

Entropic entanglement criteria in phase space

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We derive entropic inseparability criteria for the phase-space representation of quantum states. In contrast to criteria involving differential entropies of marginal phase-space distributions, our criteria are based on a joint distribution known as the Husimi Q distribution. This distribution is experimentally accessible in cold atoms, circuit QED architectures, and photonic systems, and bears practical advantages compared to the detection of marginals. We exemplify the strengths of our entropic approach by considering several classes of non-Gaussian states where second-order criteria fail. We show that our criteria certify entanglement in previously undetectable regions, highlighting the strength of using the Husimi Q distribution for entanglement detection.

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I. INTRODUCTION

Entanglement is the distinguishing feature of quantum systems and its detection is critical for characterizing them [1]. Fundamental problems such as the thermalization of isolated quantum systems and characterizing quantum phase transitions strongly rely on a deep understanding of how entanglement manifests and evolves [2,3]. However, a central problem in studying entanglement is being able to derive experimentally accessible witnesses to detect it [4,5]. The complexity of such a problem depends upon the Hilbert space size, such that continuous variable systems with an infinite-dimensional Hilbert space pose a particular challenge.

For continuous quantum variables, many entanglement criteria rely on measuring the second-order moments of two marginal distributions [6–11]. These criteria are most powerful when the state is Gaussian but are often insensitive elsewhere [12,13]. This can pose significant problems as there are many important classes of highly entangled non-Gaussian states that cannot be witnessed by second-order criteria [14–16].

To capture higher-order moments of measured distributions [17,18], one can use differential entropies of measured marginal distributions [19,20]. Differential entropies reach beyond the scope of second-order criteria since they are a functional of the full probability density function. Examples include criteria that rely on entropic uncertainty relations [21–25] as well as the complexity-based criterion [26]. Other approaches are predicated on entropic uncertainty relations with (quantum) memory [27,28] or as entropic steering

inequalities [29,30]. Entropic criteria have also been derived and experimentally tested for discrete variables [31–36].

One disadvantage to these methods is that measuring marginals of a distribution is often costly and impractical as it requires angle tomography. This is particularly difficult in ultracold quantum gas experiments where statistics are limited [37]. In this paper, we take a conceptually different approach and characterize the inseparability of a given bipartite state not by its marginal distributions, but by its joint probability distribution. This distribution, known as the Husimi Q distribution, is a quasiprobability distribution that contains the full information about the state [38–42]. Unlike the Wigner function, it is non-negative and hence has an associated entropy known as the Wehrl entropy [43,44].

The detection of marginal distributions (via Wigner) and the simultaneous detection of phase-space variables (via Husimi) for quantum state tomography are considered as two complementary, but in principle equally powerful, approaches [45]. For jointly measured observables, the full information about correlations between different directions in phase space is simultaneously available; for sequentially measured marginals, a direction needs to be preselected. This suggests the Husimi Q distribution offers unexplored opportunities to derive entanglement witnesses for systems where experimental statistics are limited.¹

Crucially, the Husimi Q distribution (and hence the Wehrl entropy) is an experimentally accessible quantity that can be measured via tomographic methods [47]. This is well established within quantum optics and has been realized in experiments [48–50]. Recently, the measurement of Husimi Q distributions has been demonstrated on a variety of other platforms, including ultracold Bose gases [51,52], atoms in optical cavities [53,54], and circuit QED architectures [55],

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¹The Wehrl entropy in the context of entanglement has only been discussed to define an entanglement monotone [46].

showing practical advantages with respect to the detection of marginals.

Here, we derive entanglement criteria in terms of entropies of the Husimi Q distribution. We show that these criteria are stronger than previously known ones for certain classes of states within the non-Gaussian regime. We discuss various experimental platforms where we expect our criteria could be most powerful in terms of implementation and state detection.

Notation. We set $\hbar = 1$ and disregard operator hats. We use capital letters for quantum operators X_j and small letters for their corresponding eigenvalues x_j and eigenvectors $|x_j\rangle$.

II. HUSIMI Q DISTRIBUTION AND WEHRL ENTROPY

We consider a set of continuous quantum variables X_j and P_j that fulfill the bosonic commutation relation $[X_j, P_k] = i\delta_{jk}$ where the subindices denote the two subsystems $j, k \in \{1, 2\}$. A canonical transformation can make rotations in the local phase spaces by angles ϑ_j ,

$$\begin{pmatrix} R_j \\ S_j \end{pmatrix} = \begin{pmatrix} \cos \vartheta_j & \sin \vartheta_j \\ -\sin \vartheta_j & \cos \vartheta_j \end{pmatrix} \begin{pmatrix} X_j \\ P_j \end{pmatrix}. \quad (1)$$

To a set of (possibly rotated) position and momentum operators R_j and S_j , we define annihilation operators $A_j = (R_j + iS_j)/\sqrt{2}$ such that coherent states are their eigenstates $A_j |\alpha\rangle = \alpha_j |\alpha\rangle$. The complex eigenvalues α_j are parametrized as $\alpha_j = (r_j + is_j)/\sqrt{2}$. One can associate a positive operator-valued measure (POVM) to pure coherent-state projectors of the form $E_\alpha = |\alpha\rangle\langle\alpha|$. This POVM is experimentally accessible through tomographic schemes [47] or the heterodyne detection protocol commonly used in quantum optical systems [12,39,40], which corresponds to measuring the state ρ in the pure coherent-state basis.

The distribution associated to the measurement outcomes for a set of conjugate quantum variables is the global Husimi Q distribution

$$Q(r_1, s_1, r_2, s_2) = \text{Tr}\{\rho E_\alpha\} = \langle\alpha|\rho|\alpha\rangle. \quad (2)$$

Due to the nonorthogonality of the coherent states, the Husimi Q distribution is a quasiprobability distribution in phase space. The readout angles ϑ_i can then be chosen in a *postmeasurement* analysis unlike measuring marginals of the Wigner distribution where these angles need to be preselected. Also, the Husimi Q distribution is bounded, $0 \leq Q \leq 1$, and normalized to unity with respect to the phase-space measure $\prod_j dr_j ds_j / (2\pi)$.

Hence, an entropy associated to the Husimi Q distribution exists, known as the Wehrl entropy [43,44],

$$S_W(Q) = - \prod_j \int \frac{dr_j ds_j}{2\pi} Q \ln Q. \quad (3)$$

The Wehrl entropy is strictly monotonous under partial trace and can therefore be regarded as a classical (differential) entropy [56].

The Wehrl-Lieb inequality bounds the Wehrl entropy from below [57,58] (for $2N$ -dimensional phase space),

$$S_W(Q) \geq N, \quad (4)$$

and can therefore be understood as an entropic uncertainty relation in the quantum mechanical phase space. In contrast to many common uncertainty relations, it is not invariant under one-mode squeezing but is instead preserved under rotations in phase space.

It is useful to make a linear transformation to nonlocal Einstein-Podolsky-Rosen (EPR)-type variables [59],

$$r_\pm = r_1 \pm r_2, \quad s_\pm = s_1 \pm s_2. \quad (5)$$

By integrating over half of these variables one obtains from Eq. (2) marginalized distributions

$$Q_\pm(r_\pm, s_\mp) = \int \frac{dr ds}{2\pi} Q(r, s, \mp r \pm r_\pm, \pm s \mp s_\mp). \quad (6)$$

These are normalized to unity with respect to the measure $dr_\pm ds_\mp / (2\pi)$ but, because of the twisted assignment (r_\pm, s_\mp) , they are not themselves Husimi Q distributions.² One can associate to them an entropy $S_M(Q_\pm)$ that is defined in analogy to Wehrl's entropy in Eq. (3) even though s_\mp is not the conjugate momentum of r_\pm .

III. INSEPARABILITY CRITERIA

We first derive criteria for pure states and show they generalize to mixed states. We consider pure separable states, for which the density operator is a product $\rho = \rho_1 \otimes \rho_2$. Here, the global Husimi Q -distribution factorizes

$$Q(r_1, s_1, r_2, s_2) = Q_1(r_1, s_1) Q_2(r_2, s_2), \quad (7)$$

where $Q_j(r_j, s_j)$ denotes the marginals of the global Husimi Q distribution.

Inserting Eq. (7) in Eq. (6) yields

$$Q_\pm(r_\pm, s_\mp) = (Q_1 * Q_2^{(\pm)})(r_\pm, s_\mp), \quad (8)$$

where $*$ denotes a convolution and $Q_2^{(\pm)} \equiv Q_2(\pm r, \mp s)$. Invoking the two-dimensional entropy power inequality [60–62]

$$e^{S(Q_A * Q_B)} \geq e^{S(Q_A)} + e^{S(Q_B)}, \quad (9)$$

for any two two-dimensional Husimi Q distributions Q_A and Q_B , as well as the invariance of the Wehrl entropy under mirror reflections in phase space allows us to write

$$e^{S_M(Q_\pm)} \geq e^{S_W(Q_1)} + e^{S_W(Q_2)}. \quad (10)$$

Thus, we find the pair of inequalities

$$S_M(Q_\pm) \geq \ln(e^{S_W(Q_1)} + e^{S_W(Q_2)}), \quad (11)$$

which provide a state-dependent lower bound on the entropies $S_M(Q_\pm)$ obeyed by all pure product states. Hence, pure states for which $S_M(Q_\pm)$ violates this bound are necessarily entangled. We call Eq. (11) the *strong* criteria.

To obtain a state-independent bound, we apply the Wehrl-Lieb inequality Eq. (4) to both subsystems, leaving us with the criteria

$$S_M(Q_\pm) \geq 1 + \ln 2. \quad (12)$$

²The appearance of the mixed variable pairs (r_+, s_-) and (r_-, s_+) can be understood when employing the positive partial transpose criterion for continuous quantum variables [6].

As the latter relations are in general less tight than Eq. (11), we call them the *weak* criteria.

We can generalize the weak criteria to mixed states by starting with a general mixed separable state $\rho = \sum_i p_i (\rho_1^i \otimes \rho_2^i)$, where $p_i \geq 0$ and $\sum_i p_i = 1$. On the level of the global Husimi Q distributions, one has an analogous decomposition, leading, via Eq. (6), to

$$Q_{\pm}(r_{\pm}, s_{\mp}) = \sum_i p_i Q_{\pm}^i(r_{\pm}, s_{\mp}). \quad (13)$$

Using concavity of the entropy $S_M(Q_{\pm})$ [63], we find

$$S_M(Q_{\pm}) \geq \sum_i p_i S_M(Q_{\pm}^i) \geq 1 + \ln 2, \quad (14)$$

where we have employed the strong pure state criteria Eq. (11) and then the Wehrl-Lieb inequality Eq. (4). Therefore, the weak Wehrl entropic criteria for pure product states Eq. (12) generalize identically to mixed states. One could derive a set of strong criteria for mixed states, however this requires the knowledge about the decomposition of ρ , which is inaccessible in experiments. The violation of inequality Eq. (14) thus flags entanglement, rendering it an inseparability criterion.

IV. EXAMPLE STATES

A. Gaussian states

An important class of states to consider is Gaussian states, which can be fully characterized by their first- and second-order moments. Since entropies are generally invariant under constant shifts of variables, we assume without loss of generality that the mean values vanish $\langle r \rangle = \langle s \rangle = 0$. Hence, we only need to specify the covariance of the state

$$\gamma = \begin{pmatrix} \langle r^2 \rangle & \langle rs \rangle \\ \langle sr \rangle & \langle s^2 \rangle \end{pmatrix} \equiv \begin{pmatrix} \sigma_r^2 & \sigma_{rs} \\ \sigma_{rs} & \sigma_s^2 \end{pmatrix}, \quad (15)$$

which is also the covariance matrix of the Wigner W distribution. The diagonal entries contain the variances of the corresponding marginal distributions, while the off-diagonal elements contain the covariance. One can always choose rotation angles ϑ_i such that $\sigma_{rs} = 0$, which aligns the coordinate axes along the principal axes.

For the Husimi Q distribution, we define the covariance matrix as

$$V_{ij} \equiv \frac{1}{2} \langle \{u_i, u_j\} \rangle_Q, \quad (16)$$

where $u = (r, s)$ and the subscript Q indicates the expectation value with respect to the Husimi Q distribution. Given that the Husimi Q distribution can be obtained via a Weierstrass transform of the Wigner W distribution with respect to the vacuum W_0 , we can reconcile the two covariance matrices Eqs. (16) and (15) as $V = \gamma + \gamma_0$. Then, expressed in terms of the twisted variables (r_{\pm}, s_{\mp}) , the Husimi Q distribution of a general Gaussian quantum state leads to

$$Q_{\pm}(r_{\pm}, s_{\mp}) = \frac{1}{Z} e^{-\frac{1}{2}(r_{\pm}, s_{\mp})^T V_{\pm}^{-1}(r_{\pm}, s_{\mp})}, \quad (17)$$

where $Z = \det^{1/2} V_{\pm}$ is a normalization constant.

The entropy S_M of a state with covariance matrix V_{\pm} is maximized by a Gaussian distribution of the form Eq. (17),

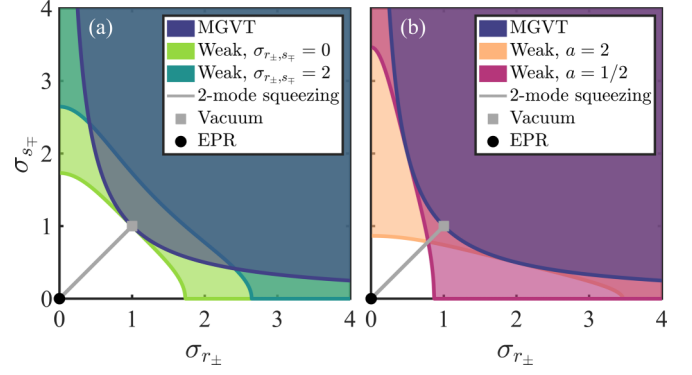


FIG. 1. Regions where the MGVT criteria Eq. (20) and the second-order criteria Eq. (19) are fulfilled, plotted against the two marginal variances $\sigma_{r_{\pm}}$ and $\sigma_{s_{\mp}}$. The regions below the covering curves indicate entanglement. (a) We vary $\sigma_{r_{\pm}, s_{\mp}}$ and fix $a = 1$. The second-order criteria Eq. (19) automatically account for an impractical alignment of the coordinate axes, while the MGVT criteria Eq. (20) can become less tight for an improper alignment. (b) We vary a and fix $\sigma_{r_{\pm}, s_{\mp}} = 0$. The MGVT criteria Eq. (20) are invariant under an equal amount of local squeezing a , whereas the second-order criteria Eq. (19) must be optimized over a . We have additionally marked the two-mode squeezed state for all squeezing parameters λ (gray line) from the vacuum state $\sigma_{r_{\pm}} = \sigma_{s_{\mp}} = 1$ (gray square) up to the fully correlated EPR state $\sigma_{r_{\pm}} = \sigma_{s_{\mp}} = 0$ (black dot).

such that

$$1 + \frac{1}{2} \ln \det V_{\pm} \geq S_M(Q_{\pm}) \geq 1 + \ln 2 \quad (18)$$

holds for all $Q_{\pm}(r_{\pm}, s_{\mp})$. Therefore, the weak entropic criteria Eq. (14) imply a set of second-order based criteria and the two are equivalent for Gaussian states.

Our criteria are invariant under rotations (see Appendix A), however they are not invariant under local squeezing. If we consider equal amounts of local squeezing $a > 0$, the second-order criteria can be rewritten as

$$(\sigma_{r_{\pm}}^2 + a^2) \left(\sigma_{s_{\mp}}^2 + \frac{1}{a^2} \right) \geq 4 + \sigma_{r_{\pm}, s_{\mp}}^2. \quad (19)$$

After optimizing over the local squeezing parameter a and choosing angles ϑ_i such that the coordinate axes are parallel to the principal axes (i.e., $\sigma_{r_{\pm}, s_{\mp}}^2 = 0$) these criteria are equivalent to the Mancini-Giovannetti-Vitali-Tombesi (MGVT) criteria [8]

$$\sigma_{r_{\pm}} \sigma_{s_{\mp}} \geq 1, \quad (20)$$

which themselves are equivalent to the entropic criteria in Ref. [24] for Gaussian states. In contrast to our second-order criteria Eq. (19), the MGVT criteria are invariant under equal amounts of local squeezing. However, they are not invariant under rotations in the \pm -phase space as they do not contain the covariances. In this sense, the two second-order criteria Eqs. (19) and (20) behave complementary under rotations and squeezing. This is summarized in Fig. 1.

Note that both second-order criteria Eqs. (19) and (20) are only sufficient criteria for inseparability. This implies that the criteria by Simon [6] and Duan *et al.* [7] (after optimization

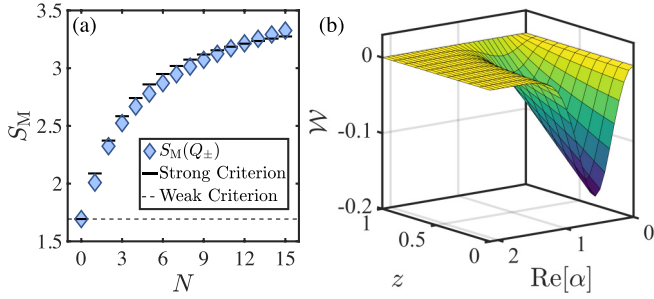


FIG. 2. (a) Wehrl entropic criteria for the first 15 NOON states. Entanglement is witnessed when the points (blue diamonds) are below the bars. The strong criterion Eq. (11) witnesses entanglement up to $N = 11$, while the weak criterion Eq. (14) does not witness any entanglement. (b) Testing the weak criterion Eq. (14) for the dephased cat state Eq. (22) with $s = 0$. Negative $\mathcal{W} = S_M(Q_{\pm}) - 1 - \ln 2$ witnesses entanglement. Entanglement is detected for $\text{Re}[\alpha] > 0$ and $z < 1$, but the difference \mathcal{W} converges quickly to zero for $\text{Re}[\alpha] \gtrsim 3/2$. In both cases $\vartheta_1 = \vartheta_2 = 0$.

over local squeezing parameters and angles) are generally stronger in the Gaussian regime.

B. Non-Gaussian states

To exemplify the strengths of our entropic criteria, we consider a set of non-Gaussian entangled states that cannot be witnessed by second-order criteria to test the weak [Eq. (14)] and strong criteria [Eq. (11)].³

First, we consider the planar NOON states that are given by

$$|\psi_N\rangle = \frac{1}{\sqrt{2(1 + \delta_{0N})}}(|N, 0\rangle + |0, N\rangle), \quad (21)$$

with $N \in \mathbb{N}_0$.

We plot the behavior for the two criteria in Fig. 2(a) up to $N = 15$ for $Q_+(r_+, s_-)$. Our strong Wehrl criteria Eq. (11) witnesses entanglement up to $N = 11$. This goes beyond the capabilities of entropic criteria based on marginal distributions. For example, the witness in Ref. [24] detects entanglement up to $N = 5$, while the generalization in Ref. [25] is capable of certifying entanglement up to $N = 6$. The weak criteria Eq. (12) do not witness any entanglement, which is analogous to the results in Refs. [24,25].

As a second example, we consider the Schrödinger cat state

$$\rho = N(\alpha)[|\alpha, \alpha\rangle\langle\alpha, \alpha| + |-\alpha, -\alpha\rangle\langle-\alpha, -\alpha| - (1-z)(|\alpha, \alpha\rangle\langle-\alpha, -\alpha| + |-\alpha, -\alpha\rangle\langle\alpha, \alpha|)], \quad (22)$$

where $0 \leq z \leq 1$ and $N(\alpha) = [1 + (1-z)e^{-4|\alpha|^2}]/2$ normalizes the state. For $z = 0$, Eq. (22) is a pure Schrödinger cat state and for $z > 0$ it is a dephased cat state that is mixed.

In Fig. 2(b), we show that entanglement is witnessed for all values of $\text{Re}[\alpha] > 0$, $\text{Im}[\alpha] = 0$, and $z < 1$ by the weak criteria Eq. (14). In principle, detecting entanglement in Eq. (22) depends on $\text{Im}[\alpha]$ too. However, one can choose arbitrary ϑ_i

such that the optimal α only depends upon its real component. The inseparability criteria in Eq. (14) are violated most in the region $0 \lesssim \text{Re}[\alpha] \lesssim 3/2$, while for larger $\text{Re}[\alpha] \gtrsim 2$, the difference between a superposition and a mixture becomes suppressed exponentially. In contrast, the entropic criteria in Refs. [24,25] certified entanglement only for $\text{Re}[\alpha] \gtrsim 5/3$ and $z < 1$ when using $\vartheta_1 = \vartheta_2 = 0$.

V. POSSIBLE EXPERIMENTAL REALIZATIONS

The protocol for applying our entropic witnesses in experiments is to measure the full Husimi Q distribution Eq. (2) and to estimate from the obtained data the entropies of the EPR-type variables Eq. (5). Techniques for measuring Husimi Q distributions are well established in quantum optics and include (i) tomographic schemes applying displacements to the prepared states before measuring its vacuum projection [64] and (ii) heterodyne measurements [48]. Recently, these schemes have been realized in other experimental systems including ultracold spinor Bose gases [52], cold atoms in cavities [53,54], and circuit QED architectures [55]. Additionally, measurements of the Q distribution via coherent displacements and measurements of the vacuum state [47] could readily be realized in trapped-ion systems [65]. The works listed here have measured (or have the potential to measure) Husimi distributions for a monopartite system (i.e., a single mode). Therefore, further work would need to be carried out to extend measurements to a bipartite system (i.e., two modes) so that the witness presented here could be applied.

Both schemes could carry practical advantages—particularly for cold atom systems—compared to the detection of marginals: Scheme (i) overcomes the problem of the high detector resolution, necessary for accurate entropy estimation, by requiring only the technically easier and more scalable task of detecting the probability of all particles being in the same state or mode (vacuum detection). Both schemes avoid determining the detection angles ϑ_i in Eq. (1), which is costly in terms of experimental runs in cold atom experiments. Additionally, tomography angles are often difficult to control precisely here. Due to these features, our entanglement criteria will potentially enable the experimental certification of entangled states beyond the reach of currently available methods.

We note that our derivation uses the Husimi Q distribution with respect to the harmonic oscillator coherent states. This accurately describes the distribution obtained from heterodyne measurements in quantum optics, however, for many of the aforementioned experimental platforms, the applicability of this description is limited due to finite particle numbers and will require a generalization to $SU(2)$ coherent states. Additionally, the extraction of entropies from experimental data is a challenging task in the presence of finite detector resolution and statistical noise [33]. The required measurement statistics for a given experimental platform, prepared state, and suitable entropy extraction scheme need to be evaluated carefully in order to make statements about the actual experimental cost and feasibility. These aspects are subject to ongoing and future research in our group.

³When the state is pure, however, the Wehrl mutual information already provides a perfect entropic witness [56].

VI. CONCLUSIONS

We have derived inseparability criteria in terms of variants of the Wehrl entropy, which can be applied when measuring the Husimi Q distribution. In contrast to most (entropic) criteria, we have shown that our criteria are invariant under rotations in phase space while depending on the local squeezing parameters. As a consequence, the criteria witnessed some entangled states that are undetectable using entropic criteria based on marginal distributions. We have discussed the implementation of our witness for a wide variety of experimental platforms and expect it to perform strongly in comparison with previous marginal criteria. Future theoretical studies should generalize the presented approach to spin operators fulfilling a $SU(2)$ algebra to formulate entropic criteria for discrete quantum spin systems.

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APPENDIX A: SYMPLECTIC TRANSFORMATIONS

To validate our second-order criteria, we consider how symplectic transformations in the \pm variables affect the Husimi Q distribution. A general symplectic transformation $S \in \text{Sp}(2, \mathbb{R})$ fulfilling $S^T \Omega S = \Omega$, with Ω being the symplectic form, can easily be applied to the original Wigner W distribution [6]. This causes the corresponding covariance matrix γ to transform as

$$\gamma \rightarrow \gamma' = S\gamma S^T. \quad (\text{A1})$$

In contrast, the distribution Q_{\pm} does not transform in a straightforward manner. We therefore restrict our analysis of the symplectic group to only Gaussian states. The

second-order criteria Eq. (18) then transform as

$$\begin{aligned} \det V_{\pm} &\rightarrow \det V'_{\pm} = \det(S\gamma_{\pm} S^T + \gamma_0) \\ &= \det[\gamma_{\pm} + \gamma_0(S^T S)^{-1}], \end{aligned} \quad (\text{A2})$$

where we used $\det S = \det S^T = 1$ and that the vacuum covariance matrix is the identity $\gamma_0 = \mathbb{1}$. This shows that invariance of $\det V_{\pm}$ is equivalent to S being an orthogonal matrix $S^T S = \mathbb{1}$ corresponding to a rotation. Therefore, the orientation of the axes is unimportant for the analysis of entanglement. This result generalizes to arbitrary marginals of Husimi Q distributions since any two-dimensional (differential) entropy is invariant under a rotation.

APPENDIX B: EXPLICIT HUSIMI Q DISTRIBUTIONS

Our first example in the main text was the set of NOON states, given in Eq. (21). The global Husimi Q distribution for $\vartheta_1 = \vartheta_2 = 0$ is

$$\begin{aligned} Q(r_1, s_1, r_2, s_2) &= \frac{e^{-\frac{1}{2}(r_1^2 + s_1^2 + r_2^2 + s_2^2)}}{2^{N+1} N! (1 + \delta_{0N})} \\ &\times [(r_1 - is_1)^N + (r_2 - is_2)^N] \\ &\times [(r_1 + is_1)^N + (r_2 + is_2)^N]. \end{aligned} \quad (\text{B1})$$

The second example that we considered was the Schrödinger cat state given in Eq. (22). Recall that for $z = 0$, Eq. (22) is a pure Schrödinger cat state and for $z > 0$ it is a dephased cat state that is mixed. The full Husimi Q distribution for $\vartheta_1 = \vartheta_2 = 0$ is

$$\begin{aligned} Q(r_1, s_1, r_2, s_2) &= N(\alpha) [e^{-\frac{1}{2}[(r-r_1)^2 + (s-s_1)^2 + (r-r_2)^2 + (s-s_2)^2]} \\ &+ e^{-\frac{1}{2}[(r+r_1)^2 + (s+s_1)^2 + (r+r_2)^2 + (s+s_2)^2]} \\ &+ 2(1-z)e^{-r^2 - s^2 - \frac{1}{2}(r_1^2 + s_1^2 + r_2^2 + s_2^2)} \\ &\times \cos[r(s_1 + s_2) - s(r_1 + r_2)]], \end{aligned} \quad (\text{B2})$$

where we use the parametrization $\alpha = (r + is)/\sqrt{2}$.

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