ab}) = \sum_i (|i\rangle_a \langle i| \otimes I_b)$. As a result, one has

$$\begin{aligned} S(\rho_{ab}||\sigma_{ab}^{(cq)}) &= \text{tr}[\rho_{ab} \log_2 \rho_{ab}] - \text{tr}[\rho_{ab} \log_2 \sigma_{ab}^{(cq)}] \\ &= S(\Pi_a(\rho_{ab})) - S(\rho_{ab}) + S(\Pi_a(\rho_{ab})||\sigma_{ab}^{(cq)}). \end{aligned} \quad (2)$$

A similar equation is established in block coherence theory [50].

C. Bures distance

The Bures distance is defined as [42,43]

$$d_B(\rho, \sigma) := \sqrt{2 - 2F(\rho, \sigma)}, \quad (3)$$

where $F(\rho, \sigma)$ is the fidelity $F(\rho, \sigma) := \text{tr} \sqrt{\sqrt{\rho} \sigma \sqrt{\rho}}$ between ρ and σ . Since $F(\rho, \sigma) \in [0, 1]$ and is unity if and only if $\rho = \sigma$, $d_B(\rho, \sigma)$ is non-negative and vanishes if and only if $\rho = \sigma$. Moreover, the monotonicity and joint concavity of fidelity [3] implies that d_B^2 is contractive and jointly convex.

Unlike relative entropy, the Bures distance is a bona fide distance on state space, that is, it is faithful, symmetric, and satisfies the triangle inequality. Moreover, it is Riemannian [51] and its metric is given in [51–53]. Quantum Fisher information, a quantum analog of Fisher information in statistics, is defined in terms of the Bures distance metric [54]. These properties of Bures distance enable it to quantify various quantum correlations including entanglement [36], discord [46,55], coherence [11], etc.

D. Entanglement

An entanglement measure E is a functional on $D(\mathcal{H})$ satisfying (i) faithfulness $E(\rho) \geq 0$, where the equality holds if and only if $\rho \in \mathcal{S}$, and (ii) monotonicity $E(\rho) \geq E(\Phi(\rho))$ for any local operations and classical communication (LOCC) operation Φ .

Bures distance of entanglement. The Bures distance of entanglement is defined as the minimal square of Bures distance to separable states [36],

$$E_B(\rho_{ab}) := \min_{\sigma_{ab} \in \mathcal{S}} d_B^2(\rho_{ab}, \sigma_{ab}). \quad (4)$$

Obviously, $E_B(\rho_{ab})$ is non-negative, and vanishes if and only if ρ_{ab} is separable. Furthermore, E_B is convex and non-increasing under LOCC operations [34,37]. Note that for the bipartite pure state $|\psi\rangle = \sum_i \sqrt{\lambda_i} |x_i\rangle_a |y_i\rangle_b$, with $\lambda_1 \geq \dots \geq \lambda_n$, $E_B(|\psi\rangle) = 2(1 - \sqrt{\lambda_1})$ [45]. In fact, if we assume that $\sigma_{ab} \in \mathcal{S}$ has a separable pure state decomposition $\sigma_{ab} = \sum_j q_j |\phi_j\rangle_{ab} \langle\phi_j|$, then

$$\begin{aligned} F(|\psi\rangle, \sigma_{ab}) &= \sqrt{\sum_j q_j |\langle\psi|\phi_j\rangle|^2} \\ &\leq \sqrt{\sum_j q_j |\langle\psi|\phi_{\max}\rangle|^2} = |\langle\psi|\phi_{\max}\rangle|, \end{aligned} \quad (5)$$

where $|\langle\psi|\phi_{\max}\rangle| := \max_j |\langle\psi|\phi_j\rangle|$ is the maximal over all j . Therefore, $E_B(|\psi\rangle) = 2(1 - \sqrt{\lambda_1})$ and the corresponding closest separable state is $|x_1, y_1\rangle_{ab}$.

Convex roof of the Bures distance of entanglement. The convex roof of the Bures distance of entanglement is defined as

$$E_B^{cr}(\rho_{ab}) := \min_{p_i, |\psi_i\rangle} \sum_i p_i E_B(|\psi_i\rangle), \quad (6)$$

where the minimal is taken over all pure state decompositions $\rho_{ab} = \sum_i p_i |\psi_i\rangle \langle\psi_i|$.

E. Quantum discord

Asymmetric quantum discord. A functional \hat{D} on $D(\mathcal{H})$ is called a discord measure (asymmetric or local) if it satisfies the following properties.

(D1) \hat{D} is faithful, i.e., $\hat{D}(\rho_{ab}) \geq 0$ and the equality holds if and only if $\rho_{ab} \in \mathcal{CQ}$.

(D2) \hat{D} is nonincreasing for any quantum operation on subsystem b , i.e., $\hat{D}(\rho_{ab}) \geq \hat{D}(I_a \otimes \Phi_b(\rho_{ab}))$ for any local operation Φ_b .

(D3) \hat{D} is invariant under local unitary transformations, i.e., $\hat{D}(U_a \otimes U_b \rho_{ab} U_a^\dagger \otimes U_b^\dagger) = \hat{D}(\rho_{ab})$ for any unitary $U_a \otimes U_b$ acting on $\mathcal{H}_a \otimes \mathcal{H}_b$.

(D4) \hat{D} reduces to an entanglement monotone [56] for pure states.

Quantum discord has been studied extensively from different viewpoints in the forms of geometric discord, measurement-induced geometric discord [57], and geometric correlated partial coherence [49]. Quantum discord obviously captures quantum correlation beyond entanglement in the sense that a separable state may have nonzero quantum discord.

Symmetric quantum discord. A functional \tilde{D} in state space is called a (symmetric or global) discord [9,10] measure if it satisfies the following properties.

(D1') \tilde{D} is faithful, i.e., $\tilde{D}(\rho_{ab}) \geq 0$ and the equality holds if and only if $\rho_{ab} \in \mathcal{CC}$.

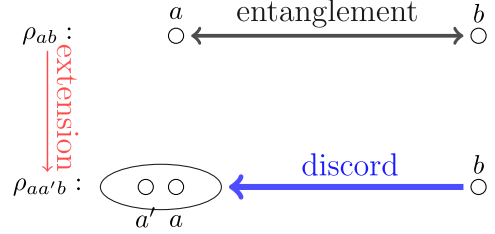


FIG. 1. The minimal discord of extended state $\rho_{aa'b}$ between parties aa' and b is a quantification of entanglement of ρ_{ab} between a and b .

(D2') \tilde{D} is invariant under local unitary transformations, i.e., $\tilde{D}(U_a \otimes U_b \rho_{ab} U_a^\dagger \otimes U_b^\dagger) = \tilde{D}(\rho_{ab})$ for any unitary acting on \mathcal{H}_a and \mathcal{H}_b .

(D3') \tilde{D} reduces to an entanglement monotone on pure states.

Bures distance of discord. The Bures distance of discord is defined as the minimal square of the Bures distance to classical-quantum states [46,55],

$$\hat{D}_B(\rho_{ab}) := \min_{\sigma_{ab} \in \mathcal{CQ}} d_B^2(\rho_{ab}, \sigma_{ab}). \quad (7)$$

It has been shown that \hat{D}_B is a discord measure, i.e., it satisfies properties (D1)–(D4).

Remark 1. Quantum interferometric power (QIP) [58] is a measure of discord-type quantum correlation defined via quantum Fisher information, which is upper bounded by local quantum uncertainty (LQU) [59]. Actually, if subsystem a is a qubit, LQU is equivalent to the Hellinger distance of discord [57,60] and is upper bounded by the Bures distance of discord. As a result, QIP is also upper bounded by the Bures distance of discord in the $C^2 \otimes C^d$ case.

III. CHARACTERIZING ENTANGLEMENT VIA ASYMMETRIC QUANTUM DISCORD

In this section we propose the framework to quantify entanglement via quantum discord (asymmetric or local) and introduce several quantifiers with geometric discord and measurement-induced geometric discord.

A. General results

In this section we explore and interpret the relationship between entanglement and quantum discord (asymmetric) over state extensions.

Definition 1. For a bipartite state $\rho_{ab} \in D(\mathcal{H})$, the minimal discord over state extensions is defined as

$$\hat{\mathcal{E}}(\rho_{ab}) := \min_{\text{tr}_{a'}[\rho_{aa'b}] = \rho_{ab}} \hat{D}(\rho_{aa'b}), \quad (8)$$

where the minimization is taken over all state extensions $\rho_{aa'b}$ of ρ_{ab} [41] along the bipartition $aa' : b$ (see Fig. 1).

Remark 2. Squashed entanglement [40] and the conditional entanglement of mutual information [61], two additive entanglement measures, are defined using the notion of over state extensions. Thus, characterizing quantum entanglement using extended systems has proved an important idea in entanglement theory.

Theorem 1. If \hat{D} is a discord measure satisfying (D1)–(D4), the corresponding minimal discord $\hat{\mathcal{E}}$ over state extensions has the following remarkable properties.

- (E1) $\hat{\mathcal{E}}(\rho_{ab}) \geq 0$ with the equality if and only if $\rho \in \mathcal{S}$.
- (E2) $\hat{\mathcal{E}}$ is invariant under local unitary transformations.
- (E3) $\hat{\mathcal{E}}$ is nonincreasing under a local partial trace

$$\hat{\mathcal{E}}(\rho_{ab}) \leq \hat{\mathcal{E}}(\rho_{aa_1b})$$

for any state extension ρ_{aa_1b} of ρ_{ab} .

- (E4) $\hat{\mathcal{E}}$ is nonincreasing under local operations in party a .
- (E5) $\hat{\mathcal{E}}$ reduces to an entanglement monotone for pure states.

Proof. (E1) The non-negativity of quantum discord implies that $\hat{\mathcal{E}}$ is always non-negative. Moreover, any separable state $\rho_{ab} = \sum_i p_i \rho_a^i \otimes \rho_b^i$ can be embedded into a larger classical-quantum state $\rho_{aa'b} = \sum_i p_i |\alpha_i\rangle_{aa'} \langle \alpha_i| \otimes \rho_b^i$ such that $\rho_{ab} = \text{tr}_{a'}[\rho_{aa'b}]$, where a' is the ancillary system pertinent to party a and $|\alpha_i\rangle_{aa'}$ is the purification of ρ_a^i for each i [30]. However, an entangled state does not admit such an extension. Therefore, $\hat{\mathcal{E}}$ is faithful in separable states.

(E2) Assuming $\rho_{aa'b}$ is the state extension of ρ_{ab} , $\text{tr}_{a'}[U_a \otimes U_b \rho_{aa'b} U_a^\dagger \otimes U_b^\dagger] = U_a \otimes U_b \rho_{ab} U_a^\dagger \otimes U_b^\dagger$ and the local unitary invariance of \hat{D} implies that $\hat{\mathcal{E}}(\rho_{ab}) \geq \hat{\mathcal{E}}(U_a \otimes U_b \rho_{ab} U_a^\dagger \otimes U_b^\dagger)$. On the contrary, we can also show $\hat{\mathcal{E}}(\rho_{ab}) \leq \hat{\mathcal{E}}(U_a \otimes U_b \rho_{ab} U_a^\dagger \otimes U_b^\dagger)$, implying that $\hat{\mathcal{E}}$ is invariant under local unitary transformation.

(E3) This follows trivially because any state extension $\rho_{aa_1a'b}$ of state ρ_{aa_1b} is also the extension of ρ_{ab} .

(E4) Using the Stinespring representation [62], the local operation in party a can be realized by adding a pure state ancilla, followed by a global unitary operation and tracing out the ancilla system, i.e., $\sum_i K_i^a \rho_{ab} K_i^{a\dagger} = \text{tr}_{a_1} U_{aa_1} (\rho_{ab} \otimes |0\rangle_{a_1} \langle 0|) U_{aa_1}^\dagger$. Therefore, one has

$$\begin{aligned} \hat{\mathcal{E}}(\rho_{ab}) &\geq \hat{\mathcal{E}}(\rho_{ab} \otimes |0\rangle_{a_1} \langle 0|) \\ &= \hat{\mathcal{E}}(U_{aa_1} \rho_{ab} \otimes |0\rangle_{a_1} \langle 0| U_{aa_1}^\dagger) \\ &\geq \hat{\mathcal{E}}\left(\sum_i K_i^a \rho_{ab} K_i^{a\dagger}\right), \end{aligned} \quad (9)$$

where the inequality in the first and the third lines follows from property (E3).

(E5) First, combining the definition of $\hat{\mathcal{E}}$ and property (E3),

$$\hat{\mathcal{E}}(|\psi\rangle) \geq \min_{|\phi\rangle} \hat{D}(|\psi\rangle \otimes |\phi\rangle) \geq \hat{D}(|\psi\rangle) = E(|\psi\rangle). \quad (10)$$

Further, considering the special case, that is, that the extension space is one dimensional, $\hat{\mathcal{E}}(|\psi\rangle) = \hat{D}(|\psi\rangle) = E(|\psi\rangle)$. ■

Remark 3. The above results provide an alternative avenue to understand quantum entanglement from the viewpoint of discord over state extensions. We can see that $\hat{\mathcal{E}}$ is a good candidate of an entanglement measure. Moreover, \hat{D} reduces to $\hat{\mathcal{E}}$ for pure states.

We call a non-negative bivariate function d on state space pseudodistance if $d(\rho, \sigma) = 0$ if and only if $\rho = \sigma$ and call d contractive if it satisfies $d(\rho, \sigma) \geq d(\Phi(\rho), \Phi(\sigma))$ for any quantum operation Φ and $\rho, \sigma \in D(\mathcal{H})$. Relative entropy and Bures distance are examples of pseudodistances. We will just write “distance” for simplicity.

Definition 2. For a bipartite state $\rho_{ab} \in D(\mathcal{H})$, the minimal geometric discord over state extensions (GDSE) is defined as

$$\hat{\mathcal{E}}_d(\rho_{ab}) := \min_{\sigma_{aa'b} \in \mathcal{C}\mathcal{Q}} \min_{\text{tr}_{a'}[\rho_{aa'b}] = \rho_{ab}} d(\rho_{aa'b}, \sigma_{aa'b}), \quad (11)$$

where the minimization is taken over all extended states $\rho_{aa'b}$ and classical-quantum states in $D(\mathcal{H}_{aa'b})$. Here d is a contractive distance in $D(\mathcal{H}_{aa'b})$. Furthermore, the minimal measurement-induced geometric discord over state extensions (MIDSE) is defined as

$$\hat{\mathcal{E}}'_d(\rho_{ab}) := \min_{\Pi_{aa'}} \min_{\text{tr}_{a'}[\rho_{aa'b}] = \rho_{ab}} d(\rho_{aa'b}, \Pi_{aa'} \otimes I_b(\rho_{aa'b})), \quad (12)$$

where the minimization is taken over extended states and local projection in subsystem aa' .

Remark 4. For both relative entropy and Bures distance, the corresponding quantifiers of discord satisfy (D1)–(D4). Theorem 1 then implies that the corresponding GDSE and MIDSE are good candidates for entanglement measures.

Next we consider the quantification of entanglement by performing a partial trace, a slight modification of Eq. (11). We define quantum correlation by

$$\check{\mathcal{E}}_d(\rho_{ab}) := \min_{\sigma_{aa'b} \in \mathcal{C}\mathcal{Q}} d(\rho_{ab}, \text{tr}_{a'} \sigma_{aa'b}), \quad (13)$$

where the minimum is taken over all classical-quantum states in $D(\mathcal{H}_{aa'b})$. Obviously, $\check{\mathcal{E}}_d$ is equivalent to $E_d := \min_{\sigma_{ab} \in \mathcal{S}} d(\rho_{ab}, \sigma_{ab})$, which is the corresponding entanglement quantification of distance d . In other words, geometric entanglement can be linked to discord over state extensions. For measurement-induced geometric discord, however, it is not the trivial case, as shown below.

Definition 3. For $\rho_{ab} \in D(\mathcal{H}_{ab})$, we define a quantity

$$\check{\mathcal{E}}'_d(\rho_{ab}) := \min_{\Pi_{aa'}} \min_{\text{tr}_{a'}[\rho_{aa'b}] = \rho_{ab}} d(\rho_{ab}, \text{tr}_{a'}[\Pi_{aa'} \otimes I_b(\rho_{aa'b})]), \quad (14)$$

where the minimal is taken over all extended states $\rho_{aa'b}$ and local projection in subsystem aa' .

This quantity is related to the previous ones, via the following inequalities.

Theorem 2. For $\rho_{ab} \in D(\mathcal{H}_{ab})$,

$$\hat{\mathcal{E}}'_d(\rho_{ab}) \geq \hat{\mathcal{E}}_d(\rho_{ab}) \geq E_d(\rho_{ab}), \quad (15)$$

$$\check{\mathcal{E}}'_d(\rho_{ab}) \geq \check{\mathcal{E}}_d(\rho_{ab}) = E_d(\rho_{ab}) \quad (16)$$

and

$$\hat{\mathcal{E}}'_d(\rho_{ab}) \geq \check{\mathcal{E}}_d^{ra}(\rho_{ab}), \quad (17)$$

$$\hat{\mathcal{E}}_d(\rho_{ab}) \geq \check{\mathcal{E}}_d(\rho_{ab}) = E_d(\rho_{ab}). \quad (18)$$

These inequalities can be derived from the definition directly.

B. Characterization with Bures distance discord

Here we prove two important theorems related to the minimal Bures distance discord $\hat{\mathcal{E}}_B$ over state extensions. In the following theorem we show that \mathcal{E}_B^a is convex.

Theorem 3. $\hat{\mathcal{E}}_B$ is convex,

$$\hat{\mathcal{E}}_B\left(\sum_i p_i \rho_{ab}^i\right) \leq \sum_i p_i \hat{\mathcal{E}}_B(\rho_{ab}^i), \quad (19)$$

where p_i are probabilities and ρ_{ab}^i are bipartite states shared between parties a and b .

For the proof of Theorem 3, see Appendix A.

Using Theorem 3, we arrive at the following theorem.

Theorem 4. For $\rho_{ab} \in D(\mathcal{H})$, the minimal Bures distance of discord over state extensions is equivalent to the Bures distance of entanglement,

$$\hat{\mathcal{E}}_B(\rho_{ab}) = E_B(\rho_{ab}). \quad (20)$$

Proof. Since $E_B(\rho_{ab}) = E_B^{cr}(\rho_{ab})$ [45], we just need to show that for each $\rho_{ab} \in D(\mathcal{H})$, we have

$$E_B(\rho_{ab}) \leq \hat{\mathcal{E}}_B(\rho_{ab}) \leq E_B^{cr}(\rho_{ab}). \quad (21)$$

■

See Appendix B for the complete proof.

Having established these results, Theorem 2 for the Bures distance together with Theorem 4 implies

$$\hat{\mathcal{E}}_B(\rho_{ab}) \geq \check{\mathcal{E}}_B(\rho_{ab}) \geq E_B(\rho_{ab}) = \hat{\mathcal{E}}_B(\rho_{ab}) = \check{\mathcal{E}}_B(\rho_{ab}). \quad (22)$$

C. Operational meaning of Bures distance of entanglement

The quest for an operational meaning or interpretation of an entanglement measure lies at the very heart of the entanglement theory. While entanglement of formation, distillable entanglement, or entanglement cost are measures of entanglement having an operational meaning [1,3], the Bures distance of entanglement is an entanglement measure defined from the geometric viewpoint and its physical meaning is not clear. By means of the equivalence in Theorem 4 and the operational meaning of the Bures distance of discord [46], we provide an operational meaning of the Bures distance of entanglement.

Let us briefly review the ambiguous quantum state discrimination (QSD) protocol [63]. Suppose Alice chooses a state ρ_i from a set of states $\{\rho_i\}$ with probability η_i and sends it to Bob, who determines which state he receives by performing a positive-operator-valued measure (POVM). Since the probability to get the result j with given state ρ_i is $p(j|i) = \text{tr}(M_j \rho_i)$, the corresponding optimal success probability is

$$P_s^{\text{opt}}(\{\rho_i, \eta_i\}_{i=1}^N) := \max_{M_i} \sum_i \eta_i \text{tr}(M_i \rho_i), \quad (23)$$

where the maximization is done over all POVMs. Similarly, $P_s^{\text{opt}(vN)}(\{\rho_i, \eta_i\}_{i=1}^N) = \max_{\Pi_i} \sum_i \eta_i \text{tr}(\Pi_i \rho_i)$ is the optimal success probability to discriminate $\{\rho_i, \eta_i\}_{i=1}^N$ with von Neumann measurement.

Based on the operational meaning of the Bures distance of asymmetric discord [46], the connection between geometric entanglement and QSD is as follows.

Corollary. For a bipartite quantum state $\rho_{ab} \in D(\mathcal{H})$, the square of the maximal fidelity to the set of separable states $F(\rho_{ab}, \mathcal{S}) := \max_{\sigma_{ab} \in \mathcal{S}} F(\rho_{ab}, \sigma_{ab})$ is equal to the optimal success probability to discriminate a set of quantum states with von Neumann measurement, i.e.,

$$F^2(\rho_{ab}, \mathcal{S}) = \max_{\rho_{aa'b}, \{\alpha_i\}} P_s^{\text{opt}(vN)}(\{\rho_i, \eta_i\}), \quad (24)$$

where $\eta_i = \text{tr}(\alpha_i | \rho_{aa'b} | \alpha_i)$, $\rho_i = \eta_i^{-1} \sqrt{\rho_{aa'b}} | \alpha_i \rangle \langle \alpha_i | \sqrt{\rho_{aa'b}}$, and the maximum is taken over all possible extended states $\rho_{aa'b}$, von Neumann measurement Π_i , and orthogonal basis $\{|\alpha_i\rangle\}$ on $\mathcal{H}_{aa'}$.

This offers an operational meaning to the Bures distance of entanglement.

D. Characterization with relative entropy discord

We define relative entropy of discord by $\hat{D}_r(\rho) := \min_{\sigma \in \mathcal{C} \otimes \mathcal{Q}} S(\rho || \sigma)$ and measurement-induced relative entropy of discord by $\hat{D}'_r(\rho) := \min_{\Pi_a} S(\rho || \Pi_a(\rho))$.

Remark 5. In [11] the authors provided a framework to quantify quantum coherence with respect to a chosen orthogonal basis. Although discord and coherence seem different, they have essential similarities. Discord can be regarded as the basis-independent coherence [64], and relative entropy of discord is equivalent to the minimal relative entropy of partial coherence [65]. Moreover, quantum discord can also be explained as an upper bound to the quantum correlations generated from partial coherence via partial incoherent operations [66].

In the following theorem, we show that \hat{D}_r and \hat{D}'_r are equivalent.

Theorem 5. For a bipartite state $\rho_{ab} \in D(\mathcal{H})$, the relative entropy of discord is equivalent to the measurement-induced relative entropy of discord, i.e.,

$$\hat{D}_r(\rho_{ab}) = \hat{D}'_r(\rho_{ab}). \quad (25)$$

Proof. Since any classical-quantum state has the form $\sigma_{ab}^{(cq)} = \sum_i p_i |i\rangle_a \langle i| \otimes \sigma_b^i$, using Eq. (2) we have

$$\begin{aligned} \hat{D}_r(\rho_{ab}) &= \min_{\pi_a^i} \min_{p_i, \sigma_b^i} S(\rho_{ab} || \sigma_{ab}^{(cq)}) \\ &= \min_{\pi_a^i} [S(\Pi_a(\rho_{ab})) - S(\rho_{ab}) + \min_{p_i, \sigma_b^i} S(\Pi_a(\rho_{ab}) || \sigma_{ab}^{(cq)})] \\ &= \min_{\pi_a^i} S(\Pi_a(\rho_{ab})) - S(\rho_{ab}) \\ &= \min_{\pi_a^i} S(\rho_{ab} || \Pi_a(\rho_{ab})) = \hat{D}'_r(\rho_{ab}), \end{aligned} \quad (26)$$

where $\Pi_a(\rho_{ab}) = \sum_i (|i\rangle_a \langle i| \otimes I_b) \rho_{ab} (|i\rangle_a \langle i| \otimes I_b)$. ■

Next, for the relative entropic quantities $E_r, \hat{\mathcal{E}}'_r, \hat{\mathcal{E}}_r, \check{\mathcal{E}}'_r,$ and $\check{\mathcal{E}}_r$, where symbols have their usual meanings, Theorems 2 and 5 lead to the following result.

Theorem 6. For a bipartite state $\rho_{ab} \in D(\mathcal{H})$,

$$\hat{\mathcal{E}}'_r(\rho_{ab}) = \hat{\mathcal{E}}_r(\rho_{ab}) \geq E_r(\rho_{ab}) = \check{\mathcal{E}}'_r(\rho_{ab}) = \check{\mathcal{E}}_r(\rho_{ab}). \quad (27)$$

Proof. From Theorem 5 we have

$$\hat{\mathcal{E}}'_r(\rho_{ab}) = \hat{\mathcal{E}}_r(\rho_{ab}) \quad (28)$$

and

$$\check{\mathcal{E}}'_r(\rho_{ab}) = \check{\mathcal{E}}_r(\rho_{ab}) = E_r(\rho_{ab}). \quad (29)$$

The inequality follows from Theorem 2. ■

Remark 6. In Ref. [30] the authors proved that $\check{\mathcal{E}}'_r(\rho_{ab}) = \min_{\text{tr}_a[\sigma_{ab}^{(\text{sep})}] = \rho_b} S(\rho_{ab} || \sigma_{ab}^{(\text{sep})}) \geq E_r(\rho_{ab})$, where the minimum is taken over all separable states $\sigma_{ab}^{(\text{sep})}$ with $\text{tr}_a[\sigma_{ab}^{(\text{sep})}] = \rho_b$. Our results affirm that they are equal.

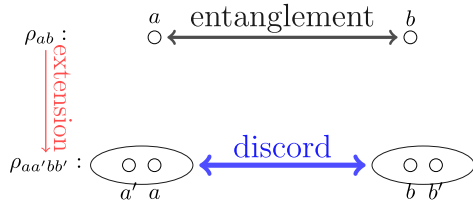


FIG. 2. The minimal discord of an extended state $\rho_{aa'bb'}$ shared between parties aa' and bb' is a quantification of entanglement of ρ_{ab} between a and b .

IV. CHARACTERIZING ENTANGLEMENT VIA SYMMETRIC DISCORD

In this section we quantify entanglement via symmetric or global quantum discord. In the symmetric or global quantum discord of a bipartite system, the local measurement is performed on both subsystems. All the content developed in the asymmetric case will be recalled here, with the only difference that now quantum measurement or other treatment will be on both parties.

A. General results

In the following, we define the minimal quantum discord over state extensions. In the asymmetric case, the state extension was in party a . Here we consider state extension in both parties.

Definition 4. For bipartite state $\rho_{ab} \in D(\mathcal{H})$, the minimal discord over state extensions is defined as

$$\tilde{\mathcal{E}}(\rho_{ab}) := \min_{\text{tr}_{a'b'}[\rho_{aa'bb'}] = \rho_{ab}} \tilde{D}(\rho_{aa'bb'}), \quad (30)$$

where the minimization is taken along the bipartition $aa' : bb'$ with $\text{tr}_{a'}\rho_{aa'bb'} = \rho_{ab}$ (see Fig. 2).

Theorem 7. If \tilde{D} is a symmetric discord measure satisfying (D1')–(D3'), the minimal discord, $\tilde{\mathcal{E}}$, over state extensions has the following desirable and remarkable properties.

- (E1') $\tilde{\mathcal{E}}(\rho_{ab}) \geq 0$ with the equality if and only if $\rho \in \mathcal{CC}$.
- (E2') $\tilde{\mathcal{E}}$ is invariant under local unitary transformations.
- (E3') $\tilde{\mathcal{E}}$ is nonincreasing under a local partial trace,

$$\tilde{\mathcal{E}}(\rho_{ab}) \leq \tilde{\mathcal{E}}(\rho_{aa_1bb_1}), \quad (31)$$

for any state extension $\rho_{aa_1bb_1}$ of ρ_{ab} .

- (E4') $\tilde{\mathcal{E}}$ is nonincreasing under local operations.
- (E5') $\tilde{\mathcal{E}}$ reduces to an entanglement monotone for pure states.

The proof is similar to that of Theorem 1.

B. Minimal Bures distance of symmetric discord

In this section we define the minimal Bures distance of symmetric discord.

Definition 5. For a bipartite state $\rho_{ab} \in \mathcal{H}$, the minimal Bures distance of discord over state extensions is defined as

$$\tilde{\mathcal{E}}_B(\rho_{ab}) := \min_{\sigma \in \mathcal{CC}} \min_{\text{tr}_{a'b'}[\rho_{aa'bb'}] = \rho_{ab}} d_B^2(\rho_{aa'bb'}, \sigma), \quad (32)$$

where the minimum is taken with respect to the bipartition $aa' : bb'$.

In the following, we state that the minimal Bures distance of symmetric discord is convex and is equivalent to the Bures distance of entanglement.

Theorem 8. $\tilde{\mathcal{E}}_B$ is convex,

$$\tilde{\mathcal{E}}_B\left(\sum_i p_i \rho_{ab}^i\right) \leq \sum_i p_i \tilde{\mathcal{E}}_B(\rho_{ab}^i), \quad (33)$$

where p_i are probabilities and ρ_{ab}^i are bipartite states shared between parties a and b .

The proof is similar to that of Theorem 3. With Theorem 8, one has the following result.

Theorem 9. For $\rho_{ab} \in D(\mathcal{H})$, $\tilde{\mathcal{E}}_B(\rho_{ab}) = E_B(\rho_{ab})$.

The proof is similar to that of Theorem 4.

Remark 7. It is interesting to note that $E_B(\rho_{ab}) = \hat{\mathcal{E}}_B(\rho_{ab}) = \tilde{\mathcal{E}}_B(\rho_{ab})$, which means that the Bures distance of entanglement is equivalent to the minimal Bures distance of discord over state extensions, both on one subsystem and two subsystems.

Remark 8. It was Luo who first proposed to quantify entanglement as the minimal quantum discord over state extensions [31]. Theorem 9 offers affirmative evidence that this kind of entanglement quantification is consistent with the previous entanglement measures. However, it is not clear whether this equivalence still holds for other entanglement measures such as entanglement measures based on distances [1] and relative entropy of entanglement [36].

C. More definitions

Definition 6. For a bipartite state $\rho_{ab} \in D(\mathcal{H})$, the minimal GDSE is defined as

$$\tilde{\mathcal{E}}_d(\rho_{ab}) := \min_{\sigma_{aa'bb'} \in \mathcal{CC}} \min_{\text{tr}_{a'b'}[\rho_{aa'bb'}] = \rho_{ab}} d(\rho_{aa'bb'}, \sigma_{aa'bb'}), \quad (34)$$

where the minimum is taken with respect to the bipartition $aa' : bb'$. Moreover, the minimal MIDSE is defined as

$$\tilde{\mathcal{E}}'_d(\rho_{ab}) := \min_{\Pi_{aa'}, \Pi_{bb'}} \min_{\text{tr}_{a'b'}[\rho_{aa'bb'}] = \rho_{ab}} \times d(\rho_{aa'bb'}, \Pi_{aa'} \otimes \Pi_{bb'}(\rho_{aa'bb'})), \quad (35)$$

where the minimum is taken over all local projections in both subsystems aa' and bb' and along the bipartition $aa' : bb'$.

Remark 9. For relative entropy or Bures distance, the corresponding geometric discord satisfies (D1')–(D3'). From Theorem 7 the corresponding quantification is a good candidate of entanglement measure.

Next let us consider the quantification by performing a partial trace,

$$\tilde{\mathcal{E}}_d(\rho_{ab}) := \min_{\sigma_{aa'bb'} \in \mathcal{CC}} d(\rho_{ab}, \text{tr}_{a'b'}\sigma_{aa'bb'}), \quad (36)$$

where the minimum is taken over all classical-classical states. It is easy to see that $\tilde{\mathcal{E}}_d$ is equivalent to the general geometric entanglement measure E_d , which is defined as the minimal distance between the state and the set of separable states. Nevertheless, it is not the case for the measurement-induced geometric discord, as shown below.

Definition 7. For $\rho_{ab} \in D(\mathcal{H}_{ab})$, we define a quantity $\tilde{\mathcal{E}}'_d(\rho_{ab}) := \min_{\Pi_{aa'}, \Pi_{bb'}} \min_{\text{tr}_{a'b'}[\rho_{aa'bb'}] = \rho_{ab}} d(\rho_{ab}, \text{tr}_{a'b'}[\Pi_{aa'} \otimes \Pi_{bb'}(\rho_{aa'bb'})])$, where the minimum is taken over all extended states $\rho_{aa'bb'}$ and local projection in both subsystems aa' and bb' .

Certainly, we obtain $\tilde{\mathcal{E}}'_d(\rho_{ab}) \geq E_d(\rho_{ab})$ for any $\rho_{ab} \in D(\mathcal{H}_{ab})$. Moreover, based on the discussion in Remark 6, the equality holds for relative entropy. Whether the equality holds for other distances as well would be an interesting investigation.

In conclusion, we have the following result.

Theorem 10. For $\rho_{ab} \in D(\mathcal{H}_{ab})$, the following inequalities are true:

$$\tilde{\mathcal{E}}'_d(\rho_{ab}) \geq \tilde{\mathcal{E}}_d(\rho_{ab}) \geq E_d(\rho_{ab}), \quad (37)$$

$$\tilde{\mathcal{E}}'_d(\rho_{ab}) \geq \tilde{\mathcal{E}}_d(\rho_{ab}) = E_d(\rho_{ab}) \quad (38)$$

and

$$\tilde{\mathcal{E}}'_d(\rho_{ab}) \geq \tilde{\mathcal{E}}'_d(\rho_{ab}), \quad (39)$$

$$\tilde{\mathcal{E}}_d(\rho_{ab}) \geq \tilde{\mathcal{E}}_d(\rho_{ab}) = E_d(\rho_{ab}). \quad (40)$$

The inequalities can be derived directly from the definitions.

D. Characterization with Bures distance discord and relative entropy discord

For the Bures distance, Theorem 9 implies

$$\tilde{\mathcal{E}}'_B(\rho_{ab}) \geq \tilde{\mathcal{E}}_B(\rho_{ab}) \geq E_B(\rho_{ab}) = \mathcal{E}_B(\rho_{ab}) = \tilde{\mathcal{E}}_B(\rho_{ab}), \quad (41)$$

and for relative entropy,

$$\tilde{\mathcal{E}}'_r(\rho_{ab}) = \tilde{\mathcal{E}}_r(\rho_{ab}) \geq E_r(\rho_{ab}) = \tilde{\mathcal{E}}'_r(\rho_{ab}) = \tilde{\mathcal{E}}_r(\rho_{ab}).$$

V. CONCLUSION

In this paper we have proposed a framework to quantify entanglement from the viewpoint of discord, which unifies previous works including [29–31]. Several quantifications of entanglement were introduced based on quantum discord over state extensions. In particular, for the Bures distance, we proved that the minimal discord (both asymmetric and symmetric) over state extensions is equivalent to the Bures distance of entanglement, which not only establishes an equivalence between this kind of entanglement quantification with existing entanglement measure but also provides an operational meaning for the Bures distance of entanglement. Moreover, for relative entropy, by proving that the corresponding discord measure is equivalent to the measurement-induced relative entropy of discord, we showed that the MIDSE (both asymmetric and symmetric) is equivalent to the relative entropy of entanglement, which reinforces the result in Ref. [30].

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APPENDIX A: PROOF OF THEOREM 3

Proof. Note that

$$\rho_{aa'a''b} := \sum_i p_i \rho_{aa'a''b}^i \otimes |i\rangle_{a''} \langle i| \quad (A1)$$

is a state extension of $\rho_{ab} = \sum_i p_i \rho_{ab}^i$ whenever $\rho_{aa'a''b}^i$ is a state extension of ρ_{ab}^i for all i . Without loss of generality, suppose $\rho_{aa'a''b}^i$ is the optimal state extension of ρ_{ab}^i for each i and σ_i^* is the corresponding closest classical-quantum state. Then we have

$$\sum_i p_i \mathcal{E}_B^a(\rho_{ab}^i) = \sum_i p_i d_B^2(\rho_{aa'a''b}^i, \sigma_i^*) \quad (A2)$$

$$= \sum_i p_i d_B^2(\rho_{aa'a''b}^i \otimes |i\rangle_{a''} \langle i|, \sigma_i^* \otimes |i\rangle_{a''} \langle i|) \quad (A3)$$

$$\geq d_B^2\left(\sum_i p_i \rho_{aa'a''b}^i \otimes |i\rangle_{a''} \langle i|, \sum_i p_i \sigma_i^* \otimes |i\rangle_{a''} \langle i|\right) \quad (A4)$$

$$\geq \hat{D}_B\left(\sum_i p_i \rho_{aa'a''b}^i \otimes |i\rangle_{a''} \langle i|\right) \quad (A5)$$

$$\geq \hat{\mathcal{E}}_B\left(\sum_i p_i \rho_{ab}^i\right), \quad (A6)$$

where the first inequality follows from the joint convexity of d_B^2 and the second inequality is based on the definition of \hat{D}_B and the fact that $\sum_i p_i \sigma_i^* \otimes |i\rangle_{a''} \langle i|$ is a classical-quantum state along the bipartition $aa'a'' : b$. The last inequality follows because $\sum_i p_i \rho_{aa'a''b}^i \otimes |i\rangle_{a''} \langle i|$ is a state extension of $\sum_i p_i \rho_{ab}^i$. ■

APPENDIX B: PROOF OF THEOREM 4

Here we restate the claim to be proved, that is, for each $\rho_{ab} \in D(\mathcal{H})$,

$$E_B(\rho_{ab}) \leq \hat{\mathcal{E}}_B(\rho_{ab}) \leq E_B^{cr}(\rho_{ab}). \quad (B1)$$

Proof. Suppose $\rho_{aa'a''b}^*$ is the optimal state extensions of ρ_{ab} and σ^* is the corresponding closest classical-quantum state. Then

$$\hat{\mathcal{E}}_B(\rho_{ab}) = d_B^2(\rho_{aa'a''b}^*, \sigma^*) \geq d_B^2(\rho_{ab}, \text{tr}_{a'} \sigma^*) \geq E_B(\rho_{ab}). \quad (B2)$$

The first inequality is the result of the contractibility of the Bures distance and the second inequality is because $\text{tr}_{a'} \sigma^*$ is a separable state. In fact, if $\sigma^* = \sum_i p_i |\alpha_i\rangle_{aa'} \langle \alpha_i| \otimes \rho_i$, where $|\alpha_i\rangle_{aa'} = \sum_k \lambda_k^i |x_k^i\rangle_a |y_k^i\rangle_{a'}$, then tracing out the subsystem a' will lead to a decomposition of the form $\text{tr}_{a'} \sigma^* = \sum_{i,k} p_i \lambda_k^i |x_k^i\rangle_a \langle x_k^i| \otimes \rho_i$, which must be a separable state.

In particular, let us consider the pure state case. Suppose $|\psi\rangle = \sum_i \sqrt{\lambda_i} |x_i\rangle_a |y_i\rangle_b$, with $\lambda_1 \geq \dots \geq \lambda_n$. Then

$$\begin{aligned} \hat{\mathcal{E}}_B(|\psi\rangle) &\leq \min_{\sigma_{aa'b} \in \mathcal{CQ}} d_B^2(|\psi\rangle_{ab} \langle \psi| \otimes |u\rangle_{a'} \langle u|, \sigma_{aa'b}) \\ &\leq d_B^2(|\psi\rangle_{ab} \langle \psi| \otimes |u\rangle_{a'} \langle u|, |x_1, y_1\rangle_{ab} \\ &\quad \times \langle x_1, y_1| \otimes |u\rangle_{a'} \langle u|) \\ &= E_B(|\psi\rangle). \end{aligned} \quad (B3)$$

Combining the above two results, one has $\hat{\mathcal{E}}_B(|\psi\rangle) = E_B(|\psi\rangle)$ for any pure state $|\psi\rangle$.

On the other hand, for any mixed state with pure state decomposition $\rho_{ab} = \sum_i p_i |\psi_i\rangle_{ab} \langle\psi_i|$, Theorem 1 tells us that

$$\hat{\mathcal{E}}_B(\rho_{ab}) \leq \sum_i p_i \hat{\mathcal{E}}_B(|\psi_i\rangle) = \sum_i p_i E_B(|\psi_i\rangle). \quad (\text{B4})$$

Taking the minimum over all pure state decompositions,

$$\hat{\mathcal{E}}_B(\rho_{ab}) \leq E_B^{cr}(\rho_{ab}). \quad (\text{B5})$$

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